Hidden Utilities and $\TeX$

Headers

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Pensieve header: Quantum sl_n following Rosso and Yamane.

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Notes.
1. Rosso is “An Analogue of P.B.W. Theorem and the Universal R-matrix for $U_\hbar(sl(N+1))$'',
Yamane is “A Poincare-Birkhoff-Witt Theorem for Quantized Universal Enveloping Algebras of Type A_N”.
2. We call our algebra $U_q(\mathrm{sl}_{\$n})$, not $U_\hbar(\mathrm{sl}(N+1))$.
3. We dot not have a name for Rosso's X_i and Y_j. Rosso's $E_{\epsilon_i-\epsilon_{j+1}}$ is our $x[i, j] \leftrightarrow$ xij. Likewise Rosso's $\eta_{\epsilon_i-\epsilon_{j+1}}$ is our $y[i, j] \leftrightarrow$ yij.
4. Rosso's H_i is our a_i and Rosso's ξ_i is our b_i.

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In[∘]:=
```
SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
<< ../Profile/Profile.m
```

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```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
```

pdf

```
This is Profile.m of http://www.drorbn.net/AcademicPensieve/Projects/Profile/.
```

pdf

```
This version: April 2020. Original version: July 1994.
```

In[∘]:=
```
MakePDF[]
```

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In[∘]:=
```
δ_{ij__} := KroneckerDelta[ij];
```

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In[∘]:=
```
SetAttributes[{SS}, HoldAll];
SS[ℰ_, op_] := Collect[Normal@Series[ℰ, {ϵ, 0, $k}], ϵ, op];
SS[ℰ_] := SS[ℰ, Together];
```

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In[∘]:=
```
CCF[ℰ_] := PP_CCF@Expand[Expand[ℰ] /. e^p_ :> e^(Factor@p)];
CF[ℰ_] := PP_CF@If[$k == ∞,
    CCF[ℰ],
    CCF[Normal@Series[ℰ /. q → e^(ħ ϵ γ), {ϵ, 0, $k}]]
   ];
```

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In[∘]:=
```
x[i_, j_] := x <> i <> j; a[i_] := a <> i; A[i_] := A <> i;
ξ[i_, j_] := ξ <> i <> j; α[i_] := α <> i; 𝒜[i_] := 𝒜 <> i;
```

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In[∘]:=
```
τ[s_Symbol] := First[Characters[SymbolName[s]]];
ι[s_Symbol] := FromDigits /@ Rest[Characters[SymbolName[s]]];
τ[s_[_]] := τ[s]; ι[s_[_]] := ι[s];
```

In[∘]:=
```
{τ[x[3, 4]], ι[x[3, 4]]} // FullForm
```

Time: 0.03 seconds Out[∘]=

```
List["x", List[3, 4]]
```

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The following ordering agrees with Rosso's; right at the start of Section 2 and then continued implicitly within the formula for *R* at the start of Section 4. Simply put, each class of variables is ordered lexicographically, and the classes are ordered xAayBb.

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In[∘]:=
```
(* q=ℯ^(ℏ γ ϵ); Ai[n]=ℯ^(ℏ ϵ ai); Bi[n]=ℯ^(ℏ γ bi) *)
```

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In[∘]:=
```
$n = 5; $k = ∞; $E := {$n, $k}
```

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In[∘]:=
```
$EPBWGens := Flatten@{
   Table[As[i, 1, $n - 1], As[j, i + 1, $n],
    x[i, j]* = ξ[i, j]; ξ[i, j]* = x[i, j]; x[i, j]],
   Table[As[i, 1, $n - 1],
    A[i]* = 𝒜[i]; 𝒜[i]* = A[i];  A[i][_]],
   Table[As[i, 1, $n - 1],
    a[i]* = α[i]; α[i]* = a[i]; a[i]]
  }
(#_U = U_0@#) & /@ $EPBWGens;
$EPBWRule := Module[{k = 0}, Flatten@Table[{g → ++k, g_i_ → {i, k}}, {g, $EPBWGens}]];
x_ ≤ y_ := OrderedQ[{x, y} /. $EPBWRule]; x_ ≺ y_ := ! OrderedQ[{y, x} /. $EPBWRule];
$PBWGens := DeleteCases[$EPBWGens, _[_]]
```

In[∘]:=
```
#* & /@ $EPBWGens
```

Time: 0.00 seconds Out[∘]=

```
{ξ12, ξ13, ξ14, ξ15, ξ23, ξ24, ξ25, ξ34,
 ξ35, ξ45, A1[_]*, A2[_]*, A3[_]*, A4[_]*, α1, α2, α3, α4}
```

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In[∘]:=
```
UCF[ℰ_] := PP_UCF@Module[{F, k}, Expand@Collect[
      ℰ //. {U_i_[l___, _[0], r___] → U_i[l, r]} //. If[$k == ∞, {}, {
          U_i_[l___, AB_[n_], r___] ⧴
           Sum[As[k, 0, $k], (ℏ^k ϵ^k)/(k!) U_i[l, Sequence @@ Table[a @@ ι[AB], k], r]]
         }],
      U_[___], F
     ] /. F → CF]
```

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In[∘]:=
```
m_s__[0] = 0;
m_s__[x_Plus] := UCF[m_s /@ x];
m_i_→j_[ℰ_] := ℰ /. U_i → U_j;
m_i_,j_→k_[c_. U_i_[x___] U_j_[]] := c U_k[x];
m_i_,j_→k_[c_. U_i_[] U_j_[y___]] := c U_k[y];
m_i_,j_→k_[c_. U_i_[xx___, x_[n1_]] U_j_[x_[n2_], yy___]] :=
  UCF[c U_i[xx, x[n1 + n2]] U_j[yy]] // m_i,j→k;
m_i_,j_→k_[c_. U_i_[xx___, x_] U_j_[y_, yy___]] := If[x ≤ y,
   c U_k[xx, x, y, yy],
   ((c U_i[xx] RR_j[x, y] // UCF // m_i,j→i) U_j[yy] // UCF // m_i,j→k) // UCF
  ];
m_s__[ℰ_] := m_s[UCF[ℰ]];
```

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In[∘]:=
```
Δ_i_→j_,k_[ℰ_] := UCF@Module[{j1, k1}, ℰ /. {
      U_i[] → U_j[] U_k[],
      U_i[f_, fs___] ⧴ m_k,k1→k@ m_j,j1→j[Δ_j,k[f] Δ_i→j1,k1[U_i[fs]]]
     }];
Δ_i_→j_,k_,l__[ℰ_] := ℰ // Δ_i→j,k // Δ_k→k,l
```

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In[∘]:=
```
S_i_[ℰ_] := UCF@Module[{i1}, ℰ /. {
      U_i[f_, fs___] ⧴ m_i1,i→i[S_i[f] S_i1[U_i1[fs]]]
     }];
```

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The 𝔸-algebra

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```
In[∘]:= RRₖ_[ai_, aj_] /; τ[ai] == τ[aj] == "a" := Uₖ[aj, ai];
RRₖ_[ai_, Aj_[n_]] /;  τ[ai] == "a" ∧ τ[Aj] == "A"  := Uₖ[Aj[n], ai];
RRₖ_[Ai_[m_], Aj_[n_]] /;  τ[Ai] == τ[Aj] == "A" := Uₖ[Aj[n], Ai[m]];
RRₖ_[am_, xij_] /; τ[am] == "a" ∧ τ[xij] == "x" := Module[{i, j, m},
   {i, j} = ι@xij; {m} = ι@am;
   Uₖ[xij, am] + γ (δm,i - δm+1,i - δm,j + δm+1,j) Uₖ[xij] / 2
  ];
RRₖ_[Am_[n_], xij_] /; τ[Am] == "A" ∧ τ[xij] == "x" := Module[{i, j, m},
  {i, j} = ι@xij; {m} = ι@Am; q^(-n (δm,i-δm+1,i-δm,j+δm+1,j)) Uₖ[xij, Am[n]]
 ]
```

```
In[∘]:= RR0[a[1], x[1, 2]]
```

Time: 0.01 seconds Out[∘]=

$\gamma\, U_0[x12] + U_0[x12, a1]$

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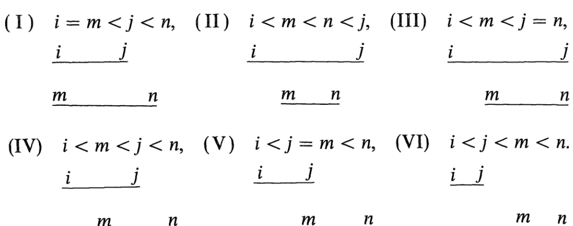
We take the reduction rules RR for xij's from Yamane page 509:



$$q^{-2}e_{ij}e_{mn} - e_{mn}e_{ij} = 0 \qquad \text{if } ((i,j),(m,n)) \in C_{(\mathrm{I})} \bigcup C_{(\mathrm{III})}.$$
$$[e_{ij}, e_{mn}] = 0 \qquad \text{if } ((i,j),(m,n)) \in C_{(\mathrm{II})} \bigcup C_{(\mathrm{VI})}.$$
$$[e_{ij}, e_{mn}] = (q^2 - q^{-2})e_{in}e_{mj} \qquad \text{if } ((i,j),(m,n)) \in C_{(\mathrm{IV})}.$$
$$q^2 e_{ij}e_{mn} - e_{mn}e_{ij} = q e_{in} \qquad \text{if } ((i,j),(m,n)) \in C_{(\mathrm{V})}.$$

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```
In[∘]:= RRₖ_[xmn_, xij_] /; τ[xij] == τ[xmn] == "x" := Module[{i, j, m, n},
  {i, j} = ι@xij; {m, n} = ι@xmn;
  Which[
   (i == m ∧ j < n) ∨ (i < m < j ∧ n == j), q^-2 Uₖ[xij, xmn], (* I∨III *)
   (i < m < j ∧ n < j) ∨ (j < m), Uₖ[xij, xmn], (* II∨VI *)
   i < m < j ∧ n > j, Uₖ[xij, xmn] + (q^-2 - q^2) Uₖ[x[i, n], x[m, j]], (* IV *)
   j == m, q^2 Uₖ[xij, xmn] - q Uₖ[x[i, n]] (* V *)
  ]
 ]
```

```
In[•]:= U1[x34] U2[x23] U3[x12] // m1,2→2
Time: 0.03 seconds Out[•]=
```

$-q\, U_2[x24]\, U_3[x12] + q^2\, U_2[x23, x34]\, U_3[x12]$

```
In[•]:= U1[x34] U2[x23] U3[x12] // m1,2→2 // m2,3→0
Time: 0.06 seconds Out[•]=
```

$q^2\, U_0[x14] - q^3\, U_0[x12, x24] - q^3\, U_0[x13, x34] + q^4\, U_0[x12, x23, x34]$

```
In[•]:= Module[{Bas = $PBWGens, P},
  DeleteCases[_ → True][Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
     P = U1[X] U2[Y] U3[Z];
     {X, Y, Z} → (P // m1,2→2 // m2,3→0) == (P // m2,3→2 // m1,2→0)
    ]]
 ]
Time: 40.53 seconds Out[•]=
 {}

In[•]:= Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]][RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ → True][Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
     P = U1[X] U2[Y] U3[Z];
     {X, Y, Z} → (P // m1,2→2 // m2,3→0) == (P // m2,3→2 // m1,2→0)
    ]]
 ]
» {x12, x13, x23, a1, a2, A1[0], A2[-2], A1[2], A2[0], A1[-1], A1[0], A2[2], A1[1], A1[1], A1[-2]}
Out[•]=
 {}

Module[{Bas = $PBWGens, P},
  DeleteCases[_ → True][Flatten@Table[As[W, Bas], As[X, Bas], As[Y, Bas], As[Z, Bas],
     P = U1[W] U2[X] U3[Y] U4[Z];
     {W, X, Y, Z} → (P // m2,3→2 // m2,4→2 // m1,2→1) ==
       (P // m3,4→3 // m2,3→2 // m1,2→1) == (P // m1,2→1 // m3,4→3 // m1,3→1) ==
       (P // m1,2→1 // m1,3→1 // m1,4→1) == (P // m2,3→2 // m1,2→1 // m1,4→1)
    ]]
 ]
» {x12, x13, x23, a1, a2}
Out[•]=
 {}
```

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The Co-Product, Co-Associativity, and the Hopf Axiom

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In[ ]:=
```
Do[As[i, 1, $n - 1],
  Δj_,k_[a[i]] = Uj[a[i]] Uk[] + Uj[] Uk[a[i]];
  Δj_,k_[A[i][n_]] = Uj[A[i][n]] Uk[A[i][n]];
  Δj_,k_[x[i, i + 1]] = Uj[x[i, i + 1]] Uk[A[i][1]] + Uj[A[i][-1]] Uk[x[i, i + 1]]
 ];
Do[As[i, 1, $n - 2], As[n, i + 2, $n],
 Δj_,k_[x[i, n]] = mk,k1→k[mj,j1→j[
    q Δj,k[x[i, n - 1]] Δj1,k1[x[n - 1, n]] - q^-1 Δj,k[x[n - 1, n]] Δj1,k1[x[i, n - 1]]
   ]]
]
```

```
In[•]:= Column@Module[{Bas = $PBWGens, P},
          Table[As[P, Ui /@ Bas], P → Δi→j,k[P]]
        ]
```

Time: 0.08 seconds Out[•]=

$U_i[x12] \to U_j[A1[-1]]\,U_k[x12] + U_j[x12]\,U_k[A1[1]]$

$U_i[x13] \to U_j[A1[-1], A2[-1]]\,U_k[x13] + \left(-\frac{1}{q^2}+q^2\right) U_j[x12, A2[-1]]\,U_k[x23, A1[1]] + U_j[x13]\,U_k[A1[1], A2[1]]$

$U_i[x14] \to U_j[A1[-1], A2[-1], A3[-1]]\,U_k[x14] + \left(-\frac{1}{q^2}+q^2\right) U_j[x12, A2[-1], A3[-1]]\,U_k[x24, A1[1]] + \left(-\frac{1}{q^2}+q^2\right) U_j[x13, A3[-1]]\,U_k[x34, A1[1], A2[1]] + U_j[x14]\,U_k[A1[1], A2[1], A3[1]]$

$U_i[x15] \to U_j[A1[-1], A2[-1], A3[-1], A4[-1]]\,U_k[x15] + \left(-\frac{1}{q^2}+q^2\right) U_j[x12, A2[-1], A3[-1], A4[-1]]\,U_k[x25, A1[1]] + \left(-\frac{1}{q^2}+q^2\right) U_j[x13, A3[-1], A4[-1]]\,U_k[x35, A1[1], A2[1]] + \left(-\frac{1}{q^2}+q^2\right) U_j[x14, A4[-1]]\,U_k[x45, A1[1], A2[1], A3[1]] + U_j[x15]\,U_k[A1[1], A2[1], A3[1], A4[1]]$

$U_i[x23] \to U_j[A2[-1]]\,U_k[x23] + U_j[x23]\,U_k[A2[1]]$

$U_i[x24] \to U_j[A2[-1], A3[-1]]\,U_k[x24] + \left(-\frac{1}{q^2}+q^2\right) U_j[x23, A3[-1]]\,U_k[x34, A2[1]] + U_j[x24]\,U_k[A2[1], A3[1]]$

$U_i[x25] \to U_j[A2[-1], A3[-1], A4[-1]]\,U_k[x25] + \left(-\frac{1}{q^2}+q^2\right) U_j[x23, A3[-1], A4[-1]]\,U_k[x35, A2[1]] + \left(-\frac{1}{q^2}+q^2\right) U_j[x24, A4[-1]]\,U_k[x45, A2[1], A3[1]] + U_j[x25]\,U_k[A2[1], A3[1], A4[1]]$

$U_i[x34] \to U_j[A3[-1]]\,U_k[x34] + U_j[x34]\,U_k[A3[1]]$

$U_i[x35] \to U_j[A3[-1], A4[-1]]\,U_k[x35] + \left(-\frac{1}{q^2}+q^2\right) U_j[x34, A4[-1]]\,U_k[x45, A3[1]] + U_j[x35]\,U_k[A3[1], A4[1]]$

$U_i[x45] \to U_j[A4[-1]]\,U_k[x45] + U_j[x45]\,U_k[A4[1]]$

$U_i[a1] \to U_j[a1]\,U_k[] + U_j[]\,U_k[a1]$

$U_i[a2] \to U_j[a2]\,U_k[] + U_j[]\,U_k[a2]$

$U_i[a3] \to U_j[a3]\,U_k[] + U_j[]\,U_k[a3]$

$U_i[a4] \to U_j[a4]\,U_k[] + U_j[]\,U_k[a4]$

In[∘]:=
```
Short[{Δ_{4,5}[a2], Δ_{7,9}[A2[77]], Δ_{1,2}[x12], Δ_{1,2}[x15]}, 8]
```

Time: 0.02 seconds Out[∘]=

$\{U_4[a2]\, U_5[\,] + U_4[\,]\, U_5[a2],\ U_7[A2[77]]\, U_9[A2[77]],$
$U_1[A1[-1]]\, U_2[x12] + U_1[x12]\, U_2[A1[1]],\ U_1[A1[-1], A2[-1], A3[-1], A4[-1]]\, U_2[x15] +$
$\left(-\frac{1}{q^2}+q^2\right) U_1[x12, A2[-1], A3[-1], A4[-1]]\, U_2[x25, A1[1]] +$
$\left(-\frac{1}{q^2}+q^2\right) U_1[x13, A3[-1], A4[-1]]\, U_2[x35, A1[1], A2[1]] +$
$\left(-\frac{1}{q^2}+q^2\right) U_1[x14, A4[-1]]\, U_2[x45, A1[1], A2[1], A3[1]] +$
$U_1[x15]\, U_2[A1[1], A2[1], A3[1], A4[1]]\}$

In[∘]:=
```
Δ_{1→1,2}[U_1[x13, a2] U_0[x34]]
```

Time: 0.03 seconds Out[∘]=

$U_0[x34]\, U_1[A1[-1], A2[-1], a2]\, U_2[x13] + U_0[x34]\, U_1[A1[-1], A2[-1]]\, U_2[x13, a2] +$
$\left(-\frac{1}{q^2}+q^2\right) U_0[x34]\, U_1[x12, A2[-1], a2]\, U_2[x23, A1[1]] + U_0[x34]\, U_1[x13, a2]\, U_2[A1[1], A2[1]] +$
$\left(-\frac{1}{q^2}+q^2\right) U_0[x34]\, U_1[x12, A2[-1]]\, U_2[x23, A1[1], a2] + U_0[x34]\, U_1[x13]\, U_2[A1[1], A2[1], a2]$

In[∘]:=
```
Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]][RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ → True][Flatten@Table[As[X, Bas],
     P = U_1[X];
     X → (P // Δ_{1→1,2} // Δ_{2→2,3}) == (P // Δ_{1→1,3} // Δ_{1→1,2})
    ]]
 ]
```

» {x12, x13, x14, x15, x23, x24, x25, x34, x35, x45, a1, a2, a3, a4,
A1[1], A4[-1], A2[1], A1[-1], A1[2], A2[-2], A2[0], A4[2], A2[-1], A1[-2]}

Out[∘]=

{}

In[∘]:=
```
Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]][RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ → True][Flatten@Table[As[X, Bas], As[Y, Bas],
     P = U_1[X] U_2[Y];
     {X, Y} → (P // m_{1,2→1} // Δ_{1→1,2}) == (P // Δ_{2→3,4} // Δ_{1→1,2} // m_{1,3→1} // m_{2,4→2})
    ]]
 ]
```

» {x12, x13, x14, x15, x23, x24, x25, x34, x35, x45, a1, a2, a3, a4,
A4[2], A2[0], A4[-1], A2[-2], A4[2], A1[0], A2[-1], A2[-2], A1[0], A4[1]}

Out[∘]=

{}

Is Δ a quantization of the co-bracket?

In[∘]:=
```
Column@Module[{Bas = $PBWGens, P},
  Table[As[P, U1 /@ Bas],
   P → UCF[ (Δ1→1,2[P] - Δ1→2,1[P]) //.
       {q^n_· :→ 1 + n ℏ / 2, Ui_[xs___, A_[n_], as___] :→ Ui[xs] + n ℏ Ui[xs, a @@ ι[A], as] / 2}
     ] /. ℏ^n_ /; n > 1 :→ 0
  ]
 ]
```

Time: 0.08 seconds Out[∘]=

$U_1[x12] \to \hbar U_1[x12] U_2[a1] - \hbar U_1[a1] U_2[x12]$

$U_1[x13] \to \hbar U_1[x13] U_2[a1] - 2\hbar U_1[x23] U_2[x12] - \hbar U_1[a1] U_2[x13] + 2\hbar U_1[x12] U_2[x23]$

$U_1[x14] \to \hbar U_1[x14] U_2[a1] - 2\hbar U_1[x24] U_2[x12] - 2\hbar U_1[x34] U_2[x13] - \hbar U_1[a1] U_2[x14] + 2\hbar U_1[x12] U_2[x24] + 2\hbar U_1[x13] U_2[x34]$

$U_1[x15] \to \hbar U_1[x15] U_2[a1] - 2\hbar U_1[x25] U_2[x12] - 2\hbar U_1[x35] U_2[x13] - 2\hbar U_1[x45] U_2[x14] - \hbar U_1[a1] U_2[x15] + 2\hbar U_1[x12] U_2[x25] + 2\hbar U_1[x13] U_2[x35] + 2\hbar U_1[x14] U_2[x45]$

$U_1[x23] \to \hbar U_1[x23] U_2[a2] - \hbar U_1[a2] U_2[x23]$

$U_1[x24] \to \hbar U_1[x24] U_2[a2] - 2\hbar U_1[x34] U_2[x23] - \hbar U_1[a2] U_2[x24] + 2\hbar U_1[x23] U_2[x34]$

$U_1[x25] \to \hbar U_1[x25] U_2[a2] - 2\hbar U_1[x35] U_2[x23] - 2\hbar U_1[x45] U_2[x24] - \hbar U_1[a2] U_2[x25] + 2\hbar U_1[x23] U_2[x35] + 2\hbar U_1[x24] U_2[x45]$

$U_1[x34] \to \hbar U_1[x34] U_2[a3] - \hbar U_1[a3] U_2[x34]$

$U_1[x35] \to \hbar U_1[x35] U_2[a3] - 2\hbar U_1[x45] U_2[x34] - \hbar U_1[a3] U_2[x35] + 2\hbar U_1[x34] U_2[x45]$

$U_1[x45] \to \hbar U_1[x45] U_2[a4] - \hbar U_1[a4] U_2[x45]$

$U_1[a1] \to 0$

$U_1[a2] \to 0$

$U_1[a3] \to 0$

$U_1[a4] \to 0$

pdf

The Antipode

pdf

In[∘]:=
```
Do[As[i, 1, $n - 1],
  Sj_[a[i]] = -Uj[a[i]];
  Sj_[A[i][n_]] = Uj[A[i][-n]];
  Sj_[x[i, i + 1]] = -q^-2 Uj[x[i, i + 1]] (*?*)
 ];
Do[As[i, 1, $n - 2], As[n, i + 2, $n],
 Sj_[x[i, n]] = mj1,j→j[
   q Sj[x[i, n - 1]] Sj1[x[n - 1, n]] - q^-1 Sj[x[n - 1, n]] Sj1[x[i, n - 1]]
  ]
]
```

In[∘]:=
```
Module[{Bas = $PBWGens, P},
 Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]][RandomInteger[{-2, 2}]], {10}]];
 DeleteCases[_ → True][Flatten@Table[As[X, Bas],
   P = U1[X];
   X → (P // Δ1→2,3 // S2 // m2,3→1)
  ]]
]
```

» {x12, x13, x14, x15, x23, x24, x25, x34, x35, x45, a1, a2, a3, a4, A3[-1], A2[0], A4[2], A4[-2], A4[2], A4[-2], A2[-1], A2[0], A2[2], A3[1]}

Out[∘]=

{x12 → 0, x13 → 0, x14 → 0, x15 → 0, x23 → 0, x24 → 0, x25 → 0, x34 → 0, x35 → 0, x45 → 0, a1 → 0, a2 → 0, a3 → 0, a4 → 0, A3[-1] → U_1[], A2[0] → U_1[], A4[2] → U_1[], A4[-2] → U_1[], A4[2] → U_1[], A4[-2] → U_1[], A2[-1] → U_1[], A2[0] → U_1[], A2[2] → U_1[], A3[1] → U_1[]}

pdf

The Generating Functions for the 𝔸 operations

In[∘]:=
```
BeginProfile[]
```

Time: 0.11 seconds Out[∘]=

ProfileRoot

In[∘]:=
```
$k = 1;
```

pdf

NonCommutativeMultiply, B, adPower, Ad, TT, Seepage

pdf

In[∘]:=
```
Supp[ℰ_] := Union@Cases[{ℰ}, U_i_[___] :→ i, ∞];
```

In[∘]:=
```
Supp[2 ℏ U1[x45] U2[x34]]
```

Time: 0.01 seconds Out[∘]=

{1, 2}

pdf

In[∘]:=
```
Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ⊠ y_ := Module[{i, is = Supp[x] ⋂ Supp[y], σ, z},
   z = x; Do[z = m_{i→σ@i}[z], {i, is}];
   z = Expand[y z]; Do[z = m_{σ@i,i→i}[z], {i, is}]; z];
UB[x_, y_] := PP_UB@UCF[x ⊠ y - y ⊠ x];
```

In[∘]:=
```
(2 ℏ1 U1[x45] U2[x34]) ⊠ (2 ℏ2 U1[x45] U2[x34])
```

Time: 0.03 seconds Out[∘]=

4 $\hbar_1$ $\hbar_2$ U_1[x45, x45] U_2[x34, x34]

pdf

In[∘]:=
```
(*adPower[x_,k_,Env_][c_*y_QU]:=UCF[c adPower[x,k,Env][y]];
adPower[x_,k_,Env_][y_Plus] := adPower[x,k,Env]/@ y;*)
adPower[_, 0, Env_][y_] := y;
(*adPower[x_,k_,Env_][0]=0;*)
adPower[x_, k_, Env_][y_] :=
  adPower[x, k, Env][y] = UCF@UB[adPower[x, k - 1, Env][y], x];
adPower[x_, k_][y_] := adPower[x, k, $E][y];
```

In[∘]:=
```
Table[As[k, 0, 5], adPower[U₀[x12], k][U₀[x23]]]
```

Time: 0.06 seconds Out[∘]=

$\{U_0[x23],\ (-1-\gamma\in\hbar)\ U_0[x13]+2\,\gamma\in\hbar\ U_0[x12,\ x23],\ 0,\ 0,\ 0,\ 0\}$

In[∘]:=
```
Column@Module[{Bas = $PBWGens, P},
  Table[As[P, Uᵢ /@ Bas], P → Δ_{i→j,k}[P]]
 ]
```

Time: 0.08 seconds Out[∘]=

$U_i[x12] \to U_j[x12]\ U_k[\,] + \in\hbar\ U_j[x12]\ U_k[a1] + U_j[\,]\ U_k[x12] + \in\hbar\ U_j[a1]\ U_k[x12]$

$U_i[x13] \to U_j[x13]\ U_k[\,] + \in\hbar\ U_j[x13]\ U_k[a1] + \in\hbar\ U_j[x13]\ U_k[a2] +$
$\quad U_j[\,]\ U_k[x13] + \in\hbar\ U_j[a1]\ U_k[x13] + \in\hbar\ U_j[a2]\ U_k[x13] + 4\,\gamma\in\hbar\ U_j[x12]\ U_k[x23]$

$U_i[x23] \to U_j[x23]\ U_k[\,] + \in\hbar\ U_j[x23]\ U_k[a2] + U_j[\,]\ U_k[x23] + \in\hbar\ U_j[a2]\ U_k[x23]$

$U_i[a1] \to U_j[a1]\ U_k[\,] + U_j[\,]\ U_k[a1]$

$U_i[a2] \to U_j[a2]\ U_k[\,] + U_j[\,]\ U_k[a2]$

pdf

In[∘]:=
```
Ad[0] = Identity;
```

pdf

In[∘]:=
```
Ad[c_. U_i_[xjk_] (EUs : U_[] ...)][y_] /; τ[xjk] == "x" := PP_Ad@Module[{k = 0, s = 0, t},
   While[(t = adPower[U_i[xjk], k][y]) =!= 0,
    s += c^k t / k!;
    ++k
   ];
   UCF[s ⊠ EUs]
  ]
```

In[∘]:=
```
Ad[∈ ħ U_j[x12] U_k[]][U_j[x23] U_k[a4]]
```

Time: 0.09 seconds Out[∘]=

$-\in\hbar\ U_j[x13]\ U_k[a4] + U_j[x23]\ U_k[a4]$

In[∘]:=
```
Ad[7 ħ U_j[x12] U_k[]][U_j[x23] U_k[a4]]
```

Time: 0.08 seconds Out[∘]=

$\left(-7\,\hbar - 7\,\gamma\in\hbar^2\right)\ U_j[x13]\ U_k[a4] + U_j[x23]\ U_k[a4] + 14\,\gamma\in\hbar^2\ U_j[x12,\ x23]\ U_k[a4]$

pdf

In[∘]:=
```
Wt_aj_[x_] := Wt_aj[x] = UB[U₀[aj], U₀[x]] / U₀[x]
```

In[∘]:=
```
Table[Wt_a[j][x23], {j, 1, $n}]
```
Time: 0.08 seconds Out[∘]=

$$\left\{-\frac{\gamma}{2}, \gamma, -\frac{\gamma}{2}, 0, 0\right\}$$

pdf

In[∘]:=
```
Ad[c_. U_i_[aj_] (EUs : U_[] ...)][y_] /; τ[aj] == "a" :=
  (y /. U_i[gs___] :→ U_i[gs] ⅇ^(c Total[Wt_aj /@ {gs}]))
```

In[∘]:=
```
Ad[π ⅈ U_j[a1]][U_j[x23] U_k[a4]]
```
Time: 0.03 seconds Out[∘]=

$$e^{-\frac{1}{2} i \pi \gamma} U_j[x23]\, U_k[a4]$$

pdf

In[∘]:=
```
TT_λ_[] = 0; (* Translated Tangent *)
TT_λ_[xs___, x_] := PP_TT@UCF[Ad[x][TT_λ[xs]] + ∂_λ x];
TT_λ_[xs_List] := TT_λ @@ xs
```

In[∘]:=
```
TT_λ[λ ξ12 U_0[x12] U_1[], λ ξ23 U_0[x23] U_1[]]
```
Time: 0.08 seconds Out[∘]=

$$\xi 12\, U_0[x12]\, U_1[] + \lambda\, \xi 12\, \xi 23\, U_0[x13]\, U_1[] + \xi 23\, U_0[x23]\, U_1[]$$

In[∘]:=
```
$PBWGens
```
Time: 0.00 seconds Out[∘]=

{x12, x13, x23, a1, a2}

In[∘]:=
```
Join[
 Cases[$PBWGens, g_ :→ Which[
    τ[g] === "x", λ (g*)_i U_k[g],
    τ[g] === "a", (g*)_i U_k[g]
   ]],
 Cases[$PBWGens, g_ :→ Which[
    τ[g] === "x", λ (g*)_j U_k[g],
    τ[g] === "a", (g*)_j U_k[g]
   ]]
]
```
Time: 0.03 seconds Out[∘]=

$$\{\lambda\, \xi 12_i\, U_k[x12], \lambda\, \xi 13_i\, U_k[x13], \lambda\, \xi 23_i\, U_k[x23], \alpha 1_i\, U_k[a1], \alpha 2_i\, U_k[a2], \lambda\, \xi 12_j\, U_k[x12], \lambda\, \xi 13_j\, U_k[x13], \lambda\, \xi 23_j\, U_k[x23], \alpha 1_j\, U_k[a1], \alpha 2_j\, U_k[a2]\}$$

In[∘]:=
```
𝕆_i_[poly_] := Total[
  CoefficientRules[poly, $PBWGens] /.
   (p_ → c_) :→ c U_i @@ Flatten@MapThread[ConstantArray, {$PBWGens, p}]
 ]
```

In[∘]:=
```
𝕆i[x12^2 x13 + 7 x23^17]
```

Time: 0.03 seconds Out[∘]=

```
Ui[x12, x12, x13] +
 7 Ui[x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23]
```

pdf

In[∘]:=
```
Seepage_i_[{}, c_] = c;
Seepage_i_[ℰs_List, P_] := Module[{lx},
  lx = $PBWGens⟦Length@ℰs⟧;
  CF@Collect[P, lx, (Ad[Last[ℰs] Ui[lx]][𝕆i[Seepage_i[Most@ℰs, #]]] /. Ui → Times) &]
 ]
```

pdf

In[∘]:=
```
Invert_i_[ℰ_] := Module[{A, B, β, j}, (* ℰ=A U[]+∊ B *)
  A = Coefficient[ℰ, Ui[]] /. ϵ → 0;
  B = UCF[(ℰ - A Ui[])];
  UCF[
   A^-1 (Ui[] + Sum[As[j, 1, $k], (-1)^j NonCommutativeMultiply @@ ConstantArray[B / A, j]])]
 ]
```

```
{$k = 1, $n = 3}
```

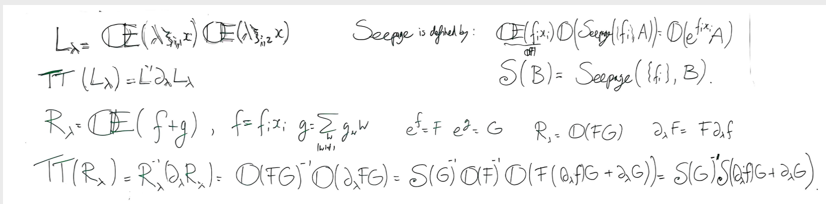
Time: 0.01 seconds Out[∘]=

```
{2, 3}
```

pdf

In[∘]:=
```
sol[kk_, nn_] := PP"sol"[kk,nn]@Block[{$k = kk, $n = nn}, Module[{(*TTL, *)},
    ES[s_][ℰ_] := (EchoLabel[s][Short[ℰ, 5]]; ℰ);
    TTL = ES["TTL"]@TTλ@
      Join[
       Cases[$PBWGens, g_ :→ Which[
          τ[g] === "x", λ (g*)i Uk[g],
          τ[g] === "a", (g*)i Uk[g]
         ]],
       Cases[$PBWGens, g_ :→ Which[
          τ[g] === "x", λ (g*)j Uk[g],
          τ[g] === "a", (g*)j Uk[g]
         ]]
      ];
    mons[-1] = {};
    mons[p_] := mons[p - 1] ⋃ Union@Cases[Coefficient[TTL, ϵ, p], Uk[___], ∞];
    ff = Table[
       Sum[ϵ^p fp,u[λ], {p, 0, $k}],
       {u, $PBWGens}
      ] /. f_p_,u_[λ] /; τ[u] == "a" :→ If[p == 0, (u*)i + (u*)j, 0];
    fs = Flatten[Table[
```

```
      f_{k,Times@@m},
      {k, 0, $k}, {m, mons[k]}
     ]];
   gg = Sum[
      ϵ^p f_{p,Times@@u}[λ] Times @@ u,
      {p, 0, $k}, {u, Complement[mons[p], U_k /@ $PBWGens]}
    ];
   G = Series[ⅇ^gg, {ϵ, 0, $k}];
```




```
   ;
   S_i_[B_] := UCF[SS@O_i[Seepage_i[ff, SS@UCF@B]]];
   SG = ES["SG"]@S_k[G];
   TTR = ES["TTR"]@UCF[Invert_k[SG] ⊠ S_k[(∂_λ (ff.$PBWGens) G + ∂_λ G)]];
   diff = ES["diff"]@UCF[TTL - TTR];
   eqns = ES["eqns"]@Flatten[Table[
       Coefficient[Coefficient[diff, m], ϵ, k] == 0,
       {k, 0, $k}, {m, mons[k]}
      ]];
   inits = (#[0] == 0) & /@ fs;
   ES["DSolve"][
    DSolve[eqns ⋃ inits, fs, λ] /. F_Function :> ReplacePart[F, 2 → CF[F[[2]]]]]
  ]]
```

In[•]:= sol[2, 2]

» TTL $\left(e^{\gamma(\alpha 1_i+\alpha 1_j)}\,\xi 12_i + e^{\gamma\,\alpha 1_j}\,\xi 12_j\right) U_k[x12]$

» $\mathcal{S}G$ $U_k[\,]$

» TTR $U_k[x12]\left(e^{\gamma(\alpha 1_i+\alpha 1_j)} f_{0,x12}'[\lambda] + e^{\gamma(\alpha 1_i+\alpha 1_j)}\,\epsilon\, f_{1,x12}'[\lambda] + e^{\gamma(\alpha 1_i+\alpha 1_j)}\,\epsilon^2 f_{2,x12}'[\lambda]\right)$

» diff
$U_k[x12]\left(e^{\gamma(\alpha 1_i+\alpha 1_j)}\,\xi 12_i + e^{\gamma\,\alpha 1_j}\,\xi 12_j - e^{\gamma(\alpha 1_i+\alpha 1_j)} f_{0,x12}'[\lambda] - e^{\gamma(\alpha 1_i+\alpha 1_j)}\,\epsilon\, f_{1,x12}'[\lambda] - e^{\gamma(\alpha 1_i+\alpha 1_j)}\,\epsilon^2 f_{2,x12}'[\lambda]\right)$

» eqns $\{e^{\gamma(\alpha 1_i+\alpha 1_j)}\,\xi 12_i + e^{\gamma\,\alpha 1_j}\,\xi 12_j - e^{\gamma(\alpha 1_i+\alpha 1_j)} f_{0,x12}'[\lambda] == 0,$
$-e^{\gamma(\alpha 1_i+\alpha 1_j)} f_{1,x12}'[\lambda] == 0,\ -e^{\gamma(\alpha 1_i+\alpha 1_j)} f_{2,x12}'[\lambda] == 0\}$

» DSolve
$\{\{f_{0,x12} \to \mathrm{Function}[\{\lambda\}, \lambda\,\xi 12_i + e^{-\gamma\,\alpha 1_i}\,\lambda\,\xi 12_j],\ f_{1,x12} \to \mathrm{Function}[\{\lambda\}, 0],\ f_{2,x12} \to \mathrm{Function}[\{\lambda\}, 0]\}\}$

Out[•]=
$\{\{f_{0,x12} \to \mathrm{Function}[\{\lambda\}, \lambda\,\xi 12_i + e^{-\gamma\,\alpha 1_i}\,\lambda\,\xi 12_j],$
$f_{1,x12} \to \mathrm{Function}[\{\lambda\}, 0],\ f_{2,x12} \to \mathrm{Function}[\{\lambda\}, 0]\}\}$

In[•]:= sol[1, 3]

» TTL $\left(e^{\frac{1}{2}\gamma(2\alpha1_i+2\alpha1_j-\alpha2_i-\alpha2_j)}\,\xi12_i+e^{\frac{1}{2}\gamma(2\alpha1_j-\alpha2_j)}\,\xi12_j\right)U_k[x12]+$

$\ll 5\gg+\Big(2\,e^{\frac{3}{2}\gamma(\alpha2_i+\alpha2_j)}\,\gamma\,\epsilon\,\lambda\,\hbar\,\xi13_i\,\xi23_i-2\,e^{-\frac{1}{2}\gamma(\alpha1_i-2\alpha2_i-3\alpha2_j)}\,\gamma\,\epsilon\,\lambda\,\hbar\,\xi13_j\,\xi23_i+$

$2\,e^{\frac{1}{2}\gamma(\alpha1_i+\alpha2_i+3\alpha2_j)}\,\gamma\,\epsilon\,\lambda\,\hbar\,\xi13_i\,\xi23_j+2\,e^{\frac{3\gamma\alpha2_j}{2}}\,\gamma\,\epsilon\,\lambda\,\hbar\,\xi13_j\,\xi23_j\Big)U_k[x13,\ x23]$

» *SG* $U_k[\,]+e^{\frac{3}{2}\gamma(\alpha1_i+\alpha1_j)}\,\epsilon\,U_k[x12,\ x13]\,f_{1,x12\,x13}[\lambda]+$

$U_k[x13,\ x13]\left(e^{\gamma(\alpha1_i+\alpha1_j+\alpha2_i+\alpha2_j)}\,\epsilon\,f_{0,x23}[\lambda]\,f_{1,x12\,x13}[\lambda]+e^{\gamma(\alpha1_i+\alpha1_j+\ll 1\gg+\alpha2_j)}\,\epsilon\,f_{1,\ll 3\gg^2}[\lambda]\right)+$

$e^{\frac{1}{2}\gamma(\alpha1_i+\alpha1_j+\alpha2_i+\alpha2_j)}\,\epsilon\,U_k[x12,\ x23]\,f_{1,x12\,x23}[\lambda]+$

$U_k[x13,\ x23]\left(e^{\frac{3}{2}\gamma(\alpha2_i+\alpha2_j)}\,\epsilon\,f_{0,x23}[\lambda]\,f_{1,x12\,x23}[\lambda]+e^{\frac{3}{2}\gamma(\alpha2_i+\alpha2_j)}\,\epsilon\,f_{1,x13\,x23}[\lambda]\right)$

» TTR $U_k[x12]\left(e^{\frac{1}{2}\gamma(2\alpha1_i+2\alpha1_j-\alpha2_i-\alpha2_j)}\,f_{0,x12}'[\lambda]+e^{\frac{1}{2}\gamma(2\alpha1_i+2\alpha1_j-\alpha2_i-\alpha2_j)}\,\epsilon\,f_{1,x12}'[\lambda]\right)+\ll 5\gg+U_k[x13,\ x23]$

$\left(2\,e^{\frac{3}{2}\gamma(\alpha2_i+\alpha2_j)}\,\gamma\,\epsilon\,\hbar\,f_{0,x23}[\lambda]\,f_{0,x13}'[\lambda]+e^{\frac{3}{2}\gamma(\alpha2_i+\alpha2_j)}\,\epsilon\,f_{0,x23}[\lambda]\,f_{1,x12\,x23}'[\lambda]+e^{\frac{3}{2}\gamma(\alpha2_i+\alpha2_j)}\,\epsilon\,f_{1,x13\,x23}'[\lambda]\right)$

» diff $U_k[x12]\Big(e^{\frac{1}{2}\gamma(2\alpha1_i+2\alpha1_j-\alpha2_i-\alpha2_j)}\,\xi12_i+e^{\frac{1}{2}\gamma(2\alpha1_j-\alpha2_j)}\,\xi12_j-$

$e^{\frac{1}{2}\gamma(2\alpha1_i+2\alpha1_j-\alpha2_i-\alpha2_j)}\,f_{0,x12}'[\lambda]-e^{\frac{1}{2}\gamma(2\alpha1_i+2\alpha1_j-\alpha2_i-\alpha2_j)}\,\epsilon\,f_{1,x12}'[\lambda]\Big)+$

$U_k[x13]\left(e^{\ll 1\gg}\,\xi13_i+e^{\ll 1\gg}\ll 1\gg+\ll 22\gg\right)+\ll 3\gg+U_k[x12,\ x23]\,(\ll 1\gg+\ll 8\gg)+$

$U_k[x13,\ x23]\left(2\,e^{\frac{3}{2}\gamma(\alpha2_i+\alpha2_j)}\,\gamma\,\epsilon\,\lambda\,\hbar\,\xi13_i\,\xi23_i-2\,e^{-\frac{1}{2}\gamma(\alpha1_i-2\ll 2\gg_i-3\alpha2_j)}\,\gamma\,\epsilon\,\lambda\,\hbar\,\xi13_j\,\xi23_i+\ll 8\gg\right)$

» eqns

$\Big\{e^{\frac{1}{2}\gamma(2\alpha1_i+2\alpha1_j-\alpha2_i-\alpha2_j)}\,\xi12_i+e^{\frac{1}{2}\gamma(2\alpha1_j-\alpha2_j)}\,\xi12_j-e^{\frac{1}{2}\gamma(2\alpha1_i+2\alpha1_j-\alpha2_i-\alpha2_j)}\,f_{0,x12}'[\lambda]==0,\ e^{\frac{1}{2}\gamma(\alpha1_i+\alpha1_j+\alpha2_i+\alpha2_j)}\,\xi13_i+$

$e^{\frac{1}{2}\gamma(\alpha1_j+\alpha2_j)}\,\xi13_j+e^{\frac{1}{2}\gamma(\alpha1_i+\alpha1_j+\alpha2_i+\alpha2_j)}\,\lambda\,\xi12_i\,\xi23_i-\ll 1\gg+\ll 1\gg+e^{\frac{1}{2}\gamma(\ll 1\gg)}\,\lambda\,\xi12_j\,\xi23_j-$

$e^{\frac{1}{2}\gamma(\alpha1_i+\alpha1_j+\alpha2_i+\alpha2_j)}\,f_{0,x23}[\lambda]\,f_{0,x12}'[\lambda]-e^{\frac{1}{2}\gamma(\alpha1_i+\alpha1_j+\alpha2_i+\alpha2_j)}\,f_{0,x13}'[\lambda]==0,\ \ll 6\gg,\ \ll 1\gg,\ \ll 1\gg\Big\}$

» DSolve $\Big\{\Big\{f_{0,x12}\to\text{Function}\Big[\{\lambda\},\ \lambda\,\xi12_i+e^{-\frac{1}{2}\gamma(2\alpha1_i-\alpha2_i)}\,\lambda\,\xi12_j\Big],$

$f_{0,x23}\to\text{Function}\Big[\{\lambda\},\ \lambda\,\xi23_i+e^{\frac{1}{2}\gamma(\alpha1_i-2\alpha2_i)}\,\lambda\,\xi23_j\Big],$

$f_{0,x13}\to\text{Function}\Big[\{\lambda\},\ \lambda\,\xi13_i+e^{-\frac{1}{2}\gamma(\alpha1_i+\alpha2_i)}\,\lambda\,\xi13_j-e^{-\frac{1}{2}\gamma(2\alpha1_i-\alpha2_i)}\,\lambda^2\,\xi12_j\,\xi23_i\Big],$

$\ll 5\gg,\ f_{1,x13^2}\to\text{Function}[\{\lambda\},\ 0],\ f_{1,x12}\to\text{Function}[\{\lambda\},\ 0]\Big\}\Big\}$

Out[#]=

$\Big\{\Big\{f_{0,x12}\to\text{Function}\Big[\{\lambda\},\ \lambda\,\xi12_i+e^{-\frac{1}{2}\gamma(2\alpha1_i-\alpha2_i)}\,\lambda\,\xi12_j\Big],$

$f_{0,x23}\to\text{Function}\Big[\{\lambda\},\ \lambda\,\xi23_i+e^{\frac{1}{2}\gamma(\alpha1_i-2\alpha2_i)}\,\lambda\,\xi23_j\Big],$

$f_{0,x13}\to\text{Function}\Big[\{\lambda\},\ \lambda\,\xi13_i+e^{-\frac{1}{2}\gamma(\alpha1_i+\alpha2_i)}\,\lambda\,\xi13_j-e^{-\frac{1}{2}\gamma(2\alpha1_i-\alpha2_i)}\,\lambda^2\,\xi12_j\,\xi23_i\Big],$

$f_{1,x23}\to\text{Function}[\{\lambda\},\ 0],\ f_{1,x13}\to\text{Function}\Big[\{\lambda\},\ -e^{-\frac{1}{2}\gamma(2\alpha1_i-\alpha2_i)}\,\gamma\,\lambda^2\,\hbar\,\xi12_j\,\xi23_i\Big],$

$f_{1,x12\,x23}\to\text{Function}\Big[\{\lambda\},\ 2\,e^{-\frac{1}{2}\gamma(2\alpha1_i-\alpha2_i)}\,\gamma\,\lambda^2\,\hbar\,\xi12_j\,\xi23_i\Big],$

$f_{1,x12\,x13}\to\text{Function}\Big[\{\lambda\},\ -2\,e^{-\frac{1}{2}\gamma(2\alpha1_i-\alpha2_i)}\,\gamma\,\lambda^2\,\hbar\,\xi12_j\,\xi13_i\Big],$

$f_{1,x13\,x23}\to\text{Function}\Big[\{\lambda\},\ -2\,e^{-\frac{1}{2}\gamma(\alpha1_i+\alpha2_i)}\,\gamma\,\lambda^2\,\hbar\,\xi13_j\,\xi23_i\Big],$

$f_{1,x13^2}\to\text{Function}[\{\lambda\},\ 0],\ f_{1,x12}\to\text{Function}[\{\lambda\},\ 0]\Big\}\Big\}$

In[ ]:=
```
sol[1, 4]
```

Time: 560.02 seconds Out[ ]=

```
$Aborted
```