Hidden TEX

Headers

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Pensieve header: The PBW multiplication tensor for $gl_{n,\epsilon}$ using the nilpotent-only λ-tangent formalism. Some material from pensive://Projects/glneps and from pensieve://Projects/UEA.

pdf

In[•]:=
```
SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
```

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```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
```

nb2tex

In[•]:=
```
Get["../nb2tex/nb2tex.m"];
nb2tex["gln", PDFWidth → 7.5, PDFFolder → "Snips"];
texfile = OpenAppend["gln.tex"]; WriteLine[texfile, "\\end{document}"]; Close[texfile];
Run@
  "\"C:\\Users\\drorb\\AppData\\Local\\Programs\\MiKTeX\\miktex\\bin\\x64\\pdflatex.exe\"
    gln.tex";
```

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Conventions

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$x_{\alpha,\beta}$ is always with $\alpha \le \beta$ and represents a basis element of the upper triangular matrices. $y_{\alpha,\beta}$ is also always with $\alpha \le \beta$ and represents a basis element of the lower triangular matrices. The adjoint of $x_{\alpha,\beta}$ is $y_{\alpha,\beta}$.

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Variable declarations:

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In[•]:=
```
VarPattern = ((y | x | η | ξ) | (y | x | η | ξ)__)[_];
```

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Greek - Latin duality:

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In[•]:=
```
{y*, x*, η*, ξ*} = {η, ξ, y, x};
(u_αβ__)* := (u*)αβ;
((u : ((y | x | η | ξ) | (y | x | η | ξ)__))[i_])* := u*[i];
(vs_List)* := (v ↦ v*) /@ vs;
```

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In[•]:=
```
{x_{1,2}, y_{2,3}, η_{1,2}[7]}^*
```

Out[•]=

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$$\{\xi_{1,2},\ \eta_{2,3},\ y_{1,2}[7]\}$$

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Weights:

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In[•]:=
```
Wt[x_{α_,β_}] := {1, α - β}; Wt[y_{α_,β_}] := {0, α - β};
Wt[ξ_{α_,β_}] := {0, β - α}; Wt[η_{α_,β_}] := {1, β - α};
Wt[u_[_]] := Wt[u];
```

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The Engine

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Canonical Forms:

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In[•]:=
```
LogReduce[ℰ_] :=
  ℰ /. c_ * Log[a_] :> Log@Factor[a^c] //. Log[a_] + Log[b_] :> Log@Factor[a b];
CF[ℰ_] := Expand@Collect[ℰ, Cases[ℰ, VarPattern, ∞], CCF] /. CCF → Factor;
CF[ℰ_𝔼] := CF /@ MapAt[LogReduce, ℰ, 1];
CF[ℰ : 𝔼__[___]] := CF /@ ℰ;
```

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𝔼 operations:

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In[•]:=
```
ℰ_𝔼[$] := Length[ℰ] - 1; 𝔼_[ℰs___][$] := 𝔼[ℰs][$];
ℰ_𝔼[k_Integer] := ℰ[[k + 1]]; 𝔼_[ℰs___][k_Integer] := {ℰs}[[k + 1]];
𝔼 /: ℰ1_𝔼 ≡ ℰ2_𝔼 := Inner[CF@#1 == CF@#2 &, ℰ1, ℰ2, And];
𝔼_{d1_→r1_}[ℰ1s___] ≡ 𝔼_{d2_→r2_}[ℰ2s___] ^:= (d1 == d2) ∧ (r1 == r2) ∧ (𝔼[ℰ1s] ≡ 𝔼[ℰ2s]);
𝔼 /: ℰ1_𝔼 * ℰ2_𝔼 := 𝔼 @@ Table[CF[ℰ1[kk] + ℰ2[kk]], {kk, 0, Min[ℰ1[$], ℰ2[$]]}];
𝔼_{d1_→r1_}[ℰ1s___] 𝔼_{d2_→r2_}[ℰ2s___] ^:= 𝔼_{(d1⋃d2)→(r1⋃r2)} @@ (𝔼[ℰ1s] 𝔼[ℰ2s]);
```

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In[•]:=
```
𝔼_{d1_→r1_}[ℰ1s___] // 𝔼_{d2_→r2_}[ℰ2s___] := Module[{is = r1 ⋂ d2, lvs, llvs},
  lvs = Cases[{ℰ1s}, (x | y)__[i_] /; MemberQ[is, i], ∞] ⋃
    Cases[{ℰ2s}, (ξ | η)__[i_] /; MemberQ[is, i], ∞]^*;
  llvs = Map[$, lvs, {2}];
  𝔼_{(d1⋃Complement[d2,is])→(r2⋃Complement[r1,is])} @@ (Zip_{llvs⋃llvs^*}[{(llvs)^*.(llvs), Times[
       𝔼[ℰ1s] /. Thread[lvs → llvs],
       𝔼[ℰ2s] /. Thread[lvs^* → llvs^*]
      ]}])
 ]
```

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Zipping! Lemmas 2 and 3 are combined, yet they must be applied first to the middle weight variables and then to the heavy and light variables.

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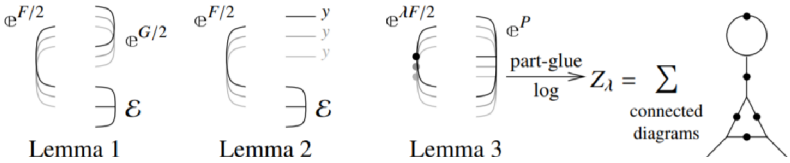

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Comment. Zip3 of the outer variables must occur after all other operations are completed, because we must allow for gluings of the weight n variables in perturbations with the weight 0 variables in the coefficients of Q.

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dvs stands for "Diagonal Variables". In gl_n they are the x_{ii}'s and the y_{ii}'s. They have weights {1,0} and {0,0}.

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In[•]:=
```
Zip_vs_[{F_, E_}] := Module[{dvs = Select[vs, (Wt[#] == {0, 0} ∨ Wt[#] == {1, 0}) &]},
   {F, E} // Zip1_vs (*//Zip2_Complement[vs,dvs]*) // Zip2_dvs //
     Zip3_Complement[vs,dvs] // Zip3_dvs
  ] // Last
```

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Getting rid of the quadratic.

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Lemma 1. With convergences left to the reader,
$$\left\langle F : \mathcal{E}\, e^{\frac{1}{2}\sum_{i,j\in B} G_{ij} z_i z_j} \right\rangle_B = \det(1-GF)^{-1/2} \left\langle F(1-GF)^{-1} : \mathcal{E} \right\rangle_B$$

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In[•]:=
```
Zip1_{} = Identity;
Zip1_vs_@{F_, 𝔼[Q_, P___]} := Module[{I, F, G, u, v},
  I = IdentityMatrix@Length@vs;
  F = Table[If[Wt[u] + Wt[v] == {1, 0}, ∂_{u,v} F, 0], {u, vs}, {v, vs}];
  G = Table[If[Wt[u] + Wt[v] == {1, 0}, ∂_{u,v} Q, 0], {u, vs}, {v, vs}];
  {CF[(vs).(F.Inverse[I - G.F]).(vs) / 2],
   𝔼[CF[Q - PowerExpand@Log[Det[I - G.F]] / 2 - vs.G.vs / 2], P]}
 ]
```

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Getting rid of linear terms.

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Lemma 2. $\left\langle F : \mathcal{E}\, e^{\sum_{i\in B} y_i z_i} \right\rangle_B = e^{\frac{1}{2}\sum_{i,j\in B} F_{ij} y_i y_j} \left\langle F : \mathcal{E}|_{z_B \to z_B + F y_B} \right\rangle_B$.

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In[•]:=
```
Zip2{} = Identity;
Zip2vs_@{ℱ_, 𝔼[Q_, P___]} := Module[{F, Y, u, v},
  F = Table[If[Wt[u] + Wt[v] == {1, 0}, CF[∂u,vℱ], 0], {u, vs}, {v, vs}];
  Y = Table[∂vQ, {v, vs}] /. Table[v → 0, {v, vs}];
  CF /@ {ℱ, 𝔼[Q - Y.vs + Y.F.Y / 2, P] /. Thread[vs → vs + F.Y]}
 ]
```

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Dealing with Feynman diagrams.

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Lemma 3. With an extra variable λ, $Z_\lambda := \log[\lambda F : e^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2}\sum_{i,j\in B} F_{ij}\left(\partial_{z_i}\partial_{z_j}Z_\lambda + (\partial_{z_i}Z_\lambda)(\partial_{z_j}Z_\lambda)\right).$$

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Note that the power m of λ is at most $k-1+\frac{2k+2}{2} = 2k$. We write $Z_\lambda = \sum Z[m]\,\lambda^m$.

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In[•]:=
```
Zip3vs_@{ℱ_, ℰ_𝔼} := Module[{F, u, v, Z, kk, jj, $m = 0, m, n},
  Do[Z[0, kk] = ℰ[[kk + 1]], {kk, 0, ℰ@$}];
  F[u_, v_] := F[u, v] = CF@If[Wt[u] + Wt[v] == {1, 0}, ∂u,vℱ, 0];
  Z[m_, kk_, u_] := Z[m, kk, u] = ∂uZ[m, kk];
  Z[m_, kk_, u_, v_] := Z[m, kk, u, v] = ∂vZ[m, kk, u];
  For[m = 0, m ≤ 2 $m, ++m, For[kk = 0, kk ≤ ℰ@$, ++kk,
    Z[m + 1, kk] = CF@Sum[
       If[F[u, v] === 0, 0, F[u, v] / (2 (m + 1)) (Z[m, kk, u, v] +
           Sum[Z[n, jj, u] * Z[m - n, kk - jj, v], {n, 0, m}, {jj, 0, kk}])],
       {u, vs}, {v, vs}];
    If[Z[m + 1, kk] =!= 0, $m = m + 1]
   ]];
  CF /@ ({
      ℱ - Sum[F[u, v] * u * v / 2, {u, vs}, {v, vs}],
      𝔼 @@ Table[Sum[Z[m, kk], {m, 0, $m}], {kk, 0, ℰ@$}]
     } /. Table[v → 0, {v, vs}])
 ]
```

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In[•]:=
```
Λ2𝔼d_→r_[Λ_] := Λ2𝔼d→r[$k, Λ];
Λ2𝔼d_→r_[$k_, Λ_] :=
  Module[{k}, 𝔼d→r @@ Table[SeriesCoefficient[Λ, {ϵ, 0, k}], {k, 0, $k}]];
```

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Prolog

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In[•]:=
```
(*BeginPackage["UEA`"];*)
Print["UEA` does computations in general universal enveloping
    algebras and PBW algebras. It is in the public domain, available
    at http://drorbn.net/AcademicPensieve/Projects/SolvablePBW/.
    Dror Bar-Natan is committed to support it within
    reason until June 1, 2027. This is version 260601."];
Print["UEA` implements / extends ",
  Sort@{"**", ϵ, $k, CF, SSQ, NilQ, B, m, SetAlgebra,
    U, UB, UProducts, USimp, UU, $Basis, $PBWRule, δ, adPower, Ad},
  "."];
(*Begin["`Private`"];*)
```

pdf

```
UEA` does computations in general universal enveloping algebras and PBW algebras. It is in the
  public domain, available at http://drorbn.net/AcademicPensieve/Projects/SolvablePBW/. Dror
  Bar-Natan is committed to support it within reason until June 1, 2027. This is version 260601.
```

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```
UEA` implements / extends {**, Ad, adPower, B, CF, m, NilQ,
  SetAlgebra, SSQ, U, UB, UProducts, USimp, UU, δ, ϵ, $Basis, $k, $PBWRule}.
```

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Utilities

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In[•]:=
```
δ_,_ := If[i == j, 1, 0];
```

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In[•]:=
```
SSQ[c_. * x_] /; MemberQ[$Basis, x] := ¬ NilQ[x]; (* Semi-Simple Q *)
NilQ[c_ * x_] /; MemberQ[$Basis, x] := NilQ[x];
SSQ[x_Plus] := And @@ (SSQ /@ (List @@ x));
NilQ[x_Plus] := And @@ (NilQ /@ (List @@ x));
```

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In[•]:=
```
$k = 1;
```

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In[•]:=
```
CF[ℰ_] := Expand[ℰ /. e^L_ :> e^Expand[L]] /. {ϵ^k_. /; k > $k → 0};
```

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Implementing general universal enveloping algebras

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```
In[•]:= B[0, _] = 0; B[_, 0] = 0;
B[c_ * x_, y_] /; MemberQ[$Basis, x] := CF[c B[x, y]];
B[y_, c_ * x_] /; MemberQ[$Basis, x] := CF[c B[y, x]];
B[x_Plus, y_] := B[#, y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
B[x_, x_] = 0;
B[y_, x_] := CF[-B[x, y]];
```

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```
In[•]:= x_ ≼ y_ := OrderedQ[{x, y} /. $PBWRule]; x_ ≺ y_ := ¬ OrderedQ[{y, x} /. $PBWRule];
UU_i_[1] := U_i[];
UU_i_[x_^p_] := UU_i @@ Table[x, {p}];
UU_i_[ℰ_] := ℰ /. {
    U[xs__] :↦ U_i[xs],
    x_ /; MemberQ[$Basis, x] :↦ U_i[x]
   };
UU_i_[x_, xs__] := UU_t1[x] UU_t2[xs] // Expand // m_{t1,t2→i};
USimp[ℰ_] := Collect[ℰ, Times[U_[__] ..], Expand];
USimp[ℰ_] := Expand[ℰ];
```

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```
In[•]:= m_s__[0] = 0;
m_s__[x_Plus] := m_s /@ x;
m_s__[sd_SeriesData] := MapAt[m_s, sd, {3, All}];
m_{i_→j_}[ℰ_] := ℰ /. U_i → U_j;
```

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```
In[•]:= m_{i_,j_→k_}[c_. U_i_[x___] U_j_[]] := c U_k[x];
m_{i_,j_→k_}[c_. U_i_[] U_j_[y___]] := c U_k[y];
m_{i_,j_→k_}[c_. U_i_[xx___, x_] U_j_[y_, yy___]] := If[x ≼ y,
   c U_k[xx, x, y, yy],
   ((U_i[xx] (U_j[y, x] + UU_j[B[x, y]]) // Expand // m_{i,j→i}) U_j[yy] // Expand // m_{i,j→k})
     c // USimp
  ];
```

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```
In[•]:= UProducts[{}, 0] = {1}; UProducts[{}, d_Integer] /; d > 0 = {};
UProducts[{i_, is___}, d_Integer] := Sort@
   Flatten@Table[(U_i @@@ Subsets[$Basis, {j}]) u, {j, 0, d}, {u, UProducts[{is}, d - j]}];
```

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```
In[•]:= Supp[ℰ_] := Union@Cases[{ℰ}, U_i_[___] :↦ i, ∞];
```

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In[•]:=
```
Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ⊠ y_ := Module[{is = Supp[x] ⋂ Supp[y], σ, z},
   z = x; Do[z = m_{i→σ@i}[z], {i, is}];
   z = Expand[y z]; Do[z = m_{σ@i,i→i}[z], {i, is}]; z];
UB[x_, y_] := USimp[x ⊠ y - y ⊠ x];
```

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Epilog

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In[•]:=
```
(*End[]; EndPackage[];*)
```

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Predefined Algebras

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sl(2)

pdf

In[•]:=
```
Print["UEA`SetAlgebra knows \"sl2\"."];
```

pdf

```
UEA`SetAlgebra knows "sl2".
```

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In[•]:=
```
SetAlgebra["sl2"] := (
   Print["In sl2: ⟨e,h,f⟩/([h,e]=2e, [h,f]=-2f, [e,f]=h)."];
   B[h, e] = 2 e; B[h, f] = -2 f; B[e, f] = h;
   $Algebra = "sl2";
   $Basis = {e, h, f};
   $PBWRule = {e → 1, h → 2, f → 3};
   NilQ[e] = NilQ[f] = True; NilQ[h] = False;
  );
```

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$gl_{n,\epsilon}$

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In[•]:=
```
Print["UEA`SetAlgebra knows the ϵ-nilpotent algebra gl_{n,ϵ}."];
```

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```
UEA`SetAlgebra knows the ϵ-nilpotent algebra gl_{n,ϵ}.
```

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In[•]:=
```
SetAlgebra[gl_{n_,ϵ}] := (
   $Algebra = gl_{n,ϵ};
   $Basis = Flatten@{
      Table[y_{α,β}, {β, 2, n}, {α, 1, β - 1}],
      Table[x_{α,β}, {β, 1, n}, {α, 1, β}]
     };
   (*$Basis=Reverse@SortBy[$Basis,If[⌗⟦1⟧===x,{0,⌗⟦3⟧-⌗⟦2⟧}, {1,⌗⟦2⟧-⌗⟦3⟧}]&];*)
   NilQ[y__] = True; NilQ[x_{α_,β_}] := (α =!= β); (* Nilpotent Q *)
   $PBWRule = Thread[$Basis → Range@Length@$Basis];
   B[x_{i_,j_}, x_{k_,l_}] := δ_{j,k} x_{i,l} - δ_{l,i} x_{k,j};
   B[y_{i_,j_}, y_{k_,l_}] := CF[-ϵ δ_{j,k} y_{i,l} + ϵ δ_{l,i} y_{k,j}];
   B[x_{i_,j_}, y_{l_,k_}] := CF[δ_{j,k} x_{i,l} - δ_{l,i} x_{k,j} /. x_{α_,β_} :↦ If[α ≤ β, ϵ x_{α,β}, y_{β,α}]];
   DefRep[0] = Normal@SparseArray[{}, {n, n}];
   DefRep[ℰ_] := ℰ /. {
      x_{α_,β_} :↦ Normal@SparseArray[{α, β} → 1, {n, n}],
      y_{α_,β_} :↦ Normal@SparseArray[{β, α} → ϵ, {n, n}]
     }
  );
```

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In[•]:=
```
adPower[x_, k_, Env_][c_ * y_] /; MemberQ[$Basis, y] := CF[c adPower[x, k, Env][y]];
adPower[x_, k_, Env_][y_Plus] := adPower[x, k, Env] /@ y;
adPower[_, 0, Env_][y_] := y;
adPower[x_, k_, Env_][0] = 0;
adPower[x_, k_, Env_][y_] /; MemberQ[$Basis, y] :=
  adPower[x, k, Env][y] = B[adPower[x, k - 1, Env][y], x];
adPower[x_, k_][y_] := adPower[x, k, {$Algebra, $k}][y];
```

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```
Ad[x_][y_] /; SSQ[x] := Expand@Module[{A, b, c},
   A = Table[Coefficient[B[b, x], c], {c, $Basis}, {b, $Basis}];
   CF[
    $Basis.Normal[Series[MatrixExp[A], {ϵ, 0, $k}]].Table[Coefficient[y, b], {b, $Basis}]]
  ]
```

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```
Ad[x_][y_] /; NilQ[x] := Expand@Module[{k = 0, s = 0},
   While[adPower[x, k][y] =!= 0,
    s += adPower[x, k][y] / k!;
    ++k
   ];
   s
  ]
```

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In[•]:=
```
AdMatrix[x_, Env_] := Module[{A, b, c, P, S, k = 0},
   A = Table[Coefficient[B[b, x], c], {c, $Basis}, {b, $Basis}];
   M = CF@If[NilQ[x],
      P = S = IdentityMatrix[Length@$Basis];
      While[P =!= 0 P, S += (P = CF[P.A]) / (++k) !]; S,
      Normal[Series[MatrixExp[A], {ϵ, 0, $k}]]
     ];
   If[False ∧ MemberQ[$Basis, x], AdMatrix[x, Env] = M, M]
  ];
AdMatrix[x_] := AdMatrix[x, {$Algebra, $k}];
Ad[x_][y_] := $Basis.AdMatrix[x].Table[Coefficient[y, b], {b, $Basis}];
```

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In[•]:=
```
λTangent[] = 0;
λTangent[xs___, x_] := Ad[x][λTangent[xs]] + ∂λ x;
λTangent[xs_List] := λTangent @@ xs
```

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In[•]:=
```
TriangularDSolve[eqns_, funs_, iv_] := Monitor[
  Module[{sol = {}, e, fun, der},
   MapThread[
    {e, fun} ↦ sol = Join[sol,
       $Mon = MapAt[CF, First@DSolve[CF[e /. Derivative[1][f_][v_] :↦ der[f[v], v] /.
              sol /. der[f_, v_] :↦ ∂v f], fun, iv], {1, 2}]
      ],
    {eqns, funs}
   ];
   sol
  ],
  $Mon
 ]
```

$gl_{2,\epsilon}$ Experiments

```
In[•]:= SetAlgebra[gl_{2,ϵ}];
        $PBWRule
        MatrixForm@Table[{b1, b2} → B[b1, b2], {b1, $Basis}, {b2, $Basis}];
        MatrixForm@
          Table[{b1, b2} → (DefRep[b1].DefRep[b2] - DefRep[b2].DefRep[b1] == DefRep[B[b1, b2]]),
           {b1, $Basis}, {b2, $Basis}];
        DeleteCases[Flatten[Table[{a, b, c} → B[a, B[b, c]] + B[b, B[c, a]] + B[c, B[a, b]],
           {a, $Basis}, {b, $Basis}, {c, $Basis}]], _ → 0]
```

```
Out[•]=
{y_{1,2} → 1, x_{1,1} → 2, x_{1,2} → 3, x_{2,2} → 4}
```

```
Out[•]=
{}
```

```
In[•]:= lhs = λTangent[Join[
            Table[If[SSQ[b1], (b1 /. x → ξ1) b1, λ (b1 /. {x → ξ1, y → η1}) b1], {b1, $Basis}],
            Table[If[SSQ[b2], (b2 /. x → ξ2) b2, λ (b2 /. {x → ξ2, y → η2}) b2], {b2, $Basis}]
           ]];
        rhs = λTangent[
           Table[If[SSQ[b], b /. x_{α_,α_} :→ (ξ1_{α,α} + ξ2_{α,α} + ϵ f_{α,α}[λ]) x_{α,α},
             (b /. {x_{αβ__} → f_{αβ}[λ], y_{αβ__} → g_{αβ}[λ]}) b], {b, $Basis}]
          ];
        eqns = Simplify[(Coefficient[lhs - rhs, #] == 0) & /@ $Basis];
        unknowns = $Basis /. {x_{αβ__} → f_{αβ}[λ], y_{αβ__} → g_{αβ}[λ]};
        {sol0} = Block[{$k = 0}, Simplify@DSolve[
            And @@ Join[
              CF /@ eqns,
              Select[$Basis, NilQ] /. {x_{αβ__} :→ f_{αβ}[0] == 0, y_{αβ__} :→ g_{αβ}[0] == 0}
             ],
            Select[$Basis, NilQ] /. {x_{αβ__} → f_{αβ}[λ], y_{αβ__} → g_{αβ}[λ]},
            λ
           ]]
```

```
Out[•]=
{{g_{1,2}[λ] → λ (η1_{1,2} + e^{-ξ1_{1,1}+ξ1_{2,2}} η2_{1,2}), f_{1,2}[λ] → e^{-ξ2_{1,1}} λ ξ1_{1,2} + e^{-ξ1_{2,2}} λ ξ2_{1,2}}}
```

In[•]:=
```
{sol1} = Block[{$k = 1}, Simplify@DSolve[
    lhs = λTangent[Join[
       Table[If[SSQ[b1], (b1 /. x → ξ1) b1, λ (b1 /. {x → ξ1, y → η1}) b1], {b1, $Basis}],
       Table[If[SSQ[b2]], (b2 /. x → ξ2) b2, λ (b2 /. {x → ξ2, y → η2}) b2], {b2, $Basis}]
      ]];
    rhs = λTangent[
      Table[If[SSQ[b],
        b /. x_α_,α_ :↦ (ξ1_α,α + ξ2_α,α + ϵ f_α,α[λ]) x_α,α,
        (b /. {x_αβ__ → f_αβ[λ], y_αβ__ → g_αβ[λ]}) b /. (sol0 /. ((a_ → b_) :↦ (a → ϵ a + b)))
       ], {b, $Basis}]
     ];
    Join[
     (Coefficient[CF[lhs - rhs], #] == 0) & /@ $Basis,
     $Basis /. {x_αβ__ :↦ f_αβ[0] == 0, y_αβ__ :↦ g_αβ[0] == 0}
    ],
    $Basis /. {x_αβ__ → f_αβ[λ], y_αβ__ → g_αβ[λ]},
    λ
   ]]
```

Out[•]=
$$\left\{\left\{f_{1,1}[\lambda] \to e^{\xi 1_{2,2}}\, \lambda^2\, \eta 2_{1,2}\, \xi 1_{1,2},\ f_{1,2}[\lambda] \to \lambda^3\, \eta 2_{1,2}\, \xi 1_{1,2}\, \xi 2_{1,2},\right.\right.$$
$$\left.\left. f_{2,2}[\lambda] \to -e^{\xi 1_{2,2}}\, \lambda^2\, \eta 2_{1,2}\, \xi 1_{1,2},\ g_{1,2}[\lambda] \to -e^{-\xi 1_{1,1}+2\,\xi 1_{2,2}}\, \lambda^3\, \eta 2_{1,2}^2\, \xi 1_{1,2}\right\}\right\}$$

In[•]:=
```
cm[i_, j_ → k_] := 𝔼_{i,j}→{k}[CF@Sum[If[SSQ[b],
       b /. x_α_,α_ :↦ (ξ1_α,α + ξ2_α,α) x_α,α[k],
       b[k] (b /. {x_αβ__ → f_αβ[λ], y_αβ__ → g_αβ[λ]} /. sol0),
      ], {b, $Basis}],
    CF@Sum[b[k] (b /. {x_αβ__ → f_αβ[λ], y_αβ__ → g_αβ[λ]} /. sol1), {b, $Basis}]
   ] /. {ξ1_αβ__ :↦ ξ_αβ[i], η1_αβ__ :↦ η_αβ[i], ξ2_αβ__ :↦ ξ_αβ[j], η2_αβ__ :↦ η_αβ[j], λ → 1};
cm[i, j → k]
```

Out[•]=
$$\mathbb{E}_{\{i,j\}\to\{k\}}\left[y_{1,2}[k]\,\eta_{1,2}[i] + e^{-\xi_{1,1}[i]+\xi_{2,2}[i]}\, y_{1,2}[k]\,\eta_{1,2}[j] + x_{1,1}[k]\,\xi_{1,1}[i] + x_{1,1}[k]\,\xi_{1,1}[j] + \right.$$
$$e^{-\xi_{1,1}[j]}\, x_{1,2}[k]\,\xi_{1,2}[i] + e^{-\xi_{2,2}[i]}\, x_{1,2}[k]\,\xi_{1,2}[j] + x_{2,2}[k]\,\xi_{2,2}[i] + x_{2,2}[k]\,\xi_{2,2}[j],$$
$$e^{\xi_{2,2}[i]}\, x_{1,1}[k]\,\eta_{1,2}[j]\,\xi_{1,2}[i] - e^{\xi_{2,2}[i]}\, x_{2,2}[k]\,\eta_{1,2}[j]\,\xi_{1,2}[i] -$$
$$\left. e^{-\xi_{1,1}[i]+2\,\xi_{2,2}[i]}\, y_{1,2}[k]\,\eta_{1,2}[j]^2\,\xi_{1,2}[i] + x_{1,2}[k]\,\eta_{1,2}[j]\,\xi_{1,2}[i]\,\xi_{1,2}[j]\right]$$

```
In[•]:= cm[1, 2 → 2] // cm[2, 3 → 1]
```

Out[•]=

$$\mathbb{E}_{\{1,2,3\}\to\{1\}}\Big[y_{1,2}[1]\,\eta_{1,2}[1]+e^{-\xi_{1,1}[1]+\xi_{2,2}[1]}\,y_{1,2}[1]\,\eta_{1,2}[2]+e^{-\xi_{1,1}[1]-\xi_{1,1}[2]+\xi_{2,2}[1]+\xi_{2,2}[2]}\,y_{1,2}[1]\,\eta_{1,2}[3]+x_{1,1}[1]\,\xi_{1,1}[1]+x_{1,1}[1]\,\xi_{1,1}[2]+x_{1,1}[1]\,\xi_{1,1}[3]+e^{-\xi_{1,1}[2]-\xi_{1,1}[3]}\,x_{1,2}[1]\,\xi_{1,2}[1]+e^{-\xi_{1,1}[3]-\xi_{2,2}[1]}\,x_{1,2}[1]\,\xi_{1,2}[2]+e^{-\xi_{2,2}[1]-\xi_{2,2}[2]}\,x_{1,2}[1]\,\xi_{1,2}[3]+x_{2,2}[1]\,\xi_{2,2}[1]+x_{2,2}[1]\,\xi_{2,2}[2]+x_{2,2}[1]\,\xi_{2,2}[3],\; e^{\xi_{2,2}[1]}\,x_{1,1}[1]\,\eta_{1,2}[2]\,\xi_{1,2}[1]-e^{\xi_{2,2}[1]}\,x_{2,2}[1]\,\eta_{1,2}[2]\,\xi_{1,2}[1]-e^{-\xi_{1,1}[1]+2\,\xi_{2,2}[1]}\,y_{1,2}[1]\,\eta_{1,2}[2]^2\,\xi_{1,2}[1]+e^{-\xi_{1,1}[2]+\xi_{2,2}[1]+\xi_{2,2}[2]}\,x_{1,1}[1]\,\eta_{1,2}[3]\,\xi_{1,2}[1]-e^{-\xi_{1,1}[2]+\xi_{2,2}[1]+\xi_{2,2}[2]}\,x_{2,2}[1]\,\eta_{1,2}[3]\,\xi_{1,2}[1]-2\,e^{-\xi_{1,1}[1]-\xi_{1,1}[2]+2\,\xi_{2,2}[1]+\xi_{2,2}[2]}\,y_{1,2}[1]\,\eta_{1,2}[2]\,\eta_{1,2}[3]\,\xi_{1,2}[1]-e^{-\xi_{1,1}[1]-2\,\xi_{1,1}[2]+2\,\xi_{2,2}[1]+2\,\xi_{2,2}[2]}\,y_{1,2}[1]\,\eta_{1,2}[3]^2\,\xi_{1,2}[1]+e^{\xi_{2,2}[2]}\,x_{1,1}[1]\,\eta_{1,2}[3]\,\xi_{1,2}[2]-e^{\xi_{2,2}[2]}\,x_{2,2}[1]\,\eta_{1,2}[3]\,\xi_{1,2}[2]-e^{-\xi_{1,1}[1]-\xi_{1,1}[2]+\xi_{2,2}[1]+2\,\xi_{2,2}[2]}\,y_{1,2}[1]\,\eta_{1,2}[3]^2\,\xi_{1,2}[2]+e^{-\xi_{1,1}[3]}\,x_{1,2}[1]\,\eta_{1,2}[2]\,\xi_{1,2}[1]\,\xi_{1,2}[2]+e^{-\xi_{2,2}[2]}\,x_{1,2}[1]\,\eta_{1,2}[2]\,\xi_{1,2}[1]\,\xi_{1,2}[3]+e^{-\xi_{1,1}[2]}\,x_{1,2}[1]\,\eta_{1,2}[3]\,\xi_{1,2}[1]\,\xi_{1,2}[3]+e^{-\xi_{2,2}[1]}\,x_{1,2}[1]\,\eta_{1,2}[3]\,\xi_{1,2}[2]\,\xi_{1,2}[3]\Big]$$

```
In[•]:= Simplify[(cm[1, 2 → 2] // cm[2, 3 → 1]) ≡ (cm[2, 3 → 2] // cm[1, 2 → 1])]
```

Out[•]=

True

$gl_{3,\epsilon}$ Experiments

```
In[•]:= SetAlgebra[gl_{3,ϵ}];
$k = 1
$PBWRule
MatrixForm@Table[{b1, b2} → B[b1, b2], {b1, $Basis}, {b2, $Basis}];
MatrixForm@
  Table[{b1, b2} → (DefRep[b1].DefRep[b2] - DefRep[b2].DefRep[b1] == DefRep[B[b1, b2]]),
   {b1, $Basis}, {b2, $Basis}];
DeleteCases[Flatten[Table[{a, b, c} → B[a, B[b, c]] + B[b, B[c, a]] + B[c, B[a, b]],
   {a, $Basis}, {b, $Basis}, {c, $Basis}]], _ → 0]
```

Out[•]=

1

Out[•]=

$$\{y_{1,2}\to 1,\; y_{1,3}\to 2,\; y_{2,3}\to 3,\; x_{1,1}\to 4,\; x_{1,2}\to 5,\; x_{2,2}\to 6,\; x_{1,3}\to 7,\; x_{2,3}\to 8,\; x_{3,3}\to 9\}$$

Out[•]=

{}

```
In[•]:= {sol0} = Block[{$k = 0},
  lhs = λTangent[Join[
     Table[If[SSQ[b1], (b1 /. x → ξ1) b1, λ (b1 /. {x → ξ1, y → η1}) b1], {b1, $Basis}],
     Table[If[SSQ[b2], (b2 /. x → ξ2) b2, λ (b2 /. {x → ξ2, y → η2}) b2], {b2, $Basis}]
    ]];
  rhs = λTangent[
    Table[If[SSQ[b], b /. x_α_,α_ :> (ξ1_α,α + ξ2_α,α + ϵ f_α,α[λ]) x_α,α,
      (b /. {x_αβ__ → f_αβ[λ], y_αβ__ → g_αβ[λ]}) b], {b, $Basis}]
   ];
  Simplify@DSolve[
    And @@ Join[
      (Coefficient[CF[lhs - rhs], #] == 0) & /@ $Basis,
      Select[$Basis, NilQ] /. {x_αβ__ :> f_αβ[0] == 0, y_αβ__ :> g_αβ[0] == 0}
     ],
    Select[$Basis, NilQ] /. {x_αβ__ → f_αβ[λ], y_αβ__ → g_αβ[λ]},
    λ
   ]]
```

```
Out[•]=
{{f_1,2[λ] → e^(-ξ2_1,1 λ) ξ1_1,2 + e^(-ξ1_2,2 λ) ξ2_1,2, g_2,3[λ] → λ (η1_2,3 + e^(ξ1_3,3 (e^(-ξ1_2,2 λ) η2_2,3 - λ η2_1,3 ξ1_1,2))),
  f_1,3[λ] → λ (e^(-ξ2_1,1 λ) ξ1_1,3 - λ ξ1_2,3 ξ2_1,2 + e^(-ξ1_3,3 λ) ξ2_1,3), f_2,3[λ] → e^(-ξ2_2,2 λ) ξ1_2,3 + e^(-ξ1_3,3 λ) ξ2_2,3,
  g_1,2[λ] → λ (η1_1,2 + e^(-ξ1_1,1+ξ1_2,2 λ) (η2_1,2 + e^(ξ1_3,3 λ) η2_1,3 ξ1_2,3)), g_1,3[λ] → λ (η1_1,3 + e^(-ξ1_1,1+ξ1_3,3 λ) η2_1,3)}}
```

```
In[•]:= $k = 1; SBasis = SortBy[$Basis, If[#[[1]] === x, {0, #[[3]] - #[[2]]}, {1, #[[2]] - #[[3]]}] &];
lhs = λTangent[Join[
     Table[If[SSQ[b1], (b1 /. x → ξ1) b1, λ (b1 /. {x → ξ1, y → η1}) b1], {b1, $Basis}],
     Table[If[SSQ[b2], (b2 /. x → ξ2) b2, λ (b2 /. {x → ξ2, y → η2}) b2], {b2, $Basis}]
    ]];
rhs = λTangent[
   Echo@Table[If[SSQ[b],
      b /. x_α_,α_ :> (ξ1_α,α + ξ2_α,α + ϵ f_α,α[λ]) x_α,α,
      (b /. {x_αβ__ → f_αβ[λ], y_αβ__ → g_αβ[λ]}) b /. (sol0 /. ((v_ → s_) :> (v → ϵ v + s)))
     ], {b, $Basis}]
  ];
eqns = Transpose@{
     (Coefficient[CF[lhs - rhs], #] == 0) & /@ SBasis,
     SBasis /. {x_αβ__ :> f_αβ[0] == 0, y_αβ__ :> g_αβ[0] == 0}
    };
unknowns = SBasis /. {x_αβ__ → f_αβ[λ], y_αβ__ → g_αβ[λ]};
(*{sol1}=DSolve[Flatten@eqns,unknowns,λ]*)
sol1 = TriangularDSolve[eqns, unknowns, λ];
```

```
» {y_1,2 (λ (η1_1,2 + e^(-ξ1_1,1+ξ1_2,2 (η2_1,2 + e^(ξ1_3,3 λ η2_1,3 ξ1_2,3))) + ϵ g_1,2[λ])),
  y_1,3 (λ (η1_1,3 + e^(-ξ1_1,1+ξ1_3,3 η2_1,3)) + ϵ g_1,3[λ]),
  y_2,3 (λ (η1_2,3 + e^(ξ1_3,3 (e^(-ξ1_2,2 λ) η2_2,3 - λ η2_1,3 ξ1_1,2))) + ϵ g_2,3[λ]),
  x_1,1 (ξ1_1,1 + ξ2_1,1 + ϵ f_1,1[λ]), x_1,2 (e^(-ξ2_1,1 λ) ξ1_1,2 + e^(-ξ1_2,2 λ) ξ2_1,2 + ϵ f_1,2[λ]),
  x_2,2 (ξ1_2,2 + ξ2_2,2 + ϵ f_2,2[λ]), x_1,3 (λ (e^(-ξ2_1,1 λ) ξ1_1,3 - λ ξ1_2,3 ξ2_1,2 + e^(-ξ1_3,3 λ) ξ2_1,3) + ϵ f_1,3[λ]),
  x_2,3 (e^(-ξ2_2,2 λ) ξ1_2,3 + e^(-ξ1_3,3 λ) ξ2_2,3 + ϵ f_2,3[λ]), x_3,3 (ξ1_3,3 + ξ2_3,3 + ϵ f_3,3[λ])}
```

In[•]:=
```
Sort@sol1
```

Out[•]=

$$\Big\{\mathsf{f}_{1,1}[\lambda]\to e^{\xi 1_{2,2}}\lambda^2\eta 2_{1,2}\xi 1_{1,2}+e^{\xi 1_{3,3}}\lambda^2\eta 2_{1,3}\xi 1_{1,3}+e^{\xi 1_{2,2}+\xi 1_{3,3}}\lambda^3\eta 2_{1,3}\xi 1_{1,2}\xi 1_{2,3},$$

$$\mathsf{f}_{1,2}[\lambda]\to e^{-\xi 1_{2,2}+\xi 1_{3,3}-\xi 2_{1,1}}\lambda^2\eta 2_{2,3}\xi 1_{1,3}-e^{\xi 1_{3,3}-\xi 2_{1,1}}\lambda^3\eta 2_{1,3}\xi 1_{1,2}\xi 1_{1,3}+$$
$$\lambda^3\eta 2_{1,2}\xi 1_{1,2}\xi 2_{1,2}-e^{-\xi 1_{2,2}+\xi 1_{3,3}}\lambda^3\eta 2_{2,3}\xi 1_{2,3}\xi 2_{1,2}+e^{\xi 1_{3,3}}\lambda^4\eta 2_{1,3}\xi 1_{1,2}\xi 1_{2,3}\xi 2_{1,2},$$

$$\mathsf{f}_{1,3}[\lambda]\to\lambda^3\eta 2_{1,2}\xi 1_{1,3}\xi 2_{1,2}+\lambda^3\eta 2_{1,3}\xi 1_{1,3}\xi 2_{1,3}+\lambda^3\eta 2_{2,3}\xi 1_{2,3}\xi 2_{1,3},$$

$$\mathsf{f}_{2,2}[\lambda]\to -e^{\xi 1_{2,2}}\lambda^2\eta 2_{1,2}\xi 1_{1,2}+e^{\xi 1_{3,3}}\lambda^2\eta 2_{2,3}\xi 1_{2,3}-e^{\xi 1_{2,2}+\xi 1_{3,3}}\lambda^3\eta 2_{1,3}\xi 1_{1,2}\xi 1_{2,3},$$

$$\mathsf{f}_{2,3}[\lambda]\to -e^{-\xi 2_{2,2}}\lambda^2\eta 2_{1,2}\xi 1_{1,3}+\lambda^3\eta 2_{1,3}\xi 1_{1,3}\xi 2_{2,3}+\lambda^3\eta 2_{2,3}\xi 1_{2,3}\xi 2_{2,3},$$

$$\mathsf{f}_{3,3}[\lambda]\to -e^{\xi 1_{3,3}}\lambda^2\eta 2_{1,3}\xi 1_{1,3}-e^{\xi 1_{3,3}}\lambda^2\eta 2_{2,3}\xi 1_{2,3},$$

$$\mathsf{g}_{1,2}[\lambda]\to -e^{-\xi 1_{1,1}+2\,\xi 1_{2,2}}\lambda^3\eta 2_{1,2}^2\xi 1_{1,2}-e^{-\xi 1_{1,1}+\xi 1_{2,2}+\xi 1_{3,3}}\lambda^3\eta 2_{1,2}\eta 2_{1,3}\xi 1_{1,3}-2\,e^{-\xi 1_{1,1}+2\,\xi 1_{2,2}+\xi 1_{3,3}}\lambda^4\eta 2_{1,2}$$
$$\eta 2_{1,3}\xi 1_{1,2}\xi 1_{2,3}-e^{-\xi 1_{1,1}+\xi 1_{2,2}+2\,\xi 1_{3,3}}\lambda^4\eta 2_{1,3}^2\xi 1_{1,3}\xi 1_{2,3}-e^{-\xi 1_{1,1}+2\,\xi 1_{2,2}+2\,\xi 1_{3,3}}\lambda^5\eta 2_{1,3}^2\xi 1_{1,2}\xi 1_{2,3}^2,$$

$$\mathsf{g}_{1,3}[\lambda]\to e^{-\xi 1_{1,1}+\xi 1_{2,2}}\lambda^2\eta 1_{2,3}\eta 2_{1,2}-e^{-\xi 1_{1,1}+\xi 1_{2,2}+\xi 1_{3,3}}\lambda^3\eta 2_{1,2}\eta 2_{1,3}\xi 1_{1,2}-e^{-\xi 1_{1,1}+2\,\xi 1_{3,3}}\lambda^3\eta 2_{1,3}^2\xi 1_{1,3}+$$
$$e^{-\xi 1_{1,1}+\xi 1_{2,2}+\xi 1_{3,3}}\lambda^3\eta 1_{2,3}\eta 2_{1,3}\xi 1_{2,3}-e^{-\xi 1_{1,1}+\xi 1_{2,2}+2\,\xi 1_{3,3}}\lambda^4\eta 2_{1,3}^2\xi 1_{1,2}\xi 1_{2,3},$$

$$\mathsf{g}_{2,3}[\lambda]\to e^{\xi 1_{3,3}}\lambda^3\eta 2_{1,2}\eta 2_{2,3}\xi 1_{1,2}-e^{-\xi 1_{2,2}+2\,\xi 1_{3,3}}\lambda^3\eta 2_{1,3}\eta 2_{2,3}\xi 1_{1,3}+$$
$$e^{2\,\xi 1_{3,3}}\lambda^4\eta 2_{1,3}^2\xi 1_{1,2}\xi 1_{1,3}-e^{-\xi 1_{2,2}+2\,\xi 1_{3,3}}\lambda^3\eta 2_{2,3}^2\xi 1_{2,3}+e^{2\,\xi 1_{3,3}}\lambda^4\eta 2_{1,3}\eta 2_{2,3}\xi 1_{1,2}\xi 1_{2,3}\Big\}$$

In[•]:=
```
cm[i_, j_ → k_] := 𝔼{i,j}→{k}[CF@Sum[If[SSQ[b],
        b /. x_α_,α_ :> (ξ1α,α + ξ2α,α) xα,α[k],
        b[k] (b /. {xαβ__ → fαβ[λ], yαβ__ → gαβ[λ]} /. sol0)
      ], {b, $Basis}],
    CF@Sum[b[k] (b /. {xαβ__ → fαβ[λ], yαβ__ → gαβ[λ]} /. sol1), {b, $Basis}]
   ] /. {ξ1αβ__ :> ξαβ[i], η1αβ__ :> ηαβ[i], ξ2αβ__ :> ξαβ[j], η2αβ__ :> ηαβ[j], λ → 1};
cm[i, j → k]
```

Out[•]=

$$
\begin{aligned}
&\mathbb{E}_{\{i,j\}\to\{k\}}\Big[y_{1,2}[k]\,\eta_{1,2}[i] + e^{-\xi_{1,1}[i]+\xi_{2,2}[i]}\,y_{1,2}[k]\,\eta_{1,2}[j] + y_{1,3}[k]\,\eta_{1,3}[i] +\\
&\quad e^{-\xi_{1,1}[i]+\xi_{3,3}[i]}\,y_{1,3}[k]\,\eta_{1,3}[j] + y_{2,3}[k]\,\eta_{2,3}[i] + e^{-\xi_{2,2}[i]+\xi_{3,3}[i]}\,y_{2,3}[k]\,\eta_{2,3}[j] +\\
&\quad x_{1,1}[k]\,\xi_{1,1}[i] + x_{1,1}[k]\,\xi_{1,1}[j] + e^{-\xi_{1,1}[j]}\,x_{1,2}[k]\,\xi_{1,2}[i] - e^{\xi_{3,3}[i]}\,y_{2,3}[k]\,\eta_{1,3}[j]\,\xi_{1,2}[i] +\\
&\quad e^{-\xi_{2,2}[i]}\,x_{1,2}[k]\,\xi_{1,2}[j] + e^{-\xi_{1,1}[j]}\,x_{1,3}[k]\,\xi_{1,3}[i] + e^{-\xi_{3,3}[i]}\,x_{1,3}[k]\,\xi_{1,3}[j] + x_{2,2}[k]\,\xi_{2,2}[i] +\\
&\quad x_{2,2}[k]\,\xi_{2,2}[j] + e^{-\xi_{2,2}[j]}\,x_{2,3}[k]\,\xi_{2,3}[i] + e^{-\xi_{1,1}[i]+\xi_{2,2}[i]+\xi_{3,3}[i]}\,y_{1,2}[k]\,\eta_{1,3}[j]\,\xi_{2,3}[i] -\\
&\quad x_{1,3}[k]\,\xi_{1,2}[j]\,\xi_{2,3}[i] + e^{-\xi_{3,3}[i]}\,x_{2,3}[k]\,\xi_{2,3}[j] + x_{3,3}[k]\,\xi_{3,3}[i] + x_{3,3}[k]\,\xi_{3,3}[j],\\
&\quad e^{-\xi_{1,1}[i]+\xi_{2,2}[i]}\,y_{1,3}[k]\,\eta_{1,2}[j]\,\eta_{2,3}[i] + e^{\xi_{2,2}[i]}\,x_{1,1}[k]\,\eta_{1,2}[j]\,\xi_{1,2}[i] -\\
&\quad e^{\xi_{2,2}[i]}\,x_{2,2}[k]\,\eta_{1,2}[j]\,\xi_{1,2}[i] - e^{-\xi_{1,1}[i]+2\,\xi_{2,2}[i]}\,y_{1,2}[k]\,\eta_{1,2}[j]^2\,\xi_{1,2}[i] -\\
&\quad e^{-\xi_{1,1}[i]+\xi_{2,2}[i]+\xi_{3,3}[i]}\,y_{1,3}[k]\,\eta_{1,2}[j]\,\eta_{1,3}[j]\,\xi_{1,2}[i] + e^{\xi_{3,3}[i]}\,y_{2,3}[k]\,\eta_{1,2}[j]\,\eta_{2,3}[j]\,\xi_{1,2}[i] +\\
&\quad x_{1,2}[k]\,\eta_{1,2}[j]\,\xi_{1,2}[i]\,\xi_{1,2}[j] - e^{-\xi_{2,2}[j]}\,x_{2,3}[k]\,\eta_{1,2}[j]\,\xi_{1,3}[i] +\\
&\quad e^{\xi_{3,3}[i]}\,x_{1,1}[k]\,\eta_{1,3}[j]\,\xi_{1,3}[i] - e^{\xi_{3,3}[i]}\,x_{3,3}[k]\,\eta_{1,3}[j]\,\xi_{1,3}[i] -\\
&\quad e^{-\xi_{1,1}[i]+\xi_{2,2}[i]+\xi_{3,3}[i]}\,y_{1,2}[k]\,\eta_{1,2}[j]\,\eta_{1,3}[j]\,\xi_{1,3}[i] - e^{-\xi_{1,1}[i]+2\,\xi_{3,3}[i]}\,y_{1,3}[k]\,\eta_{1,3}[j]^2\,\xi_{1,3}[i] +\\
&\quad e^{-\xi_{1,1}[j]-\xi_{2,2}[i]+\xi_{3,3}[i]}\,x_{1,2}[k]\,\eta_{2,3}[j]\,\xi_{1,3}[i] - e^{-\xi_{2,2}[i]+2\,\xi_{3,3}[i]}\,y_{2,3}[k]\,\eta_{1,3}[j]\,\eta_{2,3}[j]\,\xi_{1,3}[i] -\\
&\quad e^{-\xi_{1,1}[j]+\xi_{3,3}[i]}\,x_{1,2}[k]\,\eta_{1,3}[j]\,\xi_{1,2}[i]\,\xi_{1,3}[i] + e^{2\,\xi_{3,3}[i]}\,y_{2,3}[k]\,\eta_{1,3}[j]^2\,\xi_{1,2}[i]\,\xi_{1,3}[i] +\\
&\quad x_{1,3}[k]\,\eta_{1,2}[j]\,\xi_{1,2}[j]\,\xi_{1,3}[i] + x_{1,3}[k]\,\eta_{1,3}[j]\,\xi_{1,3}[i]\,\xi_{1,3}[j] +\\
&\quad e^{-\xi_{1,1}[i]+\xi_{2,2}[i]+\xi_{3,3}[i]}\,y_{1,3}[k]\,\eta_{1,3}[j]\,\eta_{2,3}[i]\,\xi_{2,3}[i] + e^{\xi_{3,3}[i]}\,x_{2,2}[k]\,\eta_{2,3}[j]\,\xi_{2,3}[i] -\\
&\quad e^{\xi_{3,3}[i]}\,x_{3,3}[k]\,\eta_{2,3}[j]\,\xi_{2,3}[i] - e^{-\xi_{2,2}[i]+2\,\xi_{3,3}[i]}\,y_{2,3}[k]\,\eta_{2,3}[j]^2\,\xi_{2,3}[i] +\\
&\quad e^{\xi_{2,2}[i]+\xi_{3,3}[i]}\,x_{1,1}[k]\,\eta_{1,3}[j]\,\xi_{1,2}[i]\,\xi_{2,3}[i] - e^{\xi_{2,2}[i]+\xi_{3,3}[i]}\,x_{2,2}[k]\,\eta_{1,3}[j]\,\xi_{1,2}[i]\,\xi_{2,3}[i] -\\
&\quad 2\,e^{-\xi_{1,1}[i]+2\,\xi_{2,2}[i]+\xi_{3,3}[i]}\,y_{1,2}[k]\,\eta_{1,2}[j]\,\eta_{1,3}[j]\,\xi_{1,2}[i]\,\xi_{2,3}[i] -\\
&\quad e^{-\xi_{1,1}[i]+\xi_{2,2}[i]+2\,\xi_{3,3}[i]}\,y_{1,3}[k]\,\eta_{1,3}[j]^2\,\xi_{1,2}[i]\,\xi_{2,3}[i] +\\
&\quad e^{2\,\xi_{3,3}[i]}\,y_{2,3}[k]\,\eta_{1,3}[j]\,\eta_{2,3}[j]\,\xi_{1,2}[i]\,\xi_{2,3}[i] -\\
&\quad e^{-\xi_{2,2}[i]+\xi_{3,3}[i]}\,x_{1,2}[k]\,\eta_{2,3}[j]\,\xi_{1,2}[j]\,\xi_{2,3}[i] + e^{\xi_{3,3}[i]}\,x_{1,2}[k]\,\eta_{1,3}[j]\,\xi_{1,2}[i]\,\xi_{1,2}[j]\,\xi_{2,3}[i] -\\
&\quad e^{-\xi_{1,1}[i]+\xi_{2,2}[i]+2\,\xi_{3,3}[i]}\,y_{1,2}[k]\,\eta_{1,3}[j]^2\,\xi_{1,3}[i]\,\xi_{2,3}[i] + x_{1,3}[k]\,\eta_{2,3}[j]\,\xi_{1,3}[j]\,\xi_{2,3}[i] -\\
&\quad e^{-\xi_{1,1}[i]+2\,\xi_{2,2}[i]+2\,\xi_{3,3}[i]}\,y_{1,2}[k]\,\eta_{1,3}[j]^2\,\xi_{1,2}[i]\,\xi_{2,3}[i]^2 +\\
&\quad x_{2,3}[k]\,\eta_{1,3}[j]\,\xi_{1,3}[i]\,\xi_{2,3}[j] + x_{2,3}[k]\,\eta_{2,3}[j]\,\xi_{2,3}[i]\,\xi_{2,3}[j]\Big]
\end{aligned}
$$

In[•]:=
```
Short[cm[1, 2 → 2] // cm[2, 3 → 1], 10]
```

Out[•]//Short=

$$\mathbb{E}_{\{1,2,3\}\to\{1\}}\Big[y_{1,2}[1]\,\eta_{1,2}[1] + e^{-\xi_{1,1}[1]+\xi_{2,2}[1]}\,y_{1,2}[1]\,\eta_{1,2}[2] + e^{-\xi_{1,1}[1]-\xi_{1,1}[2]+\xi_{2,2}[1]+\xi_{2,2}[2]}\,y_{1,2}[1]\,\eta_{1,2}[3] + y_{1,3}[1]\,\eta_{1,3}[1] + e^{-\xi_{1,1}[1]+\xi_{3,3}[1]}\,y_{1,3}[1]\,\eta_{1,3}[2] + e^{-\xi_{1,1}[1]-\xi_{1,1}[2]+\xi_{3,3}[1]+\xi_{3,3}[2]}\,y_{1,3}[1]\,\eta_{1,3}[3] + y_{2,3}[1]\,\eta_{2,3}[1] + e^{-\xi_{2,2}[1]+\xi_{3,3}[1]}\,y_{2,3}[1]\,\eta_{2,3}[2] + e^{-\xi_{2,2}[1]-\xi_{2,2}[2]+\xi_{3,3}[1]+\xi_{3,3}[2]}\,y_{2,3}[1]\,\eta_{2,3}[3] + x_{1,1}[1]\,\xi_{1,1}[1] + x_{1,1}[1]\,\xi_{1,1}[2] + x_{1,1}[1]\,\xi_{1,1}[3] + \ll 14 \gg + x_{2,2}[1]\,\xi_{2,2}[3] + e^{-\xi_{2,2}[2]-\xi_{2,2}[3]}\,x_{2,3}[1]\,\xi_{2,3}[1] + e^{-\xi_{1,1}[1]+\xi_{2,2}[1]+\xi_{3,3}[1]}\,y_{1,2}[1]\,\eta_{1,3}[2]\,\xi_{2,3}[1] + e^{-\xi_{1,1}[1]-\xi_{1,1}[2]+\xi_{2,2}[1]+\xi_{3,3}[1]+\xi_{3,3}[2]}\,y_{1,2}[1]\,\eta_{1,3}[3]\,\xi_{2,3}[1] - e^{-\xi_{1,1}[3]}\,x_{1,3}[1]\,\xi_{1,2}[2]\,\xi_{2,3}[1] - e^{-\xi_{2,2}[2]}\,x_{1,3}[1]\,\xi_{1,2}[3]\,\xi_{2,3}[1] + e^{-\xi_{3,3}[1]}\,x_{2,3}[1]\,\xi_{2,3}[2] + e^{-\xi_{1,1}[1]-\xi_{1,1}[2]+\xi_{2,2}[1]+\xi_{2,2}[2]+\xi_{3,3}[2]}\,y_{1,2}[1]\,\eta_{1,3}[3]\,\xi_{2,3}[2] - e^{-\xi_{3,3}[1]}\,x_{1,3}[1]\,\xi_{1,2}[3]\,\xi_{2,3}[2] + e^{-\xi_{3,3}[1]-\xi_{3,3}[2]}\,x_{2,3}[1]\,\xi_{2,3}[3] + x_{3,3}[1]\,\xi_{3,3}[1] + x_{3,3}[1]\,\xi_{3,3}[2] + x_{3,3}[1]\,\xi_{3,3}[3],\ \ll 1 \gg \Big]$$

In[•]:=
```
Simplify[(cm[1, 2 → 2] // cm[2, 3 → 1]) ≡ (cm[2, 3 → 2] // cm[1, 2 → 1])]
```

Out[•]=

True

In[•]:=
```
sol0
```

Out[•]=

$$\Big\{f_{1,2}[\lambda] \to e^{-\xi 2_{1,1}}\,\lambda\,\xi 1_{1,2} + e^{-\xi 1_{2,2}}\,\lambda\,\xi 2_{1,2},\ g_{2,3}[\lambda] \to \lambda\left(\eta 1_{2,3} + e^{\xi 1_{3,3}}\left(e^{-\xi 1_{2,2}}\,\eta 2_{2,3} - \lambda\,\eta 2_{1,3}\,\xi 1_{1,2}\right)\right),$$
$$f_{1,3}[\lambda] \to \lambda\left(e^{-\xi 2_{1,1}}\,\xi 1_{1,3} - \lambda\,\xi 1_{2,3}\,\xi 2_{1,2} + e^{-\xi 1_{3,3}}\,\xi 2_{1,3}\right),\ f_{2,3}[\lambda] \to e^{-\xi 2_{2,2}}\,\lambda\,\xi 1_{2,3} + e^{-\xi 1_{3,3}}\,\lambda\,\xi 2_{2,3},$$
$$g_{1,2}[\lambda] \to \lambda\left(\eta 1_{1,2} + e^{-\xi 1_{1,1}+\xi 1_{2,2}}\left(\eta 2_{1,2} + e^{\xi 1_{3,3}}\,\lambda\,\eta 2_{1,3}\,\xi 1_{2,3}\right)\right),\ g_{1,3}[\lambda] \to \lambda\left(\eta 1_{1,3} + e^{-\xi 1_{1,1}+\xi 1_{3,3}}\,\eta 2_{1,3}\right)\Big\}$$

In[•]:=
```
CF[sol1 /. λ → 1]
```

Out[•]=

$$\Big\{f_{1,1}[1] \to e^{\xi 1_{2,2}}\,\eta 2_{1,2}\,\xi 1_{1,2} + e^{\xi 1_{3,3}}\,\eta 2_{1,3}\,\xi 1_{1,3} + e^{\xi 1_{2,2}+\xi 1_{3,3}}\,\eta 2_{1,3}\,\xi 1_{1,2}\,\xi 1_{2,3},$$
$$f_{2,2}[1] \to -e^{\xi 1_{2,2}}\,\eta 2_{1,2}\,\xi 1_{1,2} + e^{\xi 1_{3,3}}\,\eta 2_{2,3}\,\xi 1_{2,3} - e^{\xi 1_{2,2}+\xi 1_{3,3}}\,\eta 2_{1,3}\,\xi 1_{1,2}\,\xi 1_{2,3},$$
$$f_{3,3}[1] \to -e^{\xi 1_{3,3}}\,\eta 2_{1,3}\,\xi 1_{1,3} - e^{\xi 1_{3,3}}\,\eta 2_{2,3}\,\xi 1_{2,3},$$
$$f_{1,2}[1] \to e^{-\xi 1_{2,2}+\xi 1_{3,3}-\xi 2_{1,1}}\,\eta 2_{2,3}\,\xi 1_{1,3} - e^{\xi 1_{3,3}-\xi 2_{1,1}}\,\eta 2_{1,3}\,\xi 1_{1,2}\,\xi 1_{1,3} + \eta 2_{1,2}\,\xi 1_{1,2}\,\xi 2_{1,2} - e^{-\xi 1_{2,2}+\xi 1_{3,3}}\,\eta 2_{2,3}\,\xi 1_{2,3}\,\xi 2_{1,2} + e^{\xi 1_{3,3}}\,\eta 2_{1,3}\,\xi 1_{1,2}\,\xi 1_{2,3}\,\xi 2_{1,2},$$
$$f_{2,3}[1] \to -e^{-\xi 2_{2,2}}\,\eta 2_{1,2}\,\xi 1_{1,3} + \eta 2_{1,3}\,\xi 1_{1,3}\,\xi 2_{2,3} + \eta 2_{2,3}\,\xi 1_{2,3}\,\xi 2_{2,3},$$
$$f_{1,3}[1] \to \eta 2_{1,2}\,\xi 1_{1,3}\,\xi 2_{1,2} + \eta 2_{1,3}\,\xi 1_{1,3}\,\xi 2_{1,3} + \eta 2_{2,3}\,\xi 1_{2,3}\,\xi 2_{1,3},$$
$$g_{1,3}[1] \to e^{-\xi 1_{1,1}+\xi 1_{2,2}}\,\eta 1_{2,3}\,\eta 2_{1,2} - e^{-\xi 1_{1,1}+\xi 1_{2,2}+\xi 1_{3,3}}\,\eta 2_{1,2}\,\eta 2_{1,3}\,\xi 1_{1,2} - e^{-\xi 1_{1,1}+2\,\xi 1_{3,3}}\,\eta 2_{1,3}^2\,\xi 1_{1,3} + e^{-\xi 1_{1,1}+\xi 1_{2,2}+\xi 1_{3,3}}\,\eta 1_{2,3}\,\eta 2_{1,3}\,\xi 1_{2,3} - e^{-\xi 1_{1,1}+\xi 1_{2,2}+2\,\xi 1_{3,3}}\,\eta 2_{1,3}^2\,\xi 1_{1,2}\,\xi 1_{2,3},\ g_{1,2}[1] \to$$
$$-e^{-\xi 1_{1,1}+2\,\xi 1_{2,2}}\,\eta 2_{1,2}^2\,\xi 1_{1,2} - e^{-\xi 1_{1,1}+\xi 1_{2,2}+\xi 1_{3,3}}\,\eta 2_{1,2}\,\eta 2_{1,3}\,\xi 1_{1,3} - 2\,e^{-\xi 1_{1,1}+2\,\xi 1_{2,2}+\xi 1_{3,3}}\,\eta 2_{1,2}\,\eta 2_{1,3}\,\xi 1_{1,2}\,\xi 1_{2,3} - e^{-\xi 1_{1,1}+\xi 1_{2,2}+2\,\xi 1_{3,3}}\,\eta 2_{1,3}^2\,\xi 1_{1,3}\,\xi 1_{2,3} - e^{-\xi 1_{1,1}+2\,\xi 1_{2,2}+2\,\xi 1_{3,3}}\,\eta 2_{1,3}^2\,\xi 1_{1,2}\,\xi 1_{2,3}^2,$$
$$g_{2,3}[1] \to e^{\xi 1_{3,3}}\,\eta 2_{1,2}\,\eta 2_{2,3}\,\xi 1_{1,2} - e^{-\xi 1_{2,2}+2\,\xi 1_{3,3}}\,\eta 2_{1,3}\,\eta 2_{2,3}\,\xi 1_{1,3} + e^{2\,\xi 1_{3,3}}\,\eta 2_{1,3}^2\,\xi 1_{1,2}\,\xi 1_{1,3} - e^{-\xi 1_{2,2}+2\,\xi 1_{3,3}}\,\eta 2_{2,3}^2\,\xi 1_{1,2} + e^{2\,\xi 1_{3,3}}\,\eta 2_{1,3}\,\eta 2_{2,3}\,\xi 1_{1,2}\,\xi 1_{2,3}\Big\}$$

pdf

$\mathfrak{gl}_{4,\epsilon}$ Experiments

pdf

In[•]:=
```
SetAlgebra[gl4,ϵ];
$k = 1
$PBWRule
MatrixForm@Table[{b1, b2} → B[b1, b2], {b1, $Basis}, {b2, $Basis}];
MatrixForm@
  Table[{b1, b2} → (DefRep[b1].DefRep[b2] - DefRep[b2].DefRep[b1] == DefRep[B[b1, b2]]),
   {b1, $Basis}, {b2, $Basis}];
DeleteCases[Flatten[Table[{a, b, c} → B[a, B[b, c]] + B[b, B[c, a]] + B[c, B[a, b]],
   {a, $Basis}, {b, $Basis}, {c, $Basis}]], _ → 0]
```

Out[•]=
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1

Out[•]=
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$$\{y_{1,2}\to 1,\ y_{1,3}\to 2,\ y_{2,3}\to 3,\ y_{1,4}\to 4,\ y_{2,4}\to 5,\ y_{3,4}\to 6,\ x_{1,1}\to 7,\ x_{1,2}\to 8,\ x_{2,2}\to 9,\ x_{1,3}\to 10,\ x_{2,3}\to 11,\ x_{3,3}\to 12,\ x_{1,4}\to 13,\ x_{2,4}\to 14,\ x_{3,4}\to 15,\ x_{4,4}\to 16\}$$

Out[•]=
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{}

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In[•]:=
```
SBasis = SortBy[$Basis, If[#[[1]] === x, {0, #[[3]] - #[[2]]}, {1, #[[2]] - #[[3]]}] &];
PBasis = SBasis;
sol[-1] = Table[If[SSQ[bb], ((bb /. x → ξ1) + (bb /. x → ξ2)), 0], {bb, PBasis}];
Do[
  Block[{$k = k},
   s0 = Thread[PBasis → sol[k - 1]];
   lhs = λTangent@Echo[
       Join[Table[If[NilQ[bb], λ, 1] bb (bb /. {x → ξ1, y → η1}), {bb, PBasis}],
        Table[If[NilQ[bb], λ, 1] bb (bb /. {x → ξ2, y → η2}), {bb, PBasis}]]
     ];
   rhs = λTangent[
     PBasis (PBasis /. {x_αβ__ :> (x_αβ /. s0) + ϵ^k f_αβ[λ], y_αβ__ :> (y_αβ /. s0) + ϵ^k g_αβ[λ]})
    ];
   Echo@Short[diff = CF[lhs - rhs], 20];
   eqns = Transpose@{
       (Coefficient[diff, #] == 0) & /@ SBasis,
       SBasis /. {x_αβ__ :> f_αβ[0] == 0, y_αβ__ :> g_αβ[0] == 0}
     };
   Echo@Length@eqns;
   unknowns = SBasis /. {x_αβ__ → f_αβ[λ], y_αβ__ → g_αβ[λ]};
   s = TriangularDSolve[eqns, unknowns, λ];
   sol[k] = CF[sol[k - 1] + ϵ^k (PBasis /. {x_αβ__ → f_αβ[λ], y_αβ__ → g_αβ[λ]} /. s)]
  ],
  {k, 0, $k}
 ];
cm[i_, j_ → k_] := Λ2𝔼_{i,j}→{k}[sol[$k].PBasis /. {x_αβ__ :> x_αβ[k], y_αβ__ :> y_αβ[k],
      ξ1_αβ__ :> ξ_αβ[i], η1_αβ__ :> η_αβ[i], ξ2_αβ__ :> ξ_αβ[j], η2_αβ__ :> η_αβ[j], λ → 1}];
Short[cm[i, j → k], 20]
```

pdf

» $\{x_{1,1}\,\xi 1_{1,1},\ x_{2,2}\,\xi 1_{2,2},\ x_{3,3}\,\xi 1_{3,3},\ x_{4,4}\,\xi 1_{4,4},\ \lambda\, x_{1,2}\,\xi 1_{1,2},\ \lambda\, x_{2,3}\,\xi 1_{2,3},\ \lambda\, x_{3,4}\,\xi 1_{3,4},\ \lambda\, x_{1,3}\,\xi 1_{1,3},$
$\lambda\, x_{2,4}\,\xi 1_{2,4},\ \lambda\, x_{1,4}\,\xi 1_{1,4},\ \lambda\, y_{1,4}\,\eta 1_{1,4},\ \lambda\, y_{1,3}\,\eta 1_{1,3},\ \lambda\, y_{2,4}\,\eta 1_{2,4},\ \lambda\, y_{1,2}\,\eta 1_{1,2},\ \lambda\, y_{2,3}\,\eta 1_{2,3},\ \lambda\, y_{3,4}\,\eta 1_{3,4},$
$x_{1,1}\,\xi 2_{1,1},\ x_{2,2}\,\xi 2_{2,2},\ x_{3,3}\,\xi 2_{3,3},\ x_{4,4}\,\xi 2_{4,4},\ \lambda\, x_{1,2}\,\xi 2_{1,2},\ \lambda\, x_{2,3}\,\xi 2_{2,3},\ \lambda\, x_{3,4}\,\xi 2_{3,4},\ \lambda\, x_{1,3}\,\xi 2_{1,3},$
$\lambda\, x_{2,4}\,\xi 2_{2,4},\ \lambda\, x_{1,4}\,\xi 2_{1,4},\ \lambda\, y_{1,4}\,\eta 2_{1,4},\ \lambda\, y_{1,3}\,\eta 2_{1,3},\ \lambda\, y_{2,4}\,\eta 2_{2,4},\ \lambda\, y_{1,2}\,\eta 2_{1,2},\ \lambda\, y_{2,3}\,\eta 2_{2,3},\ \lambda\, y_{3,4}\,\eta 2_{3,4}\}$

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» $e^{\xi 2_{1,1}-\xi 2_{2,2}}\, y_{1,2}\, \eta 1_{1,2} + e^{\xi 2_{1,1}-\xi 2_{3,3}}\, y_{1,3}\, \eta 1_{1,3} + e^{\xi 2_{1,1}-\xi 2_{4,4}}\, y_{1,4}\, \eta 1_{1,4} + e^{\xi 2_{2,2}-\xi 2_{3,3}}\, y_{2,3}\, \eta 1_{2,3} + e^{\xi 2_{2,2}-\xi 2_{4,4}}\, y_{2,4}\, \eta 1_{2,4} +$
$e^{\xi 2_{3,3}-\xi 2_{4,4}}\, y_{3,4}\, \eta 1_{3,4} + y_{1,2}\, \eta 2_{1,2} + y_{1,3}\, \eta 2_{1,3} + y_{1,4}\, \eta 2_{1,4} + y_{2,3}\, \eta 2_{2,3} + y_{2,4}\, \eta 2_{2,4} + y_{3,4}\, \eta 2_{3,4} +$
$e^{-\xi 2_{1,1}+\xi 2_{2,2}}\, x_{1,2}\, \xi 1_{1,2} - e^{\xi 2_{2,2}-\xi 2_{3,3}}\, \lambda\, y_{2,3}\, \eta 1_{1,3}\, \xi 1_{1,2} - e^{\xi 2_{2,2}-\xi 2_{4,4}}\, \lambda\, y_{2,4}\, \eta 1_{1,4}\, \xi 1_{1,2} - e^{-\xi 2_{1,1}+\xi 2_{2,2}}\, \lambda\, y_{2,3}\, \eta 2_{1,3}\, \xi 1_{1,2} -$
$e^{-\xi 2_{1,1}+\xi 2_{2,2}}\, \lambda\, y_{2,4}\, \eta 2_{1,4}\, \xi 1_{1,2} + e^{-\xi 2_{1,1}+\xi 2_{3,3}}\, x_{1,3}\, \xi 1_{1,3} - e^{\xi 2_{3,3}-\xi 2_{4,4}}\, \lambda\, y_{3,4}\, \eta 1_{1,4}\, \xi 1_{1,3} - e^{-\xi 2_{1,1}+\xi 2_{3,3}}\, \lambda\, y_{3,4}\, \eta 2_{1,4}\, \xi 1_{1,3} +$
$e^{-\xi 2_{1,1}+\xi 2_{4,4}}\, x_{1,4}\, \xi 1_{1,4} + e^{-\xi 2_{2,2}+\xi 2_{3,3}}\, x_{2,3}\, \xi 1_{2,3} + e^{\xi 2_{1,1}-\xi 2_{2,2}}\, \lambda\, y_{1,2}\, \eta 1_{1,3}\, \xi 1_{2,3} - e^{\xi 2_{3,3}-\xi 2_{4,4}}\, \lambda\, y_{3,4}\, \eta 1_{2,4}\, \xi 1_{2,3} +$
$e^{-\xi 2_{2,2}+\xi 2_{3,3}}\, \lambda\, y_{1,2}\, \eta 2_{1,3}\, \xi 1_{2,3} - e^{-\xi 2_{2,2}+\xi 2_{3,3}}\, \lambda\, y_{3,4}\, \eta 2_{2,4}\, \xi 1_{2,3} + e^{-\xi 2_{1,1}+\xi 2_{3,3}}\, \lambda\, x_{1,3}\, \xi 1_{1,2}\, \xi 1_{2,3} -$
$e^{\xi 2_{3,3}-\xi 2_{4,4}}\, \lambda^2\, y_{3,4}\, \eta 1_{1,4}\, \xi 1_{1,2}\, \xi 1_{2,3} - e^{-\xi 2_{1,1}+\xi 2_{3,3}}\, \lambda^2\, y_{3,4}\, \eta 2_{1,4}\, \xi 1_{1,2}\, \xi 1_{2,3} + e^{-\xi 2_{2,2}+\xi 2_{4,4}}\, x_{2,4}\, \xi 1_{2,4} +$
$e^{\xi 2_{1,1}-\xi 2_{2,2}}\, \lambda\, y_{1,2}\, \eta 1_{1,4}\, \xi 1_{2,4} + e^{-\xi 2_{2,2}+\xi 2_{4,4}}\, \lambda\, y_{1,2}\, \eta 2_{1,4}\, \xi 1_{2,4} + e^{-\xi 2_{1,1}+\xi 2_{4,4}}\, \lambda\, x_{1,4}\, \xi 1_{1,2}\, \xi 1_{2,4} +$
$e^{-\xi 2_{3,3}+\xi 2_{4,4}}\, x_{3,4}\, \xi 1_{3,4} + e^{\xi 2_{1,1}-\xi 2_{3,3}}\, \lambda\, y_{1,3}\, \eta 1_{1,4}\, \xi 1_{3,4} + \ll 162 \gg + x_{1,3}\, f_{1,3}[\lambda]\, f_{3,3}'[\lambda] + x_{2,3}\, f_{2,3}[\lambda]\, f_{3,3}'[\lambda] -$
$x_{3,4}\, f_{3,4}[\lambda]\, f_{3,3}'[\lambda] + x_{1,4}\, f_{1,3}[\lambda]\, f_{3,4}[\lambda]\, f_{3,3}'[\lambda] + x_{2,4}\, f_{2,3}[\lambda]\, f_{3,4}[\lambda]\, f_{3,3}'[\lambda] - y_{1,3}\, g_{1,3}[\lambda]\, f_{3,3}'[\lambda] +$
$y_{1,2}\, f_{2,3}[\lambda]\, g_{1,3}[\lambda]\, f_{3,3}'[\lambda] - y_{3,4}\, f_{1,3}[\lambda]\, g_{1,4}[\lambda]\, f_{3,3}'[\lambda] - y_{1,3}\, f_{3,4}[\lambda]\, g_{1,4}[\lambda]\, f_{3,3}'[\lambda] +$
$y_{1,2}\, f_{2,3}[\lambda]\, f_{3,4}[\lambda]\, g_{1,4}[\lambda]\, f_{3,3}'[\lambda] - y_{2,3}\, g_{2,3}[\lambda]\, f_{3,3}'[\lambda] - y_{3,4}\, f_{2,3}[\lambda]\, g_{2,4}[\lambda]\, f_{3,3}'[\lambda] -$
$y_{2,3}\, f_{3,4}[\lambda]\, g_{2,4}[\lambda]\, f_{3,3}'[\lambda] + y_{3,4}\, g_{3,4}[\lambda]\, f_{3,3}'[\lambda] - x_{3,4}\, f_{3,4}'[\lambda] + x_{1,4}\, f_{1,3}[\lambda]\, f_{3,4}'[\lambda] -$
$y_{1,3}\, g_{1,4}[\lambda]\, f_{3,4}'[\lambda] - y_{2,3}\, g_{2,4}[\lambda]\, f_{3,4}'[\lambda] - x_{4,4}\, f_{4,4}'[\lambda] + x_{1,4}\, f_{1,4}[\lambda]\, f_{4,4}'[\lambda] + x_{2,4}\, f_{2,4}[\lambda]\, f_{4,4}'[\lambda] +$
$x_{3,4}\, f_{3,4}[\lambda]\, f_{4,4}'[\lambda] - x_{1,4}\, f_{1,3}[\lambda]\, f_{3,4}[\lambda]\, f_{4,4}'[\lambda] - y_{1,4}\, g_{1,4}[\lambda]\, f_{4,4}'[\lambda] + y_{1,2}\, f_{2,4}[\lambda]\, g_{1,4}[\lambda]\, f_{4,4}'[\lambda] +$
$y_{1,3}\, f_{3,4}[\lambda]\, g_{1,4}[\lambda]\, f_{4,4}'[\lambda] - y_{2,4}\, g_{2,4}[\lambda]\, f_{4,4}'[\lambda] + y_{2,3}\, f_{3,4}[\lambda]\, g_{2,4}[\lambda]\, f_{4,4}'[\lambda] - y_{3,4}\, g_{3,4}[\lambda]\, f_{4,4}'[\lambda] -$
$y_{1,2}\, g_{1,2}'[\lambda] - y_{1,3}\, g_{1,3}'[\lambda] - y_{1,4}\, g_{1,4}'[\lambda] - y_{2,3}\, g_{2,3}'[\lambda] - y_{2,4}\, g_{2,4}'[\lambda] - y_{3,4}\, g_{3,4}'[\lambda]$

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» 16

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» $\{x_{1,1}\, \xi 1_{1,1},\ x_{2,2}\, \xi 1_{2,2},\ x_{3,3}\, \xi 1_{3,3},\ x_{4,4}\, \xi 1_{4,4},\ \lambda\, x_{1,2}\, \xi 1_{1,2},\ \lambda\, x_{2,3}\, \xi 1_{2,3},\ \lambda\, x_{3,4}\, \xi 1_{3,4},\ \lambda\, x_{1,3}\, \xi 1_{1,3},$
$\lambda\, x_{2,4}\, \xi 1_{2,4},\ \lambda\, x_{1,4}\, \xi 1_{1,4},\ \lambda\, y_{1,4}\, \eta 1_{1,4},\ \lambda\, y_{1,3}\, \eta 1_{1,3},\ \lambda\, y_{2,4}\, \eta 1_{2,4},\ \lambda\, y_{1,2}\, \eta 1_{1,2},\ \lambda\, y_{2,3}\, \eta 1_{2,3},\ \lambda\, y_{3,4}\, \eta 1_{3,4},$
$x_{1,1}\, \xi 2_{1,1},\ x_{2,2}\, \xi 2_{2,2},\ x_{3,3}\, \xi 2_{3,3},\ x_{4,4}\, \xi 2_{4,4},\ \lambda\, x_{1,2}\, \xi 2_{1,2},\ \lambda\, x_{2,3}\, \xi 2_{2,3},\ \lambda\, x_{3,4}\, \xi 2_{3,4},\ \lambda\, x_{1,3}\, \xi 2_{1,3},$
$\lambda\, x_{2,4}\, \xi 2_{2,4},\ \lambda\, x_{1,4}\, \xi 2_{1,4},\ \lambda\, y_{1,4}\, \eta 2_{1,4},\ \lambda\, y_{1,3}\, \eta 2_{1,3},\ \lambda\, y_{2,4}\, \eta 2_{2,4},\ \lambda\, y_{1,2}\, \eta 2_{1,2},\ \lambda\, y_{2,3}\, \eta 2_{2,3},\ \lambda\, y_{3,4}\, \eta 2_{3,4}\}$

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» $2\, e^{\xi 2_{2,2}-\xi 2_{3,3}} \in \lambda\, y_{1,3}\, \eta 1_{2,3}\, \eta 2_{1,2} + 2\, e^{\xi 2_{3,3}-\xi 2_{4,4}} \in \lambda\, y_{1,4}\, \eta 1_{3,4}\, \eta 2_{1,3} + 2\, e^{\xi 2_{3,3}-\xi 2_{4,4}} \in \lambda\, y_{2,4}\, \eta 1_{3,4}\, \eta 2_{2,3} -$
$2\, e^{\xi 2_{1,1}-\xi 2_{2,2}} \in \lambda\, y_{1,4}\, \eta 1_{1,2}\, \eta 2_{2,4} - e^{-\xi 2_{1,1}+2\, \xi 2_{2,2}-\xi 2_{3,3}} \in \lambda^2\, y_{2,3}\, \eta 1_{2,3}\, \eta 2_{1,2}\, \xi 1_{1,2} -$
$e^{-\xi 2_{1,1}+\xi 2_{2,2}+\xi 2_{3,3}-\xi 2_{4,4}} \in \lambda^2\, y_{2,4}\, \eta 1_{3,4}\, \eta 2_{1,3}\, \xi 1_{1,2} + \epsilon\, \lambda^2\, y_{2,4}\, \eta 1_{1,2}\, \eta 2_{2,4}\, \xi 1_{1,2} -$
$e^{-\xi 2_{1,1}+2\, \xi 2_{3,3}-\xi 2_{4,4}} \in \lambda^2\, y_{3,4}\, \eta 1_{3,4}\, \eta 2_{1,3}\, \xi 1_{1,3} + e^{-\xi 2_{2,2}+\xi 2_{3,3}} \in \lambda^2\, y_{3,4}\, \eta 1_{1,2}\, \eta 2_{2,4}\, \xi 1_{1,3} + \epsilon\, \lambda^2\, y_{1,2}\, \eta 1_{2,3}\, \eta 2_{1,2}\, \xi 1_{2,3} -$
$e^{-\xi 2_{2,2}+2\, \xi 2_{3,3}-\xi 2_{4,4}} \in \lambda^2\, y_{3,4}\, \eta 1_{3,4}\, \eta 2_{2,3}\, \xi 1_{2,3} - e^{-\xi 2_{1,1}+2\, \xi 2_{3,3}-\xi 2_{4,4}} \in \lambda^3\, y_{3,4}\, \eta 1_{3,4}\, \eta 2_{1,3}\, \xi 1_{1,2}\, \xi 1_{2,3} +$
$e^{-\xi 2_{2,2}+\xi 2_{3,3}} \in \lambda^3\, y_{3,4}\, \eta 1_{1,2}\, \eta 2_{2,4}\, \xi 1_{1,2}\, \xi 1_{2,3} + e^{-\xi 2_{2,2}+\xi 2_{3,3}} \in \lambda^2\, y_{1,2}\, \eta 1_{3,4}\, \eta 2_{1,3}\, \xi 1_{2,4} -$
$e^{\xi 2_{1,1}-2\, \xi 2_{2,2}+\xi 2_{4,4}} \in \lambda^2\, y_{1,2}\, \eta 1_{1,2}\, \eta 2_{2,4}\, \xi 1_{2,4} + \epsilon\, \lambda^2\, y_{1,3}\, \eta 1_{3,4}\, \eta 2_{1,3}\, \xi 1_{3,4} + \epsilon\, \lambda^2\, y_{2,3}\, \eta 1_{3,4}\, \eta 2_{2,3}\, \xi 1_{3,4} -$
$e^{\xi 2_{1,1}-\xi 2_{2,2}-\xi 2_{3,3}+\xi 2_{4,4}} \in \lambda^2\, y_{1,3}\, \eta 1_{1,2}\, \eta 2_{2,4}\, \xi 1_{3,4} + e^{-\xi 2_{2,2}+\xi 2_{3,3}} \in \lambda^3\, y_{1,2}\, \eta 1_{3,4}\, \eta 2_{1,3}\, \xi 1_{2,3}\, \xi 1_{3,4} -$
$e^{\xi 2_{1,1}-2\, \xi 2_{2,2}+\xi 2_{4,4}} \in \lambda^3\, y_{1,2}\, \eta 1_{1,2}\, \eta 2_{2,4}\, \xi 1_{2,3}\, \xi 1_{3,4} - 2\, e^{\xi 2_{1,1}-\xi 2_{2,2}} \in \lambda\, x_{1,1}\, \eta 1_{1,2}\, \xi 2_{1,2} +$
$2\, e^{\xi 2_{1,1}-\xi 2_{2,2}} \in \lambda\, x_{2,2}\, \eta 1_{1,2}\, \xi 2_{1,2} + e^{2\, \xi 2_{1,1}-2\, \xi 2_{2,2}} \in \lambda^2\, y_{1,2}\, \eta 1_{1,2}^2\, \xi 2_{1,2} + \ll 1810 \gg + \epsilon\, \lambda\, x_{1,4}\, \xi 2_{1,4}\, f_{4,4}'[\lambda] -$
$e^{\xi 2_{2,2}-\xi 2_{4,4}} \in \lambda^2\, y_{3,4}\, \eta 1_{2,4}\, \xi 2_{2,3}\, f_{4,4}'[\lambda] - e^{-\xi 2_{3,3}+\xi 2_{4,4}} \in \lambda^2\, x_{2,4}\, \xi 1_{3,4}\, \xi 2_{2,3}\, f_{4,4}'[\lambda] -$
$e^{\xi 2_{1,1}-\xi 2_{3,3}} \in \lambda^3\, y_{1,2}\, \eta 1_{1,4}\, \xi 1_{3,4}\, \xi 2_{2,3}\, f_{4,4}'[\lambda] - e^{-\xi 2_{3,3}+\xi 2_{4,4}} \in \lambda^3\, y_{1,2}\, \eta 2_{1,4}\, \xi 1_{3,4}\, \xi 2_{2,3}\, f_{4,4}'[\lambda] -$
$e^{\xi 2_{1,1}-\xi 2_{4,4}} \in \lambda^3\, y_{3,4}\, \eta 1_{1,4}\, \xi 2_{1,2}\, \xi 2_{2,3}\, f_{4,4}'[\lambda] + \epsilon\, \lambda\, x_{2,4}\, \xi 2_{2,4}\, f_{4,4}'[\lambda] + e^{\xi 2_{1,1}-\xi 2_{4,4}} \in \lambda^2\, y_{1,2}\, \eta 1_{1,4}\, \xi 2_{2,4}\, f_{4,4}'[\lambda] +$
$\epsilon\, \lambda^2\, y_{1,2}\, \eta 2_{1,4}\, \xi 2_{2,4}\, f_{4,4}'[\lambda] + \epsilon\, \lambda\, x_{3,4}\, \xi 2_{3,4}\, f_{4,4}'[\lambda] + e^{\xi 2_{1,1}-\xi 2_{4,4}} \in \lambda^2\, y_{1,3}\, \eta 1_{1,4}\, \xi 2_{3,4}\, f_{4,4}'[\lambda] +$
$e^{\xi 2_{2,2}-\xi 2_{4,4}} \in \lambda^2\, y_{2,3}\, \eta 1_{2,4}\, \xi 2_{3,4}\, f_{4,4}'[\lambda] + \epsilon\, \lambda^2\, y_{1,3}\, \eta 2_{1,4}\, \xi 2_{3,4}\, f_{4,4}'[\lambda] + \epsilon\, \lambda^2\, y_{2,3}\, \eta 2_{2,4}\, \xi 2_{3,4}\, f_{4,4}'[\lambda] +$
$e^{\xi 2_{1,1}-\xi 2_{4,4}} \in \lambda^3\, y_{2,3}\, \eta 1_{1,4}\, \xi 2_{1,2}\, \xi 2_{3,4}\, f_{4,4}'[\lambda] - \epsilon\, \lambda^2\, x_{1,4}\, \xi 2_{1,3}\, \xi 2_{3,4}\, f_{4,4}'[\lambda] - \epsilon\, y_{1,2}\, g_{1,2}'[\lambda] -$
$\epsilon\, y_{1,3}\, g_{1,3}'[\lambda] - \epsilon\, y_{1,4}\, g_{1,4}'[\lambda] - \epsilon\, y_{2,3}\, g_{2,3}'[\lambda] - \epsilon\, y_{2,4}\, g_{2,4}'[\lambda] - \epsilon\, y_{3,4}\, g_{3,4}'[\lambda]$

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» 16

Out[•]//Short=

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$$\mathbb{E}_{\{i,j\}\to\{k\}}\Big[y_{1,4}[k]\,\Big(e^{\xi_{1,1}[j]-\xi_{4,4}[j]}\,\eta_{1,4}[i]+\eta_{1,4}[j]\Big)+ x_{1,1}[k]\,(\xi_{1,1}[i]+\xi_{1,1}[j])+x_{1,2}[k]\,\Big(e^{-\xi_{1,1}[j]+\xi_{2,2}[j]}\,\xi_{1,2}[i]+\xi_{1,2}[j]\Big)+ y_{2,4}[k]\,\Big(e^{\xi_{2,2}[j]-\xi_{4,4}[j]}\,\eta_{2,4}[i]+\eta_{2,4}[j]+e^{\xi_{1,1}[j]-\xi_{4,4}[j]}\,\eta_{1,4}[i]\,\xi_{1,2}[j]\Big)+ x_{2,2}[k]\,(\xi_{2,2}[i]+\xi_{2,2}[j])+ x_{1,3}[k]\,\Big(e^{-\xi_{1,1}[j]+\xi_{3,3}[j]}\,\xi_{1,3}[i]+\xi_{1,3}[j]-e^{-\xi_{2,2}[j]+\xi_{3,3}[j]}\,\xi_{1,2}[j]\,\xi_{2,3}[i]\Big)+ x_{2,3}[k]\,\Big(e^{-\xi_{2,2}[j]+\xi_{3,3}[j]}\,\xi_{2,3}[i]+\xi_{2,3}[j]\Big)+e^{-\xi_{4,4}[j]}\,y_{3,4}[k]\,\Big(e^{\xi_{3,3}[j]}\,\eta_{3,4}[i]+e^{\xi_{4,4}[j]}\,\eta_{3,4}[j]+ e^{\xi_{1,1}[j]}\,\eta_{1,4}[i]\,\xi_{1,3}[j]+e^{\xi_{2,2}[j]}\,\eta_{2,4}[i]\,\xi_{2,3}[j]+e^{\xi_{1,1}[j]}\,\eta_{1,4}[i]\,\xi_{1,2}[j]\,\xi_{2,3}[j]\Big)+y_{1,2}[k]\,\Big(e^{\xi_{1,1}[j]-\xi_{2,2}[j]}\,\eta_{1,2}[i]+\eta_{1,2}[j]-e^{\xi_{1,1}[j]-\xi_{3,3}[j]}\,\eta_{1,3}[i]\,\xi_{2,3}[j]-e^{\xi_{1,1}[j]-\xi_{4,4}[j]}\,\eta_{1,4}[i]\,\xi_{2,4}[j]\Big)+ x_{3,3}[k]\,(\xi_{3,3}[i]+\xi_{3,3}[j])+ x_{2,4}[k]\,\Big(e^{-\xi_{2,2}[j]+\xi_{4,4}[j]}\,\xi_{2,4}[i]+\xi_{2,4}[j]-e^{-\xi_{3,3}[j]+\xi_{4,4}[j]}\,\xi_{2,3}[j]\,\xi_{3,4}[i]\Big)+ x_{3,4}[k]\,\Big(e^{-\xi_{3,3}[j]+\xi_{4,4}[j]}\,\xi_{3,4}[i]+\xi_{3,4}[j]\Big)+ y_{1,3}[k]\,\Big(e^{\xi_{1,1}[j]-\xi_{3,3}[j]}\,\eta_{1,3}[i]+\eta_{1,3}[j]-e^{\xi_{1,1}[j]-\xi_{4,4}[j]}\,\eta_{1,4}[i]\,\xi_{3,4}[j]\Big)+ y_{2,3}[k]\,\Big(e^{\xi_{2,2}[j]-\xi_{3,3}[j]}\,\eta_{2,3}[i]+\eta_{2,3}[j]+e^{\xi_{1,1}[j]-\xi_{3,3}[j]}\,\eta_{1,3}[i]\,\xi_{1,2}[j]- e^{\xi_{2,2}[j]-\xi_{4,4}[j]}\,\eta_{2,4}[i]\,\xi_{3,4}[j]-e^{\xi_{1,1}[j]-\xi_{4,4}[j]}\,\eta_{1,4}[i]\,\xi_{1,2}[j]\,\xi_{3,4}[j]\Big)+ x_{1,4}[k]\,\Big(e^{-\xi_{1,1}[j]+\xi_{4,4}[j]}\,\xi_{1,4}[i]+\xi_{1,4}[j]-e^{-\xi_{2,2}[j]+\xi_{4,4}[j]}\,\xi_{1,2}[j]\,\xi_{2,4}[i]- e^{-\xi_{2,2}[j]+\xi_{4,4}[j]}\,\xi_{1,2}[j]\,\xi_{2,3}[i]\,\xi_{3,4}[i]+e^{-\xi_{1,1}[j]+\xi_{3,3}[j]}\,\xi_{1,3}[i]\,\xi_{3,4}[j]- e^{-\xi_{2,2}[j]+\xi_{3,3}[j]}\,\xi_{1,2}[j]\,\xi_{2,3}[i]\,\xi_{3,4}[j]\Big)+x_{4,4}[k]\,(\xi_{4,4}[i]+\xi_{4,4}[j]),\; x_{2,2}[k]\,\Big(e^{\xi_{1,1}[j]-\xi_{2,2}[j]}\,\eta_{1,2}[i]\,\xi_{1,2}[j]-e^{\xi_{2,2}[j]-\xi_{3,3}[j]}\,\eta_{2,3}[i]\,\xi_{2,3}[j]-e^{\xi_{1,1}[j]-\xi_{3,3}[j]}\,\eta_{1,3}[i]\,\xi_{1,2}[j]\,\xi_{2,3}[j]-e^{\xi_{2,2}[j]-\xi_{4,4}[j]}\,\eta_{2,4}[i]\,\xi_{2,4}[j]-e^{\xi_{1,1}[j]-\xi_{4,4}[j]}\,\eta_{1,4}[i]\,\xi_{1,2}[j]\,\xi_{2,4}[j]\Big)+ x_{1,1}[k]\,\Big(-e^{\xi_{1,1}[j]-\xi_{2,2}[j]}\,\eta_{1,2}[i]\,\xi_{1,2}[j]-e^{\xi_{1,1}[j]-\xi_{3,3}[j]}\,\eta_{1,3}[i]\,\xi_{1,3}[j]- e^{\xi_{1,1}[j]-\xi_{4,4}[j]}\,\eta_{1,4}[i]\,\xi_{1,4}[j]+e^{\xi_{1,1}[j]-\xi_{4,4}[j]}\,\eta_{1,4}[i]\,\xi_{1,3}[j]\,\xi_{3,4}[j]\Big)+ \ll 14\gg + x_{3,4}[k]\,\Big(e^{\xi_{1,1}[j]-\xi_{\ll 1\gg}[j]}\,\eta_{1,3}[i]\,\xi_{1,4}[j]+\ll 21\gg\Big)+ x_{1,4}[k]\,\Big(e^{-\xi_{2,2}[j]+\xi_{4,4}[j]}\,\eta_{1,2}[i]\,\xi_{1,2}[j]\,\xi_{1,4}[i]+e^{-\xi_{3,3}[j]+\xi_{4,4}[j]}\,\eta_{1,3}[i]\,\xi_{1,3}[j]\,\xi_{1,4}[i]+ \eta_{1,3}[i]\,\xi_{1,3}[i]\,\xi_{1,4}[j]+e^{\xi_{1,1}[j]-\xi_{3,3}[j]}\,\eta_{1,3}[i]\,\xi_{1,3}[j]\,\xi_{1,4}[j]+\ll 82\gg\Big)\Big]$$

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In[•]:=
```
(cm[1, 2 → 2] // cm[2, 3 → 1]) ≡ (cm[2, 3 → 2] // cm[1, 2 → 1])
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Out[•]=

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True