

Hidden T_EX

Headers

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Pensieve header: The PBW multiplication tensor for $\$gl_{\{n,\epsilon\}}\$$ using the nilpotent-only $\$ \lambda$ -tangent formalism. Some material from pensieve://Projects/glneps and from pensieve://Projects/UEA.

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```
In[ ]:= SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
```

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Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at <http://katlas.org/wiki/KnotTheory>.

nb2tex

```
In[ ]:= Get["./nb2tex/nb2tex.m"];
nb2tex["gln", PDFWidth -> 7.5, PDFFolder -> "Snips"];
texfile = OpenAppend["gln.tex"]; WriteLine[texfile, "\\end{document}"]; Close[texfile];
Run@
  "\"C:\\Users\\drorb\\AppData\\Local\\Programs\\MiKTeX\\miktex\\bin\\x64\\pdflatex.exe\"
  gln.tex";
```

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Conventions

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$x_{\alpha,\beta}$ is always with $\alpha \leq \beta$ and represents a basis element of the upper triangular matrices. $y_{\alpha,\beta}$ is also always with $\alpha \leq \beta$ and represents a basis element of the lower triangular matrices. The adjoint of $x_{\alpha,\beta}$ is $y_{\alpha,\beta}$.

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Variable declarations:

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```
In[ ]:= VarPattern = ((y | x | η | ξ) | (y | x | η | ξ)__) [ _ ];
```

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Greek - Latin duality:

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```
In[ ]:= {y*, x*, η*, ξ*} = {η, ξ, y, x};
(u_{αβ})^* := (u^*)_{αβ};
((u : ((y | x | η | ξ) | (y | x | η | ξ)__)) [i_])^* := u^*[i];
(vs_List)^* := (v ↦ v^*) /@ vs;
```

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```
In[*]:= {x1,2, y2,3, η1,2[7]}*
```

Out[*]=

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```
{ξ1,2, η2,3, y1,2[7]}
```

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Weights:

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```
In[*]:= Wt[xα,β] := {1, α - β}; Wt[yα,β] := {0, α - β};
Wt[ξα,β] := {0, β - α}; Wt[ηα,β] := {1, β - α};
Wt[u-] := Wt[u];
```

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The Engine

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Canonical Forms:

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```
In[*]:= LogReduce[ε_] :=
  ε /. c_ * Log[a_] :=> Log@Factor[a^c] /. Log[a_] + Log[b_] :=> Log@Factor[a b];
CF[ε_] := Expand@Collect[ε, Cases[ε, VarPattern, ∞], CCF] /. CCF -> Factor;
CF[ε_E] := CF /@ MapAt[LogReduce, ε, 1];
CF[ε : E___] := CF /@ ε;
```

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E operations:

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```
In[*]:= ε_E[$] := Length[ε] - 1; E_[εs___][$] := E[εs][$];
ε_E[k_Integer] := ε[[k + 1]]; E_[εs___][k_Integer] := {εs}[[k + 1]];
E /: ε1_E ≡ ε2_E := Inner[CF@#1 == CF@#2 &, ε1, ε2, And];
Ed1→r1[ε1s___] ≡ Ed2→r2[ε2s___] ^:= (d1 == d2) ∧ (r1 == r2) ∧ (E[ε1s] ≡ E[ε2s]);
E /: ε1_E * ε2_E := E @@ Table[CF[ε1[kk] + ε2[kk]], {kk, 0, Min[ε1[$], ε2[$]]};
Ed1→r1[ε1s___] Ed2→r2[ε2s___] ^:= E(d1∪d2)→(r1∪r2) @@ (E[ε1s] E[ε2s]);
```

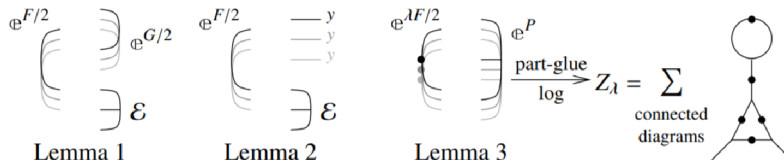
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```
In[*]:= Ed1→r1[ε1s___] // Ed2→r2[ε2s___] := Module[{is = r1 ∩ d2, lvs, llvs},
  lvs = Cases[{ε1s}, (x | y)_[i_] /; MemberQ[is, i], ∞] ∪
  Cases[{ε2s}, (ξ | η)_[i_] /; MemberQ[is, i], ∞]*;
  llvs = Map[$, lvs, {2}];
  E(d1∪Complement[d2,is])→(r2∪Complement[r1,is]) @@ (Zipllvs∪llvs [{(llvs)*.(llvs)}, Times[
    E[ε1s] /. Thread[lvs → llvs],
    E[ε2s] /. Thread[lvs* → llvs*]
  ]])
]
```

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Zippping! Lemmas 2 and 3 are combined, yet they must be applied first to the middle weight variables and then to the heavy and light variables.

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Comment. Zip3 of the outer variables must occur after all other operations are completed, because we must allow for gluings of the weight n variables in perturbations with the weight 0 variables in the coefficients of Q .

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dvs stands for “Diagonal Variables”. In gl_n they are the x_{ii} ’s and the y_{ii} ’s. They have weights $\{1,0\}$ and $\{0,0\}$.

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```
In[*]:= Zipvs [{ $\mathcal{F}$ _,  $\mathcal{E}$ _}] := Module[{dvs = Select[vs, (Wt[#] == {0, 0}  $\vee$  Wt[#] == {1, 0}) &]},
  { $\mathcal{F}$ _,  $\mathcal{E}$ _} // Zip1vs (* // Zip2Complement[vs,dvs] *) // Zip2dvs //
  Zip3Complement[vs,dvs] // Zip3dvs
] // Last
```

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Getting rid of the quadratic.

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Lemma 1. With convergences left to the reader,

$$\left\langle F : \mathcal{E} \otimes^{\frac{1}{2} \sum_{i,j \in B} G_{ij} z_i z_j} \right\rangle_B = \det(1 - GF)^{-1/2} \left\langle F(1 - GF)^{-1} : \mathcal{E} \right\rangle_B$$

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```
In[*]:= Zip1{} = Identity;
Zip1vs @ { $\mathcal{F}$ _,  $\mathbb{E}[Q_, P\_]$ } := Module[{ $\mathcal{I}$ , F, G, u, v},
   $\mathcal{I}$  = IdentityMatrix@Length@vs;
  F = Table[If[Wt[u] + Wt[v] == {1, 0},  $\partial_{u,v} \mathcal{F}$ , 0], {u, vs}, {v, vs}];
  G = Table[If[Wt[u] + Wt[v] == {1, 0},  $\partial_{u,v} Q$ , 0], {u, vs}, {v, vs}];
  {CF[(vs) . (F.Inverse[ $\mathcal{I}$  - G.F]) . (vs) / 2],
   $\mathbb{E}[CF[Q - PowerExpand@Log[Det[ $\mathcal{I}$  - G.F]] / 2 - vs.G.vs / 2], P]}
]$ 
```

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Getting rid of linear terms.

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Lemma 2. $\left\langle F : \mathcal{E} \otimes^{\sum_{i \in B} y_i z_i} \right\rangle_B = \otimes^{\frac{1}{2} \sum_{i,j \in B} F_{ij} y_i y_j} \left\langle F : \mathcal{E}|_{z_B \rightarrow z_B + F y_B} \right\rangle_B$.

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```

In[*]:= Zip2_{ } = Identity;
Zip2_{vs_} @ {F_, E[Q_, P___]} := Module[{F, Y, u, v},
  F = Table[If[Wt[u] + Wt[v] == {1, 0}, CF[∂_{u,v}F], 0], {u, vs}, {v, vs}];
  Y = Table[∂_v Q, {v, vs}] /. Table[v → 0, {v, vs}];
  CF /@ {F, E[Q - Y.vs + Y.F.Y / 2, P] /. Thread[vs → vs + F.Y]}
]

```

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Dealing with Feynman diagrams.

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Lemma 3. With an extra variable λ , $Z_\lambda := \log[\lambda F : \mathbb{P}]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2} \sum_{i,j \in B} F_{ij} (\partial_{z_i} \partial_{z_j} Z_\lambda + (\partial_{z_i} Z_\lambda)(\partial_{z_j} Z_\lambda)).$$

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Note that the power m of λ is at most $k - 1 + \frac{2k+2}{2} = 2k$. We write $Z_\lambda = \sum Z[m] \lambda^m$.

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```

In[*]:= Zip3_{vs_} @ {F_, E[E_]} := Module[{F, u, v, Z, kk, jj, $m = 0, m, n},
  Do[Z[0, kk] = E[kk + 1], {kk, 0, E@$}];
  F[u_, v_] := F[u, v] = CF@If[Wt[u] + Wt[v] == {1, 0}, ∂_{u,v}F, 0];
  Z[m_, kk_, u_] := Z[m, kk, u] = ∂_u Z[m, kk];
  Z[m_, kk_, u_, v_] := Z[m, kk, u, v] = ∂_v Z[m, kk, u];
  For[m = 0, m ≤ 2 $m, ++m, For[kk = 0, kk ≤ E@$ , ++kk,
    Z[m + 1, kk] = CF@Sum[
      If[F[u, v] == 0, 0, F[u, v] / (2 (m + 1)) (Z[m, kk, u, v] +
        Sum[Z[n, jj, u] * Z[m - n, kk - jj, v], {n, 0, m}, {jj, 0, kk}])],
      {u, vs}, {v, vs}];
    If[Z[m + 1, kk] != 0, $m = m + 1]
  ]];
  CF /@ ({
    F - Sum[F[u, v] * u * v / 2, {u, vs}, {v, vs}],
    E@@Table[Sum[Z[m, kk], {m, 0, $m}], {kk, 0, E@$}]
  } /. Table[v → 0, {v, vs}])
]

```

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```

In[*]:= Λ2E_{d→r_}[A_] := Λ2E_{d→r_}[$k, A];
Λ2E_{d→r_}[$k_, A_] :=
  Module[{k}, E_{d→r_}@@Table[SeriesCoefficient[A, {ε, 0, k}], {k, 0, $k}]]];

```

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Prolog

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```
In[ ]:= (*BeginPackage["UEA`"];*)
Print["UEA` does computations in general universal enveloping
algebras and PBW algebras. It is in the public domain, available
at http://drorbn.net/AcademicPensieve/Projects/SolvablePBW/.
Dror Bar-Natan is committed to support it within
reason until June 1, 2027. This is version 260601."];
Print["UEA` implements / extends ",
Sort@{"**",  $\epsilon$ , $k, CF, SSQ, NilQ, B, m, SetAlgebra, U,
UB, UProducts, USimp, UU, $Basis, $PBWRule,  $\delta$ , adPower, adExp},
"."];
(*Begin["`Private`"];*)
```

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UEA` does computations in general universal enveloping algebras and PBW algebras. It is in the public domain, available at <http://drorbn.net/AcademicPensieve/Projects/SolvablePBW/>. Dror Bar-Natan is committed to support it within reason until June 1, 2027. This is version 260601.

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UEA` implements / extends {**, adExp, adPower, B, CF, m, NilQ, SetAlgebra, SSQ, U, UB, UProducts, USimp, UU, δ , ϵ , \$Basis, \$k, \$PBWRule}.

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Utilities

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```
In[ ]:=  $\delta_{i,j}$  := If[i == j, 1, 0];
```

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```
In[ ]:= SSQ[c_.*x_] /; MemberQ[$Basis, x] :=  $\neg$  NilQ[x]; (* Semi-Simple Q *)
NilQ[c_.*x_] /; MemberQ[$Basis, x] := NilQ[x];
SSQ[x_Plus] := And @@ (SSQ /@ (List @@ x));
NilQ[x_Plus] := And @@ (NilQ /@ (List @@ x));
```

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```
In[ ]:= $k = 1;
```

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```
In[ ]:= CF[ $\mathcal{E}$ _] := Expand[ $\mathcal{E}$  /.  $e^{L-} \rightarrow e^{\text{Expand}[L]}$ ] /. { $e^{k-}$  /;  $k > \$k \rightarrow 0$ };
```

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Implementing general universal enveloping algebras

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```
In[*]:= B[0, _] = 0; B[_ , 0] = 0;
B[c_*x_, y_] /; MemberQ[$Basis, x] := CF[c B[x, y]];
B[y_, c_*x_] /; MemberQ[$Basis, x] := CF[c B[y, x]];
B[x_Plus, y_] := B[#, y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
B[x_, x_] = 0;
B[y_, x_] := CF[-B[x, y]];
```

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```
In[*]:= x_ ≤ y_ := OrderedQ[{x, y} /. $PBWRule]; x_ < y_ := ¬ OrderedQ[{y, x} /. $PBWRule];
UU_i_[1] := U_i[];
UU_i_[x^p_] := UU_i_@@Table[x, {p}];
UU_i_[ε_] := ε /. {
  U[xs__] => U_i[xs],
  x_ /; MemberQ[$Basis, x] => U_i[x]
};
UU_i_[x_, xs__] := UU_t1[x] UU_t2[xs] // Expand // m_t1,t2->i;
USimp[ε_] := Collect[ε, Times[U[___] ..], Expand];
USimp[ε_] := Expand[ε];
```

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```
In[*]:= m_s__[0] = 0;
m_s__[x_Plus] := m_s_/@x;
m_s__[sd_SeriesData] := MapAt[m_s, sd, {3, All}];
m_i->j_[ε_] := ε /. U_i->U_j;
```

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```
In[*]:= m_i,j->k_[c_. U_i_[x__] U_j_[y__]] := c U_k[x];
m_i,j->k_[c_. U_i_[y__] U_j_[x__]] := c U_k[y];
m_i,j->k_[c_. U_i_[xx__, x_] U_j_[y_, yy__]] := If[x ≤ y,
  c U_k[xx, x, y, yy],
  ((U_i[xx] (U_j[y, x] + UU_j[B[x, y]])) // Expand // m_i,j->i) U_j[yy] // Expand // m_i,j->k)
  c // USimp
];
```

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```
In[*]:= UProducts[{}, 0] = {1}; UProducts[{}, d_Integer] /; d > 0 = {};
UProducts[{i_, is__}, d_Integer] := Sort@
  Flatten@Table[(U_i@@@Subsets[$Basis, {j}]) u, {j, 0, d}, {u, UProducts[{is, d-j}]}];
```

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```
In[*]:= Supp[ε_] := Union@Cases[{ε}, U_i[___] => i, ∞];
```

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```
In[ ]:= Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ⋈ y_ := Module[{is = Supp[x] ∩ Supp[y], σ, z},
  z = x; Do[z = mi→σ@i[z], {i, is}];
  z = Expand[y z]; Do[z = mσ@i, i→i[z], {i, is}]; z];
UB[x_, y_] := USimp[x ⋈ y - y ⋈ x];
```

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Epilog

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```
In[ ]:= (*End[]; EndPackage[];*)
```

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Predefined Algebras

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sl(2)

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```
In[ ]:= Print["UEA`SetAlgebra knows \"sl2\"."];
```

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UEA`SetAlgebra knows "sl2".

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```
In[ ]:= SetAlgebra["sl2"] := (
  Print["In sl2: ⟨e,h,f⟩ / ([h,e]=2e, [h,f]=-2f, [e,f]=h)."];
  B[h, e] = 2 e; B[h, f] = -2 f; B[e, f] = h;
  $Algebra = "sl2";
  $Basis = {e, h, f};
  $PBWRule = {e → 1, h → 2, f → 3};
  NilQ[e] = NilQ[f] = True; NilQ[h] = False;
);
```

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$gl_{n,\epsilon}$

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```
In[ ]:= Print["UEA`SetAlgebra knows the ε-nilpotent algebra gln,ε."];
```

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UEA`SetAlgebra knows the ϵ -nilpotent algebra $gl_{n,\epsilon}$.

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```

In[ ]:= SetAlgebra[gln,ε] := (
  $Algebra = gln,ε;
  $Basis = Flatten@{
    Table[yα,β, {β, 2, n}, {α, 1, β-1}],
    Table[xα,β, {β, 1, n}, {α, 1, β}]
  };
  (*$Basis=Reverse@SortBy[$Basis,If[#[[1]]==x,{0,#[[3]]-#[[2]]},{1,#[[2]]-#[[3]]}&];*)
  NilQ[y_] = True; NilQ[xα,β] := (α != β); (* Nilpotent Q *)
  $PBWRule = Thread[$Basis → Range@Length@$Basis];
  B[xi,j, xk,l] := δj,k xi,l - δl,i xk,j;
  B[yi,j, yk,l] := CF[-ε δj,k yi,l + ε δl,i yk,j];
  B[xi,j, yk,l] := CF[δj,k xi,l - δl,i xk,j /. xα,β → If[α ≤ β, ε xα,β, yβ,α]];
  DefRep[0] = Normal@SparseArray[{}, {n, n}];
  DefRep[ε_] := ε /. {
    xα,β → Normal@SparseArray[{α, β} → 1, {n, n}],
    yα,β → Normal@SparseArray[{β, α} → ε, {n, n}]
  }
);

```

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```

In[ ]:= adPower[x_, k_, Env_] [c_ * y_] /; MemberQ[$Basis, y] := CF[c adPower[x, k, Env] [y]];
adPower[x_, k_, Env_] [y_Plus] := adPower[x, k, Env] /@ y;
adPower[_, 0, Env_] [y_] := y;
adPower[x_, k_, Env_] [0] = 0;
adPower[x_, k_, Env_] [y_] /; MemberQ[$Basis, y] :=
  adPower[x, k, Env] [y] = B[adPower[x, k-1, Env] [y], x];
adPower[x_, k_] [y_] := adPower[x, k, {$Algebra, $k}] [y];

```

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```

In[ ]:= adExp[x_] [y_] /; SSQ[x] := Expand@Module[{A, b, c},
  A = Table[Coefficient[B[b, x], c], {c, $Basis}, {b, $Basis}];
  CF[$Basis.Normal[Series[MatrixExp[A], {ε, 0, $k}]]].
  Table[Coefficient[y, b], {b, $Basis}]
]

```

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```

In[ ]:= adExp[x_] [y_] /; NilQ[x] := Expand@Module[{k = 0, s = 0},
  While[adPower[x, k] [y] != 0,
    s += adPower[x, k] [y] / k!;
    ++k
  ];
  s
]

```

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```
In[ ]:= λTangent[] = 0;
λTangent[xs___, x_] := adExp[x][λTangent[xs]] + ∂λx;
λTangent[xs_List] := λTangent@@xs
```

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```
In[ ]:= TriangularDSolve[eqns_, funs_, iv_] := Monitor[
Module[{sol = {}, e, fun, der},
MapThread[
{e, fun} ↦ sol = Join[sol,
$Mon = MapAt[CF, First@DSolve[CF[e /. Derivative[1][f_][v_] => der[f[v], v] /.
sol /. der[f_, v_] => ∂vf], fun, iv], {1, 2}]
],
{eqns, funs}
];
sol
],
$Mon
]
```

gl_{2,ε} Experiments

```
In[ ]:= SetAlgebra[gl2,ε];
$PBWRule
MatrixForm@Table[{b1, b2} → B[b1, b2], {b1, $Basis}, {b2, $Basis}];
MatrixForm@
Table[{b1, b2} → (DefRep[b1].DefRep[b2] - DefRep[b2].DefRep[b1] == DefRep[B[b1, b2]]),
{b1, $Basis}, {b2, $Basis}];
DeleteCases[Flatten[Table[{a, b, c} → B[a, B[b, c]] + B[b, B[c, a]] + B[c, B[a, b]],
{a, $Basis}, {b, $Basis}, {c, $Basis}]], _ → 0]
```

Out[]:=

```
{y1,2 → 1, x1,1 → 2, x1,2 → 3, x2,2 → 4}
```

Out[]:=

```
{}
```

```

In[*]:= lhs = λTangent[Join[
  Table[If[SSQ[b1], (b1 /. x → ξ1) b1, λ (b1 /. {x → ξ1, y → η1}) b1], {b1, $Basis}],
  Table[If[SSQ[b2], (b2 /. x → ξ2) b2, λ (b2 /. {x → ξ2, y → η2}) b2], {b2, $Basis}]
]];
rhs = λTangent[
  Table[If[SSQ[b], b /. xα,α ⇒ (ξ1α,α + ξ2α,α + e fα,α[λ]) xα,α,
    (b /. {xαβ → fαβ[λ], yαβ → gαβ[λ]}) b], {b, $Basis}]
];
eqns = Simplify[(Coefficient[lhs - rhs, #] == 0) & /@ $Basis];
unknowns = $Basis /. {xαβ → fαβ[λ], yαβ → gαβ[λ]};
{sol0} = Block[{ $k = 0 }, Simplify@DSolve[
  And @@ Join[
    CF /@ eqns,
    Select[$Basis, NilQ] /. {xαβ ⇒ fαβ[0] == 0, yαβ ⇒ gαβ[0] == 0}
  ],
  Select[$Basis, NilQ] /. {xαβ → fαβ[λ], yαβ → gαβ[λ]},
  λ
]]

Out[*]= { {g1,2[λ] → λ (η1,2 + e-ξ1,1+ξ1,2 η2,2), f1,2[λ] → e-ξ2,1 λ ξ1,2 + e-ξ1,2 λ ξ2,2} }

In[*]:= {sol1} = Block[{ $k = 1 }, Simplify@DSolve[
  lhs = λTangent[Join[
    Table[If[SSQ[b1], (b1 /. x → ξ1) b1, λ (b1 /. {x → ξ1, y → η1}) b1], {b1, $Basis}],
    Table[If[SSQ[b2], (b2 /. x → ξ2) b2, λ (b2 /. {x → ξ2, y → η2}) b2], {b2, $Basis}]
  ]];
  rhs = λTangent[
    Table[If[SSQ[b],
      b /. xα,α ⇒ (ξ1α,α + ξ2α,α + e fα,α[λ]) xα,α,
      (b /. {xαβ → fαβ[λ], yαβ → gαβ[λ]}) b /. (sol0 /. ((a → b) ⇒ (a → e a + b)))
    ], {b, $Basis}]
  ];
  Join[
    (Coefficient[CF[lhs - rhs], #] == 0) & /@ $Basis,
    $Basis /. {xαβ ⇒ fαβ[0] == 0, yαβ ⇒ gαβ[0] == 0}
  ],
  $Basis /. {xαβ → fαβ[λ], yαβ → gαβ[λ]},
  λ
]]

Out[*]= { {f1,1[λ] → eξ1,2 λ2 η2,2 ξ1,2, f1,2[λ] → λ3 η2,2 ξ1,2 ξ2,2,
  f2,2[λ] → -eξ1,2 λ2 η2,2 ξ1,2, g1,2[λ] → -e-ξ1,1+2 ξ1,2 λ3 η2,2 ξ1,2} }

```

```

In[*]:= cm[i_, j_ → k_] := E_{i,j}→{k} [CF@Sum[If[SSQ[b],
    b /. x_{α,α} → (ξ1_{α,α} + ξ2_{α,α}) x_{α,α}[k],
    b[k] (b /. {x_{αβ} → f_{αβ}[λ], y_{αβ} → g_{αβ}[λ]} /. sol10),
], {b, $Basis}],
    CF@Sum[b[k] (b /. {x_{αβ} → f_{αβ}[λ], y_{αβ} → g_{αβ}[λ]} /. sol1), {b, $Basis}]
] /. {ξ1_{αβ} → ξ_{αβ}[i], η1_{αβ} → η_{αβ}[i], ξ2_{αβ} → ξ_{αβ}[j], η2_{αβ} → η_{αβ}[j], λ → 1};
cm[i, j → k]

Out[*]=
E_{i,j}→{k} [y_{1,2}[k] η_{1,2}[i] + e^{-ξ_{1,1}[i]+ξ_{2,2}[i]} y_{1,2}[k] η_{1,2}[j] + x_{1,1}[k] ξ_{1,1}[i] + x_{1,1}[k] ξ_{1,1}[j] +
    e^{-ξ_{1,1}[j]} x_{1,2}[k] ξ_{1,2}[i] + e^{-ξ_{2,2}[i]} x_{1,2}[k] ξ_{1,2}[j] + x_{2,2}[k] ξ_{2,2}[i] + x_{2,2}[k] ξ_{2,2}[j],
    e^{ξ_{2,2}[i]} x_{1,1}[k] η_{1,2}[j] ξ_{1,2}[i] - e^{ξ_{2,2}[i]} x_{2,2}[k] η_{1,2}[j] ξ_{1,2}[i] -
    e^{-ξ_{1,1}[i]+2 ξ_{2,2}[i]} y_{1,2}[k] η_{1,2}[j]^2 ξ_{1,2}[i] + x_{1,2}[k] η_{1,2}[j] ξ_{1,2}[i] ξ_{1,2}[j]]

In[*]:= cm[1, 2 → 2] // cm[2, 3 → 1]

Out[*]=
E_{1,2,3}→{1} [y_{1,2}[1] η_{1,2}[1] + e^{-ξ_{1,1}[1]+ξ_{2,2}[1]} y_{1,2}[1] η_{1,2}[2] +
    e^{-ξ_{1,1}[1]-ξ_{1,1}[2]+ξ_{2,2}[1]+ξ_{2,2}[2]} y_{1,2}[1] η_{1,2}[3] + x_{1,1}[1] ξ_{1,1}[1] + x_{1,1}[1] ξ_{1,1}[2] +
    x_{1,1}[1] ξ_{1,1}[3] + e^{-ξ_{1,1}[2]-ξ_{1,1}[3]} x_{1,2}[1] ξ_{1,2}[1] + e^{-ξ_{1,1}[3]-ξ_{2,2}[1]} x_{1,2}[1] ξ_{1,2}[2] +
    e^{-ξ_{2,2}[1]-ξ_{2,2}[2]} x_{1,2}[1] ξ_{1,2}[3] + x_{2,2}[1] ξ_{2,2}[1] + x_{2,2}[1] ξ_{2,2}[2] + x_{2,2}[1] ξ_{2,2}[3],
    e^{ξ_{2,2}[1]} x_{1,1}[1] η_{1,2}[2] ξ_{1,2}[1] - e^{ξ_{2,2}[1]} x_{2,2}[1] η_{1,2}[2] ξ_{1,2}[1] -
    e^{-ξ_{1,1}[1]+2 ξ_{2,2}[1]} y_{1,2}[1] η_{1,2}[2]^2 ξ_{1,2}[1] + e^{-ξ_{1,1}[2]+ξ_{2,2}[1]+ξ_{2,2}[2]} x_{1,1}[1] η_{1,2}[3] ξ_{1,2}[1] -
    e^{-ξ_{1,1}[2]+ξ_{2,2}[1]+ξ_{2,2}[2]} x_{2,2}[1] η_{1,2}[3] ξ_{1,2}[1] -
    2 e^{-ξ_{1,1}[1]-ξ_{1,1}[2]+2 ξ_{2,2}[1]+ξ_{2,2}[2]} y_{1,2}[1] η_{1,2}[2] η_{1,2}[3] ξ_{1,2}[1] -
    e^{-ξ_{1,1}[1]-2 ξ_{1,1}[2]+2 ξ_{2,2}[1]+2 ξ_{2,2}[2]} y_{1,2}[1] η_{1,2}[3]^2 ξ_{1,2}[1] + e^{ξ_{2,2}[2]} x_{1,1}[1] η_{1,2}[3] ξ_{1,2}[2] -
    e^{ξ_{2,2}[2]} x_{2,2}[1] η_{1,2}[3] ξ_{1,2}[2] - e^{-ξ_{1,1}[1]-ξ_{1,1}[2]+ξ_{2,2}[1]+2 ξ_{2,2}[2]} y_{1,2}[1] η_{1,2}[3]^2 ξ_{1,2}[2] +
    e^{-ξ_{1,1}[3]} x_{1,2}[1] η_{1,2}[2] ξ_{1,2}[1] ξ_{1,2}[2] + e^{-ξ_{2,2}[2]} x_{1,2}[1] η_{1,2}[2] ξ_{1,2}[1] ξ_{1,2}[3] +
    e^{-ξ_{1,1}[2]} x_{1,2}[1] η_{1,2}[3] ξ_{1,2}[1] ξ_{1,2}[3] + e^{-ξ_{2,2}[1]} x_{1,2}[1] η_{1,2}[3] ξ_{1,2}[2] ξ_{1,2}[3]]

In[*]:= Simplify[(cm[1, 2 → 2] // cm[2, 3 → 1]) ≡ (cm[2, 3 → 2] // cm[1, 2 → 1])]

Out[*]=
True

```

gl_{3,ε} Experiments

```

In[ ]:= SetAlgebra[gl3,ε];
$k = 1
$PBWRule
MatrixForm@Table[{b1, b2} → B[b1, b2], {b1, $Basis}, {b2, $Basis}];
MatrixForm@
  Table[{b1, b2} → (DefRep[b1].DefRep[b2] - DefRep[b2].DefRep[b1] == DefRep[B[b1, b2]]),
    {b1, $Basis}, {b2, $Basis}];
DeleteCases[Flatten[Table[{a, b, c} → B[a, B[b, c]] + B[b, B[c, a]] + B[c, B[a, b]],
  {a, $Basis}, {b, $Basis}, {c, $Basis}]], _ → 0]

Out[ ]:=
1

Out[ ]:=
{y1,2 → 1, y1,3 → 2, y2,3 → 3, x1,1 → 4, x1,2 → 5, x2,2 → 6, x1,3 → 7, x2,3 → 8, x3,3 → 9}

Out[ ]:=
{}

In[ ]:= {sol0} = Block[{ $k = 0},
  lhs = λTangent[Join[
    Table[If[SSQ[b1], (b1 /. x → ξ1) b1, λ (b1 /. {x → ξ1, y → η1}) b1], {b1, $Basis}],
    Table[If[SSQ[b2], (b2 /. x → ξ2) b2, λ (b2 /. {x → ξ2, y → η2}) b2], {b2, $Basis}]
  ]];
  rhs = λTangent[
    Table[If[SSQ[b], b /. xα,α → (ξ1α,α + ξ2α,α + ε fα,α[λ]) xα,α,
      (b /. {xαβ → fαβ[λ], yαβ → gαβ[λ]}) b], {b, $Basis}]
  ];
  Simplify@DSolve[
    And@@Join[
      (Coefficient[CF[lhs - rhs], #] == 0) & /@ $Basis,
      Select[$Basis, NilQ] /. {xαβ → fαβ[0] == 0, yαβ → gαβ[0] == 0}
    ],
    Select[$Basis, NilQ] /. {xαβ → fαβ[λ], yαβ → gαβ[λ]},
    λ
  ]
]

Out[ ]:=
{ {f1,2[λ] → e-ξ1,1 λ ξ1,2 + e-ξ1,2 λ ξ2,2, g2,3[λ] → λ (η1,3 + eξ1,3 (e-ξ1,2 η2,3 - λ η2,3 ξ1,2)) },
  f1,3[λ] → λ (e-ξ1,1 ξ1,3 - λ ξ1,3 ξ2,2 + e-ξ1,3 ξ2,3), f2,3[λ] → e-ξ2,2 λ ξ1,3 + e-ξ1,3 λ ξ2,3,
  g1,2[λ] → λ (η1,2 + e-ξ1,1+ξ1,2 (η2,2 + eξ1,3 λ η2,3 ξ1,3)), g1,3[λ] → λ (η1,3 + e-ξ1,1+ξ1,3 η2,3) } }

```

```

In[*]:= $k = 1; SBasis = SortBy[$Basis, If[#[[1]] == x, {0, #[[3]] - #[[2]]}, {1, #[[2]] - #[[3]]}] &];
lhs = λTangent[Join[
  Table[If[SSQ[b1], (b1 /. x → ξ1) b1, λ (b1 /. {x → ξ1, y → η1}) b1], {b1, $Basis}],
  Table[If[SSQ[b2], (b2 /. x → ξ2) b2, λ (b2 /. {x → ξ2, y → η2}) b2], {b2, $Basis}]
]];
rhs = λTangent[
  Echo@Table[If[SSQ[b],
    b /. xα,α ⇒ (ξ1,α + ξ2,α + e fα,α[λ]) xα,α,
    (b /. {xαβ ⇒ fαβ[λ], yαβ ⇒ gαβ[λ]}) b /. (sol0 /. ((v- → s-) ⇒ (v → e v + s)))
  ], {b, $Basis}]
];
eqns = Transpose@{
  (Coefficient[CF[lhs - rhs], #] == 0) & /@ SBasis,
  SBasis /. {xαβ ⇒ fαβ[0] == 0, yαβ ⇒ gαβ[0] == 0}
};
unknowns = SBasis /. {xαβ → fαβ[λ], yαβ → gαβ[λ]};
(*{sol1}=DSolve[Flatten@eqns, unknowns, λ]*)
sol1 = TriangularDSolve[eqns, unknowns, λ];
» {y1,2 (λ (η1,2 + e-ξ1,1+ξ1,2 (η2,2 + eξ1,3 λ η2,3 ξ1,2,3)) + e g1,2[λ]),
  y1,3 (λ (η1,3 + e-ξ1,1+ξ1,3 η2,3)) + e g1,3[λ]),
  y2,3 (λ (η2,3 + eξ1,3 (e-ξ1,2 η2,3 - λ η2,3 ξ1,2)) + e g2,3[λ]),
  x1,1 (ξ1,1 + ξ2,1 + e f1,1[λ]), x1,2 (e-ξ2,1 λ ξ1,2 + e-ξ1,2 λ ξ2,2 + e f1,2[λ]),
  x2,2 (ξ1,2 + ξ2,2 + e f2,2[λ]), x1,3 (λ (e-ξ2,1 ξ1,3 - λ ξ1,3 ξ2,2 + e-ξ1,3 ξ2,3)) + e f1,3[λ]),
  x2,3 (e-ξ2,2 λ ξ1,3 + e-ξ1,3 λ ξ2,3 + e f2,3[λ]), x3,3 (ξ1,3 + ξ2,3 + e f3,3[λ])}

In[*]:= Sort@sol1
Out[*]=
{f1,1[λ] → eξ1,2 λ2 η2,2 ξ1,2 + eξ1,3 λ2 η2,3 ξ1,3 + eξ1,2+ξ1,3 λ3 η2,3 ξ1,2 ξ1,3,
f1,2[λ] → e-ξ1,2+ξ1,3-ξ2,1 λ2 η2,3 ξ1,3 - eξ1,3-ξ2,1 λ3 η2,3 ξ1,2 ξ1,3 +
λ3 η2,2 ξ1,2 ξ2,2 - e-ξ1,2+ξ1,3 λ3 η2,3 ξ1,2 ξ2,2 + eξ1,3 λ4 η2,3 ξ1,2 ξ1,3 ξ2,2,
f1,3[λ] → λ3 η2,2 ξ1,3 ξ2,2 + λ3 η2,3 ξ1,3 ξ2,3 + λ3 η2,3 ξ1,2 ξ2,3,
f2,2[λ] → -eξ1,2 λ2 η2,2 ξ1,2 + eξ1,3 λ2 η2,3 ξ1,2 - eξ1,2+ξ1,3 λ3 η2,3 ξ1,2 ξ1,3,
f2,3[λ] → -e-ξ2,2 λ2 η2,2 ξ1,3 + λ3 η2,3 ξ1,3 ξ2,3 + λ3 η2,3 ξ1,2 ξ2,3,
f3,3[λ] → -eξ1,3 λ2 η2,3 ξ1,3 - eξ1,3 λ2 η2,3 ξ1,2,
g1,2[λ] → -e-ξ1,1+2 ξ1,2 λ3 η2,2 ξ1,2 - e-ξ1,1+ξ1,2+ξ1,3 λ3 η2,2 η2,3 ξ1,3 - 2 e-ξ1,1+2 ξ1,2+ξ1,3 λ4 η2,2
η2,3 ξ1,2 ξ1,3 - e-ξ1,1+ξ1,2+2 ξ1,3 λ4 η2,32 ξ1,3 ξ1,2 - e-ξ1,1+2 ξ1,2+2 ξ1,3 λ5 η2,32 ξ1,2 ξ1,3,
g1,3[λ] → e-ξ1,1+ξ1,2 λ2 η2,3 η2,2 - e-ξ1,1+ξ1,2+ξ1,3 λ3 η2,2 η2,3 ξ1,2 - e-ξ1,1+2 ξ1,3 λ3 η2,32 ξ1,3 +
e-ξ1,1+ξ1,2+ξ1,3 λ3 η2,3 η2,3 ξ1,2 - e-ξ1,1+ξ1,2+2 ξ1,3 λ4 η2,32 ξ1,2 ξ1,3,
g2,3[λ] → eξ1,3 λ3 η2,2 η2,3 ξ1,2 - e-ξ1,2+2 ξ1,3 λ3 η2,3 η2,3 ξ1,3 +
e2 ξ1,3 λ4 η2,32 ξ1,2 ξ1,3 - e-ξ1,2+2 ξ1,3 λ3 η2,32 ξ1,2 + e2 ξ1,3 λ4 η2,3 η2,3 ξ1,2 ξ1,3}

```

```

In[*]:= cm[i_, j_ → k_] := E_{i,j}→{k} [CF@Sum[If[SSQ[b],
    b /. x_{α,α} → (ξ1_{α,α} + ξ2_{α,α}) x_{α,α}[k],
    b[k] (b /. {x_{αβ} → f_{αβ}[λ], y_{αβ} → g_{αβ}[λ]} /. sol10)
], {b, $Basis}],
    CF@Sum[b[k] (b /. {x_{αβ} → f_{αβ}[λ], y_{αβ} → g_{αβ}[λ]} /. sol1), {b, $Basis}]
] /. {ξ1_{αβ} → ξ_{αβ}[i], η1_{αβ} → η_{αβ}[i], ξ2_{αβ} → ξ_{αβ}[j], η2_{αβ} → η_{αβ}[j], λ → 1};
cm[i, j → k]

Out[*]=
E_{i,j}→{k} [y1,2[k] η1,2[i] + e^{-ξ1,1[i]+ξ2,2[i]} y1,2[k] η1,2[j] + y1,3[k] η1,3[i] +
    e^{-ξ1,1[i]+ξ3,3[i]} y1,3[k] η1,3[j] + y2,3[k] η2,3[i] + e^{-ξ2,2[i]+ξ3,3[i]} y2,3[k] η2,3[j] +
    x1,1[k] ξ1,1[i] + x1,1[k] ξ1,1[j] + e^{-ξ1,1[j]} x1,2[k] ξ1,2[i] - e^{ξ3,3[i]} y2,3[k] η1,3[j] ξ1,2[i] +
    e^{-ξ2,2[i]} x1,2[k] ξ1,2[j] + e^{-ξ1,1[j]} x1,3[k] ξ1,3[i] + e^{-ξ3,3[i]} x1,3[k] ξ1,3[j] + x2,2[k] ξ2,2[i] +
    x2,2[k] ξ2,2[j] + e^{-ξ2,2[j]} x2,3[k] ξ2,3[i] + e^{-ξ1,1[i]+ξ2,2[i]+ξ3,3[i]} y1,2[k] η1,3[j] ξ2,3[i] -
    x1,3[k] ξ1,2[j] ξ2,3[i] + e^{-ξ3,3[i]} x2,3[k] ξ2,3[j] + x3,3[k] ξ3,3[i] + x3,3[k] ξ3,3[j],
    e^{-ξ1,1[i]+ξ2,2[i]} y1,3[k] η1,2[j] η2,3[i] + e^{ξ2,2[i]} x1,1[k] η1,2[j] ξ1,2[i] -
    e^{ξ2,2[i]} x2,2[k] η1,2[j] ξ1,2[i] - e^{-ξ1,1[i]+2 ξ2,2[i]} y1,2[k] η1,2[j]^2 ξ1,2[i] -
    e^{-ξ1,1[i]+ξ2,2[i]+ξ3,3[i]} y1,3[k] η1,2[j] η1,3[j] ξ1,2[i] + e^{ξ3,3[i]} y2,3[k] η1,2[j] η2,3[j] ξ1,2[i] +
    x1,2[k] η1,2[j] ξ1,2[i] ξ1,2[j] - e^{-ξ2,2[j]} x2,3[k] η1,2[j] ξ1,3[i] +
    e^{ξ3,3[i]} x1,1[k] η1,3[j] ξ1,3[i] - e^{ξ3,3[i]} x3,3[k] η1,3[j] ξ1,3[i] -
    e^{-ξ1,1[i]+ξ2,2[i]+ξ3,3[i]} y1,2[k] η1,2[j] η1,3[j] ξ1,3[i] - e^{-ξ1,1[i]+2 ξ3,3[i]} y1,3[k] η1,3[j]^2 ξ1,3[i] +
    e^{-ξ1,1[j]-ξ2,2[i]+ξ3,3[i]} x1,2[k] η2,3[j] ξ1,3[i] - e^{-ξ2,2[i]+2 ξ3,3[i]} y2,3[k] η1,3[j] η2,3[j] ξ1,3[i] -
    e^{-ξ1,1[j]+ξ3,3[i]} x1,2[k] η1,3[j] ξ1,2[i] ξ1,3[i] + e^{2 ξ3,3[i]} y2,3[k] η1,3[j]^2 ξ1,2[i] ξ1,3[i] +
    x1,3[k] η1,2[j] ξ1,2[j] ξ1,3[i] + x1,3[k] η1,3[j] ξ1,3[i] ξ1,3[j] +
    e^{-ξ1,1[i]+ξ2,2[i]+ξ3,3[i]} y1,3[k] η1,3[j] η2,3[i] ξ2,3[i] + e^{ξ3,3[i]} x2,2[k] η2,3[j] ξ2,3[i] -
    e^{ξ3,3[i]} x3,3[k] η2,3[j] ξ2,3[i] - e^{-ξ2,2[i]+2 ξ3,3[i]} y2,3[k] η2,3[j]^2 ξ2,3[i] +
    e^{ξ2,2[i]+ξ3,3[i]} x1,1[k] η1,3[j] ξ1,2[i] ξ2,3[i] - e^{ξ2,2[i]+ξ3,3[i]} x2,2[k] η1,3[j] ξ1,2[i] ξ2,3[i] -
    2 e^{-ξ1,1[i]+2 ξ2,2[i]+ξ3,3[i]} y1,2[k] η1,2[j] η1,3[j] ξ1,2[i] ξ2,3[i] -
    e^{-ξ1,1[i]+ξ2,2[i]+2 ξ3,3[i]} y1,3[k] η1,3[j]^2 ξ1,2[i] ξ2,3[i] +
    e^{2 ξ3,3[i]} y2,3[k] η1,3[j] η2,3[j] ξ1,2[i] ξ2,3[i] -
    e^{-ξ2,2[i]+ξ3,3[i]} x1,2[k] η2,3[j] ξ1,2[j] ξ2,3[i] + e^{ξ3,3[i]} x1,2[k] η1,3[j] ξ1,2[i] ξ1,2[j] ξ2,3[i] -
    e^{-ξ1,1[i]+ξ2,2[i]+2 ξ3,3[i]} y1,2[k] η1,3[j]^2 ξ1,3[i] ξ2,3[i] + x1,3[k] η2,3[j] ξ1,3[j] ξ2,3[i] -
    e^{-ξ1,1[i]+2 ξ2,2[i]+2 ξ3,3[i]} y1,2[k] η1,3[j]^2 ξ1,2[i] ξ2,3[i]^2 +
    x2,3[k] η1,3[j] ξ1,3[i] ξ2,3[j] + x2,3[k] η2,3[j] ξ2,3[i] ξ2,3[j]]

```

In[*]:= Short[cm[1, 2 → 2] // cm[2, 3 → 1], 10]

Out[*]//Short=

$$\begin{aligned} & \mathbb{E}_{\{1,2,3\} \rightarrow \{1\}} \left[y_{1,2}[1] \eta_{1,2}[1] + e^{-\xi_{1,1}[1] + \xi_{2,2}[1]} y_{1,2}[1] \eta_{1,2}[2] + \right. \\ & e^{-\xi_{1,1}[1] - \xi_{1,1}[2] + \xi_{2,2}[1] + \xi_{2,2}[2]} y_{1,2}[1] \eta_{1,2}[3] + y_{1,3}[1] \eta_{1,3}[1] + e^{-\xi_{1,1}[1] + \xi_{3,3}[1]} y_{1,3}[1] \eta_{1,3}[2] + \\ & e^{-\xi_{1,1}[1] - \xi_{1,1}[2] + \xi_{3,3}[1] + \xi_{3,3}[2]} y_{1,3}[1] \eta_{1,3}[3] + y_{2,3}[1] \eta_{2,3}[1] + e^{-\xi_{2,2}[1] + \xi_{3,3}[1]} y_{2,3}[1] \eta_{2,3}[2] + \\ & e^{-\xi_{2,2}[1] - \xi_{2,2}[2] + \xi_{3,3}[1] + \xi_{3,3}[2]} y_{2,3}[1] \eta_{2,3}[3] + x_{1,1}[1] \xi_{1,1}[1] + x_{1,1}[1] \xi_{1,1}[2] + \\ & x_{1,1}[1] \xi_{1,1}[3] + e^{-\xi_{1,1}[2] - \xi_{1,1}[3]} x_{1,2}[1] \xi_{1,2}[1] - e^{\xi_{3,3}[1]} y_{2,3}[1] \eta_{1,3}[2] \xi_{1,2}[1] - \\ & e^{-\xi_{1,1}[2] + \xi_{3,3}[1] + \xi_{3,3}[2]} y_{2,3}[1] \eta_{1,3}[3] \xi_{1,2}[1] + \langle\langle 7 \rangle\rangle + x_{2,2}[1] \xi_{2,2}[1] + x_{2,2}[1] \xi_{2,2}[2] + \\ & x_{2,2}[1] \xi_{2,2}[3] + e^{-\xi_{2,2}[2] - \xi_{2,2}[3]} x_{2,3}[1] \xi_{2,3}[1] + e^{-\xi_{1,1}[1] + \xi_{2,2}[1] + \xi_{3,3}[1]} y_{1,2}[1] \eta_{1,3}[2] \xi_{2,3}[1] + \\ & e^{-\xi_{1,1}[1] - \xi_{1,1}[2] + \xi_{2,2}[1] + \xi_{3,3}[1] + \xi_{3,3}[2]} y_{1,2}[1] \eta_{1,3}[3] \xi_{2,3}[1] - e^{-\xi_{1,1}[3]} x_{1,3}[1] \xi_{1,2}[2] \xi_{2,3}[1] - \\ & e^{-\xi_{2,2}[2]} x_{1,3}[1] \xi_{1,2}[3] \xi_{2,3}[1] + e^{-\xi_{2,2}[3] - \xi_{3,3}[1]} x_{2,3}[1] \xi_{2,3}[2] + \\ & e^{-\xi_{1,1}[1] - \xi_{1,1}[2] + \xi_{2,2}[1] + \xi_{2,2}[2] + \xi_{3,3}[2]} y_{1,2}[1] \eta_{1,3}[3] \xi_{2,3}[2] - e^{-\xi_{3,3}[1]} x_{1,3}[1] \xi_{1,2}[3] \xi_{2,3}[2] + \\ & \left. e^{-\xi_{3,3}[1] - \xi_{3,3}[2]} x_{2,3}[1] \xi_{2,3}[3] + x_{3,3}[1] \xi_{3,3}[1] + x_{3,3}[1] \xi_{3,3}[2] + x_{3,3}[1] \xi_{3,3}[3], \langle\langle 1 \rangle\rangle \right] \end{aligned}$$

In[*]:= Simplify[(cm[1, 2 → 2] // cm[2, 3 → 1]) ≡ (cm[2, 3 → 2] // cm[1, 2 → 1])]

Out[*]=

True

$$\begin{aligned} \text{In[*]} := \text{osol} = & \mathbb{E}_{\{i,j\} \rightarrow \{k\}} \left[y_{1,2}[k] \eta_{1,2}[i] + e^{-\xi_{1,1}[i] + \xi_{2,2}[i]} y_{1,2}[k] \eta_{1,2}[j] + y_{1,3}[k] \eta_{1,3}[i] + \right. \\ & e^{-\xi_{1,1}[i] + \xi_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j] + y_{2,3}[k] \eta_{2,3}[i] + e^{-\xi_{2,2}[i] + \xi_{3,3}[i]} y_{2,3}[k] \eta_{2,3}[j] + \\ & x_{1,1}[k] \xi_{1,1}[i] + x_{1,1}[k] \xi_{1,1}[j] + e^{-\xi_{1,1}[j]} x_{1,2}[k] \xi_{1,2}[i] - e^{\xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j] \xi_{1,2}[i] + \\ & e^{-\xi_{2,2}[i]} x_{1,2}[k] \xi_{1,2}[j] + e^{-\xi_{1,1}[j]} x_{1,3}[k] \xi_{1,3}[i] + e^{-\xi_{3,3}[i]} x_{1,3}[k] \xi_{1,3}[j] + x_{2,2}[k] \xi_{2,2}[i] + \\ & x_{2,2}[k] \xi_{2,2}[j] + e^{-\xi_{2,2}[j]} x_{2,3}[k] \xi_{2,3}[i] + e^{-\xi_{1,1}[i] + \xi_{2,2}[i] + \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,3}[j] \xi_{2,3}[i] - \\ & x_{1,3}[k] \xi_{1,2}[j] \xi_{2,3}[i] + e^{-\xi_{3,3}[i]} x_{2,3}[k] \xi_{2,3}[j] + x_{3,3}[k] \xi_{3,3}[i] + x_{3,3}[k] \xi_{3,3}[j], \\ & -e^{-\xi_{2,2}[i] + \xi_{3,3}[i]} y_{1,3}[k] \eta_{1,2}[i] \eta_{2,3}[j] + e^{\xi_{2,2}[i]} x_{1,1}[k] \eta_{1,2}[j] \xi_{1,2}[i] - \\ & e^{\xi_{2,2}[i]} x_{2,2}[k] \eta_{1,2}[j] \xi_{1,2}[i] - e^{-\xi_{1,1}[i] + 2\xi_{2,2}[i]} y_{1,2}[k] \eta_{1,2}[j]^2 \xi_{1,2}[i] + \\ & e^{\xi_{3,3}[i]} y_{1,3}[k] \eta_{1,2}[i] \eta_{1,3}[j] \xi_{1,2}[i] + x_{1,2}[k] \eta_{1,2}[j] \xi_{1,2}[i] \xi_{1,2}[j] - \\ & e^{-\xi_{2,2}[j]} x_{2,3}[k] \eta_{1,2}[j] \xi_{1,3}[i] + e^{\xi_{3,3}[i]} x_{1,1}[k] \eta_{1,3}[j] \xi_{1,3}[i] - e^{\xi_{3,3}[i]} x_{3,3}[k] \eta_{1,3}[j] \xi_{1,3}[i] - \\ & e^{-\xi_{1,1}[i] + \xi_{2,2}[i] + \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \xi_{1,3}[i] - e^{-\xi_{1,1}[i] + 2\xi_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j]^2 \xi_{1,3}[i] + \\ & e^{-\xi_{1,1}[j] - \xi_{2,2}[i] + \xi_{3,3}[i]} x_{1,2}[k] \eta_{2,3}[j] \xi_{1,3}[i] - e^{-\xi_{2,2}[i] + 2\xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,3}[i] - \\ & e^{-\xi_{1,1}[j] + \xi_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{1,3}[i] + e^{2\xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j]^2 \xi_{1,2}[i] \xi_{1,3}[i] + \\ & x_{1,3}[k] \eta_{1,2}[j] \xi_{1,2}[j] \xi_{1,3}[i] + x_{1,3}[k] \eta_{1,3}[j] \xi_{1,3}[i] \xi_{1,3}[j] + e^{\xi_{3,3}[i]} x_{2,2}[k] \eta_{2,3}[j] \xi_{2,3}[i] - \\ & e^{\xi_{3,3}[i]} x_{3,3}[k] \eta_{2,3}[j] \xi_{2,3}[i] + e^{-\xi_{1,1}[i] + \xi_{2,2}[i] + \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{2,3}[j] \xi_{2,3}[i] - \\ & e^{-\xi_{1,1}[i] + 2\xi_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{2,3}[i] - e^{-\xi_{2,2}[i] + 2\xi_{3,3}[i]} y_{2,3}[k] \eta_{2,3}[j]^2 \xi_{2,3}[i] + \\ & e^{\xi_{2,2}[i] + \xi_{3,3}[i]} x_{1,1}[k] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] - e^{\xi_{2,2}[i] + \xi_{3,3}[i]} x_{2,2}[k] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] - \\ & 2e^{-\xi_{1,1}[i] + 2\xi_{2,2}[i] + \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] + \\ & e^{2\xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] - \\ & e^{-\xi_{2,2}[i] + \xi_{3,3}[i]} x_{1,2}[k] \eta_{2,3}[j] \xi_{1,2}[j] \xi_{2,3}[i] + e^{\xi_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{1,2}[j] \xi_{2,3}[i] - \\ & e^{-\xi_{1,1}[i] + \xi_{2,2}[i] + 2\xi_{3,3}[i]} y_{1,2}[k] \eta_{1,3}[j]^2 \xi_{1,3}[i] \xi_{2,3}[i] + x_{1,3}[k] \eta_{2,3}[j] \xi_{1,3}[j] \xi_{2,3}[i] - \\ & e^{-\xi_{1,1}[i] + 2\xi_{2,2}[i] + 2\xi_{3,3}[i]} y_{1,2}[k] \eta_{1,3}[j]^2 \xi_{1,2}[i] \xi_{2,3}[i]^2 + \\ & x_{2,3}[k] \eta_{1,3}[j] \xi_{1,3}[i] \xi_{2,3}[j] + x_{2,3}[k] \eta_{2,3}[j] \xi_{2,3}[i] \xi_{2,3}[j], \\ & -\frac{1}{2} e^{2\xi_{2,2}[i]} x_{1,1}[k] \eta_{1,2}[j]^2 \xi_{1,2}[i]^2 + \frac{1}{2} e^{2\xi_{2,2}[i]} x_{2,2}[k] \eta_{1,2}[j]^2 \xi_{1,2}[i]^2 + \\ & e^{-\xi_{1,1}[i] + 3\xi_{2,2}[i]} y_{1,2}[k] \eta_{1,2}[j]^3 \xi_{1,2}[i]^2 + e^{\xi_{3,3}[i]} x_{1,1}[k] \eta_{1,2}[j] \eta_{2,3}[j] \xi_{1,3}[i] - \\ & \left. e^{\xi_{3,3}[i]} x_{2,2}[k] \eta_{1,2}[j] \eta_{2,3}[j] \xi_{1,3}[i] - e^{-\xi_{1,1}[i] + \xi_{2,2}[i] + \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j]^2 \eta_{2,3}[j] \xi_{1,3}[i] + \right. \end{aligned}$$

$$\begin{aligned}
& e^{-\xi_{2,2}[i]+2\xi_{3,3}[i]} y_{1,3}[k] \eta_{1,2}[i] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,3}[i] - e^{\xi_{2,2}[i]+\xi_{3,3}[i]} x_{1,1}[k] \eta_{1,2}[j] \\
& \eta_{1,3}[j] \xi_{1,2}[i] \xi_{1,3}[i] + e^{\xi_{2,2}[i]+\xi_{3,3}[i]} x_{2,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{1,3}[i] + \\
& 2 e^{-\xi_{1,1}[i]+2\xi_{2,2}[i]+\xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j]^2 \eta_{1,3}[j] \xi_{1,2}[i] \xi_{1,3}[i] - e^{2\xi_{3,3}[i]} y_{1,3}[k] \\
& \eta_{1,2}[i] \eta_{1,3}[j]^2 \xi_{1,2}[i] \xi_{1,3}[i] + e^{-\xi_{2,2}[i]+\xi_{3,3}[i]} x_{1,2}[k] \eta_{1,2}[j] \eta_{2,3}[j] \xi_{1,2}[j] \xi_{1,3}[i] - \\
& e^{\xi_{3,3}[i]} x_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{1,2}[j] \xi_{1,3}[i] - \frac{1}{2} e^{2\xi_{3,3}[i]} x_{1,1}[k] \eta_{1,3}[j]^2 \xi_{1,3}[i]^2 + \\
& \frac{1}{2} e^{2\xi_{3,3}[i]} x_{3,3}[k] \eta_{1,3}[j]^2 \xi_{1,3}[i]^2 + e^{-\xi_{1,1}[i]+\xi_{2,2}[i]+2\xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j]^2 \xi_{1,3}[i]^2 + \\
& e^{-\xi_{1,1}[i]+3\xi_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j]^3 \xi_{1,3}[i]^2 - e^{-\xi_{1,1}[j]-\xi_{2,2}[i]+2\xi_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,3}[i]^2 + \\
& e^{-\xi_{2,2}[i]+3\xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j]^2 \eta_{2,3}[j] \xi_{1,3}[i]^2 + \\
& e^{-\xi_{1,1}[j]+2\xi_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j]^2 \xi_{1,2}[i] \xi_{1,3}[i]^2 - e^{3\xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j]^3 \xi_{1,2}[i] \xi_{1,3}[i]^2 + \\
& e^{-\xi_{2,2}[i]+2\xi_{3,3}[i]} y_{1,3}[k] \eta_{1,2}[i] \eta_{2,3}[j]^2 \xi_{2,3}[i] + e^{\xi_{2,2}[i]+\xi_{3,3}[i]} x_{1,1}[k] \eta_{1,2}[j] \\
& \eta_{2,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] - e^{\xi_{2,2}[i]+\xi_{3,3}[i]} x_{2,2}[k] \eta_{1,2}[j] \eta_{2,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] - \\
& 2 e^{-\xi_{1,1}[i]+2\xi_{2,2}[i]+\xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j]^2 \eta_{2,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] - \\
& e^{2\xi_{3,3}[i]} y_{1,3}[k] \eta_{1,2}[i] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] - e^{2\xi_{2,2}[i]+\xi_{3,3}[i]} x_{1,1}[k] \eta_{1,2}[j] \\
& \eta_{1,3}[j] \xi_{1,2}[i]^2 \xi_{2,3}[i] + e^{2\xi_{2,2}[i]+\xi_{3,3}[i]} x_{2,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \xi_{1,2}[i]^2 \xi_{2,3}[i] + \\
& 3 e^{-\xi_{1,1}[i]+3\xi_{2,2}[i]+\xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j]^2 \eta_{1,3}[j] \xi_{1,2}[i]^2 \xi_{2,3}[i] - \\
& e^{2\xi_{3,3}[i]} x_{2,2}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,3}[i] \xi_{2,3}[i] + e^{2\xi_{3,3}[i]} x_{3,3}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,3}[i] \xi_{2,3}[i] - \\
& 2 e^{-\xi_{1,1}[i]+\xi_{2,2}[i]+2\xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,3}[i] \xi_{2,3}[i] + \\
& 2 e^{-\xi_{1,1}[i]+3\xi_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j]^2 \eta_{2,3}[j] \xi_{1,3}[i] \xi_{2,3}[i] - \\
& e^{-\xi_{1,1}[j]-\xi_{2,2}[i]+2\xi_{3,3}[i]} x_{1,2}[k] \eta_{2,3}[j]^2 \xi_{1,3}[i] \xi_{2,3}[i] + \\
& 2 e^{-\xi_{2,2}[i]+3\xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j] \eta_{2,3}[j]^2 \xi_{1,3}[i] \xi_{2,3}[i] - e^{\xi_{2,2}[i]+2\xi_{3,3}[i]} x_{1,1}[k] \eta_{1,3}[j]^2 \\
& \xi_{1,2}[i] \xi_{1,3}[i] \xi_{2,3}[i] + e^{\xi_{2,2}[i]+2\xi_{3,3}[i]} x_{2,2}[k] \eta_{1,3}[j]^2 \xi_{1,2}[i] \xi_{1,3}[i] \xi_{2,3}[i] + \\
& 4 e^{-\xi_{1,1}[i]+2\xi_{2,2}[i]+2\xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j]^2 \xi_{1,2}[i] \xi_{1,3}[i] \xi_{2,3}[i] + \\
& e^{-\xi_{1,1}[j]+2\xi_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,2}[i] \xi_{1,3}[i] \xi_{2,3}[i] - \\
& 2 e^{3\xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j]^2 \eta_{2,3}[j] \xi_{1,2}[i] \xi_{1,3}[i] \xi_{2,3}[i] + \\
& e^{-\xi_{2,2}[i]+2\xi_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,2}[j] \xi_{1,3}[i] \xi_{2,3}[i] - \\
& e^{2\xi_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j]^2 \xi_{1,2}[i] \xi_{1,2}[j] \xi_{1,3}[i] \xi_{2,3}[i] + \\
& e^{-\xi_{1,1}[i]+\xi_{2,2}[i]+3\xi_{3,3}[i]} y_{1,2}[k] \eta_{1,3}[j]^3 \xi_{1,3}[i]^2 \xi_{2,3}[i] - \\
& \frac{1}{2} e^{2\xi_{3,3}[i]} x_{2,2}[k] \eta_{2,3}[j]^2 \xi_{2,3}[i]^2 + \frac{1}{2} e^{2\xi_{3,3}[i]} x_{3,3}[k] \eta_{2,3}[j]^2 \xi_{2,3}[i]^2 + \\
& e^{-\xi_{1,1}[i]+3\xi_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j] \eta_{2,3}[j]^2 \xi_{2,3}[i]^2 + e^{-\xi_{2,2}[i]+3\xi_{3,3}[i]} y_{2,3}[k] \eta_{2,3}[j]^3 \xi_{2,3}[i]^2 - \\
& 2 e^{-\xi_{1,1}[i]+2\xi_{2,2}[i]+2\xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,2}[i] \xi_{2,3}[i]^2 - \\
& e^{3\xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j] \eta_{2,3}[j]^2 \xi_{1,2}[i] \xi_{2,3}[i]^2 - \frac{1}{2} e^{2\xi_{2,2}[i]+2\xi_{3,3}[i]} x_{1,1}[k] \\
& \eta_{1,3}[j]^2 \xi_{1,2}[i]^2 \xi_{2,3}[i]^2 + \frac{1}{2} e^{2\xi_{2,2}[i]+2\xi_{3,3}[i]} x_{2,2}[k] \eta_{1,3}[j]^2 \xi_{1,2}[i]^2 \xi_{2,3}[i]^2 + \\
& 3 e^{-\xi_{1,1}[i]+3\xi_{2,2}[i]+2\xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j]^2 \xi_{1,2}[i]^2 \xi_{2,3}[i]^2 + \\
& e^{-\xi_{2,2}[i]+2\xi_{3,3}[i]} x_{1,2}[k] \eta_{2,3}[j]^2 \xi_{1,2}[j] \xi_{2,3}[i]^2 - e^{2\xi_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,2}[i] \\
& \xi_{1,2}[j] \xi_{2,3}[i]^2 + 2 e^{-\xi_{1,1}[i]+2\xi_{2,2}[i]+3\xi_{3,3}[i]} y_{1,2}[k] \eta_{1,3}[j]^3 \xi_{1,2}[i] \xi_{1,3}[i] \xi_{2,3}[i]^2 + \\
& e^{-\xi_{1,1}[i]+3\xi_{2,2}[i]+3\xi_{3,3}[i]} y_{1,2}[k] \eta_{1,3}[j]^3 \xi_{1,2}[i]^2 \xi_{2,3}[i]^3 \Big];
\end{aligned}$$

osol = (List@@osol).Table[e^k, {k, 0, Length[osol] - 1}]

Out[8]=

$$\begin{aligned}
& y_{1,2}[k] \eta_{1,2}[i] + e^{-\xi_{1,1}[i]+\xi_{2,2}[i]} y_{1,2}[k] \eta_{1,2}[j] + y_{1,3}[k] \eta_{1,3}[i] + e^{-\xi_{1,1}[i]+\xi_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j] + \\
& y_{2,3}[k] \eta_{2,3}[i] + e^{-\xi_{2,2}[i]+\xi_{3,3}[i]} y_{2,3}[k] \eta_{2,3}[j] + x_{1,1}[k] \xi_{1,1}[i] + x_{1,1}[k] \xi_{1,1}[j] +
\end{aligned}$$

$$\begin{aligned}
& e^{-\xi_{1,1}[j]} x_{1,2}[k] \xi_{1,2}[i] - e^{\xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j] \xi_{1,2}[i] + e^{-\xi_{2,2}[i]} x_{1,2}[k] \xi_{1,2}[j] + \\
& e^{-\xi_{1,1}[j]} x_{1,3}[k] \xi_{1,3}[i] + e^{-\xi_{3,3}[i]} x_{1,3}[k] \xi_{1,3}[j] + x_{2,2}[k] \xi_{2,2}[i] + x_{2,2}[k] \xi_{2,2}[j] + \\
& e^{-\xi_{2,2}[j]} x_{2,3}[k] \xi_{2,3}[i] + e^{-\xi_{1,1}[i] + \xi_{2,2}[i] + \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,3}[j] \xi_{2,3}[i] - x_{1,3}[k] \xi_{1,2}[j] \xi_{2,3}[i] + \\
& \epsilon^2 \left(-\frac{1}{2} e^2 \xi_{2,2}[i] x_{1,1}[k] \eta_{1,2}[j]^2 \xi_{1,2}[i]^2 + \frac{1}{2} e^2 \xi_{2,2}[i] x_{2,2}[k] \eta_{1,2}[j]^2 \xi_{1,2}[i]^2 + \right. \\
& e^{-\xi_{1,1}[i] + 3 \xi_{2,2}[i]} y_{1,2}[k] \eta_{1,2}[j]^3 \xi_{1,2}[i]^2 + e^{\xi_{3,3}[i]} x_{1,1}[k] \eta_{1,2}[j] \eta_{2,3}[j] \xi_{1,3}[i] - \\
& e^{\xi_{3,3}[i]} x_{2,2}[k] \eta_{1,2}[j] \eta_{2,3}[j] \xi_{1,3}[i] - e^{-\xi_{1,1}[i] + \xi_{2,2}[i] + \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j]^2 \eta_{2,3}[j] \xi_{1,3}[i] + \\
& e^{-\xi_{2,2}[i] + 2 \xi_{3,3}[i]} y_{1,3}[k] \eta_{1,2}[i] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,3}[i] - e^{\xi_{2,2}[i] + \xi_{3,3}[i]} x_{1,1}[k] \eta_{1,2}[j] \\
& \eta_{1,3}[j] \xi_{1,2}[i] \xi_{1,3}[i] + e^{\xi_{2,2}[i] + \xi_{3,3}[i]} x_{2,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{1,3}[i] + \\
& 2 e^{-\xi_{1,1}[i] + 2 \xi_{2,2}[i] + \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j]^2 \eta_{1,3}[j] \xi_{1,2}[i] \xi_{1,3}[i] - e^{2 \xi_{3,3}[i]} y_{1,3}[k] \\
& \eta_{1,2}[i] \eta_{1,3}[j]^2 \xi_{1,2}[i] \xi_{1,3}[i] + e^{-\xi_{2,2}[i] + \xi_{3,3}[i]} x_{1,2}[k] \eta_{1,2}[j] \eta_{2,3}[j] \xi_{1,2}[j] \xi_{1,3}[i] - \\
& e^{\xi_{3,3}[i]} x_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{1,2}[j] \xi_{1,3}[i] - \frac{1}{2} e^{2 \xi_{3,3}[i]} x_{1,1}[k] \eta_{1,3}[j]^2 \xi_{1,3}[i]^2 + \\
& \frac{1}{2} e^{2 \xi_{3,3}[i]} x_{3,3}[k] \eta_{1,3}[j]^2 \xi_{1,3}[i]^2 + e^{-\xi_{1,1}[i] + \xi_{2,2}[i] + 2 \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j]^2 \xi_{1,3}[i]^2 + \\
& e^{-\xi_{1,1}[i] + 3 \xi_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j]^3 \xi_{1,3}[i]^2 - e^{-\xi_{1,1}[j] - \xi_{2,2}[i] + 2 \xi_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,3}[i]^2 + \\
& e^{-\xi_{2,2}[i] + 3 \xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j]^2 \eta_{2,3}[j] \xi_{1,3}[i]^2 + \\
& e^{-\xi_{1,1}[j] + 2 \xi_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j]^2 \xi_{1,2}[i] \xi_{1,3}[i]^2 - e^{3 \xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j]^3 \xi_{1,2}[i] \xi_{1,3}[i]^2 + \\
& e^{-\xi_{2,2}[i] + 2 \xi_{3,3}[i]} y_{1,3}[k] \eta_{1,2}[i] \eta_{2,3}[j]^2 \xi_{2,3}[i] + e^{\xi_{2,2}[i] + \xi_{3,3}[i]} x_{1,1}[k] \eta_{1,2}[j] \\
& \eta_{2,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] - e^{\xi_{2,2}[i] + \xi_{3,3}[i]} x_{2,2}[k] \eta_{1,2}[j] \eta_{2,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] - \\
& 2 e^{-\xi_{1,1}[i] + 2 \xi_{2,2}[i] + \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j]^2 \eta_{2,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] - \\
& e^{2 \xi_{3,3}[i]} y_{1,3}[k] \eta_{1,2}[i] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] - e^{2 \xi_{2,2}[i] + \xi_{3,3}[i]} x_{1,1}[k] \eta_{1,2}[j] \\
& \eta_{1,3}[j] \xi_{1,2}[i]^2 \xi_{2,3}[i] + e^{2 \xi_{2,2}[i] + \xi_{3,3}[i]} x_{2,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \xi_{1,2}[i]^2 \xi_{2,3}[i] + \\
& 3 e^{-\xi_{1,1}[i] + 3 \xi_{2,2}[i] + \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j]^2 \eta_{1,3}[j] \xi_{1,2}[i]^2 \xi_{2,3}[i] - \\
& e^{2 \xi_{3,3}[i]} x_{2,2}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,3}[i] \xi_{2,3}[i] + e^{2 \xi_{3,3}[i]} x_{3,3}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,3}[i] \xi_{2,3}[i] - \\
& 2 e^{-\xi_{1,1}[i] + \xi_{2,2}[i] + 2 \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,3}[i] \xi_{2,3}[i] + \\
& 2 e^{-\xi_{1,1}[i] + 3 \xi_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j]^2 \eta_{2,3}[j] \xi_{1,3}[i] \xi_{2,3}[i] - \\
& e^{-\xi_{1,1}[j] - \xi_{2,2}[i] + 2 \xi_{3,3}[i]} x_{1,2}[k] \eta_{2,3}[j]^2 \xi_{1,3}[i] \xi_{2,3}[i] + \\
& 2 e^{-\xi_{2,2}[i] + 3 \xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j] \eta_{2,3}[j]^2 \xi_{1,3}[i] \xi_{2,3}[i] - e^{\xi_{2,2}[i] + 2 \xi_{3,3}[i]} x_{1,1}[k] \eta_{1,3}[j]^2 \\
& \xi_{1,2}[i] \xi_{1,3}[i] \xi_{2,3}[i] + e^{\xi_{2,2}[i] + 2 \xi_{3,3}[i]} x_{2,2}[k] \eta_{1,3}[j]^2 \xi_{1,2}[i] \xi_{1,3}[i] \xi_{2,3}[i] + \\
& 4 e^{-\xi_{1,1}[i] + 2 \xi_{2,2}[i] + 2 \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j]^2 \xi_{1,2}[i] \xi_{1,3}[i] \xi_{2,3}[i] + \\
& e^{-\xi_{1,1}[j] + 2 \xi_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,2}[i] \xi_{1,3}[i] \xi_{2,3}[i] - \\
& 2 e^{3 \xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j]^2 \eta_{2,3}[j] \xi_{1,2}[i] \xi_{1,3}[i] \xi_{2,3}[i] + \\
& e^{-\xi_{2,2}[i] + 2 \xi_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,2}[j] \xi_{1,3}[i] \xi_{2,3}[i] - e^{2 \xi_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j]^2 \\
& \xi_{1,2}[i] \xi_{1,2}[j] \xi_{1,3}[i] \xi_{2,3}[i] + e^{-\xi_{1,1}[i] + \xi_{2,2}[i] + 3 \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,3}[j]^3 \xi_{1,3}[i]^2 \xi_{2,3}[i] - \\
& \frac{1}{2} e^{2 \xi_{3,3}[i]} x_{2,2}[k] \eta_{2,3}[j]^2 \xi_{2,3}[i]^2 + \frac{1}{2} e^{2 \xi_{3,3}[i]} x_{3,3}[k] \eta_{2,3}[j]^2 \xi_{2,3}[i]^2 + \\
& e^{-\xi_{1,1}[i] + 3 \xi_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j] \eta_{2,3}[j]^2 \xi_{2,3}[i]^2 + e^{-\xi_{2,2}[i] + 3 \xi_{3,3}[i]} y_{2,3}[k] \eta_{2,3}[j]^3 \xi_{2,3}[i]^2 - \\
& 2 e^{-\xi_{1,1}[i] + 2 \xi_{2,2}[i] + 2 \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,2}[i] \xi_{2,3}[i]^2 - \\
& e^{3 \xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j] \eta_{2,3}[j]^2 \xi_{1,2}[i] \xi_{2,3}[i]^2 - \frac{1}{2} e^{2 \xi_{2,2}[i] + 2 \xi_{3,3}[i]} x_{1,1}[k] \\
& \eta_{1,3}[j]^2 \xi_{1,2}[i]^2 \xi_{2,3}[i]^2 + \frac{1}{2} e^{2 \xi_{2,2}[i] + 2 \xi_{3,3}[i]} x_{2,2}[k] \eta_{1,3}[j]^2 \xi_{1,2}[i]^2 \xi_{2,3}[i]^2 + \\
& 3 e^{-\xi_{1,1}[i] + 3 \xi_{2,2}[i] + 2 \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j]^2 \xi_{1,2}[i]^2 \xi_{2,3}[i]^2 + \\
& e^{-\xi_{2,2}[i] + 2 \xi_{3,3}[i]} x_{1,2}[k] \eta_{2,3}[j]^2 \xi_{1,2}[j] \xi_{2,3}[i]^2 - e^{2 \xi_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,2}[i]
\end{aligned}$$

$$\begin{aligned}
& \xi_{1,2}[j] \xi_{2,3}[i]^2 + 2 e^{-\xi_{1,1}[i] + 2 \xi_{2,2}[i] + 3 \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,3}[j]^3 \xi_{1,2}[i] \xi_{1,3}[i] \xi_{2,3}[i]^2 + \\
& e^{-\xi_{1,1}[i] + 3 \xi_{2,2}[i] + 3 \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,3}[j]^3 \xi_{1,2}[i]^2 \xi_{2,3}[i]^3 \Big) + e^{-\xi_{3,3}[i]} x_{2,3}[k] \xi_{2,3}[j] + \\
& \in \left(-e^{-\xi_{2,2}[i] + \xi_{3,3}[i]} y_{1,3}[k] \eta_{1,2}[i] \eta_{2,3}[j] + e^{\xi_{2,2}[i]} x_{1,1}[k] \eta_{1,2}[j] \xi_{1,2}[i] - \right. \\
& e^{\xi_{2,2}[i]} x_{2,2}[k] \eta_{1,2}[j] \xi_{1,2}[i] - e^{-\xi_{1,1}[i] + 2 \xi_{2,2}[i]} y_{1,2}[k] \eta_{1,2}[j]^2 \xi_{1,2}[i] + \\
& e^{\xi_{3,3}[i]} y_{1,3}[k] \eta_{1,2}[i] \eta_{1,3}[j] \xi_{1,2}[i] + x_{1,2}[k] \eta_{1,2}[j] \xi_{1,2}[i] \xi_{1,2}[j] - \\
& e^{-\xi_{2,2}[j]} x_{2,3}[k] \eta_{1,2}[j] \xi_{1,3}[i] + e^{\xi_{3,3}[i]} x_{1,1}[k] \eta_{1,3}[j] \xi_{1,3}[i] - e^{\xi_{3,3}[i]} x_{3,3}[k] \eta_{1,3}[j] \xi_{1,3}[i] - \\
& e^{-\xi_{1,1}[i] + \xi_{2,2}[i] + \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \xi_{1,3}[i] - e^{-\xi_{1,1}[i] + 2 \xi_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j]^2 \xi_{1,3}[i] + \\
& e^{-\xi_{1,1}[j] - \xi_{2,2}[i] + \xi_{3,3}[i]} x_{1,2}[k] \eta_{2,3}[j] \xi_{1,3}[i] - e^{-\xi_{2,2}[i] + 2 \xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,3}[i] - \\
& e^{-\xi_{1,1}[j] + \xi_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{1,3}[i] + e^{2 \xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j]^2 \xi_{1,2}[i] \xi_{1,3}[i] + \\
& x_{1,3}[k] \eta_{1,2}[j] \xi_{1,2}[j] \xi_{1,3}[i] + x_{1,3}[k] \eta_{1,3}[j] \xi_{1,3}[i] \xi_{1,3}[j] + e^{\xi_{3,3}[i]} x_{2,2}[k] \eta_{2,3}[j] \xi_{2,3}[i] - \\
& e^{\xi_{3,3}[i]} x_{3,3}[k] \eta_{2,3}[j] \xi_{2,3}[i] + e^{-\xi_{1,1}[i] + \xi_{2,2}[i] + \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{2,3}[j] \xi_{2,3}[i] - \\
& e^{-\xi_{1,1}[i] + 2 \xi_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{2,3}[i] - e^{-\xi_{2,2}[i] + 2 \xi_{3,3}[i]} y_{2,3}[k] \eta_{2,3}[j]^2 \xi_{2,3}[i] + \\
& e^{\xi_{2,2}[i] + \xi_{3,3}[i]} x_{1,1}[k] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] - e^{\xi_{2,2}[i] + \xi_{3,3}[i]} x_{2,2}[k] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] - \\
& 2 e^{-\xi_{1,1}[i] + 2 \xi_{2,2}[i] + \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] + \\
& e^{2 \xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] - \\
& e^{-\xi_{2,2}[i] + \xi_{3,3}[i]} x_{1,2}[k] \eta_{2,3}[j] \xi_{1,2}[j] \xi_{2,3}[i] + e^{\xi_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{1,2}[j] \xi_{2,3}[i] - \\
& e^{-\xi_{1,1}[i] + \xi_{2,2}[i] + 2 \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,3}[j]^2 \xi_{1,3}[i] \xi_{2,3}[i] + x_{1,3}[k] \eta_{2,3}[j] \xi_{1,3}[j] \xi_{2,3}[i] - \\
& e^{-\xi_{1,1}[i] + 2 \xi_{2,2}[i] + 2 \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,3}[j]^2 \xi_{1,2}[i] \xi_{2,3}[i]^2 + x_{2,3}[k] \eta_{1,3}[j] \xi_{1,3}[i] \xi_{2,3}[j] + \\
& x_{2,3}[k] \eta_{2,3}[j] \xi_{2,3}[i] \xi_{2,3}[j] \Big) + x_{3,3}[k] \xi_{3,3}[i] + x_{3,3}[k] \xi_{3,3}[j]
\end{aligned}$$

In[*]:= **sol0**

Out[*]=

$$\begin{aligned}
& \{f_{1,2}[\lambda] \rightarrow e^{-\xi_{2,1}^{1,1}} \lambda \xi_{1,2} + e^{-\xi_{1,2}^{1,2}} \lambda \xi_{2,1,2}, g_{2,3}[\lambda] \rightarrow \lambda (\eta_{1,2,3} + e^{\xi_{1,3}^{1,3}} (e^{-\xi_{1,2}^{1,2}} \eta_{2,2,3} - \lambda \eta_{2,1,3} \xi_{1,2,2})), \\
& f_{1,3}[\lambda] \rightarrow \lambda (e^{-\xi_{2,1}^{1,1}} \xi_{1,3} - \lambda \xi_{1,2,3} \xi_{2,1,2} + e^{-\xi_{1,3}^{1,3}} \xi_{2,1,3}), f_{2,3}[\lambda] \rightarrow e^{-\xi_{2,2}^{1,2}} \lambda \xi_{1,2,3} + e^{-\xi_{1,3}^{1,3}} \lambda \xi_{2,2,3}, \\
& g_{1,2}[\lambda] \rightarrow \lambda (\eta_{1,1,2} + e^{-\xi_{1,1}^{1,1} + \xi_{1,2}^{1,2}} (\eta_{2,1,2} + e^{\xi_{1,3}^{1,3}} \lambda \eta_{2,1,3} \xi_{1,2,3})), g_{1,3}[\lambda] \rightarrow \lambda (\eta_{1,1,3} + e^{-\xi_{1,1}^{1,1} + \xi_{1,3}^{1,3}} \eta_{2,1,3})\}
\end{aligned}$$

In[*]:= **CF[sol1 /. λ → 1]**

Out[*]=

$$\begin{aligned}
& \{f_{1,1}[1] \rightarrow e^{\xi_{1,2}^{1,2}} \eta_{2,1,2} \xi_{1,1,2} + e^{\xi_{1,3}^{1,3}} \eta_{2,1,3} \xi_{1,1,3} + e^{\xi_{1,2,2}^{1,2} + \xi_{1,3}^{1,3}} \eta_{2,1,3} \xi_{1,1,2} \xi_{1,2,3}, \\
& f_{2,2}[1] \rightarrow -e^{\xi_{1,2}^{1,2}} \eta_{2,1,2} \xi_{1,1,2} + e^{\xi_{1,3}^{1,3}} \eta_{2,2,3} \xi_{1,2,3} - e^{\xi_{1,2,2}^{1,2} + \xi_{1,3}^{1,3}} \eta_{2,1,3} \xi_{1,1,2} \xi_{1,2,3}, \\
& f_{3,3}[1] \rightarrow -e^{\xi_{1,3}^{1,3}} \eta_{2,1,3} \xi_{1,1,3} - e^{\xi_{1,3}^{1,3}} \eta_{2,2,3} \xi_{1,2,3}, \\
& f_{1,2}[1] \rightarrow e^{-\xi_{1,2,2}^{1,2} + \xi_{1,3}^{1,3} - \xi_{2,1}^{1,1}} \eta_{2,2,3} \xi_{1,1,3} - e^{\xi_{1,3}^{1,3} - \xi_{2,1}^{1,1}} \eta_{2,1,3} \xi_{1,1,2} \xi_{1,1,3} + \\
& \quad \eta_{2,1,2} \xi_{1,1,2} \xi_{2,1,2} - e^{-\xi_{1,2,2}^{1,2} + \xi_{1,3}^{1,3}} \eta_{2,2,3} \xi_{1,2,3} \xi_{2,1,2} + e^{\xi_{1,3}^{1,3}} \eta_{2,1,3} \xi_{1,1,2} \xi_{1,2,3} \xi_{2,1,2}, \\
& f_{2,3}[1] \rightarrow -e^{-\xi_{2,2}^{1,2}} \eta_{2,1,2} \xi_{1,1,3} + \eta_{2,1,3} \xi_{1,1,3} \xi_{2,2,3} + \eta_{2,2,3} \xi_{1,2,3} \xi_{2,2,3}, \\
& f_{1,3}[1] \rightarrow \eta_{2,1,2} \xi_{1,1,3} \xi_{2,1,2} + \eta_{2,1,3} \xi_{1,1,3} \xi_{2,1,3} + \eta_{2,2,3} \xi_{1,2,3} \xi_{2,1,3}, \\
& g_{1,3}[1] \rightarrow e^{-\xi_{1,1}^{1,1} + \xi_{1,2}^{1,2}} \eta_{1,2,3} \eta_{2,1,2} - e^{-\xi_{1,1}^{1,1} + \xi_{1,2,2}^{1,2} + \xi_{1,3}^{1,3}} \eta_{2,1,2} \eta_{2,1,3} \xi_{1,1,2} - e^{-\xi_{1,1}^{1,1} + 2 \xi_{1,3}^{1,3}} \eta_{2,1,3}^2 \xi_{1,1,3} + \\
& \quad e^{-\xi_{1,1}^{1,1} + \xi_{1,2,2}^{1,2} + \xi_{1,3}^{1,3}} \eta_{1,2,3} \eta_{2,1,3} \xi_{1,2,3} - e^{-\xi_{1,1}^{1,1} + \xi_{1,2,2}^{1,2} + 2 \xi_{1,3}^{1,3}} \eta_{2,1,3}^2 \xi_{1,1,2} \xi_{1,2,3}, g_{1,2}[1] \rightarrow \\
& \quad -e^{-\xi_{1,1}^{1,1} + 2 \xi_{1,2}^{1,2}} \eta_{2,1,2}^2 \xi_{1,1,2} - e^{-\xi_{1,1}^{1,1} + \xi_{1,2,2}^{1,2} + \xi_{1,3}^{1,3}} \eta_{2,1,2} \eta_{2,1,3} \xi_{1,1,3} - 2 e^{-\xi_{1,1}^{1,1} + 2 \xi_{1,2,2}^{1,2} + \xi_{1,3}^{1,3}} \eta_{2,1,2} \eta_{2,1,3} \xi_{1,1,2} \xi_{1,2,3} - \\
& \quad e^{-\xi_{1,1}^{1,1} + \xi_{1,2,2}^{1,2} + 2 \xi_{1,3}^{1,3}} \eta_{2,1,3}^2 \xi_{1,1,3} \xi_{1,2,3} - e^{-\xi_{1,1}^{1,1} + 2 \xi_{1,2,2}^{1,2} + 2 \xi_{1,3}^{1,3}} \eta_{2,1,3}^2 \xi_{1,1,2} \xi_{1,2,3}, \\
& g_{2,3}[1] \rightarrow e^{\xi_{1,3}^{1,3}} \eta_{2,1,2} \eta_{2,2,3} \xi_{1,1,2} - e^{-\xi_{1,2,2}^{1,2} + 2 \xi_{1,3}^{1,3}} \eta_{2,1,3} \eta_{2,2,3} \xi_{1,1,3} + e^{2 \xi_{1,3}^{1,3}} \eta_{2,1,3}^2 \xi_{1,1,2} \xi_{1,1,3} - \\
& \quad e^{-\xi_{1,2,2}^{1,2} + 2 \xi_{1,3}^{1,3}} \eta_{2,2,3}^2 \xi_{1,2,3} + e^{2 \xi_{1,3}^{1,3}} \eta_{2,1,3} \eta_{2,2,3} \xi_{1,1,2} \xi_{1,2,3}\}
\end{aligned}$$

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In[*]:= Coefficient[osol, x1,1[k] e]
Out[*]:=
eε2,2[i] η1,2[j] ε1,2[i] + eε3,3[i] η1,3[j] ε1,3[i] + eε2,2[i]+ε3,3[i] η1,3[j] ε1,2[i] ε2,3[i]

In[*]:= CF[(Expand@sol[$k].$Basis /. {ξ1αβ → ξαβ[i], η1αβ → ηαβ[i],
      ξ2αβ → ξαβ[j], η2αβ → ηαβ[j], λ → 1}) - (osol /. v_ [k] → v)]
Out[*]:=
sol[1].{y1,2, y1,3, y2,3, x1,1, x1,2, x2,2, x1,3, x2,3, x3,3} - y1,2 η1,2[i] -
e-ε1,1[i]+ε2,2[i] y1,2 η1,2[j] - y1,3 η1,3[i] - e-ε1,1[i]+ε3,3[i] y1,3 η1,3[j] - y2,3 η2,3[i] -
e-ε2,2[i]+ε3,3[i] y2,3 η2,3[j] + e-ε2,2[i]+ε3,3[i] e y1,3 η1,2[i] η2,3[j] - x1,1 ε1,1[i] - x1,1 ε1,1[j] -
e-ε1,1[j] x1,2 ε1,2[i] - eε2,2[i] e x1,1 η1,2[j] ε1,2[i] + eε2,2[i] e x2,2 η1,2[j] ε1,2[i] +
e-ε1,1[i]+2ε2,2[i] e y1,2 η1,2[j]2 ε1,2[i] + eε3,3[i] y2,3 η1,3[j] ε1,2[i] -
eε3,3[i] e y1,3 η1,2[i] η1,3[j] ε1,2[i] - e-ε2,2[i] x1,2 ε1,2[j] - e x1,2 η1,2[j] ε1,2[i] ε1,2[j] -
e-ε1,1[j] x1,3 ε1,3[i] + e-ε2,2[j] e x2,3 η1,2[j] ε1,3[i] - eε3,3[i] e x1,1 η1,3[j] ε1,3[i] +
eε3,3[i] e x3,3 η1,3[j] ε1,3[i] + e-ε1,1[i]+ε2,2[i]+ε3,3[i] e y1,2 η1,2[j] η1,3[j] ε1,3[i] +
e-ε1,1[i]+2ε3,3[i] e y1,3 η1,3[j]2 ε1,3[i] - e-ε1,1[j]-ε2,2[i]+ε3,3[i] e x1,2 η2,3[j] ε1,3[i] +
e-ε2,2[i]+2ε3,3[i] e y2,3 η1,3[j] η2,3[j] ε1,3[i] + e-ε1,1[j]+ε3,3[i] e x1,2 η1,3[j] ε1,2[i] ε1,3[i] -
e2ε3,3[i] e y2,3 η1,3[j]2 ε1,2[i] ε1,3[i] - e x1,3 η1,2[j] ε1,2[j] ε1,3[i] - e-ε3,3[i] x1,3 ε1,3[j] -
e x1,3 η1,3[j] ε1,3[i] ε1,3[j] - x2,2 ε2,2[i] - x2,2 ε2,2[j] - e-ε2,2[j] x2,3 ε2,3[i] -
e-ε1,1[i]+ε2,2[i]+ε3,3[i] y1,2 η1,3[j] ε2,3[i] - eε3,3[i] e x2,2 η2,3[j] ε2,3[i] +
eε3,3[i] e x3,3 η2,3[j] ε2,3[i] - e-ε1,1[i]+ε2,2[i]+ε3,3[i] e y1,2 η1,2[j] η2,3[j] ε2,3[i] +
e-ε1,1[i]+2ε3,3[i] e y1,3 η1,3[j] η2,3[j] ε2,3[i] + e-ε2,2[i]+2ε3,3[i] e y2,3 η2,3[j]2 ε2,3[i] -
eε2,2[i]+ε3,3[i] e x1,1 η1,3[j] ε1,2[i] ε2,3[i] + eε2,2[i]+ε3,3[i] e x2,2 η1,3[j] ε1,2[i] ε2,3[i] +
2 e-ε1,1[i]+2ε2,2[i]+ε3,3[i] e y1,2 η1,2[j] η1,3[j] ε1,2[i] ε2,3[i] -
e2ε3,3[i] e y2,3 η1,3[j] η2,3[j] ε1,2[i] ε2,3[i] + x1,3 ε1,2[j] ε2,3[i] +
e-ε2,2[i]+ε3,3[i] e x1,2 η2,3[j] ε1,2[j] ε2,3[i] - eε3,3[i] e x1,2 η1,3[j] ε1,2[i] ε1,2[j] ε2,3[i] +
e-ε1,1[i]+ε2,2[i]+2ε3,3[i] e y1,2 η1,3[j]2 ε1,3[i] ε2,3[i] - e x1,3 η2,3[j] ε1,3[j] ε2,3[i] +
e-ε1,1[i]+2ε2,2[i]+2ε3,3[i] e y1,2 η1,3[j]2 ε1,2[i] ε2,3[i]2 - e-ε3,3[i] x2,3 ε2,3[j] -
e x2,3 η1,3[j] ε1,3[i] ε2,3[j] - e x2,3 η2,3[j] ε2,3[i] ε2,3[j] - x3,3 ε3,3[i] - x3,3 ε3,3[j]

```

pdf

$gl_{4,\epsilon}$ Experiments

pdf

```

In[ ]:= SetAlgebra[gl4,ε];
$k = 1
$PBWRule
MatrixForm@Table[{b1, b2} → B[b1, b2], {b1, $Basis}, {b2, $Basis}];
MatrixForm@
  Table[{b1, b2} → (DefRep[b1].DefRep[b2] - DefRep[b2].DefRep[b1] == DefRep[B[b1, b2]]),
    {b1, $Basis}, {b2, $Basis}];
DeleteCases[Flatten[Table[{a, b, c} → B[a, B[b, c]] + B[b, B[c, a]] + B[c, B[a, b]],
  {a, $Basis}, {b, $Basis}, {c, $Basis}]], _ → 0]

```

Out[]=
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1

Out[]=
pdf

```

{y1,2 → 1, y1,3 → 2, y2,3 → 3, y1,4 → 4, y2,4 → 5, y3,4 → 6, x1,1 → 7, x1,2 → 8,
  x2,2 → 9, x1,3 → 10, x2,3 → 11, x3,3 → 12, x1,4 → 13, x2,4 → 14, x3,4 → 15, x4,4 → 16}

```

Out[]=
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{ }

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```

In[ ]:= SBasis = SortBy[$Basis, If[#[[1]] == x, {0, #[[3]] - #[[2]]}, {1, #[[2]] - #[[3]]}] &];
sol[-1] = Table[If[SSQ[bb], ((bb /. x -> ξ1) + (bb /. x -> ξ2)), 0], {bb, $Basis}];
Do[
  Block[{ $k = k},
    s0 = Thread[$Basis -> sol[k - 1]];
    lhs = λTangent[
      Join[Table[If[NilQ[bb], λ, 1] bb (bb /. {x -> ξ1, y -> η1}), {bb, $Basis}],
        Table[If[NilQ[bb], λ, 1] bb (bb /. {x -> ξ2, y -> η2}), {bb, $Basis}]]];
    rhs = λTangent[
      $Basis ($Basis /. {xαβ -> (xαβ /. s0) + εk fαβ[λ], yαβ -> (yαβ /. s0) + εk gαβ[λ]})];
    eqns = Transpose@{
      (Coefficient[CF[lhs - rhs], #] == 0) & /@ SBasis,
      SBasis /. {xαβ -> fαβ[0], yαβ -> gαβ[0]}];
    unknowns = SBasis /. {xαβ -> fαβ[λ], yαβ -> gαβ[λ]};
    s = TriangularDSolve[eqns, unknowns, λ];
    sol[k] = CF[sol[k - 1] + εk ($Basis /. {xαβ -> fαβ[λ], yαβ -> gαβ[λ]} /. s)]];
  {k, 0, $k}
];
cm[i_, j_ -> k_] := Λ2E{i,j} -> {k}[sol[$k].$Basis /. {xαβ -> xαβ[k], yαβ -> yαβ[k],
  ξ1αβ -> ξαβ[i], η1αβ -> ηαβ[i], ξ2αβ -> ξαβ[j], η2αβ -> ηαβ[j], λ -> 1}];
Short[cm[i, j -> k], 20]

```

Out[]//Short=
pdf

$$\begin{aligned}
& \mathbb{E}_{\{i,j\} \rightarrow \{k\}} \left[y_{1,4}[k] \left(\eta_{1,4}[i] + e^{-\xi_{1,1}[i] + \xi_{4,4}[i]} \eta_{1,4}[j] \right) + x_{1,1}[k] \left(\xi_{1,1}[i] + \xi_{1,1}[j] \right) + \right. \\
& y_{2,4}[k] \left(\eta_{2,4}[i] + e^{-\xi_{2,2}[i] + \xi_{4,4}[i]} \eta_{2,4}[j] - e^{\xi_{4,4}[i]} \eta_{1,4}[j] \xi_{1,2}[i] \right) + \\
& x_{1,2}[k] \left(e^{-\xi_{1,1}[j]} \xi_{1,2}[i] + e^{-\xi_{2,2}[i]} \xi_{1,2}[j] \right) + x_{2,2}[k] \left(\xi_{2,2}[i] + \xi_{2,2}[j] \right) + \\
& y_{3,4}[k] \left(\eta_{3,4}[i] + e^{-\xi_{3,3}[i] + \xi_{4,4}[i]} \eta_{3,4}[j] - e^{\xi_{4,4}[i]} \eta_{1,4}[j] \xi_{1,3}[i] - e^{\xi_{4,4}[i]} \eta_{2,4}[j] \xi_{2,3}[i] \right) + \\
& x_{1,3}[k] \left(e^{-\xi_{1,1}[j]} \xi_{1,3}[i] + e^{-\xi_{3,3}[i]} \xi_{1,3}[j] - \xi_{1,2}[j] \xi_{2,3}[i] \right) + \\
& x_{2,3}[k] \left(e^{-\xi_{2,2}[j]} \xi_{2,3}[i] + e^{-\xi_{3,3}[i]} \xi_{2,3}[j] \right) + x_{3,3}[k] \left(\xi_{3,3}[i] + \xi_{3,3}[j] \right) + \\
& y_{1,3}[k] \left(\eta_{1,3}[i] + e^{-\xi_{1,1}[i] + \xi_{3,3}[i]} \eta_{1,3}[j] + e^{-\xi_{1,1}[i] + \xi_{3,3}[i] + \xi_{4,4}[i]} \eta_{1,4}[j] \xi_{3,4}[i] \right) + \\
& y_{2,3}[k] \left(\eta_{2,3}[i] + e^{-\xi_{2,2}[i] + \xi_{3,3}[i]} \eta_{2,3}[j] - e^{\xi_{3,3}[i]} \eta_{1,3}[j] \xi_{1,2}[i] + \right. \\
& \quad \left. e^{-\xi_{2,2}[i] + \xi_{3,3}[i] + \xi_{4,4}[i]} \eta_{2,4}[j] \xi_{3,4}[i] - e^{\xi_{3,3}[i] + \xi_{4,4}[i]} \eta_{1,4}[j] \xi_{1,2}[i] \xi_{3,4}[i] \right) + \\
& x_{1,4}[k] \left(e^{-\xi_{1,1}[j]} \xi_{1,4}[i] + e^{-\xi_{4,4}[i]} \xi_{1,4}[j] - \xi_{1,2}[j] \xi_{2,4}[i] - \xi_{1,3}[j] \xi_{3,4}[i] \right) + \\
& e^{-\xi_{1,1}[i]} y_{1,2}[k] \left(e^{\xi_{1,1}[i]} \eta_{1,2}[i] + e^{\xi_{2,2}[i]} \eta_{1,2}[j] + e^{\xi_{2,2}[i] + \xi_{3,3}[i]} \eta_{1,3}[j] \xi_{2,3}[i] + \right. \\
& \quad \left. e^{\xi_{2,2}[i] + \xi_{4,4}[i]} \eta_{1,4}[j] \xi_{2,4}[i] + e^{\xi_{2,2}[i] + \xi_{3,3}[i] + \xi_{4,4}[i]} \eta_{1,4}[j] \xi_{2,3}[i] \xi_{3,4}[i] \right) + \\
& x_{2,4}[k] \left(e^{-\xi_{2,2}[j]} \xi_{2,4}[i] + e^{-\xi_{4,4}[i]} \xi_{2,4}[j] - \xi_{2,3}[j] \xi_{3,4}[i] \right) + \\
& x_{3,4}[k] \left(e^{-\xi_{3,3}[j]} \xi_{3,4}[i] + e^{-\xi_{4,4}[i]} \xi_{3,4}[j] \right) + x_{4,4}[k] \left(\xi_{4,4}[i] + \xi_{4,4}[j] \right), \\
& -e^{\xi_{4,4}[i]} x_{4,4}[k] \left(\eta_{1,4}[j] \xi_{1,4}[i] + \eta_{2,4}[j] \xi_{2,4}[i] + \eta_{3,4}[j] \xi_{3,4}[i] \right) + \\
& \ll 18 \gg + x_{3,4}[k] \left(-e^{-\xi_{3,3}[j]} \eta_{1,3}[j] \xi_{1,4}[i] - e^{-\xi_{3,3}[j]} \eta_{2,3}[j] \xi_{2,4}[i] + \right. \\
& \quad \left. \eta_{1,4}[j] \xi_{1,4}[i] \xi_{3,4}[j] + \eta_{2,4}[j] \xi_{2,4}[i] \xi_{3,4}[j] + \eta_{3,4}[j] \xi_{3,4}[i] \xi_{3,4}[j] \right) \Big]
\end{aligned}$$

pdf

In[]:= (cm[1, 2 → 2] // cm[2, 3 → 1]) ≡ (cm[2, 3 → 2] // cm[1, 2 → 1])

Out[]=
pdf

True