

Pensieve header: The PBW multiplication tensor for $\mathfrak{sl}_n$ using the nilpotent-only λ -tangent formalism. Some material from pensieve://Projects/glneps and from pensieve://Projects/UEA.

```
In[ ]:= SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
```

Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at <http://katlas.org/wiki/KnotTheory>.

Conventions

$x_{\alpha,\beta}$ is always with $\alpha \leq \beta$ and represents a basis element of the upper triangular matrices. $y_{\alpha,\beta}$ is always with $\beta \leq \alpha$ and represents a basis element of the lower triangular matrices. The adjoint of $x_{\alpha,\beta}$ is $y_{\beta,\alpha}$.

Variable declarations:

```
In[ ]:= VarPattern = ((y | x | η | ξ) | (y | x | η | ξ)_) [ _ ];
```

Greek - Latin duality:

```
In[ ]:= {y*, x*, η*, ξ*} = {η, ξ, y, x};
(u_{αβ})^* := (u^*)_{αβ};
((u : (y | x | η | ξ) | (y | x | η | ξ)_) [i_])^* := u^*[i];
(vs_List)^* := (v ↦ v^*) /@ vs;
```

```
In[ ]:= {x_{1,2}, y_{2,3}, η_{1,2}[7]}^*
Out[ ]:= {ξ_{1,2}, η_{2,3}, y_{1,2}[7]}
```

Weights:

```
In[ ]:= Wt[x_{α,β}] := {1, α - β}; Wt[y_{α,β}] := {0, α - β};
Wt[ξ_{α,β}] := {0, β - α}; Wt[η_{α,β}] := {1, β - α};
Wt[u_] := Wt[u];
```

The Engine

Canonical Forms:

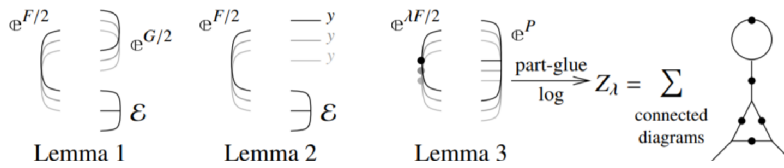
```
In[ ]:= LogReduce[ε_] :=
  ε /. c_ * Log[a_] => Log@Factor[a^c] /. Log[a_] + Log[b_] => Log@Factor[a b];
CF[ε_] := Expand@Collect[ε, Cases[ε, VarPattern, ∞], CCF] /. CCF -> Factor;
CF[ε_E] := CF /@ MapAt[LogReduce, ε, 1];
CF[ε : E_][_] := CF /@ ε;
```

E operations:

```
In[*]:=
 $\mathcal{E}_E[\$] := \text{Length}[\mathcal{E}] - 1; \mathbb{E}_-[\mathcal{E}\_\_\_\_\_\_][\$] := \mathbb{E}[\mathcal{E}][\$];$ 
 $\mathcal{E}_E[k\_Integer] := \mathcal{E}[\![k+1]\!]; \mathbb{E}_-[\mathcal{E}\_\_\_\_\_\_][k\_Integer] := \{\mathcal{E}\}[\![k+1]\!];$ 
 $\mathbb{E} /: \mathcal{E}_1\_E \equiv \mathcal{E}_2\_E := \text{Inner}[\text{CF}@\#1 == \text{CF}@\#2 \ \&, \mathcal{E}_1, \mathcal{E}_2, \text{And}];$ 
 $\mathbb{E}_{d1 \rightarrow r1}[\mathcal{E}1s\_]\equiv \mathbb{E}_{d2 \rightarrow r2}[\mathcal{E}2s\_]\wedge := (d1 == d2) \wedge (r1 == r2) \wedge (\mathbb{E}[\mathcal{E}1s] \equiv \mathbb{E}[\mathcal{E}2s]);$ 
 $\mathbb{E} /: \mathcal{E}_1\_E * \mathcal{E}_2\_E := \mathbb{E}@\text{Table}[\text{CF}[\mathcal{E}1[\text{kk}] + \mathcal{E}2[\text{kk}]], \{\text{kk}, 0, \text{Min}[\mathcal{E}1[\$], \mathcal{E}2[\$]]\}];$ 
 $\mathbb{E}_{d1 \rightarrow r1}[\mathcal{E}1s\_]\mathbb{E}_{d2 \rightarrow r2}[\mathcal{E}2s\_]\wedge := \mathbb{E}_{(d1 \cup d2) \rightarrow (r1 \cup r2)}@\text{Table}[\text{CF}[\mathcal{E}1[\text{kk}] + \mathcal{E}2[\text{kk}]], \{\text{kk}, 0, \text{Min}[\mathcal{E}1[\$], \mathcal{E}2[\$]]\}];$ 
```

```
In[*]:=
 $\mathbb{E}_{d1 \rightarrow r1}[\mathcal{E}1s\_]\text{ // } \mathbb{E}_{d2 \rightarrow r2}[\mathcal{E}2s\_]\text{ := Module}[\{\text{is} = r1 \cap d2, \text{lvs}, \text{llvs}\},$ 
 $\text{lvs} = \text{Cases}[\{\mathcal{E}1s\}, (\text{x} | \text{y})\_[\text{i}_\_] /; \text{MemberQ}[\text{is}, \text{i}], \infty] \cup$ 
 $\text{Cases}[\{\mathcal{E}2s\}, (\text{x} | \text{y})\_[\text{i}_\_] /; \text{MemberQ}[\text{is}, \text{i}], \infty]^*;$ 
 $\text{llvs} = \text{Map}[\$, \text{lvs}, \{2\}];$ 
 $\mathbb{E}_{(d1 \cup \text{Complement}[d2, \text{is}]) \rightarrow (r2 \cup \text{Complement}[r1, \text{is}])}@\text{Table}[\text{CF}[\mathcal{E}1[\text{kk}] + \mathcal{E}2[\text{kk}]], \{\text{kk}, 0, \text{Min}[\mathcal{E}1[\$], \mathcal{E}2[\$]]\}];$ 
 $\text{Times}[\mathbb{E}[\mathcal{E}1s] /. \text{Thread}[\text{lvs} \rightarrow \text{llvs}],$ 
 $\mathbb{E}[\mathcal{E}2s] /. \text{Thread}[\text{lvs}^* \rightarrow \text{llvs}^*]$ 
 $\text{ } ] ] ]$ 
```

Zippping! Lemmas 2 and 3 are combined, yet they must be applied first to the middle weight variables and then to the heavy and light variables.



Comment. Zip3 of the outer variables must occur after all other operations are completed, because we must allow for gluings of the weight n variables in perturbations with the weight 0 variables in the coefficients of Q .

dvs stands for “Diagonal Variables”. In gl_n they are the x_{ij} ’s and the y_{ij} ’s. They have weights $\{1,0\}$ and $\{0,0\}$.

```
In[*]:=
 $\text{Zip}_{vs\_}[\{\mathcal{F}\_, \mathcal{E}\_]\text{ := Module}[\{\text{dvs} = \text{Select}[\text{vs}, (\text{Wt}[\#] == \{0, 0\} \vee \text{Wt}[\#] == \{1, 0\}) \ \&]\},$ 
 $\{\mathcal{F}, \mathcal{E}\} \text{ // Zip1}_{vs}(* // \text{Zip2}_{\text{Complement}[\text{vs}, \text{dvs}]}*) \text{ // Zip2}_{\text{dvs}} \text{ //}$ 
 $\text{Zip3}_{\text{Complement}[\text{vs}, \text{dvs}]} \text{ // Zip3}_{\text{dvs}}$ 
 $\text{ } ] \text{ // Last}$ 
```

Getting rid of the quadratic.

Lemma 1. With convergences left to the reader,

$$\left\langle F : \mathcal{E} \oplus^{\frac{1}{2}} \sum_{i,j \in B} G_{ij} z_i z_j \right\rangle_B = \det(1 - GF)^{-1/2} \left\langle F(1 - GF)^{-1} : \mathcal{E} \right\rangle_B$$

```

In[*]:= Zip1_{ } = Identity;
Zip1_{vs_} @ {f_, E[Q_, P___]} := Module[{I, F, G, u, v},
  I = IdentityMatrix@Length@vs;
  F = Table[If[Wt[u] + Wt[v] == {1, 0}, D[u, v] f, 0], {u, vs}, {v, vs}];
  G = Table[If[Wt[u] + Wt[v] == {1, 0}, D[u, v] Q, 0], {u, vs}, {v, vs}];
  {CF[(vs).(F.Inverse[I - G.F]).(vs)/2],
   E[CF[Q - PowerExpand@Log[Det[I - G.F]]/2 - vs.G.vs/2], P]}
]

```

Getting rid of linear terms.

Lemma 2. $\left\langle F: \mathcal{E}_{\oplus \sum_{i \in B} y_i z_i} \right\rangle_B = \mathbb{C}^{\frac{1}{2} \sum_{i,j \in B} F_{ij} y_i y_j} \left\langle F: \mathcal{E}_{|z_B \rightarrow z_B + F y_B} \right\rangle_B$.

```

In[*]:= Zip2_{ } = Identity;
Zip2_{vs_} @ {f_, E[Q_, P___]} := Module[{F, Y, u, v},
  F = Table[If[Wt[u] + Wt[v] == {1, 0}, CF[D[u, v] f], 0], {u, vs}, {v, vs}];
  Y = Table[D[v, Q], {v, vs}] /. Table[v -> 0, {v, vs}];
  CF /@ {f, E[Q - Y.vs + Y.F.Y/2, P] /. Thread[vs -> vs + F.Y]}
]

```

Dealing with Feynman diagrams.

Lemma 3. With an extra variable λ , $Z_\lambda := \log[\lambda F: \oplus^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2} \sum_{i,j \in B} F_{ij} \left(\partial_{z_i} \partial_{z_j} Z_\lambda + (\partial_{z_i} Z_\lambda)(\partial_{z_j} Z_\lambda) \right).$$

Note that the power m of λ is at most $k - 1 + \frac{2k+2}{2} = 2k$. We write $Z_\lambda = \sum Z[m] \lambda^m$.

```

In[*]:= Zip3vs_@{ $\mathcal{F}_-$ ,  $\mathcal{E}_-$ } := Module[
{F, u, v, Z, kk, jj, $m = 0, m, n},
  Do[Z[0, kk] =  $\mathcal{E}$ [[kk + 1]], {kk, 0,  $\mathcal{E}$ @$}];
  F[u_, v_] := F[u, v] = CF@If[Wt[u] + Wt[v] == {1, 0},  $\partial_{u,v}\mathcal{F}$ , 0];
  Z[m_, kk_, u_] := Z[m, kk, u] =  $\partial_u Z[m, kk]$ ;
  Z[m_, kk_, u_, v_] := Z[m, kk, u, v] =  $\partial_v Z[m, kk, u]$ ;
  For[m = 0, m ≤ 2 $m, ++m, For[kk = 0, kk ≤  $\mathcal{E}$ @$, ++kk,
    Z[m + 1, kk] = CF@Sum[
      If[F[u, v] == 0, 0,  $\frac{F[u, v]}{2(m+1)} (Z[m, kk, u, v] +$ 
        Sum[Z[n, jj, u] * Z[m - n, kk - jj, v], {n, 0, m}, {jj, 0, kk}]]],
      {u, vs}, {v, vs}];
    If[Z[m + 1, kk] != 0, $m = m + 1]
  ]];
  CF /@ ({
     $\mathcal{F}$  - Sum[F[u, v] * u * v / 2, {u, vs}, {v, vs}],
     $\mathcal{E}$  @@ Table[Sum[Z[m, kk], {m, 0, $m}], {kk, 0,  $\mathcal{E}$ @$}]
  } /. Table[v → 0, {v, vs}])
]

```

```

In[*]:=  $\Lambda^2 \mathbb{E}_{d \rightarrow r_-}[\mathcal{A}_-]$  :=  $\Lambda^2 \mathbb{E}_{d \rightarrow r}[\$k, \mathcal{A}]$ ;
 $\Lambda^2 \mathbb{E}_{d \rightarrow r_-}[\$k_-, \mathcal{A}_-]$  :=
Module[{k},  $\mathbb{E}_{d \rightarrow r}$  @@ Table[SeriesCoefficient[ $\mathcal{A}$ , { $\epsilon$ , 0, k}], {k, 0, $k}]]];

```

Prolog

```

In[*]:= (*BeginPackage["UEA`"];*)
Print["UEA` does computations in general universal enveloping
algebras and PBW algebras. It is in the public domain, available
at http://drorbn.net/AcademicPensieve/Projects/SolvablePBW/.
Dror Bar-Natan is committed to support it within
reason until June 1, 2027. This is version 260601."];
Print["UEA` implements / extends ",
Sort@{"**",  $\epsilon$ , $k, CF, SSQ, NilQ, B, m, SetAlgebra, U,
UB, UProducts, USimp, UU, $Basis, $PBWRule,  $\delta$ , adPower, adExp},
"."];
(*Begin["`Private`"];*)

```

UEA` does computations in general universal enveloping algebras and PBW algebras. It is in the public domain, available at <http://drorbn.net/AcademicPensieve/Projects/SolvablePBW/>. Dror Bar-Natan is committed to support it within reason until June 1, 2027. This is version 260601.

UEA` implements / extends `{**, adExp, adPower, B, CF, m, NilQ, SetAlgebra, SSQ, U, UB, UProducts, USimp, UU, δ , ϵ , $Basis, $k, $PBWRule}`.

Utilities

```
In[*]:=  $\delta_{i,j} := \text{If}[i == j, 1, 0];$ 
```

```
In[*]:= SSQ[c_.*x_] /; MemberQ[$Basis, x] :=  $\neg$  NilQ[x]; (* Semi-Simple Q *)
NilQ[c_.*x_] /; MemberQ[$Basis, x] := NilQ[x];
SSQ[x_Plus] := And @@ (SSQ /@ (List @@ x));
NilQ[x_Plus] := And @@ (NilQ /@ (List @@ x));
```

```
In[*]:= $k = 1;
```

```
In[*]:= CF[ $\mathcal{E}$ _] := Expand[ $\mathcal{E}$  /.  $e^{L-} \Rightarrow e^{\text{Expand}[L]}$ ] /. { $e^{k-} /; k > $k \rightarrow 0$ };
```

Implementing general universal enveloping algebras

```
In[*]:= B[0, _] = 0; B[_, 0] = 0;
B[c_.*x_, y_] /; MemberQ[$Basis, x] := CF[c B[x, y]];
B[y_, c_.*x_] /; MemberQ[$Basis, x] := CF[c B[y, x]];
B[x_Plus, y_] := B[#, y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
B[x_, x_] = 0;
B[y_, x_] := CF[-B[x, y]];

In[*]:= x_ <= y_ := OrderedQ[{x, y} /. $PBWRule]; x_ < y_ := ! OrderedQ[{y, x} /. $PBWRule];
UU_i[1] := U_i[];
UU_i[x_P_] := UU_i @@ Table[x, {p}];
UU_i[ $\mathcal{E}$ _] :=  $\mathcal{E}$  /. {
  U[xs_] => U_i[xs],
  x_ /; MemberQ[$Basis, x] => U_i[x]
};
UU_i[x_, xs_] := UU_t1[x] UU_t2[xs] // Expand // m_t1,t2->i;
USimp[ $\mathcal{E}$ _] := Collect[ $\mathcal{E}$ , Times[U[___] ..], Expand];
USimp[ $\mathcal{E}$ _] := Expand[ $\mathcal{E}$ ];
```

```
In[ ]:= m_s__[0] = 0;
m_s__[x_Plus] := m_s / @ x;
m_s__[sd_SeriesData] := MapAt[m_s, sd, {3, All}];
m_i→j_[ε_] := ε / . U_i → U_j;
```

```
In[ ]:= m_i,j→k_[c_. U_i_[x____] U_j_[ ] ] := c U_k[x];
m_i,j→k_[c_. U_i_[ ] U_j_[y____] ] := c U_k[y];
m_i,j→k_[c_. U_i_[xx____, x_] U_j_[y_, yy____] ] := If[x ≤ y,
  c U_k[xx, x, y, yy],
  ((U_i[xx] (U_j[y, x] + U U_j[B[x, y]])) // Expand // m_i,j→i U_j[yy] // Expand // m_i,j→k)
  c // USimp
];
```

```
In[ ]:= UProducts[{ }, 0] = {1}; UProducts[{ }, d_Integer] /; d > 0 = { };
UProducts[{i_, is____}, d_Integer] := Sort@
  Flatten@Table[(U_i @@@ Subsets[$Basis, {j}]) u, {j, 0, d}, {u, UProducts[{is}, d - j]}];
```

```
In[ ]:= Supp[ε_] := Union@Cases[{ε}, U_i[____] → i, ∞];
```

```
In[ ]:= Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ⌞ y_ := Module[{is = Supp[x] ∩ Supp[y], σ, z},
  z = x; Do[z = m_i→σei[z], {i, is}];
  z = Expand[y z]; Do[z = m_σei,i→i[z], {i, is}]; z];
UB[x_, y_] := USimp[x ⌞ y - y ⌞ x];
```

Epilog

```
In[ ]:= (*End[]; EndPackage[];*)
```

Predefined Algebras

sl(2)

```
In[ ]:= Print["UEA`SetAlgebra knows \"sl2\"."];
```

UEA`SetAlgebra knows "sl2".

```
In[*]:= SetAlgebra["s12"] := (
  Print["In s12: <e,h,f>/([h,e]=2e, [h,f]=-2f, [e,f]=h)."];
  B[h, e] = 2 e; B[h, f] = -2 f; B[e, f] = h;
  $Algebra = "s12";
  $Basis = {e, h, f};
  $PBWRule = {e → 1, h → 2, f → 3};
  NilQ[e] = NilQ[f] = True; NilQ[h] = False;
);
```

$gl_{n,\epsilon}$

```
In[*]:= Print["UEA`SetAlgebra knows the  $\epsilon$ -nilpotent algebra  $gl_{n,\epsilon}$ ."];
```

UEA`SetAlgebra knows the ϵ -nilpotent algebra $gl_{n,\epsilon}$.

```
In[*]:= SetAlgebra[gln,ε] := (
  $Algebra = gln,ε;
  $Basis = Flatten@{
    Table[yα,β, {β, 2, n}, {α, 1, β - 1}],
    Table[xα,β, {β, 1, n}, {α, 1, β}]
  };
  (*$Basis=Reverse@SortBy[$Basis,If[#[[1]]==x,{0,#[[3]]-#[[2]]},{1,#[[2]]-#[[3]]}]&];*)
  NilQ[y_] = True; NilQ[xα,β] := (α != β); (* Nilpotent Q *)
  $PBWRule = Thread[$Basis → Range@Length@$Basis];
  B[xi,j, xk,l] := δj,k xi,l - δl,i xk,j;
  B[yi,j, yk,l] := CF[-ε δj,k yi,l + ε δl,i yk,j];
  B[xi,j, yk,l] := CF[δj,k xi,l - δl,i xk,j /. xα,β → If[α ≤ β, ε xα,β, yβ,α]];
  DefRep[0] = Normal@SparseArray[{}, {n, n}];
  DefRep[ε_] := ε /. {
    xα,β → Normal@SparseArray[{α, β} → 1, {n, n}],
    yα,β → Normal@SparseArray[{β, α} → ε, {n, n}]
  }
);
```

```
In[*]:= adPower[x_, k_, Env_] [c_ * y_] /; MemberQ[$Basis, y] := CF[c adPower[x, k, Env] [y]];
adPower[x_, k_, Env_] [y_Plus] := adPower[x, k, Env] /@ y;
adPower[_, 0, Env_] [y_] := y;
adPower[x_, k_, Env_] [0] = 0;
adPower[x_, k_, Env_] [y_] /; MemberQ[$Basis, y] :=
  adPower[x, k, Env] [y] = B[adPower[x, k - 1, Env] [y], x];
adPower[x_, k_] [y_] := adPower[x, k, {$Algebra, $k}] [y];
```

```

In[*]:= adExp[x_][y_] /; SSQ[x] := Expand@Module[{A, b, c},
  A = Table[Coefficient[B[b, x], c], {c, $Basis}, {b, $Basis}];
  CF[$Basis.Normal[Series[MatrixExp[A], {ϵ, 0, $k}]]].
  Table[Coefficient[y, b], {b, $Basis}]
]

```

```

In[*]:= adExp[x_][y_] /; NilQ[x] := Expand@Module[{k = 0, s = 0},
  While[adPower[x, k][y] != 0,
    s += adPower[x, k][y] / k!;
    ++k
  ];
  s
]

```

```

In[*]:= λTangent[] = 0;
λTangent[xs___, x_] := adExp[x][λTangent[xs]] + ∂λx;
λTangent[xs_List] := λTangent@@xs

```

```

In[*]:= TriangularDSolve[eqns_, funs_, iv_] := Module[{sol = {}, e, fun, der},
  MapThread[
    {e, fun} ↦ sol = Join[sol,
      Echo@MapAt[CF, First@DSolve[CF[e /. Derivative[1][f_][v_] => der[f[v], v] /. sol /.
        der[f_, v_] => ∂vf], fun, iv], {1, 2}]
    ],
    {eqns, funs}
  ];
  sol
]

```

gl_{2,ε} Experiments

```

In[*]:= SetAlgebra[gl2,ε];
$PBWRule
MatrixForm@Table[{b1, b2} → B[b1, b2], {b1, $Basis}, {b2, $Basis}]
MatrixForm@
  Table[{b1, b2} → (DefRep[b1].DefRep[b2] - DefRep[b2].DefRep[b1] == DefRep[B[b1, b2]]),
    {b1, $Basis}, {b2, $Basis}]

Out[*]=
{y1,2 → 1, x1,1 → 2, x1,2 → 3, x2,2 → 4}

Out[*]//MatrixForm=

$$\left( \begin{array}{cccc} \{y_{1,2}, y_{1,2}\} \rightarrow 0 & \{y_{1,2}, x_{1,1}\} \rightarrow y_{1,2} & \{y_{1,2}, x_{1,2}\} \rightarrow -\epsilon x_{1,1} + \epsilon x_{2,2} & \{y_{1,2}, x_{2,2}\} \rightarrow -y_{1,2} \\ \{x_{1,1}, y_{1,2}\} \rightarrow -y_{1,2} & \{x_{1,1}, x_{1,1}\} \rightarrow 0 & \{x_{1,1}, x_{1,2}\} \rightarrow x_{1,2} & \{x_{1,1}, x_{2,2}\} \rightarrow 0 \\ \{x_{1,2}, y_{1,2}\} \rightarrow \epsilon x_{1,1} - \epsilon x_{2,2} & \{x_{1,2}, x_{1,1}\} \rightarrow -x_{1,2} & \{x_{1,2}, x_{1,2}\} \rightarrow 0 & \{x_{1,2}, x_{2,2}\} \rightarrow x_{1,2} \\ \{x_{2,2}, y_{1,2}\} \rightarrow y_{1,2} & \{x_{2,2}, x_{1,1}\} \rightarrow 0 & \{x_{2,2}, x_{1,2}\} \rightarrow -x_{1,2} & \{x_{2,2}, x_{2,2}\} \rightarrow 0 \end{array} \right)$$


Out[*]//MatrixForm=

$$\left( \begin{array}{cccc} \{y_{1,2}, y_{1,2}\} \rightarrow \text{True} & \{y_{1,2}, x_{1,1}\} \rightarrow \text{True} & \{y_{1,2}, x_{1,2}\} \rightarrow \text{True} & \{y_{1,2}, x_{2,2}\} \rightarrow \text{True} \\ \{x_{1,1}, y_{1,2}\} \rightarrow \text{True} & \{x_{1,1}, x_{1,1}\} \rightarrow \text{True} & \{x_{1,1}, x_{1,2}\} \rightarrow \text{True} & \{x_{1,1}, x_{2,2}\} \rightarrow \text{True} \\ \{x_{1,2}, y_{1,2}\} \rightarrow \text{True} & \{x_{1,2}, x_{1,1}\} \rightarrow \text{True} & \{x_{1,2}, x_{1,2}\} \rightarrow \text{True} & \{x_{1,2}, x_{2,2}\} \rightarrow \text{True} \\ \{x_{2,2}, y_{1,2}\} \rightarrow \text{True} & \{x_{2,2}, x_{1,1}\} \rightarrow \text{True} & \{x_{2,2}, x_{1,2}\} \rightarrow \text{True} & \{x_{2,2}, x_{2,2}\} \rightarrow \text{True} \end{array} \right)$$


In[*]:= DeleteCases[Flatten[Table[{a, b, c} → B[a, B[b, c]] + B[b, B[c, a]] + B[c, B[a, b]],
  {a, $Basis}, {b, $Basis}, {c, $Basis}]], _ → 0]

Out[*]=
{}

In[*]:= lhs = λTangent[Join[
  Table[If[SSQ[b1], (b1 /. x → ξ1) b1, λ (b1 /. {x → ξ1, y → η1}) b1], {b1, $Basis}],
  Table[If[SSQ[b2], (b2 /. x → ξ2) b2, λ (b2 /. {x → ξ2, y → η2}) b2], {b2, $Basis}]
]]

Out[*]=

$$\begin{aligned} & e^{\epsilon^{1,1} - \epsilon^{1,2} + \epsilon^{2,1} - \epsilon^{2,2}} y_{1,2} \eta_{1,2} + e^{\epsilon^{2,1} - \epsilon^{2,2}} y_{1,2} \eta_{2,2} + e^{\epsilon^{1,2} - \epsilon^{2,1} + \epsilon^{2,2}} x_{1,2} \xi_{1,2} - e^{\epsilon^{1,1}} \epsilon \lambda x_{1,1} \eta_{1,2} \xi_{1,2} + \\ & e^{\epsilon^{1,1}} \epsilon \lambda x_{2,2} \eta_{1,2} \xi_{1,2} + e^{\epsilon^{1,2}} \epsilon \lambda x_{1,1} \eta_{2,2} \xi_{1,2} - e^{\epsilon^{1,2}} \epsilon \lambda x_{2,2} \eta_{2,2} \xi_{1,2} + \\ & 2 e^{\epsilon^{1,1} + \epsilon^{2,1} - \epsilon^{2,2}} \epsilon \lambda^2 y_{1,2} \eta_{1,2} \eta_{2,2} \xi_{1,2} - e^{\epsilon^{1,2} + \epsilon^{2,1} - \epsilon^{2,2}} \epsilon \lambda^2 y_{1,2} \eta_{2,2}^2 \xi_{1,2} - \\ & e^{\epsilon^{1,1} + \epsilon^{1,2} - \epsilon^{2,1} + \epsilon^{2,2}} \epsilon \lambda^2 x_{1,2} \eta_{1,2} \xi_{1,2}^2 + e^{\epsilon^{2,2}} x_{1,2} \xi_{2,2} - e^{\epsilon^{1,1} - \epsilon^{1,2} + \epsilon^{2,1}} \epsilon \lambda x_{1,1} \eta_{1,2} \xi_{2,2} + \\ & e^{\epsilon^{1,1} - \epsilon^{1,2} + \epsilon^{2,1}} \epsilon \lambda x_{2,2} \eta_{1,2} \xi_{2,2} - e^{\epsilon^{2,1}} \epsilon \lambda x_{1,1} \eta_{2,2} \xi_{2,2} + e^{\epsilon^{2,1}} \epsilon \lambda x_{2,2} \eta_{2,2} \xi_{2,2} - \\ & 2 e^{\epsilon^{1,1} + \epsilon^{2,2}} \epsilon \lambda^2 x_{1,2} \eta_{1,2} \xi_{1,2} \xi_{2,2} + 2 e^{\epsilon^{1,2} + \epsilon^{2,2}} \epsilon \lambda^2 x_{1,2} \eta_{2,2} \xi_{1,2} \xi_{2,2} - \\ & e^{\epsilon^{1,1} - \epsilon^{1,2} + \epsilon^{2,1} + \epsilon^{2,2}} \epsilon \lambda^2 x_{1,2} \eta_{1,2} \xi_{2,2}^2 - e^{\epsilon^{2,1} + \epsilon^{2,2}} \epsilon \lambda^2 x_{1,2} \eta_{2,2} \xi_{2,2}^2 \end{aligned}$$


```

```

In[*]:= rhs = λTangent[
  Table[If[SSQ[b], b /. xα,α => (ξ1,α + ξ2,α + e fα,α[λ]) xα,α,
    (b /. {xαβ -> fαβ[λ], yαβ -> gαβ[λ]}) b], {b, $Basis}]
]

Out[*]=
e x1,1 f1,1'[λ] + eξ1,2+ξ2,2 e x1,2 f1,2[λ] f1,1'[λ] +
eξ1,2+ξ2,2 x1,2 f1,2'[λ] + eξ1,2+ξ2,2 e x1,2 f2,2[λ] f1,2'[λ] + e x2,2 f2,2'[λ] +
eξ1,1-ξ1,2+ξ2,1-ξ2,2 y1,2 g1,2'[λ] + eξ1,1-ξ1,2+ξ2,1-ξ2,2 e y1,2 f1,1[λ] g1,2'[λ] -
eξ1,1+ξ2,1 e x1,1 f1,2[λ] g1,2'[λ] + eξ1,1+ξ2,1 e x2,2 f1,2[λ] g1,2'[λ] -
eξ1,1+ξ1,2+ξ2,1+ξ2,2 e x1,2 f1,2[λ]2 g1,2'[λ] - eξ1,1-ξ1,2+ξ2,1-ξ2,2 e y1,2 f2,2[λ] g1,2'[λ]

In[*]:= eqns = Simplify[(Coefficient[lhs - rhs, #] == 0) & /@ $Basis]

Out[*]=
{e-ξ1,2+ξ2,1-ξ2,2 (-eξ1,2 η2,2 + e2 ξ1,2 e λ2 η2,22 ξ1,2 -
eξ1,1 η1,2 (1 + 2 eξ1,2 e λ2 η2,2 ξ1,2) + eξ1,1 (1 + e f1,1[λ] - e f2,2[λ]) g1,2'[λ]) == 0,
e (λ η2,2 (eξ1,2 ξ1,2 - eξ2,1 ξ2,2) - eξ1,1-ξ1,2 λ η1,2 (eξ1,2 ξ1,2 + eξ2,1 ξ2,2) - f1,1'[λ] +
eξ1,1+ξ2,1 f1,2[λ] g1,2'[λ]) == 0, e-ξ1,2-ξ2,1+ξ2,2 (-eξ1,1+2 ξ1,2 e λ2 η1,2 ξ1,22 +
eξ1,2 ξ1,2 (-2 eξ1,1+ξ2,1 e λ2 η1,2 ξ2,2 + eξ1,2 (1 + 2 eξ2,1 e λ2 η2,2 ξ2,2)) +
eξ2,1 (eξ1,2 ξ2,2 - eξ2,1 e λ2 (eξ1,1 η1,2 + eξ1,2 η2,2) ξ2,22 +
e2 ξ1,2 (-e f1,2[λ] f1,1'[λ] - (1 + e f2,2[λ]) f1,2'[λ] + eξ1,1+ξ2,1 e f1,2[λ]2 g1,2'[λ])) ==
0, eξ1,1-ξ1,2 e λ η1,2 (eξ1,2 ξ1,2 + eξ2,1 ξ2,2) ==
e (λ η2,2 (eξ1,2 ξ1,2 - eξ2,1 ξ2,2) + f2,2'[λ] + eξ1,1+ξ2,1 f1,2[λ] g1,2'[λ])}

In[*]:= Length@eqns

Out[*]=
4

In[*]:= unknowns = $Basis /. {xαβ -> fαβ[λ], yαβ -> gαβ[λ]}

Out[*]=
{g1,2[λ], f1,1[λ], f1,2[λ], f2,2[λ]}

In[*]:= {sol0} = Block[{ $k = 0 }, Simplify@DSolve[
  And@@Join[
    CF /@ eqns,
    Select[$Basis, NilQ] /. {xαβ -> fαβ[0] == 0, yαβ -> gαβ[0] == 0}
  ],
  Select[$Basis, NilQ] /. {xαβ -> fαβ[λ], yαβ -> gαβ[λ]},
  λ
]]

Out[*]=
{{g1,2[λ] -> λ (η1,2 + e-ξ1,1+ξ1,2 η2,2), f1,2[λ] -> e-ξ2,1 λ ξ1,2 + e-ξ1,2 λ ξ2,2}}

```

```

In[*]:= {sol1} = Block[{ $k = 1}, Simplify@DSolve[
  lhs = λTangent[Join[
    Table[If[SSQ[b1], (b1 /. x → ξ1) b1, λ (b1 /. {x → ξ1, y → η1}) b1], {b1, $Basis}],
    Table[If[SSQ[b2], (b2 /. x → ξ2) b2, λ (b2 /. {x → ξ2, y → η2}) b2], {b2, $Basis}]
  ]];
  rhs = λTangent[
    Table[If[SSQ[b],
      b /. xα,α → (ξ1α,α + ξ2α,α + ε fα,α[λ]) xα,α,
      (b /. {xαβ → fαβ[λ], yαβ → gαβ[λ]}) b /. (sol0 /. ((a- → b-) → (a → ε a + b)))
    ], {b, $Basis}]
  ];
  Join[
    (Coefficient[CF[lhs - rhs], #] == 0) & /@ $Basis,
    $Basis /. {xαβ → fαβ[0] == 0, yαβ → gαβ[0] == 0}
  ],
  $Basis /. {xαβ → fαβ[λ], yαβ → gαβ[λ]},
  λ
]

```

```

Out[*]= { {f1,1[λ] → eξ1,2 λ2 η1,2 ξ1,2, f1,2[λ] → λ3 η1,2 ξ1,2 ξ2,2,
  f2,2[λ] → -eξ1,2 λ2 η1,2 ξ1,2, g1,2[λ] → -e-ξ1,1+2 ξ1,2 λ3 η1,2 ξ1,2} }

```

```

In[*]:= cm[i-, j- → k-] := E{i,j}→{k}[CF@Sum[If[SSQ[b],
  b /. xα,α → (ξ1α,α + ξ2α,α) xα,α[k],
  b[k] (b /. {xαβ → fαβ[λ], yαβ → gαβ[λ]}) /. sol0],
  ], {b, $Basis}],
  CF@Sum[b[k] (b /. {xαβ → fαβ[λ], yαβ → gαβ[λ]}) /. sol1], {b, $Basis}]
  ] /. {ξ1αβ → ξαβ[i], η1αβ → ηαβ[i], ξ2αβ → ξαβ[j], η2αβ → ηαβ[j], λ → 1};
cm[i, j → k]

```

```

Out[*]= E{i,j}→{k}[y1,2[k] η1,2[i] + e-ξ1,1[i]+ξ2,2[i] y1,2[k] η1,2[j] + x1,1[k] ξ1,1[i] + x1,1[k] ξ1,1[j] +
  e-ξ1,1[j] x1,2[k] ξ1,2[i] + e-ξ2,2[i] x1,2[k] ξ1,2[j] + x2,2[k] ξ2,2[i] + x2,2[k] ξ2,2[j],
  eξ2,2[i] x1,1[k] η1,2[j] ξ1,2[i] - eξ2,2[i] x2,2[k] η1,2[j] ξ1,2[i] -
  e-ξ1,1[i]+2 ξ2,2[i] y1,2[k] η1,2[j]2 ξ1,2[i] + x1,2[k] η1,2[j] ξ1,2[i] ξ1,2[j]]

```

In[*]:= **cm**[1, 2 → 2] // **cm**[2, 3 → 1]

Out[*]=

$$\begin{aligned} & E_{\{1,2,3\} \rightarrow \{1\}} \left[y_{1,2}[1] \eta_{1,2}[1] + e^{-\xi_{1,1}[1] + \xi_{2,2}[1]} y_{1,2}[1] \eta_{1,2}[2] + \right. \\ & e^{-\xi_{1,1}[1] - \xi_{1,1}[2] + \xi_{2,2}[1] + \xi_{2,2}[2]} y_{1,2}[1] \eta_{1,2}[3] + x_{1,1}[1] \xi_{1,1}[1] + x_{1,1}[1] \xi_{1,1}[2] + \\ & x_{1,1}[1] \xi_{1,1}[3] + e^{-\xi_{1,1}[2] - \xi_{1,1}[3]} x_{1,2}[1] \xi_{1,2}[1] + e^{-\xi_{1,1}[3] - \xi_{2,2}[1]} x_{1,2}[1] \xi_{1,2}[2] + \\ & e^{-\xi_{2,2}[1] - \xi_{2,2}[2]} x_{1,2}[1] \xi_{1,2}[3] + x_{2,2}[1] \xi_{2,2}[1] + x_{2,2}[1] \xi_{2,2}[2] + x_{2,2}[1] \xi_{2,2}[3], \\ & e^{\xi_{2,2}[1]} x_{1,1}[1] \eta_{1,2}[2] \xi_{1,2}[1] - e^{\xi_{2,2}[1]} x_{2,2}[1] \eta_{1,2}[2] \xi_{1,2}[1] - \\ & e^{-\xi_{1,1}[1] + 2 \xi_{2,2}[1]} y_{1,2}[1] \eta_{1,2}[2]^2 \xi_{1,2}[1] + e^{-\xi_{1,1}[2] + \xi_{2,2}[1] + \xi_{2,2}[2]} x_{1,1}[1] \eta_{1,2}[3] \xi_{1,2}[1] - \\ & e^{-\xi_{1,1}[2] + \xi_{2,2}[1] + \xi_{2,2}[2]} x_{2,2}[1] \eta_{1,2}[3] \xi_{1,2}[1] - \\ & 2 e^{-\xi_{1,1}[1] - \xi_{1,1}[2] + 2 \xi_{2,2}[1] + \xi_{2,2}[2]} y_{1,2}[1] \eta_{1,2}[2] \eta_{1,2}[3] \xi_{1,2}[1] - \\ & e^{-\xi_{1,1}[1] - 2 \xi_{1,1}[2] + 2 \xi_{2,2}[1] + 2 \xi_{2,2}[2]} y_{1,2}[1] \eta_{1,2}[3]^2 \xi_{1,2}[1] + e^{\xi_{2,2}[2]} x_{1,1}[1] \eta_{1,2}[3] \xi_{1,2}[2] - \\ & e^{\xi_{2,2}[2]} x_{2,2}[1] \eta_{1,2}[3] \xi_{1,2}[2] - e^{-\xi_{1,1}[1] - \xi_{1,1}[2] + \xi_{2,2}[1] + 2 \xi_{2,2}[2]} y_{1,2}[1] \eta_{1,2}[3]^2 \xi_{1,2}[2] + \\ & e^{-\xi_{1,1}[3]} x_{1,2}[1] \eta_{1,2}[2] \xi_{1,2}[1] \xi_{1,2}[2] + e^{-\xi_{2,2}[2]} x_{1,2}[1] \eta_{1,2}[2] \xi_{1,2}[1] \xi_{1,2}[3] + \\ & \left. e^{-\xi_{1,1}[2]} x_{1,2}[1] \eta_{1,2}[3] \xi_{1,2}[1] \xi_{1,2}[3] + e^{-\xi_{2,2}[1]} x_{1,2}[1] \eta_{1,2}[3] \xi_{1,2}[2] \xi_{1,2}[3] \right] \end{aligned}$$

In[*]:= **Simplify**[(**cm**[1, 2 → 2] // **cm**[2, 3 → 1]) ≡ (**cm**[2, 3 → 2] // **cm**[1, 2 → 1])]

Out[*]=

True

gl_{3,ε} Experiments

```

In[*]:= SetAlgebra[gl3,ε];
$k = 1
$PBWRule
MatrixForm@Table[{b1, b2} → B[b1, b2], {b1, $Basis}, {b2, $Basis}]
MatrixForm@
  Table[{b1, b2} → (DefRep[b1].DefRep[b2] - DefRep[b2].DefRep[b1] == DefRep[B[b1, b2]]),
    {b1, $Basis}, {b2, $Basis}]

Out[*]=
1

Out[*]=
{y1,2 → 1, y1,3 → 2, y2,3 → 3, x1,1 → 4, x1,2 → 5, x2,2 → 6, x1,3 → 7, x2,3 → 8, x3,3 → 9}

Out[*]//MatrixForm=

$$\begin{pmatrix}
\{y_{1,2}, y_{1,2}\} \rightarrow 0 & \{y_{1,2}, y_{1,3}\} \rightarrow 0 & \{y_{1,2}, y_{2,3}\} \rightarrow -\epsilon y_{1,3} & \{y_{1,2}, x_{1,1}\} \rightarrow y_{1,1} \\
\{y_{1,3}, y_{1,2}\} \rightarrow 0 & \{y_{1,3}, y_{1,3}\} \rightarrow 0 & \{y_{1,3}, y_{2,3}\} \rightarrow 0 & \{y_{1,3}, x_{1,1}\} \rightarrow y_{1,3} \\
\{y_{2,3}, y_{1,2}\} \rightarrow \epsilon y_{1,3} & \{y_{2,3}, y_{1,3}\} \rightarrow 0 & \{y_{2,3}, y_{2,3}\} \rightarrow 0 & \{y_{2,3}, x_{1,1}\} \rightarrow y_{2,3} \\
\{x_{1,1}, y_{1,2}\} \rightarrow -y_{1,2} & \{x_{1,1}, y_{1,3}\} \rightarrow -y_{1,3} & \{x_{1,1}, y_{2,3}\} \rightarrow 0 & \{x_{1,1}, x_{1,1}\} \rightarrow 0 \\
\{x_{1,2}, y_{1,2}\} \rightarrow \epsilon x_{1,1} - \epsilon x_{2,2} & \{x_{1,2}, y_{1,3}\} \rightarrow -y_{2,3} & \{x_{1,2}, y_{2,3}\} \rightarrow 0 & \{x_{1,2}, x_{1,1}\} \rightarrow -y_{1,2} \\
\{x_{2,2}, y_{1,2}\} \rightarrow y_{1,2} & \{x_{2,2}, y_{1,3}\} \rightarrow 0 & \{x_{2,2}, y_{2,3}\} \rightarrow -y_{2,3} & \{x_{2,2}, x_{1,1}\} \rightarrow y_{2,3} \\
\{x_{1,3}, y_{1,2}\} \rightarrow -\epsilon x_{2,3} & \{x_{1,3}, y_{1,3}\} \rightarrow \epsilon x_{1,1} - \epsilon x_{3,3} & \{x_{1,3}, y_{2,3}\} \rightarrow \epsilon x_{1,2} & \{x_{1,3}, x_{1,1}\} \rightarrow -\epsilon x_{2,3} \\
\{x_{2,3}, y_{1,2}\} \rightarrow 0 & \{x_{2,3}, y_{1,3}\} \rightarrow y_{1,2} & \{x_{2,3}, y_{2,3}\} \rightarrow \epsilon x_{2,2} - \epsilon x_{3,3} & \{x_{2,3}, x_{1,1}\} \rightarrow y_{1,2} \\
\{x_{3,3}, y_{1,2}\} \rightarrow 0 & \{x_{3,3}, y_{1,3}\} \rightarrow y_{1,3} & \{x_{3,3}, y_{2,3}\} \rightarrow y_{2,3} & \{x_{3,3}, x_{1,1}\} \rightarrow y_{2,3}
\end{pmatrix}$$


Out[*]//MatrixForm=

$$\begin{pmatrix}
\{y_{1,2}, y_{1,2}\} \rightarrow \text{True} & \{y_{1,2}, y_{1,3}\} \rightarrow \text{True} & \{y_{1,2}, y_{2,3}\} \rightarrow \text{True} & \{y_{1,2}, x_{1,1}\} \rightarrow \text{True} & \{y_{1,2}, x_{1,2}\} \rightarrow \text{True} & \{y_{1,2}, x_{1,3}\} \rightarrow \text{True} \\
\{y_{1,3}, y_{1,2}\} \rightarrow \text{True} & \{y_{1,3}, y_{1,3}\} \rightarrow \text{True} & \{y_{1,3}, y_{2,3}\} \rightarrow \text{True} & \{y_{1,3}, x_{1,1}\} \rightarrow \text{True} & \{y_{1,3}, x_{1,2}\} \rightarrow \text{True} & \{y_{1,3}, x_{1,3}\} \rightarrow \text{True} \\
\{y_{2,3}, y_{1,2}\} \rightarrow \text{True} & \{y_{2,3}, y_{1,3}\} \rightarrow \text{True} & \{y_{2,3}, y_{2,3}\} \rightarrow \text{True} & \{y_{2,3}, x_{1,1}\} \rightarrow \text{True} & \{y_{2,3}, x_{1,2}\} \rightarrow \text{True} & \{y_{2,3}, x_{1,3}\} \rightarrow \text{True} \\
\{x_{1,1}, y_{1,2}\} \rightarrow \text{True} & \{x_{1,1}, y_{1,3}\} \rightarrow \text{True} & \{x_{1,1}, y_{2,3}\} \rightarrow \text{True} & \{x_{1,1}, x_{1,1}\} \rightarrow \text{True} & \{x_{1,1}, x_{1,2}\} \rightarrow \text{True} & \{x_{1,1}, x_{1,3}\} \rightarrow \text{True} \\
\{x_{1,2}, y_{1,2}\} \rightarrow \text{True} & \{x_{1,2}, y_{1,3}\} \rightarrow \text{True} & \{x_{1,2}, y_{2,3}\} \rightarrow \text{True} & \{x_{1,2}, x_{1,1}\} \rightarrow \text{True} & \{x_{1,2}, x_{1,2}\} \rightarrow \text{True} & \{x_{1,2}, x_{1,3}\} \rightarrow \text{True} \\
\{x_{2,2}, y_{1,2}\} \rightarrow \text{True} & \{x_{2,2}, y_{1,3}\} \rightarrow \text{True} & \{x_{2,2}, y_{2,3}\} \rightarrow \text{True} & \{x_{2,2}, x_{1,1}\} \rightarrow \text{True} & \{x_{2,2}, x_{1,2}\} \rightarrow \text{True} & \{x_{2,2}, x_{1,3}\} \rightarrow \text{True} \\
\{x_{1,3}, y_{1,2}\} \rightarrow \text{True} & \{x_{1,3}, y_{1,3}\} \rightarrow \text{True} & \{x_{1,3}, y_{2,3}\} \rightarrow \text{True} & \{x_{1,3}, x_{1,1}\} \rightarrow \text{True} & \{x_{1,3}, x_{1,2}\} \rightarrow \text{True} & \{x_{1,3}, x_{1,3}\} \rightarrow \text{True} \\
\{x_{2,3}, y_{1,2}\} \rightarrow \text{True} & \{x_{2,3}, y_{1,3}\} \rightarrow \text{True} & \{x_{2,3}, y_{2,3}\} \rightarrow \text{True} & \{x_{2,3}, x_{1,1}\} \rightarrow \text{True} & \{x_{2,3}, x_{1,2}\} \rightarrow \text{True} & \{x_{2,3}, x_{1,3}\} \rightarrow \text{True} \\
\{x_{3,3}, y_{1,2}\} \rightarrow \text{True} & \{x_{3,3}, y_{1,3}\} \rightarrow \text{True} & \{x_{3,3}, y_{2,3}\} \rightarrow \text{True} & \{x_{3,3}, x_{1,1}\} \rightarrow \text{True} & \{x_{3,3}, x_{1,2}\} \rightarrow \text{True} & \{x_{3,3}, x_{1,3}\} \rightarrow \text{True}
\end{pmatrix}$$


In[*]:= DeleteCases[Flatten[Table[{a, b, c} → B[a, B[b, c]] + B[b, B[c, a]] + B[c, B[a, b]],
  {a, $Basis}, {b, $Basis}, {c, $Basis}]], _ → 0]

Out[*]=
{}

```

```

In[*]:= {sol0} = Block[{ $k = 0},
  lhs = λTangent[Join[
    Table[If[SSQ[b1], (b1 /. x → ξ1) b1, λ (b1 /. {x → ξ1, y → η1}) b1], {b1, $Basis}],
    Table[If[SSQ[b2], (b2 /. x → ξ2) b2, λ (b2 /. {x → ξ2, y → η2}) b2], {b2, $Basis}]
  ]];
  rhs = λTangent[
    Table[If[SSQ[b], b /. xα,α ⇒ (ξ1α,α + ξ2α,α + ε fα,α[λ]) xα,α,
      (b /. {xαβ → fαβ[λ], yαβ → gαβ[λ]}) b], {b, $Basis}]
  ];
  Simplify@DSolve[
    And@@Join[
      (Coefficient[CF[lhs - rhs], #] == 0) & /@ $Basis,
      Select[$Basis, NilQ] /. {xαβ ⇒ fαβ[0] == 0, yαβ ⇒ gαβ[0] == 0}
    ],
    Select[$Basis, NilQ] /. {xαβ → fαβ[λ], yαβ → gαβ[λ]},
    λ
  ]
];

Out[*]= { {f1,2[λ] → e-ξ2,1 λ ξ1,2 + e-ξ1,2 λ ξ2,2, g2,3[λ] → λ (η1,3 + eξ1,3 (e-ξ1,2 η2,3 - λ η2,3 ξ1,2))},
  {f1,3[λ] → λ (e-ξ2,1 ξ1,3 - λ ξ1,3 ξ2,2 + e-ξ1,3 ξ2,3), f2,3[λ] → e-ξ2,2 λ ξ1,3 + e-ξ1,3 λ ξ2,3,
  {g1,2[λ] → λ (η1,2 + e-ξ1,1 + ξ1,2 (η2,2 + eξ1,3 λ η2,3 ξ1,3)), g1,3[λ] → λ (η1,3 + e-ξ1,1 + ξ1,3 η2,3)} }

In[*]:= $k = 1; SBasis = SortBy[$Basis, If[#[[1]] === x, {0, #[[3]] - #[[2]]}, {1, #[[2]] - #[[3]]}] &];
  lhs = λTangent[Join[
    Table[If[SSQ[b1], (b1 /. x → ξ1) b1, λ (b1 /. {x → ξ1, y → η1}) b1], {b1, $Basis}],
    Table[If[SSQ[b2], (b2 /. x → ξ2) b2, λ (b2 /. {x → ξ2, y → η2}) b2], {b2, $Basis}]
  ]];
  rhs = λTangent[
    Echo@Table[If[SSQ[b],
      b /. xα,α ⇒ (ξ1α,α + ξ2α,α + ε fα,α[λ]) xα,α,
      (b /. {xαβ → fαβ[λ], yαβ → gαβ[λ]}) b /. (sol0 /. ((v- → s-) ⇒ (v → ε v + s)))
    ], {b, $Basis}]
  ];
  eqns = Transpose@{
    (Coefficient[CF[lhs - rhs], #] == 0) & /@ SBasis,
    SBasis /. {xαβ ⇒ fαβ[0] == 0, yαβ ⇒ gαβ[0] == 0}
  };
  unknowns = SBasis /. {xαβ → fαβ[λ], yαβ → gαβ[λ]};
  (* {sol1} = DSolve[Flatten@eqns, unknowns, λ] *)
  sol1 = TriangularDSolve[eqns, unknowns, λ];

```

- $$\begin{aligned} & \gg \{y_{1,2} (\lambda (\eta_{1,2} + e^{-\xi_{1,1}+\xi_{1,2}} (\eta_{2,2} + e^{\xi_{1,3}} \lambda \eta_{2,3} \xi_{1,2,3})) + e_{g_{1,2}}[\lambda]), \\ & y_{1,3} (\lambda (\eta_{1,3} + e^{-\xi_{1,1}+\xi_{1,3}} \eta_{2,3})) + e_{g_{1,3}}[\lambda]), \\ & y_{2,3} (\lambda (\eta_{1,2} + e^{\xi_{1,3}} (e^{-\xi_{1,2}} \eta_{2,3} - \lambda \eta_{2,3} \xi_{1,2})) + e_{g_{2,3}}[\lambda]), \\ & x_{1,1} (\xi_{1,1} + \xi_{2,1} + e_{f_{1,1}}[\lambda]), x_{1,2} (e^{-\xi_{2,1}} \lambda \xi_{1,2} + e^{-\xi_{1,2}} \lambda \xi_{2,1,2} + e_{f_{1,2}}[\lambda]), \\ & x_{2,2} (\xi_{1,2} + \xi_{2,2} + e_{f_{2,2}}[\lambda]), x_{1,3} (\lambda (e^{-\xi_{2,1}} \xi_{1,3} - \lambda \xi_{1,2,3} \xi_{2,1,2} + e^{-\xi_{1,3}} \xi_{2,1,3}) + e_{f_{1,3}}[\lambda]), \\ & x_{2,3} (e^{-\xi_{2,2}} \lambda \xi_{1,2,3} + e^{-\xi_{1,3}} \lambda \xi_{2,2,3} + e_{f_{2,3}}[\lambda]), x_{3,3} (\xi_{1,3} + \xi_{2,3} + e_{f_{3,3}}[\lambda]) \} \end{aligned}$$
- $$\gg \{2 e^{\xi_{1,2}} \in \lambda \eta_{1,2} \xi_{1,1,2} + 2 e^{\xi_{1,3}} \in \lambda \eta_{2,3} \xi_{1,1,3} + 3 e^{\xi_{1,2}+\xi_{1,3}} \in \lambda^2 \eta_{2,3} \xi_{1,1,2} \xi_{1,2,3} - e_{f_{1,1}'}[\lambda] = 0, f_{1,1}[0] = 0\}$$
- $$\gg \{f_{1,1}[\lambda] \rightarrow e^{\xi_{1,2}} \lambda^2 \eta_{1,2} \xi_{1,1,2} + e^{\xi_{1,3}} \lambda^2 \eta_{2,3} \xi_{1,1,3} + e^{\xi_{1,2}+\xi_{1,3}} \lambda^3 \eta_{2,3} \xi_{1,1,2} \xi_{1,2,3}\}$$
- $$\gg \{-2 e^{\xi_{1,2}} \in \lambda \eta_{1,2} \xi_{1,1,2} + 2 e^{\xi_{1,3}} \in \lambda \eta_{2,3} \xi_{1,2,3} - 3 e^{\xi_{1,2}+\xi_{1,3}} \in \lambda^2 \eta_{2,3} \xi_{1,1,2} \xi_{1,2,3} - e_{f_{2,2}'}[\lambda] = 0, f_{2,2}[0] = 0\}$$
- $$\gg \{f_{2,2}[\lambda] \rightarrow -e^{\xi_{1,2}} \lambda^2 \eta_{1,2} \xi_{1,1,2} + e^{\xi_{1,3}} \lambda^2 \eta_{2,3} \xi_{1,2,3} - e^{\xi_{1,2}+\xi_{1,3}} \lambda^3 \eta_{2,3} \xi_{1,1,2} \xi_{1,2,3}\}$$
- $$\gg \{-2 e^{\xi_{1,3}} \in \lambda \eta_{2,3} \xi_{1,1,3} - 2 e^{\xi_{1,3}} \in \lambda \eta_{2,3} \xi_{1,2,3} - e_{f_{3,3}'}[\lambda] = 0, f_{3,3}[0] = 0\}$$
- $$\gg \{f_{3,3}[\lambda] \rightarrow -e^{\xi_{1,3}} \lambda^2 \eta_{2,3} \xi_{1,1,3} - e^{\xi_{1,3}} \lambda^2 \eta_{2,3} \xi_{1,2,3}\}$$
- $$\begin{aligned} & \gg \{2 e^{\xi_{1,3}-\xi_{2,1}+\xi_{2,2}} \in \lambda \eta_{2,3} \xi_{1,1,3} - 3 e^{\xi_{1,2}+\xi_{1,3}-\xi_{2,1}+\xi_{2,2}} \in \lambda^2 \eta_{2,3} \xi_{1,1,2} \xi_{1,1,3} + \\ & 3 e^{\xi_{1,2}+\xi_{2,2}} \in \lambda^2 \eta_{1,2} \xi_{1,1,2} \xi_{2,1,2} - 3 e^{\xi_{1,3}+\xi_{2,2}} \in \lambda^2 \eta_{2,3} \xi_{1,2,3} \xi_{2,1,2} + \\ & 4 e^{\xi_{1,2}+\xi_{1,3}+\xi_{2,2}} \in \lambda^3 \eta_{2,3} \xi_{1,1,2} \xi_{1,2,3} \xi_{2,1,2} - e^{\xi_{1,2}+\xi_{2,2}} \in f_{1,2}'[\lambda] = 0, f_{1,2}[0] = 0\} \end{aligned}$$
- $$\begin{aligned} & \gg \{f_{1,2}[\lambda] \rightarrow e^{-\xi_{1,2}+\xi_{1,3}-\xi_{2,1}} \lambda^2 \eta_{2,3} \xi_{1,1,3} - e^{\xi_{1,3}-\xi_{2,1}} \lambda^3 \eta_{2,3} \xi_{1,1,2} \xi_{1,1,3} + \\ & \lambda^3 \eta_{1,2} \xi_{1,1,2} \xi_{2,1,2} - e^{-\xi_{1,2}+\xi_{1,3}} \lambda^3 \eta_{2,3} \xi_{1,2,3} \xi_{2,1,2} + e^{\xi_{1,3}} \lambda^4 \eta_{2,3} \xi_{1,1,2} \xi_{1,2,3} \xi_{2,1,2}\} \end{aligned}$$
- $$\begin{aligned} & \gg \{-2 e^{\xi_{1,3}-\xi_{2,2}+\xi_{2,3}} \in \lambda \eta_{1,2} \xi_{1,1,3} + 3 e^{\xi_{1,3}+\xi_{2,3}} \in \lambda^2 \eta_{2,3} \xi_{1,1,3} \xi_{2,2,3} + \\ & 3 e^{\xi_{1,3}+\xi_{2,3}} \in \lambda^2 \eta_{2,3} \xi_{1,2,3} \xi_{2,2,3} - e^{\xi_{1,3}+\xi_{2,3}} \in f_{2,3}'[\lambda] = 0, f_{2,3}[0] = 0\} \end{aligned}$$
- $$\gg \{f_{2,3}[\lambda] \rightarrow -e^{-\xi_{2,2}} \lambda^2 \eta_{1,2} \xi_{1,1,3} + \lambda^3 \eta_{2,3} \xi_{1,1,3} \xi_{2,2,3} + \lambda^3 \eta_{2,3} \xi_{1,2,3} \xi_{2,2,3}\}$$
- $$\begin{aligned} & \gg \{3 e^{\xi_{1,3}+\xi_{2,3}} \in \lambda^2 \eta_{1,2} \xi_{1,1,3} \xi_{2,1,2} + 3 e^{\xi_{1,3}+\xi_{2,3}} \in \lambda^2 \eta_{2,3} \xi_{1,1,3} \xi_{2,1,3} + \\ & 3 e^{\xi_{1,3}+\xi_{2,3}} \in \lambda^2 \eta_{2,3} \xi_{1,2,3} \xi_{2,1,3} - e^{\xi_{1,3}+\xi_{2,3}} \in f_{1,3}'[\lambda] = 0, f_{1,3}[0] = 0\} \end{aligned}$$
- $$\gg \{f_{1,3}[\lambda] \rightarrow \lambda^3 \eta_{1,2} \xi_{1,1,3} \xi_{2,1,2} + \lambda^3 \eta_{2,3} \xi_{1,1,3} \xi_{2,1,3} + \lambda^3 \eta_{2,3} \xi_{1,2,3} \xi_{2,1,3}\}$$
- $$\begin{aligned} & \gg \{2 e^{\xi_{1,2}-\xi_{1,3}+\xi_{2,1}-\xi_{2,3}} \in \lambda \eta_{1,2,3} \eta_{2,1,2} - 3 e^{\xi_{1,2}+\xi_{2,1}-\xi_{2,3}} \in \lambda^2 \eta_{1,2} \eta_{2,1,3} \xi_{1,1,2} - \\ & 3 e^{\xi_{1,3}+\xi_{2,1}-\xi_{2,3}} \in \lambda^2 \eta_{1,3}^2 \xi_{1,1,3} + 3 e^{\xi_{1,2}+\xi_{2,1}-\xi_{2,3}} \in \lambda^2 \eta_{1,2,3} \eta_{2,1,3} \xi_{1,2,3} - \\ & 4 e^{\xi_{1,2}+\xi_{1,3}+\xi_{2,1}-\xi_{2,3}} \in \lambda^3 \eta_{2,3}^2 \xi_{1,1,2} \xi_{1,2,3} - e^{\xi_{1,1}-\xi_{1,3}+\xi_{2,1}-\xi_{2,3}} \in g_{1,3}'[\lambda] = 0, g_{1,3}[0] = 0\} \end{aligned}$$
- $$\begin{aligned} & \gg \{g_{1,3}[\lambda] \rightarrow e^{-\xi_{1,1}+\xi_{1,2}} \lambda^2 \eta_{1,2,3} \eta_{2,1,2} - e^{-\xi_{1,1}+\xi_{1,2}+\xi_{1,3}} \lambda^3 \eta_{1,2} \eta_{2,1,3} \xi_{1,1,2} - \\ & e^{-\xi_{1,1}+2\xi_{1,3}} \lambda^3 \eta_{2,1,3}^2 \xi_{1,1,3} + e^{-\xi_{1,1}+\xi_{1,2}+\xi_{1,3}} \lambda^3 \eta_{1,2,3} \eta_{2,1,3} \xi_{1,2,3} - e^{-\xi_{1,1}+\xi_{1,2}+2\xi_{1,3}} \lambda^4 \eta_{2,1,3}^2 \xi_{1,1,2} \xi_{1,2,3}\} \end{aligned}$$
- $$\begin{aligned} & \gg \{-3 e^{\xi_{1,2}+\xi_{2,1}-\xi_{2,2}} \in \lambda^2 \eta_{1,2}^2 \xi_{1,1,2} - 3 e^{\xi_{1,3}+\xi_{2,1}-\xi_{2,2}} \in \lambda^2 \eta_{1,2} \eta_{2,1,3} \xi_{1,1,3} - \\ & 8 e^{\xi_{1,2}+\xi_{1,3}+\xi_{2,1}-\xi_{2,2}} \in \lambda^3 \eta_{1,2} \eta_{2,1,3} \xi_{1,1,2} \xi_{1,2,3} - 4 e^{2\xi_{1,3}+\xi_{2,1}-\xi_{2,2}} \in \lambda^3 \eta_{2,1,3}^2 \xi_{1,1,3} \xi_{1,2,3} - \\ & 5 e^{\xi_{1,2}+2\xi_{1,3}+\xi_{2,1}-\xi_{2,2}} \in \lambda^4 \eta_{2,1,3}^2 \xi_{1,1,2} \xi_{1,2,3} - e^{\xi_{1,1}-\xi_{1,2}+\xi_{2,1}-\xi_{2,2}} \in g_{1,2}'[\lambda] = 0, g_{1,2}[0] = 0\} \end{aligned}$$
- $$\begin{aligned} & \gg \{g_{1,2}[\lambda] \rightarrow \\ & -e^{-\xi_{1,1}+2\xi_{1,2}} \lambda^3 \eta_{2,1,2}^2 \xi_{1,1,2} - e^{-\xi_{1,1}+\xi_{1,2}+\xi_{1,3}} \lambda^3 \eta_{1,2} \eta_{2,1,3} \xi_{1,1,3} - 2 e^{-\xi_{1,1}+2\xi_{1,2}+\xi_{1,3}} \lambda^4 \eta_{2,1,2} \eta_{2,1,3} \xi_{1,1,2} \xi_{1,2,3} - \\ & e^{-\xi_{1,1}+\xi_{1,2}+2\xi_{1,3}} \lambda^4 \eta_{2,1,3}^2 \xi_{1,1,3} \xi_{1,2,3} - e^{-\xi_{1,1}+2\xi_{1,2}+2\xi_{1,3}} \lambda^5 \eta_{2,1,3}^2 \xi_{1,1,2} \xi_{1,2,3}\} \end{aligned}$$
- $$\begin{aligned} & \gg \{3 e^{\xi_{1,2}+\xi_{2,2}-\xi_{2,3}} \in \lambda^2 \eta_{1,2} \eta_{2,2,3} \xi_{1,1,2} - 3 e^{\xi_{1,3}+\xi_{2,2}-\xi_{2,3}} \in \lambda^2 \eta_{2,1,3} \eta_{2,2,3} \xi_{1,1,3} + \\ & 4 e^{\xi_{1,2}+\xi_{1,3}+\xi_{2,2}-\xi_{2,3}} \in \lambda^3 \eta_{2,1,3}^2 \xi_{1,1,2} \xi_{1,1,3} - 3 e^{\xi_{1,3}+\xi_{2,2}-\xi_{2,3}} \in \lambda^2 \eta_{2,2,3}^2 \xi_{1,2,3} + \\ & 4 e^{\xi_{1,2}+\xi_{1,3}+\xi_{2,2}-\xi_{2,3}} \in \lambda^3 \eta_{2,1,3} \eta_{2,2,3} \xi_{1,1,2} \xi_{1,2,3} - e^{\xi_{1,2}-\xi_{1,3}+\xi_{2,2}-\xi_{2,3}} \in g_{2,3}'[\lambda] = 0, g_{2,3}[0] = 0\} \end{aligned}$$
- $$\begin{aligned} & \gg \{g_{2,3}[\lambda] \rightarrow e^{\xi_{1,3}} \lambda^3 \eta_{1,2} \eta_{2,2,3} \xi_{1,1,2} - e^{-\xi_{1,2}+2\xi_{1,3}} \lambda^3 \eta_{2,1,3} \eta_{2,2,3} \xi_{1,1,3} + \\ & e^{2\xi_{1,3}} \lambda^4 \eta_{2,1,3}^2 \xi_{1,1,2} \xi_{1,1,3} - e^{-\xi_{1,2}+2\xi_{1,3}} \lambda^3 \eta_{2,2,3}^2 \xi_{1,2,3} + e^{2\xi_{1,3}} \lambda^4 \eta_{2,1,3} \eta_{2,2,3} \xi_{1,1,2} \xi_{1,2,3}\} \end{aligned}$$

In[*]:= **Sort@sol1**

Out[*]=

$$\begin{aligned}
& \{f_{1,1}[\lambda] \rightarrow e^{\xi_{1,2,2}} \lambda^2 \eta_{2,1,2} \xi_{1,1,2} + e^{\xi_{1,3,3}} \lambda^2 \eta_{2,1,3} \xi_{1,1,3} + e^{\xi_{1,2,2} + \xi_{1,3,3}} \lambda^3 \eta_{2,1,3} \xi_{1,1,2} \xi_{1,2,3}, \\
& f_{1,2}[\lambda] \rightarrow e^{-\xi_{1,2,2} + \xi_{1,3,3} - \xi_{2,1,1}} \lambda^2 \eta_{2,2,3} \xi_{1,1,3} - e^{\xi_{1,3,3} - \xi_{2,1,1}} \lambda^3 \eta_{2,1,3} \xi_{1,1,2} \xi_{1,1,3} + \\
& \quad \lambda^3 \eta_{2,1,2} \xi_{1,1,2} \xi_{2,1,2} - e^{-\xi_{1,2,2} + \xi_{1,3,3}} \lambda^3 \eta_{2,2,3} \xi_{1,2,3} \xi_{2,1,2} + e^{\xi_{1,3,3}} \lambda^4 \eta_{2,1,3} \xi_{1,1,2} \xi_{1,2,3} \xi_{2,1,2}, \\
& f_{1,3}[\lambda] \rightarrow \lambda^3 \eta_{2,1,2} \xi_{1,1,3} \xi_{2,1,2} + \lambda^3 \eta_{2,1,3} \xi_{1,1,3} \xi_{2,1,3} + \lambda^3 \eta_{2,2,3} \xi_{1,2,3} \xi_{2,1,3}, \\
& f_{2,2}[\lambda] \rightarrow -e^{\xi_{1,2,2}} \lambda^2 \eta_{2,1,2} \xi_{1,1,2} + e^{\xi_{1,3,3}} \lambda^2 \eta_{2,2,3} \xi_{1,2,3} - e^{\xi_{1,2,2} + \xi_{1,3,3}} \lambda^3 \eta_{2,1,3} \xi_{1,1,2} \xi_{1,2,3}, \\
& f_{2,3}[\lambda] \rightarrow -e^{-\xi_{2,2,2}} \lambda^2 \eta_{2,1,2} \xi_{1,1,3} + \lambda^3 \eta_{2,1,3} \xi_{1,1,3} \xi_{2,2,3} + \lambda^3 \eta_{2,2,3} \xi_{1,2,3} \xi_{2,2,3}, \\
& f_{3,3}[\lambda] \rightarrow -e^{\xi_{1,3,3}} \lambda^2 \eta_{2,1,3} \xi_{1,1,3} - e^{\xi_{1,3,3}} \lambda^2 \eta_{2,2,3} \xi_{1,2,3}, \\
& g_{1,2}[\lambda] \rightarrow -e^{-\xi_{1,1,1} + 2\xi_{1,2,2}} \lambda^3 \eta_{2,1,2}^2 \xi_{1,1,2} - e^{-\xi_{1,1,1} + \xi_{1,2,2} + \xi_{1,3,3}} \lambda^3 \eta_{2,1,2} \eta_{2,1,3} \xi_{1,1,3} - 2e^{-\xi_{1,1,1} + 2\xi_{1,2,2} + \xi_{1,3,3}} \lambda^4 \eta_{2,1,2} \\
& \quad \eta_{2,1,3} \xi_{1,1,2} \xi_{1,2,3} - e^{-\xi_{1,1,1} + \xi_{1,2,2} + 2\xi_{1,3,3}} \lambda^4 \eta_{2,1,3}^2 \xi_{1,1,3} \xi_{1,2,3} - e^{-\xi_{1,1,1} + 2\xi_{1,2,2} + 2\xi_{1,3,3}} \lambda^5 \eta_{2,1,3}^2 \xi_{1,1,2} \xi_{1,2,3}, \\
& g_{1,3}[\lambda] \rightarrow e^{-\xi_{1,1,1} + \xi_{1,2,2}} \lambda^2 \eta_{1,2,3} \eta_{2,1,2} - e^{-\xi_{1,1,1} + \xi_{1,2,2} + \xi_{1,3,3}} \lambda^3 \eta_{2,1,2} \eta_{2,1,3} \xi_{1,1,2} - e^{-\xi_{1,1,1} + 2\xi_{1,3,3}} \lambda^3 \eta_{2,1,3}^2 \xi_{1,1,3} + \\
& \quad e^{-\xi_{1,1,1} + \xi_{1,2,2} + \xi_{1,3,3}} \lambda^3 \eta_{1,2,3} \eta_{2,1,3} \xi_{1,2,3} - e^{-\xi_{1,1,1} + \xi_{1,2,2} + 2\xi_{1,3,3}} \lambda^4 \eta_{2,1,3}^2 \xi_{1,1,2} \xi_{1,2,3}, \\
& g_{2,3}[\lambda] \rightarrow e^{\xi_{1,3,3}} \lambda^3 \eta_{2,1,2} \eta_{2,2,3} \xi_{1,1,2} - e^{-\xi_{1,2,2} + 2\xi_{1,3,3}} \lambda^3 \eta_{2,1,3} \eta_{2,2,3} \xi_{1,1,3} + \\
& \quad e^{2\xi_{1,3,3}} \lambda^4 \eta_{2,1,3}^2 \xi_{1,1,2} \xi_{1,1,3} - e^{-\xi_{1,2,2} + 2\xi_{1,3,3}} \lambda^3 \eta_{2,2,3}^2 \xi_{1,2,3} + e^{2\xi_{1,3,3}} \lambda^4 \eta_{2,1,3} \eta_{2,2,3} \xi_{1,1,2} \xi_{1,2,3}\}
\end{aligned}$$

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In[*]:= cm[i_, j_ → k_] := E_{i,j}→{k} [CF@Sum[If[SSQ[b],
    b /. x_{α,α} → (ξ1_{α,α} + ξ2_{α,α}) x_{α,α}[k],
    b[k] (b /. {x_{αβ} → f_{αβ}[λ], y_{αβ} → g_{αβ}[λ]} /. sol10)
], {b, $Basis}],
    CF@Sum[b[k] (b /. {x_{αβ} → f_{αβ}[λ], y_{αβ} → g_{αβ}[λ]} /. sol1), {b, $Basis}]
] /. {ξ1_{αβ} → ξ_{αβ}[i], η1_{αβ} → η_{αβ}[i], ξ2_{αβ} → ξ_{αβ}[j], η2_{αβ} → η_{αβ}[j], λ → 1};
cm[i, j → k]

Out[*]=
E_{i,j}→{k} [y1,2[k] η1,2[i] + e^{-ξ1,1[i]+ξ2,2[i]} y1,2[k] η1,2[j] + y1,3[k] η1,3[i] +
    e^{-ξ1,1[i]+ξ3,3[i]} y1,3[k] η1,3[j] + y2,3[k] η2,3[i] + e^{-ξ2,2[i]+ξ3,3[i]} y2,3[k] η2,3[j] +
    x1,1[k] ξ1,1[i] + x1,1[k] ξ1,1[j] + e^{-ξ1,1[j]} x1,2[k] ξ1,2[i] - e^{ξ3,3[i]} y2,3[k] η1,3[j] ξ1,2[i] +
    e^{-ξ2,2[i]} x1,2[k] ξ1,2[j] + e^{-ξ1,1[j]} x1,3[k] ξ1,3[i] + e^{-ξ3,3[i]} x1,3[k] ξ1,3[j] + x2,2[k] ξ2,2[i] +
    x2,2[k] ξ2,2[j] + e^{-ξ2,2[j]} x2,3[k] ξ2,3[i] + e^{-ξ1,1[i]+ξ2,2[i]+ξ3,3[i]} y1,2[k] η1,3[j] ξ2,3[i] -
    x1,3[k] ξ1,2[j] ξ2,3[i] + e^{-ξ3,3[i]} x2,3[k] ξ2,3[j] + x3,3[k] ξ3,3[i] + x3,3[k] ξ3,3[j],
    e^{-ξ1,1[i]+ξ2,2[i]} y1,3[k] η1,2[j] η2,3[i] + e^{ξ2,2[i]} x1,1[k] η1,2[j] ξ1,2[i] -
    e^{ξ2,2[i]} x2,2[k] η1,2[j] ξ1,2[i] - e^{-ξ1,1[i]+2 ξ2,2[i]} y1,2[k] η1,2[j]^2 ξ1,2[i] -
    e^{-ξ1,1[i]+ξ2,2[i]+ξ3,3[i]} y1,3[k] η1,2[j] η1,3[j] ξ1,2[i] + e^{ξ3,3[i]} y2,3[k] η1,2[j] η2,3[j] ξ1,2[i] +
    x1,2[k] η1,2[j] ξ1,2[i] ξ1,2[j] - e^{-ξ2,2[j]} x2,3[k] η1,2[j] ξ1,3[i] +
    e^{ξ3,3[i]} x1,1[k] η1,3[j] ξ1,3[i] - e^{ξ3,3[i]} x3,3[k] η1,3[j] ξ1,3[i] -
    e^{-ξ1,1[i]+ξ2,2[i]+ξ3,3[i]} y1,2[k] η1,2[j] η1,3[j] ξ1,3[i] - e^{-ξ1,1[i]+2 ξ3,3[i]} y1,3[k] η1,3[j]^2 ξ1,3[i] +
    e^{-ξ1,1[j]-ξ2,2[i]+ξ3,3[i]} x1,2[k] η2,3[j] ξ1,3[i] - e^{-ξ2,2[i]+2 ξ3,3[i]} y2,3[k] η1,3[j] η2,3[j] ξ1,3[i] -
    e^{-ξ1,1[j]+ξ3,3[i]} x1,2[k] η1,3[j] ξ1,2[i] ξ1,3[i] + e^{2 ξ3,3[i]} y2,3[k] η1,3[j]^2 ξ1,2[i] ξ1,3[i] +
    x1,3[k] η1,2[j] ξ1,2[j] ξ1,3[i] + x1,3[k] η1,3[j] ξ1,3[i] ξ1,3[j] +
    e^{-ξ1,1[i]+ξ2,2[i]+ξ3,3[i]} y1,3[k] η1,3[j] η2,3[i] ξ2,3[i] + e^{ξ3,3[i]} x2,2[k] η2,3[j] ξ2,3[i] -
    e^{ξ3,3[i]} x3,3[k] η2,3[j] ξ2,3[i] - e^{-ξ2,2[i]+2 ξ3,3[i]} y2,3[k] η2,3[j]^2 ξ2,3[i] +
    e^{ξ2,2[i]+ξ3,3[i]} x1,1[k] η1,3[j] ξ1,2[i] ξ2,3[i] - e^{ξ2,2[i]+ξ3,3[i]} x2,2[k] η1,3[j] ξ1,2[i] ξ2,3[i] -
    2 e^{-ξ1,1[i]+2 ξ2,2[i]+ξ3,3[i]} y1,2[k] η1,2[j] η1,3[j] ξ1,2[i] ξ2,3[i] -
    e^{-ξ1,1[i]+ξ2,2[i]+2 ξ3,3[i]} y1,3[k] η1,3[j]^2 ξ1,2[i] ξ2,3[i] +
    e^{2 ξ3,3[i]} y2,3[k] η1,3[j] η2,3[j] ξ1,2[i] ξ2,3[i] -
    e^{-ξ2,2[i]+ξ3,3[i]} x1,2[k] η2,3[j] ξ1,2[j] ξ2,3[i] + e^{ξ3,3[i]} x1,2[k] η1,3[j] ξ1,2[i] ξ1,2[j] ξ2,3[i] -
    e^{-ξ1,1[i]+ξ2,2[i]+2 ξ3,3[i]} y1,2[k] η1,3[j]^2 ξ1,3[i] ξ2,3[i] + x1,3[k] η2,3[j] ξ1,3[j] ξ2,3[i] -
    e^{-ξ1,1[i]+2 ξ2,2[i]+2 ξ3,3[i]} y1,2[k] η1,3[j]^2 ξ1,2[i] ξ2,3[i]^2 +
    x2,3[k] η1,3[j] ξ1,3[i] ξ2,3[j] + x2,3[k] η2,3[j] ξ2,3[i] ξ2,3[j]]

```

In[*]:= Short[cm[1, 2 → 2] // cm[2, 3 → 1], 10]

Out[*]//Short=

$$\begin{aligned} & \mathbb{E}_{\{1,2,3\} \rightarrow \{1\}} \left[y_{1,2}[1] \eta_{1,2}[1] + e^{-\xi_{1,1}[1] + \xi_{2,2}[1]} y_{1,2}[1] \eta_{1,2}[2] + \right. \\ & e^{-\xi_{1,1}[1] - \xi_{1,1}[2] + \xi_{2,2}[1] + \xi_{2,2}[2]} y_{1,2}[1] \eta_{1,2}[3] + y_{1,3}[1] \eta_{1,3}[1] + e^{-\xi_{1,1}[1] + \xi_{3,3}[1]} y_{1,3}[1] \eta_{1,3}[2] + \\ & e^{-\xi_{1,1}[1] - \xi_{1,1}[2] + \xi_{3,3}[1] + \xi_{3,3}[2]} y_{1,3}[1] \eta_{1,3}[3] + y_{2,3}[1] \eta_{2,3}[1] + e^{-\xi_{2,2}[1] + \xi_{3,3}[1]} y_{2,3}[1] \eta_{2,3}[2] + \\ & e^{-\xi_{2,2}[1] - \xi_{2,2}[2] + \xi_{3,3}[1] + \xi_{3,3}[2]} y_{2,3}[1] \eta_{2,3}[3] + x_{1,1}[1] \xi_{1,1}[1] + x_{1,1}[1] \xi_{1,1}[2] + \\ & x_{1,1}[1] \xi_{1,1}[3] + e^{-\xi_{1,1}[2] - \xi_{1,1}[3]} x_{1,2}[1] \xi_{1,2}[1] - e^{\xi_{3,3}[1]} y_{2,3}[1] \eta_{1,3}[2] \xi_{1,2}[1] - \\ & e^{-\xi_{1,1}[2] + \xi_{3,3}[1] + \xi_{3,3}[2]} y_{2,3}[1] \eta_{1,3}[3] \xi_{1,2}[1] + e^{-\xi_{1,1}[3] - \xi_{2,2}[1]} x_{1,2}[1] \xi_{1,2}[2] - \\ & e^{-\xi_{2,2}[1] + \xi_{3,3}[1] + \xi_{3,3}[2]} y_{2,3}[1] \eta_{1,3}[3] \xi_{1,2}[2] + e^{-\xi_{1,1}[1] - \xi_{1,1}[2] - \xi_{1,1}[3]} x_{1,3}[1] \xi_{1,3}[1] + \\ & e^{-\xi_{1,1}[3] - \xi_{3,3}[1]} x_{1,3}[1] \xi_{1,3}[2] + e^{-\xi_{3,3}[1] - \xi_{3,3}[2]} x_{1,3}[1] \xi_{1,3}[3] + x_{2,2}[1] \xi_{2,2}[1] + x_{2,2}[1] \xi_{2,2}[2] + \\ & x_{2,2}[1] \xi_{2,2}[3] + e^{-\xi_{2,2}[2] - \xi_{2,2}[3]} x_{2,3}[1] \xi_{2,3}[1] + e^{-\xi_{1,1}[1] + \xi_{2,2}[1] + \xi_{3,3}[1]} y_{1,2}[1] \eta_{1,3}[2] \xi_{2,3}[1] + \\ & e^{-\xi_{1,1}[1] - \xi_{1,1}[2] + \xi_{2,2}[1] + \xi_{3,3}[1] + \xi_{3,3}[2]} y_{1,2}[1] \eta_{1,3}[3] \xi_{2,3}[1] - e^{-\xi_{1,1}[3]} x_{1,3}[1] \xi_{1,2}[2] \xi_{2,3}[1] - \\ & e^{-\xi_{2,2}[2]} x_{1,3}[1] \xi_{1,2}[3] \xi_{2,3}[1] + e^{-\xi_{2,2}[3] - \xi_{3,3}[1]} x_{2,3}[1] \xi_{2,3}[2] + \\ & e^{-\xi_{1,1}[1] - \xi_{1,1}[2] + \xi_{2,2}[1] + \xi_{2,2}[2] + \xi_{3,3}[2]} y_{1,2}[1] \eta_{1,3}[3] \xi_{2,3}[2] - e^{-\xi_{3,3}[1]} x_{1,3}[1] \xi_{1,2}[3] \xi_{2,3}[2] + \\ & e^{-\xi_{3,3}[1] - \xi_{3,3}[2]} x_{2,3}[1] \xi_{2,3}[3] + x_{3,3}[1] \xi_{3,3}[1] + x_{3,3}[1] \xi_{3,3}[2] + x_{3,3}[1] \xi_{3,3}[3], <<1>> \end{aligned}$$

In[*]:= Simplify[(cm[1, 2 → 2] // cm[2, 3 → 1]) ≡ (cm[2, 3 → 2] // cm[1, 2 → 1])]

Out[*]=

True

$$\begin{aligned} & \text{In[*]:= osol} = \mathbb{E}_{\{i,j\} \rightarrow \{k\}} \left[y_{1,2}[k] \eta_{1,2}[i] + e^{-\xi_{1,1}[i] + \xi_{2,2}[i]} y_{1,2}[k] \eta_{1,2}[j] + y_{1,3}[k] \eta_{1,3}[i] + \right. \\ & e^{-\xi_{1,1}[i] + \xi_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j] + y_{2,3}[k] \eta_{2,3}[i] + e^{-\xi_{2,2}[i] + \xi_{3,3}[i]} y_{2,3}[k] \eta_{2,3}[j] + \\ & x_{1,1}[k] \xi_{1,1}[i] + x_{1,1}[k] \xi_{1,1}[j] + e^{-\xi_{1,1}[j]} x_{1,2}[k] \xi_{1,2}[i] - e^{\xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j] \xi_{1,2}[i] + \\ & e^{-\xi_{2,2}[i]} x_{1,2}[k] \xi_{1,2}[j] + e^{-\xi_{1,1}[j]} x_{1,3}[k] \xi_{1,3}[i] + e^{-\xi_{3,3}[i]} x_{1,3}[k] \xi_{1,3}[j] + x_{2,2}[k] \xi_{2,2}[i] + \\ & x_{2,2}[k] \xi_{2,2}[j] + e^{-\xi_{2,2}[j]} x_{2,3}[k] \xi_{2,3}[i] + e^{-\xi_{1,1}[i] + \xi_{2,2}[i] + \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,3}[j] \xi_{2,3}[i] - \\ & x_{1,3}[k] \xi_{1,2}[j] \xi_{2,3}[i] + e^{-\xi_{3,3}[i]} x_{2,3}[k] \xi_{2,3}[j] + x_{3,3}[k] \xi_{3,3}[i] + x_{3,3}[k] \xi_{3,3}[j], \\ & - e^{-\xi_{2,2}[i] + \xi_{3,3}[i]} y_{1,3}[k] \eta_{1,2}[i] \eta_{2,3}[j] + e^{\xi_{2,2}[i]} x_{1,1}[k] \eta_{1,2}[j] \xi_{1,2}[i] - \\ & e^{\xi_{2,2}[i]} x_{2,2}[k] \eta_{1,2}[j] \xi_{1,2}[i] - e^{-\xi_{1,1}[i] + 2 \xi_{2,2}[i]} y_{1,2}[k] \eta_{1,2}[j]^2 \xi_{1,2}[i] + \\ & e^{\xi_{3,3}[i]} y_{1,3}[k] \eta_{1,2}[i] \eta_{1,3}[j] \xi_{1,2}[i] + x_{1,2}[k] \eta_{1,2}[j] \xi_{1,2}[i] \xi_{1,2}[j] - \\ & e^{-\xi_{2,2}[j]} x_{2,3}[k] \eta_{1,2}[j] \xi_{1,3}[i] + e^{\xi_{3,3}[i]} x_{1,1}[k] \eta_{1,3}[j] \xi_{1,3}[i] - e^{\xi_{3,3}[i]} x_{3,3}[k] \eta_{1,3}[j] \xi_{1,3}[i] - \\ & e^{-\xi_{1,1}[i] + \xi_{2,2}[i] + \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \xi_{1,3}[i] - e^{-\xi_{1,1}[i] + 2 \xi_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j]^2 \xi_{1,3}[i] + \\ & e^{-\xi_{1,1}[j] - \xi_{2,2}[i] + \xi_{3,3}[i]} x_{1,2}[k] \eta_{2,3}[j] \xi_{1,3}[i] - e^{-\xi_{2,2}[i] + 2 \xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,3}[i] - \\ & e^{-\xi_{1,1}[j] + \xi_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{1,3}[i] + e^{2 \xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j]^2 \xi_{1,2}[i] \xi_{1,3}[i] + \\ & x_{1,3}[k] \eta_{1,2}[j] \xi_{1,2}[j] \xi_{1,3}[i] + x_{1,3}[k] \eta_{1,3}[j] \xi_{1,3}[i] \xi_{1,3}[j] + e^{\xi_{3,3}[i]} x_{2,2}[k] \eta_{2,3}[j] \xi_{2,3}[i] - \\ & e^{\xi_{3,3}[i]} x_{3,3}[k] \eta_{2,3}[j] \xi_{2,3}[i] + e^{-\xi_{1,1}[i] + \xi_{2,2}[i] + \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{2,3}[j] \xi_{2,3}[i] - \\ & e^{-\xi_{1,1}[i] + 2 \xi_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{2,3}[i] - e^{-\xi_{2,2}[i] + 2 \xi_{3,3}[i]} y_{2,3}[k] \eta_{2,3}[j]^2 \xi_{2,3}[i] + \\ & e^{\xi_{2,2}[i] + \xi_{3,3}[i]} x_{1,1}[k] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] - e^{\xi_{2,2}[i] + \xi_{3,3}[i]} x_{2,2}[k] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] - \\ & 2 e^{-\xi_{1,1}[i] + 2 \xi_{2,2}[i] + \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] + \\ & e^{2 \xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] - \\ & e^{-\xi_{2,2}[i] + \xi_{3,3}[i]} x_{1,2}[k] \eta_{2,3}[j] \xi_{1,2}[j] \xi_{2,3}[i] + e^{\xi_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{1,2}[j] \xi_{2,3}[i] - \\ & e^{-\xi_{1,1}[i] + \xi_{2,2}[i] + 2 \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,3}[j]^2 \xi_{1,3}[i] \xi_{2,3}[i] + x_{1,3}[k] \eta_{2,3}[j] \xi_{1,3}[j] \xi_{2,3}[i] - \\ & e^{-\xi_{1,1}[i] + 2 \xi_{2,2}[i] + 2 \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,3}[j]^2 \xi_{1,2}[i] \xi_{2,3}[i]^2 + \\ & x_{2,3}[k] \eta_{1,3}[j] \xi_{1,3}[i] \xi_{2,3}[j] + x_{2,3}[k] \eta_{2,3}[j] \xi_{2,3}[i] \xi_{2,3}[j], \\ & - \frac{1}{2} e^{2 \xi_{2,2}[i]} x_{1,1}[k] \eta_{1,2}[j]^2 \xi_{1,2}[i]^2 + \frac{1}{2} e^{2 \xi_{2,2}[i]} x_{2,2}[k] \eta_{1,2}[j]^2 \xi_{1,2}[i]^2 + \end{aligned}$$

$$\begin{aligned}
& e^{-\epsilon_{1,1}[i]+3\epsilon_{2,2}[i]} y_{1,2}[k] \eta_{1,2}[j]^3 \epsilon_{1,2}[i]^2 + e^{\epsilon_{3,3}[i]} x_{1,1}[k] \eta_{1,2}[j] \eta_{2,3}[j] \epsilon_{1,3}[i] - \\
& e^{\epsilon_{3,3}[i]} x_{2,2}[k] \eta_{1,2}[j] \eta_{2,3}[j] \epsilon_{1,3}[i] - e^{-\epsilon_{1,1}[i]+\epsilon_{2,2}[i]+\epsilon_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j]^2 \eta_{2,3}[j] \epsilon_{1,3}[i] + \\
& e^{-\epsilon_{2,2}[i]+2\epsilon_{3,3}[i]} y_{1,3}[k] \eta_{1,2}[i] \eta_{1,3}[j] \eta_{2,3}[j] \epsilon_{1,3}[i] - e^{\epsilon_{2,2}[i]+\epsilon_{3,3}[i]} x_{1,1}[k] \eta_{1,2}[j] \\
& \eta_{1,3}[j] \epsilon_{1,2}[i] \epsilon_{1,3}[i] + e^{\epsilon_{2,2}[i]+\epsilon_{3,3}[i]} x_{2,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \epsilon_{1,2}[i] \epsilon_{1,3}[i] + \\
& 2 e^{-\epsilon_{1,1}[i]+2\epsilon_{2,2}[i]+\epsilon_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j]^2 \eta_{1,3}[j] \epsilon_{1,2}[i] \epsilon_{1,3}[i] - e^{2\epsilon_{3,3}[i]} y_{1,3}[k] \\
& \eta_{1,2}[i] \eta_{1,3}[j]^2 \epsilon_{1,2}[i] \epsilon_{1,3}[i] + e^{-\epsilon_{2,2}[i]+\epsilon_{3,3}[i]} x_{1,2}[k] \eta_{1,2}[j] \eta_{2,3}[j] \epsilon_{1,2}[j] \epsilon_{1,3}[i] - \\
& e^{\epsilon_{3,3}[i]} x_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \epsilon_{1,2}[i] \epsilon_{1,2}[j] \epsilon_{1,3}[i] - \frac{1}{2} e^{2\epsilon_{3,3}[i]} x_{1,1}[k] \eta_{1,3}[j]^2 \epsilon_{1,3}[i]^2 + \\
& \frac{1}{2} e^{2\epsilon_{3,3}[i]} x_{3,3}[k] \eta_{1,3}[j]^2 \epsilon_{1,3}[i]^2 + e^{-\epsilon_{1,1}[i]+\epsilon_{2,2}[i]+2\epsilon_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j]^2 \epsilon_{1,3}[i]^2 + \\
& e^{-\epsilon_{1,1}[i]+3\epsilon_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j]^3 \epsilon_{1,3}[i]^2 - e^{-\epsilon_{1,1}[j]-\epsilon_{2,2}[i]+2\epsilon_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j] \eta_{2,3}[j] \epsilon_{1,3}[i]^2 + \\
& e^{-\epsilon_{2,2}[i]+3\epsilon_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j]^2 \eta_{2,3}[j] \epsilon_{1,3}[i]^2 + \\
& e^{-\epsilon_{1,1}[j]+2\epsilon_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j]^2 \epsilon_{1,2}[i] \epsilon_{1,3}[i]^2 - e^{3\epsilon_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j]^3 \epsilon_{1,2}[i] \epsilon_{1,3}[i]^2 + \\
& e^{-\epsilon_{2,2}[i]+2\epsilon_{3,3}[i]} y_{1,3}[k] \eta_{1,2}[i] \eta_{2,3}[j]^2 \epsilon_{2,3}[i] + e^{\epsilon_{2,2}[i]+\epsilon_{3,3}[i]} x_{1,1}[k] \eta_{1,2}[j] \\
& \eta_{2,3}[j] \epsilon_{1,2}[i] \epsilon_{2,3}[i] - e^{\epsilon_{2,2}[i]+\epsilon_{3,3}[i]} x_{2,2}[k] \eta_{1,2}[j] \eta_{2,3}[j] \epsilon_{1,2}[i] \epsilon_{2,3}[i] - \\
& 2 e^{-\epsilon_{1,1}[i]+2\epsilon_{2,2}[i]+\epsilon_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j]^2 \eta_{2,3}[j] \epsilon_{1,2}[i] \epsilon_{2,3}[i] - \\
& e^{2\epsilon_{3,3}[i]} y_{1,3}[k] \eta_{1,2}[i] \eta_{1,3}[j] \eta_{2,3}[j] \epsilon_{1,2}[i] \epsilon_{2,3}[i] - e^{2\epsilon_{2,2}[i]+\epsilon_{3,3}[i]} x_{1,1}[k] \eta_{1,2}[j] \\
& \eta_{1,3}[j] \epsilon_{1,2}[i]^2 \epsilon_{2,3}[i] + e^{2\epsilon_{2,2}[i]+\epsilon_{3,3}[i]} x_{2,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \epsilon_{1,2}[i]^2 \epsilon_{2,3}[i] + \\
& 3 e^{-\epsilon_{1,1}[i]+3\epsilon_{2,2}[i]+\epsilon_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j]^2 \eta_{1,3}[j] \epsilon_{1,2}[i]^2 \epsilon_{2,3}[i] - \\
& e^{2\epsilon_{3,3}[i]} x_{2,2}[k] \eta_{1,3}[j] \eta_{2,3}[j] \epsilon_{1,3}[i] \epsilon_{2,3}[i] + e^{2\epsilon_{3,3}[i]} x_{3,3}[k] \eta_{1,3}[j] \eta_{2,3}[j] \epsilon_{1,3}[i] \epsilon_{2,3}[i] - \\
& 2 e^{-\epsilon_{1,1}[i]+\epsilon_{2,2}[i]+2\epsilon_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \eta_{2,3}[j] \epsilon_{1,3}[i] \epsilon_{2,3}[i] + \\
& 2 e^{-\epsilon_{1,1}[i]+3\epsilon_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j]^2 \eta_{2,3}[j] \epsilon_{1,3}[i] \epsilon_{2,3}[i] - \\
& e^{-\epsilon_{1,1}[j]-\epsilon_{2,2}[i]+2\epsilon_{3,3}[i]} x_{1,2}[k] \eta_{2,3}[j]^2 \epsilon_{1,3}[i] \epsilon_{2,3}[i] + \\
& 2 e^{-\epsilon_{2,2}[i]+3\epsilon_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j] \eta_{2,3}[j]^2 \epsilon_{1,3}[i] \epsilon_{2,3}[i] - e^{\epsilon_{2,2}[i]+2\epsilon_{3,3}[i]} x_{1,1}[k] \eta_{1,3}[j]^2 \\
& \epsilon_{1,2}[i] \epsilon_{1,3}[i] \epsilon_{2,3}[i] + e^{\epsilon_{2,2}[i]+2\epsilon_{3,3}[i]} x_{2,2}[k] \eta_{1,3}[j]^2 \epsilon_{1,2}[i] \epsilon_{1,3}[i] \epsilon_{2,3}[i] + \\
& 4 e^{-\epsilon_{1,1}[i]+2\epsilon_{2,2}[i]+2\epsilon_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j]^2 \epsilon_{1,2}[i] \epsilon_{1,3}[i] \epsilon_{2,3}[i] + \\
& e^{-\epsilon_{1,1}[j]+2\epsilon_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j] \eta_{2,3}[j] \epsilon_{1,2}[i] \epsilon_{1,3}[i] \epsilon_{2,3}[i] - \\
& 2 e^{3\epsilon_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j]^2 \eta_{2,3}[j] \epsilon_{1,2}[i] \epsilon_{1,3}[i] \epsilon_{2,3}[i] + \\
& e^{-\epsilon_{2,2}[i]+2\epsilon_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j] \eta_{2,3}[j] \epsilon_{1,2}[j] \epsilon_{1,3}[i] \epsilon_{2,3}[i] - \\
& e^{2\epsilon_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j]^2 \epsilon_{1,2}[i] \epsilon_{1,2}[j] \epsilon_{1,3}[i] \epsilon_{2,3}[i] + \\
& e^{-\epsilon_{1,1}[i]+\epsilon_{2,2}[i]+3\epsilon_{3,3}[i]} y_{1,2}[k] \eta_{1,3}[j]^3 \epsilon_{1,3}[i]^2 \epsilon_{2,3}[i] - \\
& \frac{1}{2} e^{2\epsilon_{3,3}[i]} x_{2,2}[k] \eta_{2,3}[j]^2 \epsilon_{2,3}[i]^2 + \frac{1}{2} e^{2\epsilon_{3,3}[i]} x_{3,3}[k] \eta_{2,3}[j]^2 \epsilon_{2,3}[i]^2 + \\
& e^{-\epsilon_{1,1}[i]+3\epsilon_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j] \eta_{2,3}[j]^2 \epsilon_{2,3}[i]^2 + e^{-\epsilon_{2,2}[i]+3\epsilon_{3,3}[i]} y_{2,3}[k] \eta_{2,3}[j]^3 \epsilon_{2,3}[i]^2 - \\
& 2 e^{-\epsilon_{1,1}[i]+2\epsilon_{2,2}[i]+2\epsilon_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \eta_{2,3}[j] \epsilon_{1,2}[i] \epsilon_{2,3}[i]^2 - \\
& e^{3\epsilon_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j] \eta_{2,3}[j]^2 \epsilon_{1,2}[i] \epsilon_{2,3}[i]^2 - \frac{1}{2} e^{2\epsilon_{2,2}[i]+2\epsilon_{3,3}[i]} x_{1,1}[k] \\
& \eta_{1,3}[j]^2 \epsilon_{1,2}[i]^2 \epsilon_{2,3}[i]^2 + \frac{1}{2} e^{2\epsilon_{2,2}[i]+2\epsilon_{3,3}[i]} x_{2,2}[k] \eta_{1,3}[j]^2 \epsilon_{1,2}[i]^2 \epsilon_{2,3}[i]^2 + \\
& 3 e^{-\epsilon_{1,1}[i]+3\epsilon_{2,2}[i]+2\epsilon_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j]^2 \epsilon_{1,2}[i]^2 \epsilon_{2,3}[i]^2 + \\
& e^{-\epsilon_{2,2}[i]+2\epsilon_{3,3}[i]} x_{1,2}[k] \eta_{2,3}[j]^2 \epsilon_{1,2}[j] \epsilon_{2,3}[i]^2 - e^{2\epsilon_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j] \eta_{2,3}[j] \epsilon_{1,2}[i]
\end{aligned}$$

$$\xi_{1,2}[j] \xi_{2,3}[i]^2 + 2 e^{-\xi_{1,1}[i] + 2 \xi_{2,2}[i] + 3 \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,3}[j]^3 \xi_{1,2}[i] \xi_{1,3}[i] \xi_{2,3}[i]^2 +$$

$$e^{-\xi_{1,1}[i] + 3 \xi_{2,2}[i] + 3 \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,3}[j]^3 \xi_{1,2}[i]^2 \xi_{2,3}[i]^3];$$

osol = (List@@osol).Table[e^k, {k, 0, Length[osol] - 1}]

Out[*]=

$$y_{1,2}[k] \eta_{1,2}[i] + e^{-\xi_{1,1}[i] + \xi_{2,2}[i]} y_{1,2}[k] \eta_{1,2}[j] + y_{1,3}[k] \eta_{1,3}[i] + e^{-\xi_{1,1}[i] + \xi_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j] +$$

$$y_{2,3}[k] \eta_{2,3}[i] + e^{-\xi_{2,2}[i] + \xi_{3,3}[i]} y_{2,3}[k] \eta_{2,3}[j] + x_{1,1}[k] \xi_{1,1}[i] + x_{1,1}[k] \xi_{1,1}[j] +$$

$$e^{-\xi_{1,1}[j]} x_{1,2}[k] \xi_{1,2}[i] - e^{\xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j] \xi_{1,2}[i] + e^{-\xi_{2,2}[i]} x_{1,2}[k] \xi_{1,2}[j] +$$

$$e^{-\xi_{1,1}[j]} x_{1,3}[k] \xi_{1,3}[i] + e^{-\xi_{3,3}[i]} x_{1,3}[k] \xi_{1,3}[j] + x_{2,2}[k] \xi_{2,2}[i] + x_{2,2}[k] \xi_{2,2}[j] +$$

$$e^{-\xi_{2,2}[j]} x_{2,3}[k] \xi_{2,3}[i] + e^{-\xi_{1,1}[i] + \xi_{2,2}[i] + \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,3}[j] \xi_{2,3}[i] - x_{1,3}[k] \xi_{1,2}[j] \xi_{2,3}[i] +$$

$$\epsilon^2 \left(-\frac{1}{2} e^{2 \xi_{2,2}[i]} x_{1,1}[k] \eta_{1,2}[j]^2 \xi_{1,2}[i]^2 + \frac{1}{2} e^{2 \xi_{2,2}[i]} x_{2,2}[k] \eta_{1,2}[j]^2 \xi_{1,2}[i]^2 + \right.$$

$$e^{-\xi_{1,1}[i] + 3 \xi_{2,2}[i]} y_{1,2}[k] \eta_{1,2}[j]^3 \xi_{1,2}[i]^2 + e^{\xi_{3,3}[i]} x_{1,1}[k] \eta_{1,2}[j] \eta_{2,3}[j] \xi_{1,3}[i] -$$

$$e^{\xi_{3,3}[i]} x_{2,2}[k] \eta_{1,2}[j] \eta_{2,3}[j] \xi_{1,3}[i] - e^{-\xi_{1,1}[i] + \xi_{2,2}[i] + \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j]^2 \eta_{2,3}[j] \xi_{1,3}[i] +$$

$$e^{-\xi_{2,2}[i] + 2 \xi_{3,3}[i]} y_{1,3}[k] \eta_{1,2}[i] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,3}[i] - e^{\xi_{2,2}[i] + \xi_{3,3}[i]} x_{1,1}[k] \eta_{1,2}[j]$$

$$\eta_{1,3}[j] \xi_{1,2}[i] \xi_{1,3}[i] + e^{\xi_{2,2}[i] + \xi_{3,3}[i]} x_{2,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{1,3}[i] +$$

$$2 e^{-\xi_{1,1}[i] + 2 \xi_{2,2}[i] + \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j]^2 \eta_{1,3}[j] \xi_{1,2}[i] \xi_{1,3}[i] - e^{2 \xi_{3,3}[i]} y_{1,3}[k]$$

$$\eta_{1,2}[i] \eta_{1,3}[j]^2 \xi_{1,2}[i] \xi_{1,3}[i] + e^{-\xi_{2,2}[i] + \xi_{3,3}[i]} x_{1,2}[k] \eta_{1,2}[j] \eta_{2,3}[j] \xi_{1,2}[j] \xi_{1,3}[i] -$$

$$e^{\xi_{3,3}[i]} x_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{1,2}[j] \xi_{1,3}[i] - \frac{1}{2} e^{2 \xi_{3,3}[i]} x_{1,1}[k] \eta_{1,3}[j]^2 \xi_{1,3}[i]^2 +$$

$$\frac{1}{2} e^{2 \xi_{3,3}[i]} x_{3,3}[k] \eta_{1,3}[j]^2 \xi_{1,3}[i]^2 + e^{-\xi_{1,1}[i] + \xi_{2,2}[i] + 2 \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j]^2 \xi_{1,3}[i]^2 +$$

$$e^{-\xi_{1,1}[i] + 3 \xi_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j]^3 \xi_{1,3}[i]^2 - e^{-\xi_{1,1}[j] - \xi_{2,2}[i] + 2 \xi_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,3}[i]^2 +$$

$$e^{-\xi_{2,2}[i] + 3 \xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j]^2 \eta_{2,3}[j] \xi_{1,3}[i]^2 +$$

$$e^{-\xi_{1,1}[j] + 2 \xi_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j]^2 \xi_{1,2}[i] \xi_{1,3}[i]^2 - e^{3 \xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j]^3 \xi_{1,2}[i] \xi_{1,3}[i]^2 +$$

$$e^{-\xi_{2,2}[i] + 2 \xi_{3,3}[i]} y_{1,3}[k] \eta_{1,2}[i] \eta_{2,3}[j]^2 \xi_{2,3}[i] + e^{\xi_{2,2}[i] + \xi_{3,3}[i]} x_{1,1}[k] \eta_{1,2}[j]$$

$$\eta_{2,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] - e^{\xi_{2,2}[i] + \xi_{3,3}[i]} x_{2,2}[k] \eta_{1,2}[j] \eta_{2,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] -$$

$$2 e^{-\xi_{1,1}[i] + 2 \xi_{2,2}[i] + \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j]^2 \eta_{2,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] -$$

$$e^{2 \xi_{3,3}[i]} y_{1,3}[k] \eta_{1,2}[i] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] - e^{2 \xi_{2,2}[i] + \xi_{3,3}[i]} x_{1,1}[k] \eta_{1,2}[j]$$

$$\eta_{1,3}[j] \xi_{1,2}[i]^2 \xi_{2,3}[i] + e^{2 \xi_{2,2}[i] + \xi_{3,3}[i]} x_{2,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \xi_{1,2}[i]^2 \xi_{2,3}[i] +$$

$$3 e^{-\xi_{1,1}[i] + 3 \xi_{2,2}[i] + \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j]^2 \eta_{1,3}[j] \xi_{1,2}[i]^2 \xi_{2,3}[i] -$$

$$e^{2 \xi_{3,3}[i]} x_{2,2}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,3}[i] \xi_{2,3}[i] + e^{2 \xi_{3,3}[i]} x_{3,3}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,3}[i] \xi_{2,3}[i] -$$

$$2 e^{-\xi_{1,1}[i] + \xi_{2,2}[i] + 2 \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,3}[i] \xi_{2,3}[i] +$$

$$2 e^{-\xi_{1,1}[i] + 3 \xi_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j]^2 \eta_{2,3}[j] \xi_{1,3}[i] \xi_{2,3}[i] -$$

$$e^{-\xi_{1,1}[j] - \xi_{2,2}[i] + 2 \xi_{3,3}[i]} x_{1,2}[k] \eta_{2,3}[j]^2 \xi_{1,3}[i] \xi_{2,3}[i] +$$

$$2 e^{-\xi_{2,2}[i] + 3 \xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j] \eta_{2,3}[j]^2 \xi_{1,3}[i] \xi_{2,3}[i] - e^{\xi_{2,2}[i] + 2 \xi_{3,3}[i]} x_{1,1}[k] \eta_{1,3}[j]^2$$

$$\xi_{1,2}[i] \xi_{1,3}[i] \xi_{2,3}[i] + e^{\xi_{2,2}[i] + 2 \xi_{3,3}[i]} x_{2,2}[k] \eta_{1,3}[j]^2 \xi_{1,2}[i] \xi_{1,3}[i] \xi_{2,3}[i] +$$

$$4 e^{-\xi_{1,1}[i] + 2 \xi_{2,2}[i] + 2 \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j]^2 \xi_{1,2}[i] \xi_{1,3}[i] \xi_{2,3}[i] +$$

$$e^{-\xi_{1,1}[j] + 2 \xi_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,2}[i] \xi_{1,3}[i] \xi_{2,3}[i] -$$

$$2 e^{3 \xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j]^2 \eta_{2,3}[j] \xi_{1,2}[i] \xi_{1,3}[i] \xi_{2,3}[i] +$$

$$e^{-\xi_{2,2}[i] + 2 \xi_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,2}[j] \xi_{1,3}[i] \xi_{2,3}[i] - e^{2 \xi_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j]^2$$

$$\xi_{1,2}[i] \xi_{1,2}[j] \xi_{1,3}[i] \xi_{2,3}[i] + e^{-\xi_{1,1}[i] + \xi_{2,2}[i] + 3 \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,3}[j]^3 \xi_{1,3}[i]^2 \xi_{2,3}[i] -$$

$$\frac{1}{2} e^{2 \xi_{3,3}[i]} x_{2,2}[k] \eta_{2,3}[j]^2 \xi_{2,3}[i]^2 + \frac{1}{2} e^{2 \xi_{3,3}[i]} x_{3,3}[k] \eta_{2,3}[j]^2 \xi_{2,3}[i]^2 +$$

$$e^{-\xi_{1,1}[i] + 3 \xi_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j] \eta_{2,3}[j]^2 \xi_{2,3}[i]^2 + e^{-\xi_{2,2}[i] + 3 \xi_{3,3}[i]} y_{2,3}[k] \eta_{2,3}[j]^3 \xi_{2,3}[i]^2 -$$

$$\begin{aligned}
& 2 e^{-\xi_{1,1}[\mathbf{i}] + 2 \xi_{2,2}[\mathbf{i}] + 2 \xi_{3,3}[\mathbf{i}]} y_{1,2}[\mathbf{k}] \eta_{1,2}[\mathbf{j}] \eta_{1,3}[\mathbf{j}] \eta_{2,3}[\mathbf{j}] \xi_{1,2}[\mathbf{i}] \xi_{2,3}[\mathbf{i}]^2 - \\
& e^{3 \xi_{3,3}[\mathbf{i}]} y_{2,3}[\mathbf{k}] \eta_{1,3}[\mathbf{j}] \eta_{2,3}[\mathbf{j}]^2 \xi_{1,2}[\mathbf{i}] \xi_{2,3}[\mathbf{i}]^2 - \frac{1}{2} e^{2 \xi_{2,2}[\mathbf{i}] + 2 \xi_{3,3}[\mathbf{i}]} x_{1,1}[\mathbf{k}] \\
& \eta_{1,3}[\mathbf{j}]^2 \xi_{1,2}[\mathbf{i}]^2 \xi_{2,3}[\mathbf{i}]^2 + \frac{1}{2} e^{2 \xi_{2,2}[\mathbf{i}] + 2 \xi_{3,3}[\mathbf{i}]} x_{2,2}[\mathbf{k}] \eta_{1,3}[\mathbf{j}]^2 \xi_{1,2}[\mathbf{i}]^2 \xi_{2,3}[\mathbf{i}]^2 + \\
& 3 e^{-\xi_{1,1}[\mathbf{i}] + 3 \xi_{2,2}[\mathbf{i}] + 2 \xi_{3,3}[\mathbf{i}]} y_{1,2}[\mathbf{k}] \eta_{1,2}[\mathbf{j}] \eta_{1,3}[\mathbf{j}]^2 \xi_{1,2}[\mathbf{i}]^2 \xi_{2,3}[\mathbf{i}]^2 + \\
& e^{-\xi_{2,2}[\mathbf{i}] + 2 \xi_{3,3}[\mathbf{i}]} x_{1,2}[\mathbf{k}] \eta_{2,3}[\mathbf{j}]^2 \xi_{1,2}[\mathbf{j}] \xi_{2,3}[\mathbf{i}]^2 - e^{2 \xi_{3,3}[\mathbf{i}]} x_{1,2}[\mathbf{k}] \eta_{1,3}[\mathbf{j}] \eta_{2,3}[\mathbf{j}] \xi_{1,2}[\mathbf{i}] \\
& \xi_{1,2}[\mathbf{j}] \xi_{2,3}[\mathbf{i}]^2 + 2 e^{-\xi_{1,1}[\mathbf{i}] + 2 \xi_{2,2}[\mathbf{i}] + 3 \xi_{3,3}[\mathbf{i}]} y_{1,2}[\mathbf{k}] \eta_{1,3}[\mathbf{j}]^3 \xi_{1,2}[\mathbf{i}] \xi_{1,3}[\mathbf{i}] \xi_{2,3}[\mathbf{i}]^2 + \\
& e^{-\xi_{1,1}[\mathbf{i}] + 3 \xi_{2,2}[\mathbf{i}] + 3 \xi_{3,3}[\mathbf{i}]} y_{1,2}[\mathbf{k}] \eta_{1,3}[\mathbf{j}]^3 \xi_{1,2}[\mathbf{i}]^2 \xi_{2,3}[\mathbf{i}]^3 \Big) + e^{-\xi_{3,3}[\mathbf{i}]} x_{2,3}[\mathbf{k}] \xi_{2,3}[\mathbf{j}] + \\
& \in \left(-e^{-\xi_{2,2}[\mathbf{i}] + \xi_{3,3}[\mathbf{i}]} y_{1,3}[\mathbf{k}] \eta_{1,2}[\mathbf{i}] \eta_{2,3}[\mathbf{j}] + e^{\xi_{2,2}[\mathbf{i}]} x_{1,1}[\mathbf{k}] \eta_{1,2}[\mathbf{j}] \xi_{1,2}[\mathbf{i}] - \right. \\
& e^{\xi_{2,2}[\mathbf{i}]} x_{2,2}[\mathbf{k}] \eta_{1,2}[\mathbf{j}] \xi_{1,2}[\mathbf{i}] - e^{-\xi_{1,1}[\mathbf{i}] + 2 \xi_{2,2}[\mathbf{i}]} y_{1,2}[\mathbf{k}] \eta_{1,2}[\mathbf{j}]^2 \xi_{1,2}[\mathbf{i}] + \\
& e^{\xi_{3,3}[\mathbf{i}]} y_{1,3}[\mathbf{k}] \eta_{1,2}[\mathbf{i}] \eta_{1,3}[\mathbf{j}] \xi_{1,2}[\mathbf{i}] + x_{1,2}[\mathbf{k}] \eta_{1,2}[\mathbf{j}] \xi_{1,2}[\mathbf{i}] \xi_{1,2}[\mathbf{j}] - \\
& e^{-\xi_{2,2}[\mathbf{j}]} x_{2,3}[\mathbf{k}] \eta_{1,2}[\mathbf{j}] \xi_{1,3}[\mathbf{i}] + e^{\xi_{3,3}[\mathbf{i}]} x_{1,1}[\mathbf{k}] \eta_{1,3}[\mathbf{j}] \xi_{1,3}[\mathbf{i}] - e^{\xi_{3,3}[\mathbf{i}]} x_{3,3}[\mathbf{k}] \eta_{1,3}[\mathbf{j}] \xi_{1,3}[\mathbf{i}] - \\
& e^{-\xi_{1,1}[\mathbf{i}] + \xi_{2,2}[\mathbf{i}] + \xi_{3,3}[\mathbf{i}]} y_{1,2}[\mathbf{k}] \eta_{1,2}[\mathbf{j}] \eta_{1,3}[\mathbf{j}] \xi_{1,3}[\mathbf{i}] - e^{-\xi_{1,1}[\mathbf{i}] + 2 \xi_{3,3}[\mathbf{i}]} y_{1,3}[\mathbf{k}] \eta_{1,3}[\mathbf{j}]^2 \xi_{1,3}[\mathbf{i}] + \\
& e^{-\xi_{1,1}[\mathbf{j}] - \xi_{2,2}[\mathbf{i}] + \xi_{3,3}[\mathbf{i}]} x_{1,2}[\mathbf{k}] \eta_{2,3}[\mathbf{j}] \xi_{1,3}[\mathbf{i}] - e^{-\xi_{2,2}[\mathbf{i}] + 2 \xi_{3,3}[\mathbf{i}]} y_{2,3}[\mathbf{k}] \eta_{1,3}[\mathbf{j}] \eta_{2,3}[\mathbf{j}] \xi_{1,3}[\mathbf{i}] - \\
& e^{-\xi_{1,1}[\mathbf{j}] + \xi_{3,3}[\mathbf{i}]} x_{1,2}[\mathbf{k}] \eta_{1,3}[\mathbf{j}] \xi_{1,2}[\mathbf{i}] \xi_{1,3}[\mathbf{i}] + e^{2 \xi_{3,3}[\mathbf{i}]} y_{2,3}[\mathbf{k}] \eta_{1,3}[\mathbf{j}]^2 \xi_{1,2}[\mathbf{i}] \xi_{1,3}[\mathbf{i}] + \\
& x_{1,3}[\mathbf{k}] \eta_{1,2}[\mathbf{j}] \xi_{1,2}[\mathbf{j}] \xi_{1,3}[\mathbf{i}] + x_{1,3}[\mathbf{k}] \eta_{1,3}[\mathbf{j}] \xi_{1,3}[\mathbf{i}] \xi_{1,3}[\mathbf{j}] + e^{\xi_{3,3}[\mathbf{i}]} x_{2,2}[\mathbf{k}] \eta_{2,3}[\mathbf{j}] \xi_{2,3}[\mathbf{i}] - \\
& e^{\xi_{3,3}[\mathbf{i}]} x_{3,3}[\mathbf{k}] \eta_{2,3}[\mathbf{j}] \xi_{2,3}[\mathbf{i}] + e^{-\xi_{1,1}[\mathbf{i}] + \xi_{2,2}[\mathbf{i}] + \xi_{3,3}[\mathbf{i}]} y_{1,2}[\mathbf{k}] \eta_{1,2}[\mathbf{j}] \eta_{2,3}[\mathbf{j}] \xi_{2,3}[\mathbf{i}] - \\
& e^{-\xi_{1,1}[\mathbf{i}] + 2 \xi_{3,3}[\mathbf{i}]} y_{1,3}[\mathbf{k}] \eta_{1,3}[\mathbf{j}] \eta_{2,3}[\mathbf{j}] \xi_{2,3}[\mathbf{i}] - e^{-\xi_{2,2}[\mathbf{i}] + 2 \xi_{3,3}[\mathbf{i}]} y_{2,3}[\mathbf{k}] \eta_{2,3}[\mathbf{j}]^2 \xi_{2,3}[\mathbf{i}] + \\
& e^{\xi_{2,2}[\mathbf{i}] + \xi_{3,3}[\mathbf{i}]} x_{1,1}[\mathbf{k}] \eta_{1,3}[\mathbf{j}] \xi_{1,2}[\mathbf{i}] \xi_{2,3}[\mathbf{i}] - e^{\xi_{2,2}[\mathbf{i}] + \xi_{3,3}[\mathbf{i}]} x_{2,2}[\mathbf{k}] \eta_{1,3}[\mathbf{j}] \xi_{1,2}[\mathbf{i}] \xi_{2,3}[\mathbf{i}] - \\
& 2 e^{-\xi_{1,1}[\mathbf{i}] + 2 \xi_{2,2}[\mathbf{i}] + \xi_{3,3}[\mathbf{i}]} y_{1,2}[\mathbf{k}] \eta_{1,2}[\mathbf{j}] \eta_{1,3}[\mathbf{j}] \xi_{1,2}[\mathbf{i}] \xi_{2,3}[\mathbf{i}] + \\
& e^{2 \xi_{3,3}[\mathbf{i}]} y_{2,3}[\mathbf{k}] \eta_{1,3}[\mathbf{j}] \eta_{2,3}[\mathbf{j}] \xi_{1,2}[\mathbf{i}] \xi_{2,3}[\mathbf{i}] - \\
& e^{-\xi_{2,2}[\mathbf{i}] + \xi_{3,3}[\mathbf{i}]} x_{1,2}[\mathbf{k}] \eta_{2,3}[\mathbf{j}] \xi_{1,2}[\mathbf{j}] \xi_{2,3}[\mathbf{i}] + e^{\xi_{3,3}[\mathbf{i}]} x_{1,2}[\mathbf{k}] \eta_{1,3}[\mathbf{j}] \xi_{1,2}[\mathbf{i}] \xi_{1,2}[\mathbf{j}] \xi_{2,3}[\mathbf{i}] - \\
& e^{-\xi_{1,1}[\mathbf{i}] + \xi_{2,2}[\mathbf{i}] + 2 \xi_{3,3}[\mathbf{i}]} y_{1,2}[\mathbf{k}] \eta_{1,3}[\mathbf{j}]^2 \xi_{1,3}[\mathbf{i}] \xi_{2,3}[\mathbf{i}] + x_{1,3}[\mathbf{k}] \eta_{2,3}[\mathbf{j}] \xi_{1,3}[\mathbf{j}] \xi_{2,3}[\mathbf{i}] - \\
& e^{-\xi_{1,1}[\mathbf{i}] + 2 \xi_{2,2}[\mathbf{i}] + 2 \xi_{3,3}[\mathbf{i}]} y_{1,2}[\mathbf{k}] \eta_{1,3}[\mathbf{j}]^2 \xi_{1,2}[\mathbf{i}] \xi_{2,3}[\mathbf{i}]^2 + x_{2,3}[\mathbf{k}] \eta_{1,3}[\mathbf{j}] \xi_{1,3}[\mathbf{i}] \xi_{2,3}[\mathbf{j}] + \\
& x_{2,3}[\mathbf{k}] \eta_{2,3}[\mathbf{j}] \xi_{2,3}[\mathbf{i}] \xi_{2,3}[\mathbf{j}] \Big) + x_{3,3}[\mathbf{k}] \xi_{3,3}[\mathbf{i}] + x_{3,3}[\mathbf{k}] \xi_{3,3}[\mathbf{j}]
\end{aligned}$$

In[*]:= sol0

Out[*]=

$$\begin{aligned}
& \{ f_{1,2}[\lambda] \rightarrow e^{-\xi_{2,1,1}} \lambda \xi_{1,2} + e^{-\xi_{1,2,2}} \lambda \xi_{2,1,2}, g_{2,3}[\lambda] \rightarrow \lambda (\eta_{1,2,3} + e^{\xi_{1,3,3}} (e^{-\xi_{1,2,2}} \eta_{2,2,3} - \lambda \eta_{2,1,3} \xi_{1,2,2})) , \\
& f_{1,3}[\lambda] \rightarrow \lambda (e^{-\xi_{2,1,1}} \xi_{1,1,3} - \lambda \xi_{1,2,3} \xi_{2,1,2} + e^{-\xi_{1,3,3}} \xi_{2,1,3}), f_{2,3}[\lambda] \rightarrow e^{-\xi_{2,2,2}} \lambda \xi_{1,2,3} + e^{-\xi_{1,3,3}} \lambda \xi_{2,2,3}, \\
& g_{1,2}[\lambda] \rightarrow \lambda (\eta_{1,1,2} + e^{-\xi_{1,1,1} + \xi_{1,2,2}} (\eta_{2,1,2} + e^{\xi_{1,3,3}} \lambda \eta_{2,1,3} \xi_{1,2,3})), g_{1,3}[\lambda] \rightarrow \lambda (\eta_{1,1,3} + e^{-\xi_{1,1,1} + \xi_{1,3,3}} \eta_{2,1,3}) \}
\end{aligned}$$

In[*]:= **CF[sol1 /. λ → 1]**

Out[*]=

$$\begin{aligned} & \{f_{1,1}[1] \rightarrow e^{\xi_{1,2}} \eta_{2,1} \xi_{1,2} + e^{\xi_{1,3}} \eta_{2,1} \xi_{1,3} + e^{\xi_{1,2}+\xi_{1,3}} \eta_{2,1} \xi_{1,2} \xi_{1,3}, \\ & f_{2,2}[1] \rightarrow -e^{\xi_{1,2}} \eta_{2,1} \xi_{1,2} + e^{\xi_{1,3}} \eta_{2,2} \xi_{1,3} - e^{\xi_{1,2}+\xi_{1,3}} \eta_{2,1} \xi_{1,2} \xi_{1,3}, \\ & f_{3,3}[1] \rightarrow -e^{\xi_{1,3}} \eta_{2,1} \xi_{1,3} - e^{\xi_{1,3}} \eta_{2,2} \xi_{1,3}, \\ & f_{1,2}[1] \rightarrow e^{-\xi_{1,2}+\xi_{1,3}-\xi_{2,1}} \eta_{2,2} \xi_{1,3} - e^{\xi_{1,3}-\xi_{2,1}} \eta_{2,1} \xi_{1,2} \xi_{1,3} + \\ & \quad \eta_{2,1} \xi_{1,2} \xi_{2,1} - e^{-\xi_{1,2}+\xi_{1,3}} \eta_{2,2} \xi_{1,3} \xi_{2,1} + e^{\xi_{1,3}} \eta_{2,1} \xi_{1,2} \xi_{1,3} \xi_{2,1}, \\ & f_{2,3}[1] \rightarrow -e^{-\xi_{2,2}} \eta_{2,1} \xi_{1,3} + \eta_{2,1} \xi_{1,3} \xi_{2,2} + \eta_{2,2} \xi_{1,3} \xi_{2,3}, \\ & f_{1,3}[1] \rightarrow \eta_{2,1} \xi_{1,3} \xi_{2,1} + \eta_{2,1} \xi_{1,3} \xi_{2,2} + \eta_{2,2} \xi_{1,3} \xi_{2,1}, \\ & g_{1,3}[1] \rightarrow e^{-\xi_{1,1}+\xi_{1,2}} \eta_{2,3} \eta_{2,1} - e^{-\xi_{1,1}+\xi_{1,2}+\xi_{1,3}} \eta_{2,1} \eta_{2,1} \xi_{1,2} - e^{-\xi_{1,1}+\xi_{1,3}} \eta_{2,1}^2 \xi_{1,3} + \\ & \quad e^{-\xi_{1,1}+\xi_{1,2}+\xi_{1,3}} \eta_{2,3} \eta_{2,1} \xi_{1,3} - e^{-\xi_{1,1}+\xi_{1,2}+\xi_{1,3}} \eta_{2,1}^2 \xi_{1,2} \xi_{1,3}, \quad g_{1,2}[1] \rightarrow \\ & \quad -e^{-\xi_{1,1}+\xi_{1,2}} \eta_{2,1}^2 \xi_{1,2} - e^{-\xi_{1,1}+\xi_{1,2}+\xi_{1,3}} \eta_{2,1} \eta_{2,1} \xi_{1,3} - 2 e^{-\xi_{1,1}+\xi_{1,2}+\xi_{1,3}} \eta_{2,1} \eta_{2,1} \xi_{1,2} \xi_{1,3} - \\ & \quad e^{-\xi_{1,1}+\xi_{1,2}+\xi_{1,3}} \eta_{2,1}^2 \xi_{1,3} \xi_{1,3} - e^{-\xi_{1,1}+\xi_{1,2}+\xi_{1,3}} \eta_{2,1}^2 \xi_{1,2} \xi_{1,3}, \\ & g_{2,3}[1] \rightarrow e^{\xi_{1,3}} \eta_{2,1} \eta_{2,2} \xi_{1,3} - e^{-\xi_{1,2}+\xi_{1,3}} \eta_{2,1} \eta_{2,2} \xi_{1,3} + e^{2\xi_{1,3}} \eta_{2,1}^2 \xi_{1,2} \xi_{1,3} - \\ & \quad e^{-\xi_{1,2}+\xi_{1,3}} \eta_{2,2}^2 \xi_{1,3} + e^{2\xi_{1,3}} \eta_{2,1} \eta_{2,2} \xi_{1,2} \xi_{1,3} \} \end{aligned}$$

In[*]:= **Coefficient[osol, x_{1,1}[k] e]**

Out[*]=

$$e^{\xi_{2,2}[i]} \eta_{1,2}[j] \xi_{1,2}[i] + e^{\xi_{3,3}[i]} \eta_{1,3}[j] \xi_{1,3}[i] + e^{\xi_{2,2}[i]+\xi_{3,3}[i]} \eta_{1,3}[j] \xi_{1,2}[i] \xi_{2,3}[i]$$

In[*]:= **CF[(Expand@sol[\$k].\$Basis /. {ξ_{1αβ} → ξ_{αβ}[i], η_{1αβ} → η_{αβ}[i],
ξ_{2αβ} → ξ_{αβ}[j], η_{2αβ} → η_{αβ}[j], λ → 1}) - (osol /. v_[k] → v)]**

Out[*]=

$$\begin{aligned} & \text{sol}[1] \cdot \{y_{1,2}, y_{1,3}, y_{2,3}, x_{1,1}, x_{1,2}, x_{2,2}, x_{1,3}, x_{2,3}, x_{3,3}\} - y_{1,2} \eta_{1,2}[i] - \\ & e^{-\xi_{1,1}[i]+\xi_{2,2}[i]} y_{1,2} \eta_{1,2}[j] - y_{1,3} \eta_{1,3}[i] - e^{-\xi_{1,1}[i]+\xi_{3,3}[i]} y_{1,3} \eta_{1,3}[j] - y_{2,3} \eta_{2,3}[i] - \\ & e^{-\xi_{2,2}[i]+\xi_{3,3}[i]} y_{2,3} \eta_{2,3}[j] + e^{-\xi_{2,2}[i]+\xi_{3,3}[i]} \in y_{1,3} \eta_{1,2}[i] \eta_{2,3}[j] - x_{1,1} \xi_{1,1}[i] - x_{1,1} \xi_{1,1}[j] - \\ & e^{-\xi_{1,1}[j]} x_{1,2} \xi_{1,2}[i] - e^{\xi_{2,2}[i]} \in x_{1,1} \eta_{1,2}[j] \xi_{1,2}[i] + e^{\xi_{2,2}[i]} \in x_{2,2} \eta_{1,2}[j] \xi_{1,2}[i] + \\ & e^{-\xi_{1,1}[i]+2\xi_{2,2}[i]} \in y_{1,2} \eta_{1,2}[j]^2 \xi_{1,2}[i] + e^{\xi_{3,3}[i]} y_{2,3} \eta_{1,3}[j] \xi_{1,2}[i] - \\ & e^{\xi_{3,3}[i]} \in y_{1,3} \eta_{1,2}[i] \eta_{1,3}[j] \xi_{1,2}[i] - e^{-\xi_{2,2}[i]} x_{1,2} \xi_{1,2}[j] - \in x_{1,2} \eta_{1,2}[j] \xi_{1,2}[i] \xi_{1,2}[j] - \\ & e^{-\xi_{1,1}[j]} x_{1,3} \xi_{1,3}[i] + e^{-\xi_{2,2}[j]} \in x_{2,3} \eta_{1,2}[j] \xi_{1,3}[i] - e^{\xi_{3,3}[i]} \in x_{1,1} \eta_{1,3}[j] \xi_{1,3}[i] + \\ & e^{\xi_{3,3}[i]} \in x_{3,3} \eta_{1,3}[j] \xi_{1,3}[i] + e^{-\xi_{1,1}[i]+\xi_{2,2}[i]+\xi_{3,3}[i]} \in y_{1,2} \eta_{1,2}[j] \eta_{1,3}[j] \xi_{1,3}[i] + \\ & e^{-\xi_{1,1}[i]+2\xi_{3,3}[i]} \in y_{1,3} \eta_{1,3}[j]^2 \xi_{1,3}[i] - e^{-\xi_{1,1}[j]-\xi_{2,2}[i]+\xi_{3,3}[i]} \in x_{1,2} \eta_{2,3}[j] \xi_{1,3}[i] + \\ & e^{-\xi_{2,2}[i]+2\xi_{3,3}[i]} \in y_{2,3} \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,3}[i] + e^{-\xi_{1,1}[j]+\xi_{3,3}[i]} \in x_{1,2} \eta_{1,3}[j] \xi_{1,2}[i] \xi_{1,3}[i] - \\ & e^{2\xi_{3,3}[i]} \in y_{2,3} \eta_{1,3}[j]^2 \xi_{1,2}[i] \xi_{1,3}[i] - \in x_{1,3} \eta_{1,2}[j] \xi_{1,2}[j] \xi_{1,3}[i] - e^{-\xi_{3,3}[i]} x_{1,3} \xi_{1,3}[j] - \\ & \in x_{1,3} \eta_{1,3}[j] \xi_{1,3}[i] \xi_{1,3}[j] - x_{2,2} \xi_{2,2}[i] - x_{2,2} \xi_{2,2}[j] - e^{-\xi_{2,2}[j]} x_{2,3} \xi_{2,3}[i] - \\ & e^{-\xi_{1,1}[i]+\xi_{2,2}[i]+\xi_{3,3}[i]} y_{1,2} \eta_{1,3}[j] \xi_{2,3}[i] - e^{\xi_{3,3}[i]} \in x_{2,2} \eta_{2,3}[j] \xi_{2,3}[i] + \\ & e^{\xi_{3,3}[i]} \in x_{3,3} \eta_{2,3}[j] \xi_{2,3}[i] - e^{-\xi_{1,1}[i]+\xi_{2,2}[i]+\xi_{3,3}[i]} \in y_{1,2} \eta_{1,2}[j] \eta_{2,3}[j] \xi_{2,3}[i] + \\ & e^{-\xi_{1,1}[i]+2\xi_{3,3}[i]} \in y_{1,3} \eta_{1,3}[j] \eta_{2,3}[j] \xi_{2,3}[i] + e^{-\xi_{2,2}[i]+2\xi_{3,3}[i]} \in y_{2,3} \eta_{2,3}[j]^2 \xi_{2,3}[i] - \\ & e^{\xi_{2,2}[i]+\xi_{3,3}[i]} \in x_{1,1} \eta_{1,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] + e^{\xi_{2,2}[i]+\xi_{3,3}[i]} \in x_{2,2} \eta_{1,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] + \\ & 2 e^{-\xi_{1,1}[i]+2\xi_{2,2}[i]+\xi_{3,3}[i]} \in y_{1,2} \eta_{1,2}[j] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] - \\ & e^{2\xi_{3,3}[i]} \in y_{2,3} \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] + x_{1,3} \xi_{1,2}[j] \xi_{2,3}[i] + \\ & e^{-\xi_{2,2}[i]+\xi_{3,3}[i]} \in x_{1,2} \eta_{2,3}[j] \xi_{1,2}[j] \xi_{2,3}[i] - e^{\xi_{3,3}[i]} \in x_{1,2} \eta_{1,3}[j] \xi_{1,2}[i] \xi_{1,2}[j] \xi_{2,3}[i] + \\ & e^{-\xi_{1,1}[i]+\xi_{2,2}[i]+2\xi_{3,3}[i]} \in y_{1,2} \eta_{1,3}[j]^2 \xi_{1,3}[i] \xi_{2,3}[i] - \in x_{1,3} \eta_{2,3}[j] \xi_{1,3}[j] \xi_{2,3}[i] + \\ & e^{-\xi_{1,1}[i]+2\xi_{2,2}[i]+2\xi_{3,3}[i]} \in y_{1,2} \eta_{1,3}[j]^2 \xi_{1,2}[i] \xi_{2,3}[i]^2 - e^{-\xi_{3,3}[i]} x_{2,3} \xi_{2,3}[j] - \\ & \in x_{2,3} \eta_{1,3}[j] \xi_{1,3}[i] \xi_{2,3}[j] - \in x_{2,3} \eta_{2,3}[j] \xi_{2,3}[i] \xi_{2,3}[j] - x_{3,3} \xi_{3,3}[i] - x_{3,3} \xi_{3,3}[j] \end{aligned}$$

gl_{4,ε} Experiments

```

In[ ]:= SetAlgebra[gl4,ε];
$k = 1
$PBWRule
MatrixForm@Table[{b1, b2} → B[b1, b2], {b1, $Basis}, {b2, $Basis}]
MatrixForm@
  Table[{b1, b2} → (DefRep[b1].DefRep[b2] - DefRep[b2].DefRep[b1] == DefRep[B[b1, b2]]),
    {b1, $Basis}, {b2, $Basis}]
DeleteCases[Flatten[Table[{a, b, c} → B[a, B[b, c]] + B[b, B[c, a]] + B[c, B[a, b]],
  {a, $Basis}, {b, $Basis}, {c, $Basis}]], _ → 0]

Out[ ]=
1

Out[ ]=
{y1,2 → 1, y1,3 → 2, y2,3 → 3, y1,4 → 4, y2,4 → 5, y3,4 → 6, x1,1 → 7, x1,2 → 8,
  x2,2 → 9, x1,3 → 10, x2,3 → 11, x3,3 → 12, x1,4 → 13, x2,4 → 14, x3,4 → 15, x4,4 → 16}

Out[ ]//MatrixForm=

$$\begin{pmatrix}
\{y_{1,2}, y_{1,2}\} \rightarrow 0 & \{y_{1,2}, y_{1,3}\} \rightarrow 0 & \{y_{1,2}, y_{2,3}\} \rightarrow -y_{1,3} & \{y_{1,2}, y_{1,4}\} \\
\{y_{1,3}, y_{1,2}\} \rightarrow 0 & \{y_{1,3}, y_{1,3}\} \rightarrow 0 & \{y_{1,3}, y_{2,3}\} \rightarrow 0 & \{y_{1,3}, y_{1,4}\} \\
\{y_{2,3}, y_{1,2}\} \rightarrow y_{1,3} & \{y_{2,3}, y_{1,3}\} \rightarrow 0 & \{y_{2,3}, y_{2,3}\} \rightarrow 0 & \{y_{2,3}, y_{1,4}\} \\
\{y_{1,4}, y_{1,2}\} \rightarrow 0 & \{y_{1,4}, y_{1,3}\} \rightarrow 0 & \{y_{1,4}, y_{2,3}\} \rightarrow 0 & \{y_{1,4}, y_{1,4}\} \\
\{y_{2,4}, y_{1,2}\} \rightarrow y_{1,4} & \{y_{2,4}, y_{1,3}\} \rightarrow 0 & \{y_{2,4}, y_{2,3}\} \rightarrow 0 & \{y_{2,4}, y_{1,4}\} \\
\{y_{3,4}, y_{1,2}\} \rightarrow 0 & \{y_{3,4}, y_{1,3}\} \rightarrow y_{1,4} & \{y_{3,4}, y_{2,3}\} \rightarrow y_{2,4} & \{y_{3,4}, y_{1,4}\} \\
\{x_{1,1}, y_{1,2}\} \rightarrow -y_{1,2} & \{x_{1,1}, y_{1,3}\} \rightarrow -y_{1,3} & \{x_{1,1}, y_{2,3}\} \rightarrow 0 & \{x_{1,1}, y_{1,4}\} \\
\{x_{1,2}, y_{1,2}\} \rightarrow y_{1,1} - x_{2,2} & \{x_{1,2}, y_{1,3}\} \rightarrow -y_{2,3} & \{x_{1,2}, y_{2,3}\} \rightarrow 0 & \{x_{1,2}, y_{1,4}\} \\
\{x_{2,2}, y_{1,2}\} \rightarrow y_{1,2} & \{x_{2,2}, y_{1,3}\} \rightarrow 0 & \{x_{2,2}, y_{2,3}\} \rightarrow -y_{2,3} & \{x_{2,2}, y_{1,4}\} \\
\{x_{1,3}, y_{1,2}\} \rightarrow -x_{2,3} & \{x_{1,3}, y_{1,3}\} \rightarrow y_{1,1} - x_{3,3} & \{x_{1,3}, y_{2,3}\} \rightarrow y_{1,2} & \{x_{1,3}, y_{1,4}\} \\
\{x_{2,3}, y_{1,2}\} \rightarrow 0 & \{x_{2,3}, y_{1,3}\} \rightarrow y_{1,2} & \{x_{2,3}, y_{2,3}\} \rightarrow y_{2,2} - x_{3,3} & \{x_{2,3}, y_{1,4}\} \\
\{x_{3,3}, y_{1,2}\} \rightarrow 0 & \{x_{3,3}, y_{1,3}\} \rightarrow y_{1,3} & \{x_{3,3}, y_{2,3}\} \rightarrow y_{2,3} & \{x_{3,3}, y_{1,4}\} \\
\{x_{1,4}, y_{1,2}\} \rightarrow -x_{2,4} & \{x_{1,4}, y_{1,3}\} \rightarrow -x_{3,4} & \{x_{1,4}, y_{2,3}\} \rightarrow 0 & \{x_{1,4}, y_{1,4}\} \rightarrow y_{1,4} \\
\{x_{2,4}, y_{1,2}\} \rightarrow 0 & \{x_{2,4}, y_{1,3}\} \rightarrow 0 & \{x_{2,4}, y_{2,3}\} \rightarrow -x_{3,4} & \{x_{2,4}, y_{1,4}\} \\
\{x_{3,4}, y_{1,2}\} \rightarrow 0 & \{x_{3,4}, y_{1,3}\} \rightarrow 0 & \{x_{3,4}, y_{2,3}\} \rightarrow 0 & \{x_{3,4}, y_{1,4}\} \\
\{x_{4,4}, y_{1,2}\} \rightarrow 0 & \{x_{4,4}, y_{1,3}\} \rightarrow 0 & \{x_{4,4}, y_{2,3}\} \rightarrow 0 & \{x_{4,4}, y_{1,4}\}
\end{pmatrix}$$


```

Out[]//MatrixForm=

{y _{1,2} , y _{1,2} } → True	{y _{1,2} , y _{1,3} } → True	{y _{1,2} , y _{2,3} } → True	{y _{1,2} , y _{1,4} } → True	{y _{1,2} , y _{2,4} } → True	{
{y _{1,3} , y _{1,2} } → True	{y _{1,3} , y _{1,3} } → True	{y _{1,3} , y _{2,3} } → True	{y _{1,3} , y _{1,4} } → True	{y _{1,3} , y _{2,4} } → True	{
{y _{2,3} , y _{1,2} } → True	{y _{2,3} , y _{1,3} } → True	{y _{2,3} , y _{2,3} } → True	{y _{2,3} , y _{1,4} } → True	{y _{2,3} , y _{2,4} } → True	{
{y _{1,4} , y _{1,2} } → True	{y _{1,4} , y _{1,3} } → True	{y _{1,4} , y _{2,3} } → True	{y _{1,4} , y _{1,4} } → True	{y _{1,4} , y _{2,4} } → True	{
{y _{2,4} , y _{1,2} } → True	{y _{2,4} , y _{1,3} } → True	{y _{2,4} , y _{2,3} } → True	{y _{2,4} , y _{1,4} } → True	{y _{2,4} , y _{2,4} } → True	{
{y _{3,4} , y _{1,2} } → True	{y _{3,4} , y _{1,3} } → True	{y _{3,4} , y _{2,3} } → True	{y _{3,4} , y _{1,4} } → True	{y _{3,4} , y _{2,4} } → True	{
{x _{1,1} , y _{1,2} } → True	{x _{1,1} , y _{1,3} } → True	{x _{1,1} , y _{2,3} } → True	{x _{1,1} , y _{1,4} } → True	{x _{1,1} , y _{2,4} } → True	{
{x _{1,2} , y _{1,2} } → True	{x _{1,2} , y _{1,3} } → True	{x _{1,2} , y _{2,3} } → True	{x _{1,2} , y _{1,4} } → True	{x _{1,2} , y _{2,4} } → True	{
{x _{2,2} , y _{1,2} } → True	{x _{2,2} , y _{1,3} } → True	{x _{2,2} , y _{2,3} } → True	{x _{2,2} , y _{1,4} } → True	{x _{2,2} , y _{2,4} } → True	{
{x _{1,3} , y _{1,2} } → True	{x _{1,3} , y _{1,3} } → True	{x _{1,3} , y _{2,3} } → True	{x _{1,3} , y _{1,4} } → True	{x _{1,3} , y _{2,4} } → True	{
{x _{2,3} , y _{1,2} } → True	{x _{2,3} , y _{1,3} } → True	{x _{2,3} , y _{2,3} } → True	{x _{2,3} , y _{1,4} } → True	{x _{2,3} , y _{2,4} } → True	{
{x _{3,3} , y _{1,2} } → True	{x _{3,3} , y _{1,3} } → True	{x _{3,3} , y _{2,3} } → True	{x _{3,3} , y _{1,4} } → True	{x _{3,3} , y _{2,4} } → True	{
{x _{1,4} , y _{1,2} } → True	{x _{1,4} , y _{1,3} } → True	{x _{1,4} , y _{2,3} } → True	{x _{1,4} , y _{1,4} } → True	{x _{1,4} , y _{2,4} } → True	{
{x _{2,4} , y _{1,2} } → True	{x _{2,4} , y _{1,3} } → True	{x _{2,4} , y _{2,3} } → True	{x _{2,4} , y _{1,4} } → True	{x _{2,4} , y _{2,4} } → True	{
{x _{3,4} , y _{1,2} } → True	{x _{3,4} , y _{1,3} } → True	{x _{3,4} , y _{2,3} } → True	{x _{3,4} , y _{1,4} } → True	{x _{3,4} , y _{2,4} } → True	{
{x _{4,4} , y _{1,2} } → True	{x _{4,4} , y _{1,3} } → True	{x _{4,4} , y _{2,3} } → True	{x _{4,4} , y _{1,4} } → True	{x _{4,4} , y _{2,4} } → True	{

Out[]=

{ }

```

In[*]:= {sol0} = Block[{ $k = 0},
  lhs = λTangent[Join[
    Table[If[SSQ[b1], (b1 /. x → ξ1) b1, λ (b1 /. {x → ξ1, y → η1}) b1], {b1, $Basis}],
    Table[If[SSQ[b2], (b2 /. x → ξ2) b2, λ (b2 /. {x → ξ2, y → η2}) b2], {b2, $Basis}]
  ]];
  rhs = λTangent[
    Table[If[SSQ[b], b /. xα,α ⇒ (ξ1,α + ξ2,α + ε fα,α[λ]) xα,α,
      (b /. {xαβ → fαβ[λ], yαβ → gαβ[λ]}) b], {b, $Basis}]
  ];
  Simplify@DSolve[
    And@@Join[
      (Coefficient[CF[lhs - rhs], #] == 0) & /@ $Basis,
      Select[$Basis, NilQ] /. {xαβ ⇒ fαβ[0] == 0, yαβ ⇒ gαβ[0] == 0}
    ],
    Select[$Basis, NilQ] /. {xαβ → fαβ[λ], yαβ → gαβ[λ]},
    λ
  ]
]

Out[*]= { {f1,2[λ] → e-ξ2,1 λ ξ1,2 + e-ξ1,2 λ ξ2,2,
  f2,3[λ] → e-ξ2,2 λ ξ1,3 + e-ξ1,3 λ ξ2,3, f1,3[λ] → λ (e-ξ2,1 ξ1,3 - λ ξ1,3 ξ2,2 + e-ξ1,3 ξ2,3),
  f3,4[λ] → e-ξ2,3 λ ξ1,4 + e-ξ1,4 λ ξ2,4, f2,4[λ] → λ (e-ξ2,2 ξ1,4 - λ ξ1,4 ξ2,3 + e-ξ1,4 ξ2,4),
  f1,4[λ] → λ (e-ξ2,1 ξ1,4 - λ ξ1,4 ξ2,2 - λ ξ1,3 ξ2,3 + e-ξ1,4 ξ2,4), g1,2[λ] →
  λ (η1,2 + e-ξ1,1+ξ1,2 (η2,2 + eξ1,3 λ η2,3 ξ1,3 + eξ1,4 λ η2,4 (ξ1,4 + eξ1,3 λ ξ1,3 ξ1,4))),
  g1,3[λ] → λ (η1,3 + e-ξ1,1+ξ1,3 (η2,3 + eξ1,4 λ η2,4 ξ1,4)), g2,3[λ] →
  λ (η1,3 + e-ξ1,2+ξ1,3 (η2,3 - eξ1,2 λ η2,3 ξ1,2 + eξ1,4 λ (η2,4 - eξ1,2 λ η2,4 ξ1,2) ξ1,4)),
  g2,4[λ] → λ (η1,4 + eξ1,4 (e-ξ1,2 η2,4 - λ η2,4 ξ1,2)),
  g3,4[λ] → λ (η1,4 + e-ξ1,3+ξ1,4 η2,4 - eξ1,4 λ (η2,4 ξ1,3 + η2,4 ξ1,3)),
  g1,4[λ] → λ (η1,4 + e-ξ1,1+ξ1,4 η2,4) } }

```

```

In[*]:= $k = 1; SBasis = SortBy[$Basis, If[#[[1]] == x, {0, #[[3]] - #[[2]]}, {1, #[[2]] - #[[3]]}] &];
lhs = λTangent[Join[
  Table[If[SSQ[b1], (b1 /. x → ξ1) b1, λ (b1 /. {x → ξ1, y → η1}) b1], {b1, $Basis}],
  Table[If[SSQ[b2], (b2 /. x → ξ2) b2, λ (b2 /. {x → ξ2, y → η2}) b2], {b2, $Basis}]
]];
rhs = λTangent[
  Echo@Table[If[SSQ[b],
    b /. xα,α → (ξ1,α + ξ2,α + e fα,α[λ]) xα,α,
    (b /. {xαβ → fαβ[λ], yαβ → gαβ[λ]}) b /. (sol0 /. ((v- → s-) → (v → e v + s)))
  ], {b, $Basis}]
];
eqns = Transpose@{
  (Coefficient[CF[lhs - rhs], #] == 0) & /@ SBasis,
  SBasis /. {xαβ → fαβ[0] == 0, yαβ → gαβ[0] == 0}
};
unknowns = SBasis /. {xαβ → fαβ[λ], yαβ → gαβ[λ]};
(*{sol1}=DSolve[Flatten@eqns, unknowns, λ]*)
sol1 = TriangularDSolve[eqns, unknowns, λ];
» {y1,2 (λ (η1,2 + e-ξ1,1+ξ1,2 (η2,2 + eξ1,3 λ η2,3 ξ1,3 + eξ1,4 λ η2,4 (ξ1,2 + eξ1,3 λ ξ1,3 ξ1,4))) + e g1,2[λ]),
  y1,3 (λ (η1,3 + e-ξ1,1+ξ1,3 (η2,3 + eξ1,4 λ η2,4 ξ1,4))) + e g1,3[λ]),
  y2,3 (λ (η2,3 + e-ξ1,2+ξ1,3 (η2,3 - eξ1,2 λ η2,3 ξ1,2 + eξ1,4 λ (η2,4 - eξ1,2 λ η2,4 ξ1,2) ξ1,3))) + e g2,3[λ]),
  y1,4 (λ (η1,4 + e-ξ1,1+ξ1,4 η2,4)) + e g1,4[λ]),
  y2,4 (λ (η2,4 + eξ1,4 (e-ξ1,2 η2,4 - λ η2,4 ξ1,2))) + e g2,4[λ]),
  y3,4 (λ (η3,4 + e-ξ1,3+ξ1,4 η2,4 - eξ1,4 λ (η2,4 ξ1,3 + η2,4 ξ1,2))) + e g3,4[λ]),
  x1,1 (ξ1,1 + ξ2,1 + e f1,1[λ]), x1,2 (e-ξ2,1 λ ξ1,2 + e-ξ1,2 λ ξ2,2 + e f1,2[λ]),
  x2,2 (ξ1,2 + ξ2,2 + e f2,2[λ]), x1,3 (λ (e-ξ2,1 ξ1,3 - λ ξ1,2 ξ2,3 + e-ξ1,3 ξ2,3)) + e f1,3[λ]),
  x2,3 (e-ξ2,2 λ ξ1,3 + e-ξ1,3 λ ξ2,3 + e f2,3[λ]), x3,3 (ξ1,3 + ξ2,3 + e f3,3[λ]),
  x1,4 (λ (e-ξ2,1 ξ1,4 - λ ξ1,2 ξ2,4 - λ ξ1,3 ξ2,3 + e-ξ1,4 ξ2,4)) + e f1,4[λ]),
  x2,4 (λ (e-ξ2,2 ξ1,4 - λ ξ1,3 ξ2,4 + e-ξ1,4 ξ2,4)) + e f2,4[λ]),
  x3,4 (e-ξ2,3 λ ξ1,4 + e-ξ1,4 λ ξ2,4 + e f3,4[λ]), x4,4 (ξ1,4 + ξ2,4 + e f4,4[λ])
},
» {f1,1[λ] → eξ1,2 λ2 η2,2 ξ1,2 + eξ1,3 λ2 η2,3 ξ1,3 + eξ1,4 λ2 η2,4 ξ1,4 + eξ1,2+ξ1,3 λ3 η2,3 ξ1,2 ξ1,3 +
  eξ1,2+ξ1,4 λ3 η2,4 ξ1,2 ξ1,4 + eξ1,3+ξ1,4 λ3 η2,4 ξ1,3 ξ1,4 + eξ1,2+ξ1,3+ξ1,4 λ4 η2,4 ξ1,2 ξ1,3 ξ1,4
},
» {f2,2[λ] → -eξ1,2 λ2 η2,2 ξ1,2 + eξ1,3 λ2 η2,3 ξ1,2 - eξ1,2+ξ1,3 λ3 η2,3 ξ1,2 ξ1,3 + eξ1,4 λ2 η2,4 ξ1,2 -
  eξ1,2+ξ1,4 λ3 η2,4 ξ1,2 ξ1,4 + eξ1,3+ξ1,4 λ3 η2,4 ξ1,3 ξ1,4 - eξ1,2+ξ1,3+ξ1,4 λ4 η2,4 ξ1,2 ξ1,3 ξ1,4
},
» {f3,3[λ] → -eξ1,3 λ2 η2,3 ξ1,3 - eξ1,3 λ2 η2,3 ξ1,2 +
  eξ1,4 λ2 η2,4 ξ1,3 - eξ1,3+ξ1,4 λ3 η2,4 ξ1,3 ξ1,4 - eξ1,3+ξ1,4 λ3 η2,4 ξ1,2 ξ1,3
},
» {f4,4[λ] → -eξ1,4 λ2 η2,4 ξ1,4 - eξ1,4 λ2 η2,4 ξ1,2 - eξ1,4 λ2 η2,4 ξ1,3
},
» {f1,2[λ] → e-ξ1,2+ξ1,3-ξ2,1 λ2 η2,3 ξ1,3 - eξ1,3-ξ2,1 λ3 η2,3 ξ1,2 ξ1,3 +
  e-ξ1,2+ξ1,4-ξ2,1 λ2 η2,4 ξ1,4 - eξ1,4-ξ2,1 λ3 η2,4 ξ1,2 ξ1,4 + e-ξ1,2+ξ1,3+ξ1,4-ξ2,1 λ3 η2,4 ξ1,3 ξ1,4 -
  eξ1,3+ξ1,4-ξ2,1 λ4 η2,4 ξ1,2 ξ1,3 ξ1,4 + λ3 η2,2 ξ1,2 ξ2,2 - e-ξ1,2+ξ1,3 λ3 η2,3 ξ1,2 ξ2,2 +
  eξ1,3 λ4 η2,3 ξ1,2 ξ2,2 - e-ξ1,2+ξ1,4 λ3 η2,4 ξ1,2 ξ2,2 + eξ1,4 λ4 η2,4 ξ1,2 ξ2,2 -
  e-ξ1,2+ξ1,3+ξ1,4 λ4 η2,4 ξ1,2 ξ2,2 + eξ1,3+ξ1,4 λ5 η2,4 ξ1,2 ξ2,2 ξ1,3 ξ1,4 ξ2,2
}

```

[illegible]

$$\begin{aligned}
& \gg \{g_{2,3}[\lambda] \rightarrow e^{\xi_{1,3}} \lambda^3 \eta_{2,1,2} \eta_{2,2,3} \xi_{1,1,2} - e^{-\xi_{1,2,2} + 2\xi_{1,3}} \lambda^3 \eta_{2,1,3} \eta_{2,2,3} \xi_{1,1,3} + \\
& \quad e^{2\xi_{1,3}} \lambda^4 \eta_{2,1,3}^2 \xi_{1,1,2} \xi_{1,1,3} - e^{-\xi_{1,2,2} + \xi_{1,3} + \xi_{1,4,4}} \lambda^3 \eta_{2,1,3} \eta_{2,2,4} \xi_{1,1,4} + e^{\xi_{1,3} + \xi_{1,4,4}} \lambda^4 \eta_{2,1,3} \eta_{2,1,4} \xi_{1,1,2} \xi_{1,1,4} - \\
& \quad e^{-\xi_{1,2,2} + 2\xi_{1,3}} \lambda^3 \eta_{2,2,3}^2 \xi_{1,2,3} + e^{2\xi_{1,3}} \lambda^4 \eta_{2,1,3} \eta_{2,2,3} \xi_{1,1,2} \xi_{1,2,3} - e^{-\xi_{1,2,2} + \xi_{1,3} + \xi_{1,4,4}} \lambda^3 \eta_{2,2,3} \eta_{2,2,4} \xi_{1,2,4} + \\
& \quad e^{\xi_{1,3} + \xi_{1,4,4}} \lambda^4 \eta_{2,1,4} \eta_{2,2,3} \xi_{1,1,2} \xi_{1,2,4} + e^{\xi_{1,3} + \xi_{1,4,4}} \lambda^4 \eta_{2,1,2} \eta_{2,2,4} \xi_{1,1,2} \xi_{1,3,4} - \\
& \quad e^{-\xi_{1,2,2} + 2\xi_{1,3} + \xi_{1,4,4}} \lambda^4 \eta_{2,1,4} \eta_{2,2,3} \xi_{1,1,3} \xi_{1,3,4} - e^{-\xi_{1,2,2} + 2\xi_{1,3} + \xi_{1,4,4}} \lambda^4 \eta_{2,1,3} \eta_{2,2,4} \xi_{1,1,3} \xi_{1,3,4} + \\
& \quad 2e^{2\xi_{1,3} + \xi_{1,4,4}} \lambda^5 \eta_{2,1,3} \eta_{2,1,4} \xi_{1,1,2} \xi_{1,1,3} \xi_{1,3,4} - e^{-\xi_{1,2,2} + \xi_{1,3} + 2\xi_{1,4,4}} \lambda^4 \eta_{2,1,4} \eta_{2,2,4} \xi_{1,1,4} \xi_{1,3,4} + \\
& \quad e^{\xi_{1,3} + 2\xi_{1,4,4}} \lambda^5 \eta_{2,1,4}^2 \xi_{1,1,2} \xi_{1,1,4} \xi_{1,3,4} - 2e^{-\xi_{1,2,2} + 2\xi_{1,3} + \xi_{1,4,4}} \lambda^4 \eta_{2,2,3} \eta_{2,2,4} \xi_{1,2,3} \xi_{1,3,4} + \\
& \quad e^{2\xi_{1,3} + \xi_{1,4,4}} \lambda^5 \eta_{2,1,4} \eta_{2,2,3} \xi_{1,1,2} \xi_{1,2,3} \xi_{1,3,4} + e^{2\xi_{1,3} + \xi_{1,4,4}} \lambda^5 \eta_{2,1,3} \eta_{2,2,4} \xi_{1,1,2} \xi_{1,2,3} \xi_{1,3,4} - \\
& \quad e^{-\xi_{1,2,2} + \xi_{1,3} + 2\xi_{1,4,4}} \lambda^4 \eta_{2,2,4}^2 \xi_{1,2,4} \xi_{1,3,4} + e^{\xi_{1,3} + 2\xi_{1,4,4}} \lambda^5 \eta_{2,1,4} \eta_{2,2,4} \xi_{1,1,2} \xi_{1,2,4} \xi_{1,3,4} - \\
& \quad e^{-\xi_{1,2,2} + 2\xi_{1,3} + 2\xi_{1,4,4}} \lambda^5 \eta_{2,1,4} \eta_{2,2,4} \xi_{1,1,3} \xi_{1,3,4}^2 + e^{2\xi_{1,3} + 2\xi_{1,4,4}} \lambda^6 \eta_{2,1,4}^2 \xi_{1,1,2} \xi_{1,1,3} \xi_{1,3,4}^2 - \\
& \quad e^{-\xi_{1,2,2} + 2\xi_{1,3} + 2\xi_{1,4,4}} \lambda^5 \eta_{2,2,4}^2 \xi_{1,2,3} \xi_{1,3,4}^2 + e^{2\xi_{1,3} + 2\xi_{1,4,4}} \lambda^6 \eta_{2,1,4} \eta_{2,2,4} \xi_{1,1,2} \xi_{1,2,3} \xi_{1,3,4}^2\} \\
& \gg \{g_{3,4}[\lambda] \rightarrow e^{\xi_{1,4,4}} \lambda^3 \eta_{2,1,2} \eta_{2,2,4} \xi_{1,1,3} + e^{\xi_{1,4,4}} \lambda^3 \eta_{2,1,3} \eta_{2,3,4} \xi_{1,1,3} - e^{-\xi_{1,3} + 2\xi_{1,4,4}} \lambda^3 \eta_{2,1,4} \eta_{2,3,4} \xi_{1,1,4} + \\
& \quad e^{2\xi_{1,4,4}} \lambda^4 \eta_{2,1,4}^2 \xi_{1,1,3} \xi_{1,1,4} + e^{\xi_{1,4,4}} \lambda^3 \eta_{2,2,3} \eta_{2,3,4} \xi_{1,2,3} + e^{2\xi_{1,4,4}} \lambda^4 \eta_{2,1,4} \eta_{2,2,4} \xi_{1,1,4} \xi_{1,2,3} - \\
& \quad e^{-\xi_{1,3} + 2\xi_{1,4,4}} \lambda^3 \eta_{2,2,4} \eta_{2,3,4} \xi_{1,2,4} + e^{2\xi_{1,4,4}} \lambda^4 \eta_{2,1,4} \eta_{2,2,4} \xi_{1,1,3} \xi_{1,2,4} + e^{2\xi_{1,4,4}} \lambda^4 \eta_{2,2,4}^2 \xi_{1,2,3} \xi_{1,2,4} - \\
& \quad e^{-\xi_{1,3} + 2\xi_{1,4,4}} \lambda^3 \eta_{2,3,4}^2 \xi_{1,3,4} + e^{2\xi_{1,4,4}} \lambda^4 \eta_{2,1,4} \eta_{2,3,4} \xi_{1,1,3} \xi_{1,3,4} + e^{2\xi_{1,4,4}} \lambda^4 \eta_{2,2,4} \eta_{2,3,4} \xi_{1,2,3} \xi_{1,3,4}\}
\end{aligned}$$

```

In[ ]:= cm[i_, j_ -> k_] := E[{i,j} -> {k}] [CF@Sum[If[SSQ[b],
    b /. x_{\alpha,\alpha} -> (\xi_{1,\alpha,\alpha} + \xi_{2,\alpha,\alpha}) x_{\alpha,\alpha}[k],
    b[k] (b /. {x_{\alpha\beta} -> f_{\alpha\beta}[\lambda], y_{\alpha\beta} -> g_{\alpha\beta}[\lambda]} /. sol0)
], {b, $Basis}],
    CF@Sum[b[k] (b /. {x_{\alpha\beta} -> f_{\alpha\beta}[\lambda], y_{\alpha\beta} -> g_{\alpha\beta}[\lambda]} /. sol1), {b, $Basis}]
] /. {\xi_{1,\alpha\beta} -> \xi_{\alpha\beta}[i], \eta_{1,\alpha\beta} -> \eta_{\alpha\beta}[i], \xi_{2,\alpha\beta} -> \xi_{\alpha\beta}[j], \eta_{2,\alpha\beta} -> \eta_{\alpha\beta}[j], \lambda -> 1};
Short[cm[i, j -> k], 10]

```

Out[]//Short=

$$\begin{aligned}
& E_{\{i,j\} \rightarrow \{k\}} [y_{1,2}[k] \eta_{1,2}[i] + e^{-\xi_{1,1}[i] + \xi_{2,2}[i]} y_{1,2}[k] \eta_{1,2}[j] + y_{1,3}[k] \eta_{1,3}[i] + \\
& \quad e^{-\xi_{1,1}[i] + \xi_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j] + y_{1,4}[k] \eta_{1,4}[i] + e^{-\xi_{1,1}[i] + \xi_{4,4}[i]} y_{1,4}[k] \eta_{1,4}[j] + y_{2,3}[k] \eta_{2,3}[i] + \\
& \quad e^{-\xi_{2,2}[i] + \xi_{3,3}[i]} y_{2,3}[k] \eta_{2,3}[j] + y_{2,4}[k] \eta_{2,4}[i] + e^{-\xi_{2,2}[i] + \xi_{4,4}[i]} y_{2,4}[k] \eta_{2,4}[j] + \\
& \quad y_{3,4}[k] \eta_{3,4}[i] + e^{-\xi_{3,3}[i] + \xi_{4,4}[i]} y_{3,4}[k] \eta_{3,4}[j] + x_{1,1}[k] \xi_{1,1}[i] + x_{1,1}[k] \xi_{1,1}[j] + \\
& \quad e^{-\xi_{1,1}[j]} x_{1,2}[k] \xi_{1,2}[i] - e^{\xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j] \xi_{1,2}[i] - e^{\xi_{4,4}[i]} y_{2,4}[k] \eta_{1,4}[j] \xi_{1,2}[i] + \\
& \quad e^{-\xi_{2,2}[i]} x_{1,2}[k] \xi_{1,2}[j] + e^{-\xi_{1,1}[j]} x_{1,3}[k] \xi_{1,3}[i] - e^{\xi_{4,4}[i]} y_{3,4}[k] \eta_{1,4}[j] \xi_{1,3}[i] + \ll 11 \gg + \\
& \quad e^{-\xi_{3,3}[i]} x_{2,3}[k] \xi_{2,3}[j] + e^{-\xi_{2,2}[j]} x_{2,4}[k] \xi_{2,4}[i] + e^{-\xi_{1,1}[i] + \xi_{2,2}[i] + \xi_{4,4}[i]} y_{1,2}[k] \eta_{1,4}[j] \xi_{2,4}[i] - \\
& \quad x_{1,4}[k] \xi_{1,2}[j] \xi_{2,4}[i] + e^{-\xi_{4,4}[i]} x_{2,4}[k] \xi_{2,4}[j] + x_{3,3}[k] \xi_{3,3}[i] + x_{3,3}[k] \xi_{3,3}[j] + \\
& \quad e^{-\xi_{3,3}[j]} x_{3,4}[k] \xi_{3,4}[i] + e^{-\xi_{1,1}[i] + \xi_{3,3}[i] + \xi_{4,4}[i]} y_{1,3}[k] \eta_{1,4}[j] \xi_{3,4}[i] + \\
& \quad e^{-\xi_{2,2}[i] + \xi_{3,3}[i] + \xi_{4,4}[i]} y_{2,3}[k] \eta_{2,4}[j] \xi_{3,4}[i] - e^{\xi_{3,3}[i] + \xi_{4,4}[i]} y_{2,3}[k] \eta_{1,4}[j] \xi_{1,2}[i] \xi_{3,4}[i] - \\
& \quad x_{1,4}[k] \xi_{1,3}[j] \xi_{3,4}[i] + e^{-\xi_{1,1}[i] + \xi_{2,2}[i] + \xi_{3,3}[i] + \xi_{4,4}[i]} y_{1,2}[k] \eta_{1,4}[j] \xi_{2,3}[i] \xi_{3,4}[i] - \\
& \quad x_{2,4}[k] \xi_{2,3}[j] \xi_{3,4}[i] + e^{-\xi_{4,4}[i]} x_{3,4}[k] \xi_{3,4}[j] + x_{4,4}[k] \xi_{4,4}[i] + x_{4,4}[k] \xi_{4,4}[j], \\
& \quad e^{-\xi_{1,1}[i] + \xi_{2,2}[i]} y_{1,3}[k] \eta_{1,2}[j] \eta_{2,3}[i] + \ll 264 \gg + x_{3,4}[k] \eta_{3,4}[j] \xi_{3,4}[i] \xi_{3,4}[j]]
\end{aligned}$$

In[*]:= Short[cm[1, 2 → 2] // cm[2, 3 → 1], 10]

Out[*]//Short=

$$\begin{aligned}
& E_{\{1,2,3\} \rightarrow \{1\}} \left[y_{1,2}[1] \eta_{1,2}[1] + e^{-\xi_{1,1}[1] + \xi_{2,2}[1]} y_{1,2}[1] \eta_{1,2}[2] + \right. \\
& e^{-\xi_{1,1}[1] - \xi_{1,1}[2] + \xi_{2,2}[1] + \xi_{2,2}[2]} y_{1,2}[1] \eta_{1,2}[3] + y_{1,3}[1] \eta_{1,3}[1] + e^{-\xi_{1,1}[1] + \xi_{3,3}[1]} y_{1,3}[1] \eta_{1,3}[2] + \\
& e^{-\xi_{1,1}[1] - \xi_{1,1}[2] + \xi_{3,3}[1] + \xi_{3,3}[2]} y_{1,3}[1] \eta_{1,3}[3] + y_{1,4}[1] \eta_{1,4}[1] + e^{-\xi_{1,1}[1] + \xi_{4,4}[1]} y_{1,4}[1] \eta_{1,4}[2] + \\
& e^{-\xi_{1,1}[1] - \xi_{1,1}[2] + \xi_{4,4}[1] + \xi_{4,4}[2]} y_{1,4}[1] \eta_{1,4}[3] + y_{2,3}[1] \eta_{2,3}[1] + e^{-\xi_{2,2}[1] + \xi_{3,3}[1]} y_{2,3}[1] \eta_{2,3}[2] + \\
& e^{-\xi_{2,2}[1] - \xi_{2,2}[2] + \xi_{3,3}[1] + \xi_{3,3}[2]} y_{2,3}[1] \eta_{2,3}[3] + y_{2,4}[1] \eta_{2,4}[1] + e^{-\xi_{2,2}[1] + \xi_{4,4}[1]} y_{2,4}[1] \eta_{2,4}[2] + \\
& e^{-\xi_{2,2}[1] - \xi_{2,2}[2] + \xi_{4,4}[1] + \xi_{4,4}[2]} y_{2,4}[1] \eta_{2,4}[3] + e^{-\xi_{1,1}[1] - \xi_{1,1}[2] + \xi_{3,3}[1] + \xi_{3,3}[2] + \xi_{4,4}[1] + \xi_{4,4}[2]} y_{1,3}[1] \eta_{1,4}[3] \xi_{3,4}[2] + \\
& e^{-\xi_{2,2}[1] - \xi_{2,2}[2] + \xi_{3,3}[1] + \xi_{3,3}[2] + \xi_{4,4}[1] + \xi_{4,4}[2]} y_{2,3}[1] \eta_{2,4}[3] \xi_{3,4}[2] - \\
& e^{-\xi_{1,1}[2] + \xi_{3,3}[1] + \xi_{3,3}[2] + \xi_{4,4}[2]} y_{2,3}[1] \eta_{1,4}[3] \xi_{1,2}[1] \xi_{3,4}[2] - \\
& e^{-\xi_{2,2}[1] + \xi_{3,3}[1] + \xi_{3,3}[2] + \xi_{4,4}[2]} y_{2,3}[1] \eta_{1,4}[3] \xi_{1,2}[2] \xi_{3,4}[2] - e^{-\xi_{4,4}[1]} x_{1,4}[1] \xi_{1,3}[3] \xi_{3,4}[2] + \\
& e^{-\xi_{1,1}[1] - \xi_{1,1}[2] + \xi_{2,2}[1] + \xi_{3,3}[1] + \xi_{3,3}[2] + \xi_{4,4}[2]} y_{1,2}[1] \eta_{1,4}[3] \xi_{2,3}[1] \xi_{3,4}[2] + \\
& e^{-\xi_{1,1}[1] - \xi_{1,1}[2] + \xi_{2,2}[1] + \xi_{2,2}[2] + \xi_{3,3}[2] + \xi_{4,4}[2]} y_{1,2}[1] \eta_{1,4}[3] \xi_{2,3}[2] \xi_{3,4}[2] - \\
& e^{-\xi_{4,4}[1]} x_{2,4}[1] \xi_{2,3}[3] \xi_{3,4}[2] + e^{-\xi_{4,4}[1] - \xi_{4,4}[2]} x_{3,4}[1] \xi_{3,4}[3] + \\
& x_{4,4}[1] \xi_{4,4}[1] + x_{4,4}[1] \xi_{4,4}[2] + x_{4,4}[1] \xi_{4,4}[3], \\
& \left. e^{-\xi_{1,1}[1] + \xi_{2,2}[1]} y_{1,3}[1] \eta_{1,2}[2] \eta_{2,3}[1] + \ll 1981 \gg + e^{-\ll 1 \gg [1]} \ll 3 \gg \xi_{\ll 1 \gg [3]} \right]
\end{aligned}$$

In[*]:= (cm[1, 2 → 2] // cm[2, 3 → 1]) ≡ (cm[2, 3 → 2] // cm[1, 2 → 1])

Out[*]=

True