

Pensieve header: The PBW multiplication tensor for $\mathfrak{sl}_n$ using the λ -tangent formalism. Some material from pensieve://Projects/UEA.

```
In[ ]:= SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
```

Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at <http://katlas.org/wiki/KnotTheory>.

Prolog

```
In[ ]:= BeginPackage["UEA`"];
Print["UEA` does computations in general universal enveloping
algebras and PBW algebras. It is in the public domain, available
at http://drorbn.net/AcademicPensieve/Projects/SolvablePBW/.
Dror Bar-Natan is committed to support it within
reason until June 1, 2027. This is version 260601."];
Print["UEA` implements / extends ",
Sort@{"**",  $\epsilon$ , $k, CF, SSQ, NilQ, B, m, SetAlgebra, U,
UB, UProducts, USimp, UU, $Basis, $PBWRule,  $\delta$ , adPower, adExp},
"."];
Begin["`Private`"];
```

UEA` does computations in general universal enveloping algebras and PBW algebras. It is in the public domain, available at <http://drorbn.net/AcademicPensieve/Projects/SolvablePBW/>. Dror Bar-Natan is committed to support it within reason until June 1, 2027. This is version 260601.

UEA` implements / extends {**, adExp, adPower, B, CF, m, NilQ, SetAlgebra, SSQ, U, UB, UProducts, USimp, UU, δ , ϵ , \$Basis, \$k, \$PBWRule}.

Utilities

```
In[ ]:=  $\delta_{i,j}$  := If[i == j, 1, 0];
```

```
In[ ]:= SSQ[c_.*x_] /; MemberQ[$Basis, x] :=  $\neg$  NilQ[x]; (* Semi-Simple Q *)
NilQ[c_.*x_] /; MemberQ[$Basis, x] := NilQ[x];
SSQ[x_Plus] := And @@ (SSQ /@ (List @@ x));
NilQ[x_Plus] := And @@ (NilQ /@ (List @@ x));
```

```
In[ ]:= $k = 1;
```

```
In[ ]:= CF[ $\mathcal{E}$ _] := Expand[ $\mathcal{E}$  /.  $e^L \rightarrow e^{\text{Expand}[L]}$ ] /. { $e^{k\cdot}$  /;  $k > \$k \rightarrow 0$ };
```

Implementing general universal enveloping algebras

```

In[ ]:= B[0, _] = 0; B[_ , 0] = 0;
B[c_ * x_, y_] /; MemberQ[$Basis, x] := CF[c B[x, y]];
B[y_, c_ * x_] /; MemberQ[$Basis, x] := CF[c B[y, x]];
B[x_Plus, y_] := B[#, y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
B[x_, x_] = 0;
B[y_, x_] := CF[-B[x, y]];

In[ ]:= x_ ≤ y_ := OrderedQ[{x, y} /. $PBWRule]; x_ < y_ := ! OrderedQ[{y, x} /. $PBWRule];
UU_i_[1] := U_i[];
UU_i_[x_P_] := UU_i_@@Table[x, {p}];
UU_i_[ε_] := ε /. {
  U[xs_] => U_i[xs],
  x_ /; MemberQ[$Basis, x] => U_i[x]
};
UU_i_[x_, xs_] := UU_t1[x] UU_t2[xs] // Expand // m_{t1,t2→i};
USimp[ε_] := Collect[ε, Times[U_[] ..], Expand];
USimp[ε_] := Expand[ε];

In[ ]:= m_s_[0] = 0;
m_s_[x_Plus] := m_s_ /@ x;
m_s_[sd_SeriesData] := MapAt[m_s, sd, {3, All}];
m_{i→j}[ε_] := ε /. U_i → U_j;

In[ ]:= m_{i,j→k}[c_. U_i[x_] U_j[]] := c U_k[x];
m_{i,j→k}[c_. U_i[] U_j[y_]] := c U_k[y];
m_{i,j→k}[c_. U_i[xx_, x_] U_j[y_, yy_]] := If[x ≤ y,
  c U_k[xx, x, y, yy],
  ((U_i[xx] (U_j[y, x] + UU_j[B[x, y]])) // Expand // m_{i,j→i} U_j[yy] // Expand // m_{i,j→k}
  c // USimp
];

In[ ]:= UProducts[{}, 0] = {1}; UProducts[{}, d_Integer] /; d > 0 = {};
UProducts[{i_, is_}, d_Integer] := Sort@
  Flatten@Table[(U_i@@@Subsets[$Basis, {j}]) u, {j, 0, d}, {u, UProducts[{is, d-j}]}];

In[ ]:= Supp[ε_] := Union@Cases[{ε}, U_i[___] => i, ∞];

```

```
In[*]:= Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ⋈ y_ := Module[{is = Supp[x] ∩ Supp[y], σ, z},
  z = x; Do[z = mi→σ@i[z], {i, is}];
  z = Expand[y z]; Do[z = mσ@i, i→i[z], {i, is}]; z];
UB[x_, y_] := USimp[x ⋈ y - y ⋈ x];
```

Epilog

```
In[*]:= End[]; EndPackage[];
```

Predefined Algebras

sl(2)

```
In[*]:= Print["UEA`SetAlgebra knows \"sl2\"."];
```

UEA`SetAlgebra knows "sl2".

```
In[*]:= SetAlgebra["sl2"] := (
  Print["In sl2: ⟨e,h,f⟩ / ([h,e]=2e, [h,f]=-2f, [e,f]=h)."];
  B[h, e] = 2 e; B[h, f] = -2 f; B[e, f] = h;
  $Algebra = "sl2";
  $Basis = {e, h, f};
  $PBWRule = {e → 1, h → 2, f → 3};
  NilQ[e] = NilQ[f] = True; NilQ[h] = False;
);
```

$gl_{n,\epsilon}$

```
In[*]:= Print["UEA`SetAlgebra knows the ε-nilpotent algebra gln,ε."];
```

UEA`SetAlgebra knows the ϵ -nilpotent algebra $gl_{n,\epsilon}$.

```

In[ ]:= SetAlgebra[gln,ε] := (
  $Algebra = gln,ε;
  $Basis = Flatten@{
    Table[yα,β, {β, 2, n}, {α, 1, β - 1}],
    Table[xα,β, {β, 1, n}, {α, 1, β}]
  };
  NilQ[y_] = True; NilQ[xα,β] := (α != β); (* Nilpotent Q *)
  $PBWRule = Thread[$Basis → Range@Length@$Basis];
  B[xi,j, xk,l] := δj,k xi,l - δl,i xk,j;
  B[yi,j, yk,l] := CF[ε δj,k yi,l - ε δl,i yk,j];
  B[xi,j, yk,l] := CF[δj,k xi,l - δl,i xk,j /. xα,β → If[α ≤ β, ε xα,β, yβ,α]];
);

```

```

SetAlgebra[gl2,ε];
$PBWRule
MatrixForm@Table[{b1, b2} → B[b1, b2], {b1, $Basis}, {b2, $Basis}]

```

Out[]:=

{y_{1,2} → 1, y_{1,3} → 2, y_{2,3} → 3, x_{1,1} → 4, x_{1,2} → 5, x_{2,2} → 6, x_{1,3} → 7, x_{2,3} → 8, x_{3,3} → 9}

Out[]//MatrixForm=

{y _{1,2} , y _{1,2} } → 0	{y _{1,2} , y _{1,3} } → 0	{y _{1,2} , y _{2,3} } → 0	{y _{1,2} , x _{1,1} } → y _{1,2}	{y _{1,2} , x _{1,2} } → 0
{y _{1,3} , y _{1,2} } → 0	{y _{1,3} , y _{1,3} } → 0	{y _{1,3} , y _{2,3} } → 0	{y _{1,3} , x _{1,1} } → y _{1,3}	{y _{1,3} , x _{1,2} } → y _{2,3}
{y _{2,3} , y _{1,2} } → 0	{y _{2,3} , y _{1,3} } → 0	{y _{2,3} , y _{2,3} } → 0	{y _{2,3} , x _{1,1} } → 0	{y _{2,3} , x _{1,2} } → 0
{x _{1,1} , y _{1,2} } → -y _{1,2}	{x _{1,1} , y _{1,3} } → -y _{1,3}	{x _{1,1} , y _{2,3} } → 0	{x _{1,1} , x _{1,1} } → 0	{x _{1,1} , x _{1,2} } → x _{1,2}
{x _{1,2} , y _{1,2} } → 0	{x _{1,2} , y _{1,3} } → -y _{2,3}	{x _{1,2} , y _{2,3} } → 0	{x _{1,2} , x _{1,1} } → -x _{1,2}	{x _{1,2} , x _{1,2} } → 0
{x _{2,2} , y _{1,2} } → y _{1,2}	{x _{2,2} , y _{1,3} } → 0	{x _{2,2} , y _{2,3} } → -y _{2,3}	{x _{2,2} , x _{1,1} } → 0	{x _{2,2} , x _{1,2} } → -x _{1,2}
{x _{1,3} , y _{1,2} } → 0	{x _{1,3} , y _{1,3} } → 0	{x _{1,3} , y _{2,3} } → 0	{x _{1,3} , x _{1,1} } → -x _{1,3}	{x _{1,3} , x _{1,2} } → 0
{x _{2,3} , y _{1,2} } → 0	{x _{2,3} , y _{1,3} } → y _{1,2}	{x _{2,3} , y _{2,3} } → 0	{x _{2,3} , x _{1,1} } → 0	{x _{2,3} , x _{1,2} } → -x _{1,3}
{x _{3,3} , y _{1,2} } → 0	{x _{3,3} , y _{1,3} } → y _{1,3}	{x _{3,3} , y _{2,3} } → y _{2,3}	{x _{3,3} , x _{1,1} } → 0	{x _{3,3} , x _{1,2} } → 0

```

In[ ]:= adPower[x_, k_, Env_] [c_ * y_] /; MemberQ[$Basis, y] := CF[c adPower[x, k, Env] [y]];
adPower[x_, k_, Env_] [y_Plus] := adPower[x, k, Env] /@ y;
adPower[_ , 0, Env_] [y_] := y;
adPower[x_, k_, Env_] [0] = 0;
adPower[x_, k_, Env_] [y_] /; MemberQ[$Basis, y] :=
  adPower[x, k, Env] [y] = B[adPower[x, k - 1, Env] [y], x];
adPower[x_, k_] [y_] := adPower[x, k, {$Algebra, $k}] [y];

```

```

In[ ]:= x0 = ($Basis /. {x → ξ0, y → η0}).$Basis
x1 = ($Basis /. {x → ξ1, y → η1}).$Basis

```

Out[]:=

y_{1,2} η_{0,2} + x_{1,1} ξ_{0,1} + x_{1,2} ξ_{0,2} + x_{2,2} ξ_{0,2}

Out[]:=

y_{1,2} η_{1,2} + x_{1,1} ξ_{1,1} + x_{1,2} ξ_{1,2} + x_{2,2} ξ_{1,2}

```
In[*]:= adExp[x_][y_] /; SSQ[x] := Expand@Module[{A, b, c},
  A = Table[Coefficient[B[b, x], c], {c, $Basis}, {b, $Basis}];
  CF[$Basis.Normal[Series[MatrixExp[A], {e, 0, $k}]]].
  Table[Coefficient[y, b], {b, $Basis}]]
]
```

```
In[*]:= SSQ[ξ0,1 x1,1]
```

```
Out[*]=
```

True

```
In[*]:= adExp[ξ0,1 x1,1][x1]
```

```
Out[*]=
```

$$e^{\xi_{0,1}} y_{1,2} \eta_{1,2} + x_{1,1} \xi_{1,1} + e^{-\xi_{0,1}} x_{1,2} \xi_{1,2} + x_{2,2} \xi_{2,2}$$

```
In[*]:= adExp[x_][y_] /; NilQ[x] := Expand@Module[{k = 0, s = 0},
  While[adPower[x, k][y] != 0,
    s += adPower[x, k][y] / k!;
    ++k
  ];
  s
]
```

```
In[*]:= adExp[ξ0,2 x1,2][x1]
```

```
Out[*]=
```

$$y_{1,2} \eta_{1,2} - e x_{1,1} \eta_{1,2} \xi_{0,2} + e x_{2,2} \eta_{1,2} \xi_{0,2} - e x_{1,2} \eta_{1,2} \xi_{0,2}^2 + \\ x_{1,1} \xi_{1,1} + x_{1,2} \xi_{0,2} \xi_{1,1} + x_{1,2} \xi_{1,2} + x_{2,2} \xi_{2,2} - x_{1,2} \xi_{0,2} \xi_{1,2}$$

```
In[*]:= λTangent[] = 0;
λTangent[xs___, x_] := adExp[x][λTangent[xs]] + ∂λx;
λTangent[xs_List] := λTangent@@xs
```

```
In[*]:= $Basis ($Basis /. {x → ξ, y → η})
```

```
Out[*]=
```

$$\{y_{1,2} \eta_{1,2}, x_{1,1} \xi_{1,1}, x_{1,2} \xi_{1,2}, x_{2,2} \xi_{2,2}\}$$

```
In[*]:= λTangent@ (λ $Basis ($Basis /. {x → ξ, y → η}))
```

```
Out[*]=
```

$$e^{\lambda \xi_{1,1} - \lambda \xi_{2,2}} y_{1,2} \eta_{1,2} + x_{1,1} \xi_{1,1} + e^{\lambda \xi_{2,2}} x_{1,2} \xi_{1,2} - e^{\lambda \xi_{1,1}} e^{\lambda} x_{1,1} \eta_{1,2} \xi_{1,2} + \\ e^{\lambda \xi_{1,1}} e^{\lambda} x_{2,2} \eta_{1,2} \xi_{1,2} + e^{\lambda \xi_{2,2}} \lambda x_{1,2} \xi_{1,1} \xi_{1,2} - e^{\lambda \xi_{1,1} + \lambda \xi_{2,2}} e^{\lambda^2} x_{1,2} \eta_{1,2} \xi_{1,2}^2 + x_{2,2} \xi_{2,2}$$

```
In[*]:= $Basis
```

```
Out[*]=
```

$$\{y_{1,2}, x_{1,1}, x_{1,2}, x_{2,2}\}$$

```

In[*]:= lhs = λTangent[
  Join[λ$Basis ($Basis /. {x → ξ1, y → η1}), λ$Basis ($Basis /. {x → ξ2, y → η2})]
]

Out[*]=
eλ ξ1,1-λ ξ1,2+λ ξ2,1-λ ξ2,2 y1,2 η1,2 + eλ ξ2,1-λ ξ2,2 y1,2 η2,2 + x1,1 ξ1,1 -
eλ ξ2,1-λ ξ2,2 λ y1,2 η2,2 ξ1,1 + eλ ξ1,2-λ ξ2,1+λ ξ2,2 x1,2 ξ1,2 - eλ ξ1,1 ∈ λ x1,1 η1,2 ξ1,2 +
eλ ξ1,1 ∈ λ x2,2 η1,2 ξ1,2 + eλ ξ1,2 ∈ λ x1,1 η2,2 ξ1,2 - eλ ξ1,2 ∈ λ x2,2 η2,2 ξ1,2 +
2 eλ ξ1,1+λ ξ2,1-λ ξ2,2 ∈ λ2 y1,2 η1,2 η2,2 ξ1,2 - eλ ξ1,2+λ ξ2,1-λ ξ2,2 ∈ λ2 y1,2 η2,22 ξ1,2 +
eλ ξ1,2-λ ξ2,1+λ ξ2,2 λ x1,2 ξ1,1 ξ1,2 + eλ ξ1,2 ∈ λ2 x1,1 η2,2 ξ1,1 ξ1,2 - eλ ξ1,2 ∈ λ2 x2,2 η2,2 ξ1,1 ξ1,2 -
eλ ξ1,2+λ ξ2,1-λ ξ2,2 ∈ λ3 y1,2 η2,22 ξ1,1 ξ1,2 - eλ ξ1,1+λ ξ1,2-λ ξ2,1+λ ξ2,2 ∈ λ2 x1,2 η1,2 ξ1,22 + x2,2 ξ1,22 +
eλ ξ2,1-λ ξ2,2 λ y1,2 η2,2 ξ1,2 + x1,1 ξ2,1 + eλ ξ2,2 x1,2 ξ2,2 - eλ ξ1,1-λ ξ1,2+λ ξ2,1 ∈ λ x1,1 η1,2 ξ2,2 +
eλ ξ1,1-λ ξ1,2+λ ξ2,1 ∈ λ x2,2 η1,2 ξ2,2 - eλ ξ2,1 ∈ λ x1,1 η2,2 ξ2,2 + eλ ξ2,1 ∈ λ x2,2 η2,2 ξ2,2 +
eλ ξ2,2 λ x1,2 ξ1,1 ξ2,2 + eλ ξ2,1 ∈ λ2 x1,1 η2,2 ξ1,1 ξ2,2 - eλ ξ2,1 ∈ λ2 x2,2 η2,2 ξ1,1 ξ2,2 -
2 eλ ξ1,1+λ ξ2,2 ∈ λ2 x1,2 η1,2 ξ1,2 ξ2,2 + 2 eλ ξ1,2+λ ξ2,2 ∈ λ2 x1,2 η2,2 ξ1,2 ξ2,2 +
2 eλ ξ1,2+λ ξ2,2 ∈ λ3 x1,2 η2,2 ξ1,1 ξ1,2 ξ2,2 - eλ ξ2,2 λ x1,2 ξ1,2 ξ2,2 -
eλ ξ2,1 ∈ λ2 x1,1 η2,2 ξ1,2 ξ2,2 + eλ ξ2,1 ∈ λ2 x2,2 η2,2 ξ1,2 ξ2,2 + eλ ξ2,2 λ x1,2 ξ2,1 ξ2,2 -
eλ ξ1,1-λ ξ1,2+λ ξ2,1+λ ξ2,2 ∈ λ2 x1,2 η1,2 ξ2,22 - eλ ξ2,1+λ ξ2,2 ∈ λ2 x1,2 η2,2 ξ2,22 +
eλ ξ2,1+λ ξ2,2 ∈ λ3 x1,2 η2,2 ξ1,1 ξ2,22 - eλ ξ2,1+λ ξ2,2 ∈ λ3 x1,2 η2,2 ξ1,2 ξ2,22 + x2,2 ξ2,2

In[*]:= rhs = λTangent@($Basis ($Basis /. {xα, α → CF[fα,α[λ]], xαβ → fαβ[λ], yαβ → gαβ[λ]}))

Out[*]=
x1,1 f1,1'[λ] + ef2,2[λ] x1,2 f1,2[λ] f1,1'[λ] + ef2,2[λ] x1,2 f1,2'[λ] +
x2,2 f2,2'[λ] + ef1,1[λ]-f2,2[λ] y1,2 g1,2'[λ] - ef1,1[λ] ∈ x1,1 f1,2[λ] g1,2'[λ] +
ef1,1[λ] ∈ x2,2 f1,2[λ] g1,2'[λ] - ef1,1[λ]+f2,2[λ] ∈ x1,2 f1,2[λ]2 g1,2'[λ]

In[*]:= SSBasis = Select[$Basis, SSQ]

Out[*]=
{x1,1, x2,2}

In[*]:= eqns = Simplify[(Coefficient[lhs - rhs, #] == 0) & /@ $Basis]

Out[*]=
{eλ (ξ1,1-ξ1,2+ξ2,1-ξ2,2) η1,2 (1 + 2 eλ ξ1,2 ∈ λ2 η2,2 ξ1,2) ==
eλ (ξ1,2+ξ2,1-ξ2,2) ∈ λ2 η2,22 (1 + λ ξ1,1) ξ1,2 +
eλ (ξ2,1-ξ2,2) η2,2 (-1 + λ ξ1,1 - λ ξ1,2) + ef1,1[λ]-f2,2[λ] g1,2'[λ],
ξ2,1 + ξ1,1 (1 + eλ η2,2 (eλ ξ1,2 ξ1,2 + eλ ξ2,1 ξ2,2)) + ef1,1[λ] ∈ f1,2[λ] g1,2'[λ] ==
eλ (ξ1,1-ξ1,2) ∈ λ η1,2 (eλ ξ1,2 ξ1,2 + eλ ξ2,1 ξ2,2) +
eλ η2,2 (-eλ ξ1,2 ξ1,2 + eλ ξ2,1 (1 + λ ξ1,2) ξ2,2) + f1,1'[λ],
e-λ (ξ1,2+ξ2,1) (-eλ (ξ1,1+2 ξ1,2+ξ2,2) ∈ λ2 η1,2 ξ1,22 + eλ (ξ1,2+ξ2,2) ξ1,2 (-2 eλ (ξ1,1+ξ2,1) ∈ λ2 η1,2
ξ2,2 + eλ ξ2,2 (1 + 2 eλ ξ2,1 ∈ λ2 η2,2 ξ2,2) + eλ ξ1,2 λ ξ1,1 (1 + 2 eλ ξ2,1 ∈ λ2 η2,2 ξ2,2)) -
eλ ξ2,1 (eλ (ξ1,2+ξ2,2) (-1 - λ ξ1,1 + λ ξ1,2 - λ ξ2,1) ξ2,2 +
eλ (ξ2,1+ξ2,2) ∈ λ2 (eλ ξ1,1 η1,2 + eλ ξ1,2 η2,2 (1 - λ ξ1,1 + λ ξ1,2)) ξ2,22 +
eλ ξ1,2+f2,2[λ] (f1,2[λ] f1,1'[λ] + f1,2'[λ] - ef1,1[λ] ∈ f1,2[λ]2 g1,2'[λ])) == 0,
ξ1,2 + eλ η1,2 (eλ ξ1,1 ξ1,2 + eλ (ξ1,1-ξ1,2+ξ2,1) ξ2,2) + ξ2,2 ==
eλ η2,2 (eλ ξ1,2 (1 + λ ξ1,1) ξ1,2 + eλ ξ2,1 (-1 + λ ξ1,1 - λ ξ1,2) ξ2,2) +
f2,2'[λ] + ef1,1[λ] ∈ f1,2[λ] g1,2'[λ]}

```

```

In[*]:= Length@eqns
Out[*]=
4

In[*]:= unknowns = $Basis /. {xαβ → fαβ[λ], yαβ → gαβ[λ]}
Out[*]=
{g1,2[λ], f1,1[λ], f1,2[λ], f2,2[λ]}

In[*]:= $Basis /. {xαβ ⇒ fαβ[0] == 0, yαβ ⇒ gαβ[0] == 0}
Out[*]=
{g1,2[0] == 0, f1,1[0] == 0, f1,2[0] == 0, f2,2[0] == 0}

In[*]:= {sol0} = Block[{ $k = 0}, Simplify@DSolve[
  lhs = λTangent[
    Join[λ $Basis ($Basis /. {x → ξ1, y → η1}), λ $Basis ($Basis /. {x → ξ2, y → η2})]
  ];
  rhs = λTangent@($Basis ($Basis /. {xα, α ⇒ CF[fα,α[λ]], xαβ → fαβ[λ], yαβ → gαβ[λ]}));
  Join[
    (Coefficient[CF[lhs - rhs], #] == 0) & /@ $Basis,
    $Basis /. {xαβ ⇒ fαβ[0] == 0, yαβ ⇒ gαβ[0] == 0}
  ],
  $Basis /. {xαβ → fαβ[λ], yαβ → gαβ[λ]},
  λ
]]
Out[*]=
{ {f1,1[λ] → λ (ξ1,1 + ξ2,1), f2,2[λ] → λ (ξ1,2 + ξ2,2),
  f1,2[λ] → e-λ ξ2,1 λ ξ1,2 + e-λ ξ1,2 λ ξ2,2, g1,2[λ] → λ (η1,2 + eλ (-ξ1,1 + ξ1,2) η2,2) } }

In[*]:= {sol1} = Block[{ $k = 1}, Simplify@DSolve[
  lhs = λTangent[
    Join[λ $Basis ($Basis /. {x → ξ1, y → η1}), λ $Basis ($Basis /. {x → ξ2, y → η2})]
  ];
  rhs = λTangent[
    $Basis ($Basis /. {xα, α ⇒ CF[fα,α[λ]], xαβ → fαβ[λ], yαβ → gαβ[λ]})) /.
    (sol0 /. ((a → b) ⇒ (a → e a + b)))
  ];
  Join[
    (Coefficient[CF[lhs - rhs], #] == 0) & /@ $Basis,
    $Basis /. {xαβ ⇒ fαβ[0] == 0, yαβ ⇒ gαβ[0] == 0}
  ],
  $Basis /. {xαβ → fαβ[λ], yαβ → gαβ[λ]},
  λ
]]
Out[*]=
{ {f1,1[λ] → eλ ξ1,2 λ2 η2,2 ξ1,2, f2,2[λ] → -eλ ξ1,2 λ2 η2,2 ξ1,2,
  f1,2[λ] → λ3 η2,2 ξ1,2 ξ2,2, g1,2[λ] → -e-λ ξ1,1 + 2 λ ξ1,2 λ3 η2,22 ξ1,2} }

```

```

In[*]:= SetAlgebra[gl3,ε];
$k = 1
$PBWRule
MatrixForm@Table[{b1, b2} → B[b1, b2], {b1, $Basis}, {b2, $Basis}]

Out[*]=
1

Out[*]=
{y1,2 → 1, y1,3 → 2, y2,3 → 3, x1,1 → 4, x1,2 → 5, x2,2 → 6, x1,3 → 7, x2,3 → 8, x3,3 → 9}

Out[*]//MatrixForm=

$$\begin{pmatrix} \{y_{1,2}, y_{1,2}\} \rightarrow 0 & \{y_{1,2}, y_{1,3}\} \rightarrow 0 & \{y_{1,2}, y_{2,3}\} \rightarrow \in y_{1,3} & \{y_{1,2}, x_{1,1}\} \rightarrow y_{1,1} \\ \{y_{1,3}, y_{1,2}\} \rightarrow 0 & \{y_{1,3}, y_{1,3}\} \rightarrow 0 & \{y_{1,3}, y_{2,3}\} \rightarrow 0 & \{y_{1,3}, x_{1,1}\} \rightarrow y_{1,3} \\ \{y_{2,3}, y_{1,2}\} \rightarrow -\in y_{1,3} & \{y_{2,3}, y_{1,3}\} \rightarrow 0 & \{y_{2,3}, y_{2,3}\} \rightarrow 0 & \{y_{2,3}, x_{1,1}\} \rightarrow y_{2,3} \\ \{x_{1,1}, y_{1,2}\} \rightarrow -y_{1,2} & \{x_{1,1}, y_{1,3}\} \rightarrow -y_{1,3} & \{x_{1,1}, y_{2,3}\} \rightarrow 0 & \{x_{1,1}, x_{1,1}\} \rightarrow 0 \\ \{x_{1,2}, y_{1,2}\} \rightarrow \in x_{1,1} - \in x_{2,2} & \{x_{1,2}, y_{1,3}\} \rightarrow -y_{2,3} & \{x_{1,2}, y_{2,3}\} \rightarrow 0 & \{x_{1,2}, x_{1,1}\} \rightarrow -x_{2,2} \\ \{x_{2,2}, y_{1,2}\} \rightarrow y_{1,2} & \{x_{2,2}, y_{1,3}\} \rightarrow 0 & \{x_{2,2}, y_{2,3}\} \rightarrow -y_{2,3} & \{x_{2,2}, x_{1,1}\} \rightarrow -x_{2,2} \\ \{x_{1,3}, y_{1,2}\} \rightarrow -\in x_{2,3} & \{x_{1,3}, y_{1,3}\} \rightarrow \in x_{1,1} - \in x_{3,3} & \{x_{1,3}, y_{2,3}\} \rightarrow \in x_{1,2} & \{x_{1,3}, x_{1,1}\} \rightarrow -x_{3,3} \\ \{x_{2,3}, y_{1,2}\} \rightarrow 0 & \{x_{2,3}, y_{1,3}\} \rightarrow y_{1,2} & \{x_{2,3}, y_{2,3}\} \rightarrow \in x_{2,2} - \in x_{3,3} & \{x_{2,3}, x_{1,1}\} \rightarrow -x_{3,3} \\ \{x_{3,3}, y_{1,2}\} \rightarrow 0 & \{x_{3,3}, y_{1,3}\} \rightarrow y_{1,3} & \{x_{3,3}, y_{2,3}\} \rightarrow y_{2,3} & \{x_{3,3}, x_{1,1}\} \rightarrow -x_{3,3} \end{pmatrix}$$


In[*]:= sol[-1] = 0 $Basis;
Do[
  Block[{ $k = k },
    s0 = Thread[$Basis → sol[k - 1]];
    lhs = λTangent[
      Join[λ $Basis ($Basis /. {x → ξ1, y → η1}), λ $Basis ($Basis /. {x → ξ2, y → η2})]
    ];
    rhs = λTangent[
      $Basis ($Basis /. {xαβ → (xαβ /. s0) + εk fαβ[λ], yαβ → (yαβ /. s0) + εk gαβ[λ]})
    ];
    Echo@"computed lhs and rhs";
    eqns = EchoLabel["eqns"]@Join[
      (Coefficient[CF[lhs - rhs], #] == 0) & /@ $Basis,
      $Basis /. {xαβ → fαβ[0] == 0, yαβ → gαβ[0] == 0}
    ];
    unknowns = $Basis /. {xαβ → fαβ[λ], yαβ → gαβ[λ]};
    {s} = DSolve[eqns, unknowns, λ];
    sol[k] = CF[sol[k - 1] + εk ($Basis /. {xαβ → fαβ[λ], yαβ → gαβ[λ]} /. s)]
  ],
  {k, 0, $k}
];
sol[$k].$Basis
» computed lhs and rhs

```

» eqns
$$\begin{aligned} & \left\{ e^{\lambda \xi_{1,1}-\lambda \xi_{1,2}+\lambda \xi_{2,1}-\lambda \xi_{2,2}} \eta_{1,1,2} + e^{\lambda \xi_{2,1}-\lambda \xi_{2,2}} \eta_{2,1,2} - e^{\lambda \xi_{2,1}-\lambda \xi_{2,2}} \lambda \eta_{2,1,2} \xi_{1,1} + \right. \\ & e^{\lambda \xi_{2,1,1}-\lambda \xi_{2,2}} \lambda \eta_{2,1,2} \xi_{1,2,2} - e^{\lambda \xi_{1,1,1}+\lambda \xi_{2,1}-\lambda \xi_{2,2}} \lambda \eta_{1,1,3} \xi_{1,2,3} + e^{\lambda \xi_{1,3}+\lambda \xi_{2,1}-\lambda \xi_{2,2}} \lambda \eta_{2,1,3} \xi_{1,2,3} + \\ & e^{\lambda \xi_{1,3}+\lambda \xi_{2,1}-\lambda \xi_{2,2}} \lambda^2 \eta_{2,1,3} \xi_{1,2,2} \xi_{1,2,3} - e^{\lambda \xi_{1,1}-\lambda \xi_{1,3}+\lambda \xi_{2,1}} \lambda \eta_{1,1,3} \xi_{2,2,3} - e^{\lambda \xi_{2,1}} \lambda \eta_{2,1,3} \xi_{2,2,3} + \\ & e^{\lambda \xi_{2,1}} \lambda^2 \eta_{2,1,3} \xi_{1,1,1} \xi_{2,2,3} - e^{\lambda \xi_{2,1}} \lambda^2 \eta_{2,1,3} \xi_{1,3,3} \xi_{2,2,3} - e^{f_{1,1}[\lambda]-f_{2,2}[\lambda]} g_{1,2}'[\lambda] + e^{f_{1,1}[\lambda]} f_{2,3}[\lambda] g_{1,3}'[\lambda] = 0, \\ & e^{\lambda \xi_{1,1}-\lambda \xi_{1,3}+\lambda \xi_{2,1}-\lambda \xi_{2,3}} \eta_{1,1,3} + e^{\lambda \xi_{2,1}-\lambda \xi_{2,3}} \eta_{2,1,3} - e^{\lambda \xi_{2,1}-\lambda \xi_{2,3}} \lambda \eta_{2,1,3} \xi_{1,1} + \\ & e^{\lambda \xi_{2,1}-\lambda \xi_{2,3}} \lambda \eta_{2,1,3} \xi_{1,3,3} - e^{f_{1,1}[\lambda]-f_{3,3}[\lambda]} g_{1,3}'[\lambda] = 0, \\ & e^{\lambda \xi_{1,2}-\lambda \xi_{1,3}+\lambda \xi_{2,2}-\lambda \xi_{2,3}} \eta_{1,2,3} + e^{\lambda \xi_{2,2}-\lambda \xi_{2,3}} \eta_{2,2,3} + e^{\lambda \xi_{1,1}+\lambda \xi_{1,2}-\lambda \xi_{1,3}+\lambda \xi_{2,2}-\lambda \xi_{2,3}} \lambda \eta_{1,1,3} \xi_{1,1,2} - \\ & e^{\lambda \xi_{1,2}+\lambda \xi_{2,2}-\lambda \xi_{2,3}} \lambda \eta_{2,1,3} \xi_{1,1,2} - e^{\lambda \xi_{1,2}+\lambda \xi_{2,2}-\lambda \xi_{2,3}} \lambda^2 \eta_{2,1,3} \xi_{1,1,1} \xi_{1,1,2} - e^{\lambda \xi_{2,2}-\lambda \xi_{2,3}} \lambda \eta_{2,2,3} \xi_{1,2,2} + \\ & e^{\lambda \xi_{2,2}-\lambda \xi_{2,3}} \lambda \eta_{2,2,3} \xi_{1,3,3} + e^{\lambda \xi_{1,1}-\lambda \xi_{1,3}+\lambda \xi_{2,1}+\lambda \xi_{2,2}-\lambda \xi_{2,3}} \lambda \eta_{1,1,3} \xi_{2,1,2} + e^{\lambda \xi_{2,1}+\lambda \xi_{2,2}-\lambda \xi_{2,3}} \lambda \eta_{2,1,3} \xi_{2,1,2} - \\ & e^{\lambda \xi_{2,1}+\lambda \xi_{2,2}-\lambda \xi_{2,3}} \lambda^2 \eta_{2,1,3} \xi_{1,1,1} \xi_{2,1,2} + e^{\lambda \xi_{2,1}+\lambda \xi_{2,2}-\lambda \xi_{2,3}} \lambda^2 \eta_{2,1,3} \xi_{1,3,3} \xi_{2,1,2} - \\ & e^{f_{1,1}[\lambda]+f_{2,2}[\lambda]-f_{3,3}[\lambda]} f_{1,2}[\lambda] g_{1,3}'[\lambda] - e^{f_{2,2}[\lambda]-f_{3,3}[\lambda]} g_{2,3}'[\lambda] = 0, \xi_{1,1,1} + \xi_{2,1,1} - f_{1,1}'[\lambda] = 0, \\ & e^{\lambda \xi_{1,2}-\lambda \xi_{2,1}+\lambda \xi_{2,2}} \xi_{1,1,2} + e^{\lambda \xi_{1,2}-\lambda \xi_{2,1}+\lambda \xi_{2,2}} \lambda \xi_{1,1,1} \xi_{1,1,2} + e^{\lambda \xi_{2,2}} \xi_{2,1,2} + e^{\lambda \xi_{2,2}} \lambda \xi_{1,1,1} \xi_{2,1,2} - \\ & e^{\lambda \xi_{2,2}} \lambda \xi_{1,2,2} \xi_{2,1,2} + e^{\lambda \xi_{2,2}} \lambda \xi_{2,1,1} \xi_{2,1,2} - e^{f_{2,2}[\lambda]} f_{1,2}[\lambda] f_{1,1}'[\lambda] - e^{f_{2,2}[\lambda]} f_{1,2}'[\lambda] = 0, \\ & \xi_{1,2,2} + \xi_{2,2,2} - f_{2,2}'[\lambda] = 0, e^{\lambda \xi_{1,3}-\lambda \xi_{2,1}+\lambda \xi_{2,3}} \xi_{1,1,3} + e^{\lambda \xi_{1,3}-\lambda \xi_{2,1}+\lambda \xi_{2,3}} \lambda \xi_{1,1,1} \xi_{1,1,3} + \\ & e^{\lambda \xi_{1,2}+\lambda \xi_{1,3}-\lambda \xi_{2,1}+\lambda \xi_{2,3}} \lambda \xi_{1,1,2} \xi_{1,2,3} + e^{\lambda \xi_{1,2}+\lambda \xi_{1,3}-\lambda \xi_{2,1}+\lambda \xi_{2,3}} \lambda^2 \xi_{1,1,1} \xi_{1,1,2} \xi_{1,2,3} - \\ & e^{\lambda \xi_{1,3}+\lambda \xi_{2,3}} \lambda \xi_{1,2,3} \xi_{2,1,2} - e^{\lambda \xi_{1,3}+\lambda \xi_{2,3}} \lambda^2 \xi_{1,2,2} \xi_{1,2,3} \xi_{2,1,2} + e^{\lambda \xi_{2,3}} \xi_{2,1,3} + e^{\lambda \xi_{2,3}} \lambda \xi_{1,1,1} \xi_{2,1,3} - \\ & e^{\lambda \xi_{2,3}} \lambda \xi_{1,3,3} \xi_{2,1,3} + e^{\lambda \xi_{2,3}} \lambda \xi_{2,1,1} \xi_{2,1,3} + e^{\lambda \xi_{1,2}-\lambda \xi_{2,1}+\lambda \xi_{2,2}+\lambda \xi_{2,3}} \lambda \xi_{1,1,2} \xi_{2,2,3} + \\ & e^{\lambda \xi_{1,2}-\lambda \xi_{2,1}+\lambda \xi_{2,2}+\lambda \xi_{2,3}} \lambda^2 \xi_{1,1,1} \xi_{1,1,2} \xi_{2,2,3} + e^{\lambda \xi_{2,2}+\lambda \xi_{2,3}} \lambda \xi_{2,1,2} \xi_{2,2,3} + e^{\lambda \xi_{2,2}+\lambda \xi_{2,3}} \lambda^2 \xi_{1,1,1} \xi_{2,1,2} \xi_{2,2,3} - \\ & e^{\lambda \xi_{2,2}+\lambda \xi_{2,3}} \lambda^2 \xi_{1,2,2} \xi_{2,1,2} \xi_{2,2,3} + e^{\lambda \xi_{2,2}+\lambda \xi_{2,3}} \lambda^2 \xi_{2,1,1} \xi_{2,1,2} \xi_{2,2,3} - e^{f_{3,3}[\lambda]} f_{1,3}[\lambda] f_{1,1}'[\lambda] - \\ & e^{f_{2,2}[\lambda]+f_{3,3}[\lambda]} f_{1,2}[\lambda] f_{2,3}[\lambda] f_{1,1}'[\lambda] - e^{f_{2,2}[\lambda]+f_{3,3}[\lambda]} f_{2,3}[\lambda] f_{1,2}'[\lambda] - e^{f_{3,3}[\lambda]} f_{1,3}'[\lambda] = 0, \\ & e^{\lambda \xi_{1,3}-\lambda \xi_{2,2}+\lambda \xi_{2,3}} \xi_{1,2,3} + e^{\lambda \xi_{1,3}-\lambda \xi_{2,2}+\lambda \xi_{2,3}} \lambda \xi_{1,2,2} \xi_{1,2,3} + e^{\lambda \xi_{2,3}} \xi_{2,2,3} + e^{\lambda \xi_{2,3}} \lambda \xi_{1,2,2} \xi_{2,2,3} - \\ & e^{\lambda \xi_{2,3}} \lambda \xi_{1,3,3} \xi_{2,2,3} + e^{\lambda \xi_{2,3}} \lambda \xi_{2,2,2} \xi_{2,2,3} - e^{f_{3,3}[\lambda]} f_{2,3}[\lambda] f_{2,2}'[\lambda] - e^{f_{3,3}[\lambda]} f_{2,3}'[\lambda] = 0, \\ & \xi_{1,3,3} + \xi_{2,3,3} - f_{3,3}'[\lambda] = 0, g_{1,2}[0] = 0, g_{1,3}[0] = 0, g_{2,3}[0] = 0, f_{1,1}[0] = 0, \\ & f_{1,2}[0] = 0, f_{2,2}[0] = 0, f_{1,3}[0] = 0, f_{2,3}[0] = 0, f_{3,3}[0] = 0 \} \end{aligned}$$

» computed lhs and rhs

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

Set: Part 7 of Asymptotics`AsymptoticRSolveValueDump`c\$15938[[Asymptotics`AsymptoticRSolveValueDump`i\$15938]] does not exist.

Set: Part 8 of Asymptotics`AsymptoticRSolveValueDump`c\$15938[[Asymptotics`AsymptoticRSolveValueDump`i\$15938]] does not exist.

Set: Part 9 of Asymptotics`AsymptoticRSolveValueDump`c\$15938[[Asymptotics`AsymptoticRSolveValueDump`i\$15938]] does not exist.

General: Further output of Set::partw will be suppressed during this calculation.

Part: Part 7 of {0, 0, 0, 0, 0, 0} does not exist.

Part: Part 8 of {0, 0, 0, 0, 0, 0} does not exist.

Part: Part 9 of {0, 0, 0, 0, 0, 0} does not exist.

General: Further output of Part::partw will be suppressed during this calculation.

Out[*]=

\$Aborted

Dot: Tensors

$\{\lambda \eta_{1,2} + e^{-\lambda \xi_{1,1} + \lambda \xi_{1,2}} \lambda \eta_{2,2} - e^{-\lambda \xi_{1,1} + 2\lambda \xi_{1,2}} \in \lambda^3 \eta_{2,2}^2 \xi_{1,2}, \lambda \xi_{1,1} + e^{\lambda \xi_{1,2}} \in \lambda^2 \eta_{2,2} \xi_{1,2} + \lambda \xi_{2,1}, e^{-\lambda \xi_{2,1}} \lambda \xi_{1,2} + e^{-\lambda \xi_{1,2}} \lambda \xi_{2,2} + \in \lambda^3 \eta_{2,2} \xi_{1,2} \xi_{2,2}, -e^{\lambda \xi_{1,2}} \in \lambda^2 \eta_{2,2} \xi_{1,2} + \lambda \xi_{1,2,2} + \lambda \xi_{2,2,2}\}$ and $\{y_{1,2}, y_{1,3}, y_{2,3}, x_{1,1}, x_{1,2}, x_{2,2}, x_{1,3}, x_{2,3}, x_{3,3}\}$ have incompatible shapes.

Out[*]=

$\{\lambda \eta_{1,2} + e^{-\lambda \xi_{1,1} + \lambda \xi_{1,2}} \lambda \eta_{2,2} - e^{-\lambda \xi_{1,1} + 2\lambda \xi_{1,2}} \in \lambda^3 \eta_{2,2}^2 \xi_{1,2}, \lambda \xi_{1,1} + e^{\lambda \xi_{1,2}} \in \lambda^2 \eta_{2,2} \xi_{1,2} + \lambda \xi_{2,1}, e^{-\lambda \xi_{2,1}} \lambda \xi_{1,2} + e^{-\lambda \xi_{1,2}} \lambda \xi_{2,2} + \in \lambda^3 \eta_{2,2} \xi_{1,2} \xi_{2,2}, -e^{\lambda \xi_{1,2}} \in \lambda^2 \eta_{2,2} \xi_{1,2} + \lambda \xi_{1,2,2} + \lambda \xi_{2,2,2}\}.$
 $\{y_{1,2}, y_{1,3}, y_{2,3}, x_{1,1}, x_{1,2}, x_{2,2}, x_{1,3}, x_{2,3}, x_{3,3}\}$

osol = $\mathbb{E}_{\{i,j\} \rightarrow \{k\}} \left[y_{1,2}[k] \eta_{1,2}[i] + e^{-\xi_{1,1}[i] + \xi_{2,2}[i]} y_{1,2}[k] \eta_{1,2}[j] + y_{1,3}[k] \eta_{1,3}[i] + e^{-\xi_{1,1}[i] + \xi_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j] + y_{2,3}[k] \eta_{2,3}[i] + e^{-\xi_{2,2}[i] + \xi_{3,3}[i]} y_{2,3}[k] \eta_{2,3}[j] + x_{1,1}[k] \xi_{1,1}[i] + x_{1,1}[k] \xi_{1,1}[j] + e^{-\xi_{1,1}[j]} x_{1,2}[k] \xi_{1,2}[i] - e^{\xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j] \xi_{1,2}[i] + e^{-\xi_{2,2}[i]} x_{1,2}[k] \xi_{1,2}[j] + e^{-\xi_{1,1}[j]} x_{1,3}[k] \xi_{1,3}[i] + e^{-\xi_{3,3}[i]} x_{1,3}[k] \xi_{1,3}[j] + x_{2,2}[k] \xi_{2,2}[i] + x_{2,2}[k] \xi_{2,2}[j] + e^{-\xi_{2,2}[j]} x_{2,3}[k] \xi_{2,3}[i] + e^{-\xi_{1,1}[i] + \xi_{2,2}[i] + \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,3}[j] \xi_{2,3}[i] - x_{1,3}[k] \xi_{1,2}[j] \xi_{2,3}[i] + e^{-\xi_{3,3}[i]} x_{2,3}[k] \xi_{2,3}[j] + x_{3,3}[k] \xi_{3,3}[i] + x_{3,3}[k] \xi_{3,3}[j], -e^{-\xi_{2,2}[i] + \xi_{3,3}[i]} y_{1,3}[k] \eta_{1,2}[i] \eta_{2,3}[j] + e^{\xi_{2,2}[i]} x_{1,1}[k] \eta_{1,2}[j] \xi_{1,2}[i] - e^{\xi_{2,2}[i]} x_{2,2}[k] \eta_{1,2}[j] \xi_{1,2}[i] - e^{-\xi_{1,1}[i] + 2\xi_{2,2}[i]} y_{1,2}[k] \eta_{1,2}[j]^2 \xi_{1,2}[i] + e^{\xi_{3,3}[i]} y_{1,3}[k] \eta_{1,2}[i] \eta_{1,3}[j] \xi_{1,2}[i] + x_{1,2}[k] \eta_{1,2}[j] \xi_{1,2}[i] \xi_{1,2}[j] - e^{-\xi_{2,2}[j]} x_{2,3}[k] \eta_{1,2}[j] \xi_{1,3}[i] + e^{\xi_{3,3}[i]} x_{1,1}[k] \eta_{1,3}[j] \xi_{1,3}[i] - e^{\xi_{3,3}[i]} x_{3,3}[k] \eta_{1,3}[j] \xi_{1,3}[i] - e^{-\xi_{1,1}[i] + \xi_{2,2}[i] + \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \xi_{1,3}[i] - e^{-\xi_{1,1}[i] + 2\xi_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j]^2 \xi_{1,3}[i] + e^{-\xi_{1,1}[j] - \xi_{2,2}[i] + \xi_{3,3}[i]} x_{1,2}[k] \eta_{2,3}[j] \xi_{1,3}[i] - e^{-\xi_{2,2}[i] + 2\xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,3}[i] - e^{-\xi_{1,1}[j] + \xi_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{1,3}[i] + e^{2\xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j]^2 \xi_{1,2}[i] \xi_{1,3}[i] + x_{1,3}[k] \eta_{1,2}[j] \xi_{1,2}[j] \xi_{1,3}[i] + x_{1,3}[k] \eta_{1,3}[j] \xi_{1,3}[i] \xi_{1,3}[j] + e^{\xi_{3,3}[i]} x_{2,2}[k] \eta_{2,3}[j] \xi_{2,3}[i] - e^{\xi_{3,3}[i]} x_{3,3}[k] \eta_{2,3}[j] \xi_{2,3}[i] + e^{-\xi_{1,1}[i] + \xi_{2,2}[i] + \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{2,3}[j] \xi_{2,3}[i] - e^{-\xi_{1,1}[i] + 2\xi_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{2,3}[i] - e^{-\xi_{2,2}[i] + 2\xi_{3,3}[i]} y_{2,3}[k] \eta_{2,3}[j]^2 \xi_{2,3}[i] + e^{\xi_{2,2}[i] + \xi_{3,3}[i]} x_{1,1}[k] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] - e^{\xi_{2,2}[i] + \xi_{3,3}[i]} x_{2,2}[k] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] - 2e^{-\xi_{1,1}[i] + 2\xi_{2,2}[i] + \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] + e^{2\xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] - e^{-\xi_{2,2}[i] + \xi_{3,3}[i]} x_{1,2}[k] \eta_{2,3}[j] \xi_{1,2}[j] \xi_{2,3}[i] + e^{\xi_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{1,2}[j] \xi_{2,3}[i] - e^{-\xi_{1,1}[i] + \xi_{2,2}[i] + 2\xi_{3,3}[i]} y_{1,2}[k] \eta_{1,3}[j]^2 \xi_{1,3}[i] \xi_{2,3}[i] + x_{1,3}[k] \eta_{2,3}[j] \xi_{1,3}[j] \xi_{2,3}[i] - e^{-\xi_{1,1}[i] + 2\xi_{2,2}[i] + 2\xi_{3,3}[i]} y_{1,2}[k] \eta_{1,3}[j]^2 \xi_{1,2}[i] \xi_{2,3}[i]^2 + x_{2,3}[k] \eta_{1,3}[j] \xi_{1,3}[i] \xi_{2,3}[j] + x_{2,3}[k] \eta_{2,3}[j] \xi_{2,3}[i] \xi_{2,3}[j],$

[illegible]

```


$$\xi_{1,2}[j] \xi_{2,3}[i]^2 + 2 e^{-\xi_{1,1}[i] + 2 \xi_{2,2}[i] + 3 \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,3}[j]^3 \xi_{1,2}[i] \xi_{1,3}[i] \xi_{2,3}[i]^2 +$$


$$e^{-\xi_{1,1}[i] + 3 \xi_{2,2}[i] + 3 \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,3}[j]^3 \xi_{1,2}[i]^2 \xi_{2,3}[i]^3];$$

osol = (List @@ osol).Table[ek, {k, 0, Length[osol] - 1}]

```

Out[*]=

```


$$y_{1,2}[k] \eta_{1,2}[i] + e^{-\xi_{1,1}[i] + \xi_{2,2}[i]} y_{1,2}[k] \eta_{1,2}[j] +$$


$$x_{1,1}[k] \xi_{1,1}[i] + x_{1,1}[k] \xi_{1,1}[j] + e^{-\xi_{1,1}[j]} x_{1,2}[k] \xi_{1,2}[i] +$$


$$e^2 \left( -\frac{1}{2} e^{2 \xi_{2,2}[i]} x_{1,1}[k] \eta_{1,2}[j]^2 \xi_{1,2}[i]^2 + \frac{1}{2} e^{2 \xi_{2,2}[i]} x_{2,2}[k] \eta_{1,2}[j]^2 \xi_{1,2}[i]^2 + \right.$$


$$\left. e^{-\xi_{1,1}[i] + 3 \xi_{2,2}[i]} y_{1,2}[k] \eta_{1,2}[j]^3 \xi_{1,2}[i]^2 \right) + e^{-\xi_{2,2}[i]} x_{1,2}[k] \xi_{1,2}[j] +$$


$$\in \left( e^{\xi_{2,2}[i]} x_{1,1}[k] \eta_{1,2}[j] \xi_{1,2}[i] - e^{\xi_{2,2}[i]} x_{2,2}[k] \eta_{1,2}[j] \xi_{1,2}[i] - e^{-\xi_{1,1}[i] + 2 \xi_{2,2}[i]} y_{1,2}[k] \right.$$


$$\left. \eta_{1,2}[j]^2 \xi_{1,2}[i] + x_{1,2}[k] \eta_{1,2}[j] \xi_{1,2}[i] \xi_{1,2}[j] \right) + x_{2,2}[k] \xi_{2,2}[i] + x_{2,2}[k] \xi_{2,2}[j]$$


```

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In[*]:= CF[(Expand@sol[$k].$Basis /. {ξ1αβ → ξαβ[i], η1αβ → ηαβ[i],
ξ2αβ → ξαβ[j], η2αβ → ηαβ[j], λ → 1}) - (osol /. v_ [k] → v)]

```

Out[*]=

0