

Pensieve header: The PBW multiplication tensor for  $\mathfrak{sl}_n$  using the  $\lambda$ -tangent formalism. Some material from pensieve://Projects/UEA.

```
In[ ]:= SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
```

Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.  
Read more at <http://katlas.org/wiki/KnotTheory>.

## Prolog

```
In[ ]:= BeginPackage["UEA`"];
Print["UEA` does computations in general universal enveloping
algebras and PBW algebras. It is in the public domain, available
at http://drorbn.net/AcademicPensieve/Projects/SolvablePBW/.
Dror Bar-Natan is committed to support it within
reason until June 1, 2027. This is version 260601."];
Print["UEA` implements / extends ",
Sort@{"**",  $\epsilon$ , $k, CF, SSQ, NilQ, B, m, SetAlgebra, U,
UB, UProducts, USimp, UU, $Basis, $PBWRule,  $\delta$ , adPower, adExp},
"."];
Begin["`Private`"];
```

UEA` does computations in general universal enveloping algebras and PBW algebras. It is in the public domain, available at <http://drorbn.net/AcademicPensieve/Projects/SolvablePBW/>. Dror Bar-Natan is committed to support it within reason until June 1, 2027. This is version 260601.

UEA` implements / extends {\*\*, adExp, adPower, B, CF, m, NilQ, SetAlgebra, SSQ, U, UB, UProducts, USimp, UU,  $\delta$ ,  $\epsilon$ , \$Basis, \$k, \$PBWRule}.

## Utilities

```
In[ ]:=  $\delta_{i,j}$  := If[i == j, 1, 0];
```

```
In[ ]:= SSQ[c_.*x_] /; MemberQ[$Basis, x] :=  $\neg$  NilQ[x]; (* Semi-Simple Q *)
NilQ[c_.*x_] /; MemberQ[$Basis, x] := NilQ[x];
SSQ[x_Plus] := And @@ (SSQ /@ (List @@ x));
NilQ[x_Plus] := And @@ (NilQ /@ (List @@ x));
```

```
In[ ]:= $k = 1;
```

```
In[ ]:= CF[ $\mathcal{E}$ _] := Expand[ $\mathcal{E}$  /.  $e^L$  ->  $e^{\text{Expand}[L]}$ ] /. { $e^{k\cdot}$  /;  $k > \$k \rightarrow 0$ };
```

## Implementing general universal enveloping algebras

```
In[*]:= B[0, _] = 0; B[_ , 0] = 0;
B[c_ * x_, y_] /; MemberQ[$Basis, x] := CF[c B[x, y]];
B[y_, c_ * x_] /; MemberQ[$Basis, x] := CF[c B[y, x]];
B[x_Plus, y_] := B[#, y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
B[x_, x_] = 0;
B[y_, x_] := CF[-B[x, y]];
```

```
In[*]:= x_ ≤ y_ := OrderedQ[{x, y} /. $PBWRule]; x_ < y_ := ! OrderedQ[{y, x} /. $PBWRule];
UU_i_[1] := U_i[];
UU_i_[x_P_] := UU_i_@@Table[x, {p}];
UU_i_[ε_] := ε /. {
  U[xs_] => U_i[xs],
  x_ /; MemberQ[$Basis, x] => U_i[x]
};
UU_i_[x_, xs_] := UU_t1[x] UU_t2[xs] // Expand // m_{t1,t2→i};
USimp[ε_] := Collect[ε, Times[U_[] ..], Expand];
USimp[ε_] := Expand[ε];
```

```
In[*]:= m_s_[0] = 0;
m_s_[x_Plus] := m_s_ /@ x;
m_s_[sd_SeriesData] := MapAt[m_s, sd, {3, All}];
m_{i→j}[ε_] := ε /. U_i → U_j;
```

```
In[*]:= m_{i,j→k}[c_. U_i[x_] U_j[]] := c U_k[x];
m_{i,j→k}[c_. U_i[] U_j[y_]] := c U_k[y];
m_{i,j→k}[c_. U_i[xx_, x_] U_j[y_, yy_]] := If[x ≤ y,
  c U_k[xx, x, y, yy],
  ((U_i[xx] (U_j[y, x] + UU_j[B[x, y]])) // Expand // m_{i,j→i} U_j[yy] // Expand // m_{i,j→k})
  c // USimp
];
```

```
In[*]:= UProducts[{}, 0] = {1}; UProducts[{}, d_Integer] /; d > 0 = {};
UProducts[{i_, is_}, d_Integer] := Sort@
  Flatten@Table[(U_i@@@Subsets[$Basis, {j}]) u, {j, 0, d}, {u, UProducts[{is, d-j}]}];
```

```
In[*]:= Supp[ε_] := Union@Cases[{ε}, U_i[___] => i, ∞];
```

```
In[*]:= Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ⋈ y_ := Module[{is = Supp[x] ∩ Supp[y], σ, z},
  z = x; Do[z = mi→σ@i[z], {i, is}];
  z = Expand[y z]; Do[z = mσ@i, i→i[z], {i, is}]; z];
UB[x_, y_] := USimp[x ⋈ y - y ⋈ x];
```

## Epilog

```
In[*]:= End[]; EndPackage[];
```

## Predefined Algebras

### sl(2)

```
In[*]:= Print["UEA`SetAlgebra knows \"sl2\"."];
```

UEA`SetAlgebra knows "sl2".

```
In[*]:= SetAlgebra["sl2"] := (
  Print["In sl2: ⟨e, h, f⟩ / ([h, e] = 2e, [h, f] = -2f, [e, f] = h)."];
  B[h, e] = 2 e; B[h, f] = -2 f; B[e, f] = h;
  $Basis = {e, h, f};
  $PBWRule = {e → 1, h → 2, f → 3};
  NilQ[e] = NilQ[f] = True; NilQ[h] = False;
);
```

### $gl_{n, \epsilon}$

```
In[*]:= Print["UEA`SetAlgebra knows the  $\epsilon$ -nilpotent algebra  $gl_{n, \epsilon}$ ."];
```

UEA`SetAlgebra knows the  $\epsilon$ -nilpotent algebra  $gl_{n, \epsilon}$ .

```

In[*]:= SetAlgebra[gln,ε] := (
  $Basis = Flatten@{
    Table[yα,β, {β, 2, n}, {α, 1, β - 1}],
    Table[xα,β, {β, 1, n}, {α, 1, β}]
  };
  NilQ[y_] = True; NilQ[xα,β] := (α != β); (* Nilpotent Q *)
  $PBWRule = Thread[$Basis → Range@Length@$Basis];
  B[xi,j, xk,l] := δj,k xi,l - δl,i xk,j;
  B[yi,j, yk,l] := CF[ε δj,k yi,l - ε δl,i yk,j];
  B[xi,j, yk,l] := CF[δj,k xi,l - δl,i xk,j /. xα,β → If[α ≤ β, ε xα,β, yβ,α]];
);

```

```

In[*]:= SetAlgebra[gl2,ε];
$PBWRule
MatrixForm@Table[{b1, b2} → B[b1, b2], {b1, $Basis}, {b2, $Basis}]

```

Out[\*]=

{y<sub>1,2</sub> → 1, x<sub>1,1</sub> → 2, x<sub>1,2</sub> → 3, x<sub>2,2</sub> → 4}

Out[\*]//MatrixForm=

$$\begin{pmatrix} \{y_{1,2}, y_{1,2}\} \rightarrow 0 & \{y_{1,2}, x_{1,1}\} \rightarrow y_{1,2} & \{y_{1,2}, x_{1,2}\} \rightarrow -\epsilon x_{1,1} + \epsilon x_{2,2} & \{y_{1,2}, x_{2,2}\} \rightarrow -y_{1,2} \\ \{x_{1,1}, y_{1,2}\} \rightarrow -y_{1,2} & \{x_{1,1}, x_{1,1}\} \rightarrow 0 & \{x_{1,1}, x_{1,2}\} \rightarrow x_{1,2} & \{x_{1,1}, x_{2,2}\} \rightarrow 0 \\ \{x_{1,2}, y_{1,2}\} \rightarrow \epsilon x_{1,1} - \epsilon x_{2,2} & \{x_{1,2}, x_{1,1}\} \rightarrow -x_{1,2} & \{x_{1,2}, x_{1,2}\} \rightarrow 0 & \{x_{1,2}, x_{2,2}\} \rightarrow x_{1,2} \\ \{x_{2,2}, y_{1,2}\} \rightarrow y_{1,2} & \{x_{2,2}, x_{1,1}\} \rightarrow 0 & \{x_{2,2}, x_{1,2}\} \rightarrow -x_{1,2} & \{x_{2,2}, x_{2,2}\} \rightarrow 0 \end{pmatrix}$$

```

In[*]:= Clear[adPower];
adPower[x_, k_][0] = 0;
adPower[x_, k_][c_ * y_] /; MemberQ[$Basis, y] := CF[c adPower[x, k][y]];
adPower[x_, k_][y_Plus] := adPower[x, k] /@ y;
adPower[_, 0][y_] := y;
adPower[x_, k_][y_] /; MemberQ[$Basis, y] :=
  adPower[x, k][y] = B[adPower[x, k - 1][y], x];

```

```

In[*]:= x0 = ($Basis /. {x → ξ0, y → η0}).$Basis
x1 = ($Basis /. {x → ξ1, y → η1}).$Basis

```

Out[\*]=

y<sub>1,2</sub> η<sub>0,2</sub> + x<sub>1,1</sub> ξ<sub>0,1</sub> + x<sub>1,2</sub> ξ<sub>0,2</sub> + x<sub>2,2</sub> ξ<sub>0,2</sub>

Out[\*]=

y<sub>1,2</sub> η<sub>1,2</sub> + x<sub>1,1</sub> ξ<sub>1,1</sub> + x<sub>1,2</sub> ξ<sub>1,2</sub> + x<sub>2,2</sub> ξ<sub>1,2</sub>

```

In[*]:= adExp[x_][y_] /; SSQ[x] := Expand@Module[{A, b, c},
  A = Table[Coefficient[B[b, x], c], {c, $Basis}, {b, $Basis}];
  CF[$Basis.Normal[Series[MatrixExp[A], {ε, 0, $k}]]].
  Table[Coefficient[y, b], {b, $Basis}]
]

```

In[\*]:= **SSQ**[ $\xi_{0,1}$   $x_{1,1}$ ]

Out[\*]=

True

In[\*]:= **adExp**[ $\xi_{0,1}$   $x_{1,1}$ ] [**x1**]

Out[\*]=

$e^{\xi_{0,1}} y_{1,2} \eta_{1,2} + x_{1,1} \xi_{1,1} + e^{-\xi_{0,1}} x_{1,2} \xi_{1,2} + x_{2,2} \xi_{1,2}$

```
adExp[x_][y_] /; NilQ[x] := Expand@Module[{k = 0, s = 0},
  While[adPower[x, k][y] != 0,
    s += adPower[x, k][y] / k!;
    ++k
  ];
  s
]
```

In[\*]:= **adExp**[ $\xi_{0,2}$   $x_{1,2}$ ] [**x1**]

Out[\*]=

$y_{1,2} \eta_{1,2} - \epsilon x_{1,1} \eta_{1,2} \xi_{0,2} + \epsilon x_{2,2} \eta_{1,2} \xi_{0,2} - \epsilon x_{1,2} \eta_{1,2} \xi_{0,2}^2 +$   
 $x_{1,1} \xi_{1,1} + x_{1,2} \xi_{0,2} \xi_{1,1} + x_{1,2} \xi_{1,2} + x_{2,2} \xi_{1,2} - x_{1,2} \xi_{0,2} \xi_{1,2}$

```
λTangent[ ] = 0;
λTangent[xs___, x_] := adExp[x][λTangent[xs]] +  $\partial_{\lambda}$  x;
λTangent[xs_List] := λTangent @@ xs
```

In[\*]:= **\$Basis** (**\$Basis** /. {**x** →  $\xi$ , **y** →  $\eta$ })

Out[\*]=

{ $y_{1,2} \eta_{1,2}$ ,  $x_{1,1} \xi_{1,1}$ ,  $x_{1,2} \xi_{1,2}$ ,  $x_{2,2} \xi_{2,2}$ }

In[\*]:= **λTangent**@(**λ** **\$Basis** (**\$Basis** /. {**x** →  $\xi$ , **y** →  $\eta$ }))

Out[\*]=

$e^{\lambda \xi_{1,1} - \lambda \xi_{2,2}} y_{1,2} \eta_{1,2} + x_{1,1} \xi_{1,1} + e^{\lambda \xi_{2,2}} x_{1,2} \xi_{1,2} - e^{\lambda \xi_{1,1}} \epsilon \lambda x_{1,1} \eta_{1,2} \xi_{1,2} +$   
 $e^{\lambda \xi_{1,1}} \epsilon \lambda x_{2,2} \eta_{1,2} \xi_{1,2} + e^{\lambda \xi_{2,2}} \lambda x_{1,2} \xi_{1,1} \xi_{1,2} - e^{\lambda \xi_{1,1} + \lambda \xi_{2,2}} \epsilon \lambda^2 x_{1,2} \eta_{1,2} \xi_{1,2}^2 + x_{2,2} \xi_{2,2}$

In[\*]:= **\$Basis**

Out[\*]=

{ $y_{1,2}$ ,  $x_{1,1}$ ,  $x_{1,2}$ ,  $x_{2,2}$ }

```

In[*]:= lhs = λTangent[
  Join[λ$Basis ($Basis /. {x → ξ1, y → η1}), λ$Basis ($Basis /. {x → ξ2, y → η2})]
]

Out[*]=
e^{λ ξ1,1-λ ξ1,2+λ ξ2,1-λ ξ2,2} y1,2 η1,2 + e^{λ ξ2,1-λ ξ2,2} y1,2 η2,2 + x1,1 ξ1,1 -
e^{λ ξ2,1-λ ξ2,2} λ y1,2 η2,2 ξ1,1 + e^{λ ξ1,2-λ ξ2,1+λ ξ2,2} x1,2 ξ1,2 - e^{λ ξ1,1} ∈ λ x1,1 η1,2 ξ1,2 +
e^{λ ξ1,1} ∈ λ x2,2 η1,2 ξ1,2 + e^{λ ξ1,2} ∈ λ x1,1 η2,2 ξ1,2 - e^{λ ξ1,2} ∈ λ x2,2 η2,2 ξ1,2 +
2 e^{λ ξ1,1+λ ξ2,1-λ ξ2,2} ∈ λ^2 y1,2 η1,2 η2,2 ξ1,2 - e^{λ ξ1,2+λ ξ2,1-λ ξ2,2} ∈ λ^2 y1,2 η2,2 ξ1,2 +
e^{λ ξ1,2-λ ξ2,1+λ ξ2,2} λ x1,2 ξ1,1 ξ1,2 + e^{λ ξ1,2} ∈ λ^2 x1,1 η2,2 ξ1,1 ξ1,2 - e^{λ ξ1,2} ∈ λ^2 x2,2 η2,2 ξ1,1 ξ1,2 -
e^{λ ξ1,2+λ ξ2,1-λ ξ2,2} ∈ λ^3 y1,2 η2,2 ξ1,1 ξ1,2 - e^{λ ξ1,1+λ ξ1,2-λ ξ2,1+λ ξ2,2} ∈ λ^2 x1,2 η1,2 ξ1,2 + x2,2 ξ1,2 +
e^{λ ξ2,1-λ ξ2,2} λ y1,2 η2,2 ξ1,2 + x1,1 ξ2,1 + e^{λ ξ2,2} x1,2 ξ2,2 - e^{λ ξ1,1-λ ξ1,2+λ ξ2,1} ∈ λ x1,1 η1,2 ξ2,2 +
e^{λ ξ1,1-λ ξ1,2+λ ξ2,1} ∈ λ x2,2 η1,2 ξ2,2 - e^{λ ξ2,1} ∈ λ x1,1 η2,2 ξ2,2 + e^{λ ξ2,1} ∈ λ x2,2 η2,2 ξ2,2 +
e^{λ ξ2,2} λ x1,2 ξ1,1 ξ2,2 + e^{λ ξ2,1} ∈ λ^2 x1,1 η2,2 ξ1,1 ξ2,2 - e^{λ ξ2,1} ∈ λ^2 x2,2 η2,2 ξ1,1 ξ2,2 -
2 e^{λ ξ1,1+λ ξ2,2} ∈ λ^2 x1,2 η1,2 ξ1,2 ξ2,2 + 2 e^{λ ξ1,2+λ ξ2,2} ∈ λ^2 x1,2 η2,2 ξ1,2 ξ2,2 +
2 e^{λ ξ1,2+λ ξ2,2} ∈ λ^3 x1,2 η2,2 ξ1,1 ξ1,2 ξ2,2 - e^{λ ξ2,2} λ x1,2 ξ1,2 ξ2,2 -
e^{λ ξ2,1} ∈ λ^2 x1,1 η2,2 ξ1,2 ξ2,2 + e^{λ ξ2,1} ∈ λ^2 x2,2 η2,2 ξ1,2 ξ2,2 + e^{λ ξ2,2} λ x1,2 ξ2,1 ξ2,2 -
e^{λ ξ1,1-λ ξ1,2+λ ξ2,1+λ ξ2,2} ∈ λ^2 x1,2 η1,2 ξ2,2 - e^{λ ξ2,1+λ ξ2,2} ∈ λ^2 x1,2 η2,2 ξ2,2 +
e^{λ ξ2,1+λ ξ2,2} ∈ λ^3 x1,2 η2,2 ξ1,1 ξ2,2 - e^{λ ξ2,1+λ ξ2,2} ∈ λ^3 x1,2 η2,2 ξ1,2 ξ2,2 + x2,2 ξ2,2

In[*]:= rhs = λTangent@($Basis ($Basis /. {x_{α,α} → CF[f_{α,α}[λ]], x_{αβ} → f_{αβ}[λ], y_{αβ} → g_{αβ}[λ]}))

Out[*]=
x1,1 f1,1'[λ] + e^{f2,2[λ]} x1,2 f1,2[λ] f1,1'[λ] + e^{f2,2[λ]} x1,2 f1,2'[λ] +
x2,2 f2,2'[λ] + e^{f1,1[λ]-f2,2[λ]} y1,2 g1,2'[λ] - e^{f1,1[λ]} ∈ x1,1 f1,2[λ] g1,2'[λ] +
e^{f1,1[λ]} ∈ x2,2 f1,2[λ] g1,2'[λ] - e^{f1,1[λ]+f2,2[λ]} ∈ x1,2 f1,2[λ]^2 g1,2'[λ]

In[*]:= SSBasis = Select[$Basis, SSQ]

Out[*]=
{x1,1, x2,2}

In[*]:= eqns = Simplify[(Coefficient[lhs - rhs, #] == 0) & /@ $Basis]

Out[*]=
{e^{λ (ξ1,1-ξ1,2+ξ2,1-ξ2,2)} η1,2 (1 + 2 e^{λ ξ1,2} ∈ λ^2 η2,2 ξ1,2) ==
e^{λ (ξ1,2+ξ2,1-ξ2,2)} ∈ λ^2 η2,2^2 (1 + λ ξ1,1) ξ1,2 +
e^{λ (ξ2,1-ξ2,2)} η2,2 (-1 + λ ξ1,1 - λ ξ1,2) + e^{f1,1[λ]-f2,2[λ]} g1,2'[λ],
ξ2,1 + ξ1,1 (1 + e^{λ^2 η2,2} (e^{λ ξ1,2} ξ1,2 + e^{λ ξ2,1} ξ2,2)) + e^{f1,1[λ]} ∈ f1,2[λ] g1,2'[λ] ==
e^{λ (ξ1,1-ξ1,2)} ∈ λ η1,2 (e^{λ ξ1,2} ξ1,2 + e^{λ ξ2,1} ξ2,2) +
∈ λ η2,2 (-e^{λ ξ1,2} ξ1,2 + e^{λ ξ2,1} (1 + λ ξ1,2) ξ2,2) + f1,1'[λ],
e^{-λ (ξ1,2+ξ2,1)} (-e^{λ (ξ1,1+2 ξ1,2+ξ2,2)} ∈ λ^2 η1,2 ξ1,2^2 + e^{λ (ξ1,2+ξ2,2)} ξ1,2 (-2 e^{λ (ξ1,1+ξ2,1)} ∈ λ^2 η1,2
ξ2,2 + e^{λ ξ2,2} (1 + 2 e^{λ ξ2,1} ∈ λ^2 η2,2 ξ2,2) + e^{λ ξ1,2} λ ξ1,1 (1 + 2 e^{λ ξ2,1} ∈ λ^2 η2,2 ξ2,2)) -
e^{λ ξ2,1} (e^{λ (ξ1,2+ξ2,2)} (-1 - λ ξ1,1 + λ ξ1,2 - λ ξ2,1) ξ2,2 +
e^{λ (ξ2,1+ξ2,2)} ∈ λ^2 (e^{λ ξ1,1} η1,2 + e^{λ ξ1,2} η2,2 (1 - λ ξ1,1 + λ ξ1,2)) ξ2,2^2 +
e^{λ ξ1,2+f2,2[λ]} (f1,2[λ] f1,1'[λ] + f1,2'[λ] - e^{f1,1[λ]} ∈ f1,2[λ]^2 g1,2'[λ])) == 0,
ξ1,2 + e^{λ η1,2} (e^{λ ξ1,1} ξ1,2 + e^{λ (ξ1,1-ξ1,2+ξ2,1)} ξ2,2) + ξ2,2 ==
∈ λ η2,2 (e^{λ ξ1,2} (1 + λ ξ1,1) ξ1,2 + e^{λ ξ2,1} (-1 + λ ξ1,1 - λ ξ1,2) ξ2,2) +
f2,2'[λ] + e^{f1,1[λ]} ∈ f1,2[λ] g1,2'[λ]}

```

```

In[*]:= Length@eqns
Out[*]=
4

In[*]:= unknowns = $Basis /. {xαβ → fαβ[λ], yαβ → gαβ[λ]}
Out[*]=
{g1,2[λ], f1,1[λ], f1,2[λ], f2,2[λ]}

In[*]:= $Basis /. {xαβ ⇒ fαβ[0], yαβ ⇒ gαβ[0] == 0}
Out[*]=
{g1,2[0] == 0, f1,1[0] == 0, f1,2[0] == 0, f2,2[0] == 0}

In[*]:= {sol0} = Block[{ $k = 0}, Simplify@DSolve[
  lhs = λTangent[
    Join[λ $Basis ($Basis /. {x → ξ1, y → η1}), λ $Basis ($Basis /. {x → ξ2, y → η2})]
  ];
  rhs = λTangent@($Basis ($Basis /. {xα, α ⇒ CF[fα,α[λ]], xαβ → fαβ[λ], yαβ → gαβ[λ]}) );
  Join[
    (Coefficient[CF[lhs - rhs], #] == 0) & /@ $Basis,
    $Basis /. {xαβ ⇒ fαβ[0] == 0, yαβ ⇒ gαβ[0] == 0}
  ],
  $Basis /. {xαβ → fαβ[λ], yαβ → gαβ[λ]},
  λ
]]
Out[*]=
{ { f1,1[λ] → λ (ξ1,1 + ξ2,1), f2,2[λ] → λ (ξ1,2 + ξ2,2),
  f1,2[λ] → e-λ ξ2,1 λ ξ1,2 + e-λ ξ1,2 λ ξ2,2, g1,2[λ] → λ (η1,2 + eλ (-ξ1,1 + ξ1,2) η2,2) } }

In[*]:= (sol0 /. (a_ → b_) ⇒ (a → e a + b))
Out[*]=
{ f1,1[λ] → λ (ξ1,1 + ξ2,1) + e f1,1[λ], f2,2[λ] → λ (ξ1,2 + ξ2,2) + e f2,2[λ],
  f1,2[λ] → e-λ ξ2,1 λ ξ1,2 + e-λ ξ1,2 λ ξ2,2 + e f1,2[λ],
  g1,2[λ] → λ (η1,2 + eλ (-ξ1,1 + ξ1,2) η2,2) + e g1,2[λ] }

```

```

In[*]:= {sol1} = Block[{ $k = 1, Simplify@DSolve[
  lhs = λTangent[
    Join[λ$Basis ($Basis /. {x → ξ1, y → η1}), λ$Basis ($Basis /. {x → ξ2, y → η2})]
  ];
  rhs = EchoLabel["rhs"]@λTangent@Echo[
    $Basis ($Basis /. {xα, α → CF[fα,α[λ]], xαβ → fαβ[λ], yαβ → gαβ[λ]}) /.
    (sol0 /. ((a → b) → (a → e a + b)))
  ];
  EchoLabel["eqns"]@Join[
    (Coefficient[CF[lhs - rhs], #] == 0) & /@ $Basis,
    $Basis /. {xαβ → fαβ[0] == 0, yαβ → gαβ[0] == 0}
  ],
  $Basis /. {xαβ → fαβ[λ], yαβ → gαβ[λ]},
  λ
]]

```

$\gg \{y_{1,2} (\lambda (\eta_{1,2} + e^{\lambda (-\xi_{1,1} + \xi_{2,2})} \eta_{2,2}) + e g_{1,2}[\lambda]), x_{1,1} (\lambda (\xi_{1,1} + \xi_{2,1}) + e f_{1,1}[\lambda]),$   
 $x_{1,2} (e^{-\lambda \xi_{2,1}} \lambda \xi_{1,2} + e^{-\lambda \xi_{1,2}} \lambda \xi_{2,2} + e f_{1,2}[\lambda]), x_{2,2} (\lambda (\xi_{1,2} + \xi_{2,2}) + e f_{2,2}[\lambda]) \}$

$\gg \text{rhs } e^{\lambda \xi_{1,1} - \lambda \xi_{1,2} + \lambda \xi_{2,1} - \lambda \xi_{2,2}} y_{1,2} \eta_{1,2} + e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} y_{1,2} \eta_{2,2} + x_{1,1} \xi_{1,1} -$   
 $e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} \lambda y_{1,2} \eta_{2,2} \xi_{1,1} + e^{\lambda \xi_{1,2} - \lambda \xi_{1,1} + \lambda \xi_{2,2}} x_{1,2} \xi_{1,2} - e^{\lambda \xi_{1,1}} e \lambda x_{1,1} \eta_{1,2} \xi_{1,2} +$   
 $e^{\lambda \xi_{1,1}} e \lambda x_{2,2} \eta_{1,2} \xi_{1,2} - e^{\lambda \xi_{1,2}} e \lambda x_{1,1} \eta_{2,2} \xi_{1,2} + e^{\lambda \xi_{1,2}} e \lambda x_{2,2} \eta_{2,2} \xi_{1,2} +$   
 $e^{\lambda \xi_{1,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} \lambda x_{1,2} \xi_{1,1} \xi_{1,2} + e^{\lambda \xi_{1,2}} e \lambda^2 x_{1,1} \eta_{2,2} \xi_{1,1} \xi_{1,2} -$   
 $e^{\lambda \xi_{1,2}} e \lambda^2 x_{2,2} \eta_{2,2} \xi_{1,1} \xi_{1,2} - e^{\lambda \xi_{1,1} + \lambda \xi_{1,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} e \lambda^2 x_{1,2} \eta_{1,2} \xi_{1,2} -$   
 $e^2 \lambda \xi_{1,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2} e \lambda^2 x_{1,2} \eta_{2,2} \xi_{1,2}^2 + e^2 \lambda \xi_{1,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2} e \lambda^3 x_{1,2} \eta_{2,2} \xi_{1,1} \xi_{1,2}^2 +$   
 $e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} \lambda y_{1,2} \eta_{2,2} \xi_{1,2} - e^{\lambda \xi_{1,2}} e \lambda^2 x_{1,1} \eta_{2,2} \xi_{1,2} \xi_{1,2} + e^{\lambda \xi_{1,2}} e \lambda^2 x_{2,2} \eta_{2,2} \xi_{1,2} \xi_{1,2} -$   
 $e^2 \lambda \xi_{1,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2} e \lambda^3 x_{1,2} \eta_{2,2} \xi_{1,2}^2 \xi_{1,2} + x_{1,1} \xi_{2,1} + e^{\lambda \xi_{2,2}} x_{1,2} \xi_{2,1} -$   
 $e^{\lambda \xi_{1,1} - \lambda \xi_{1,2} + \lambda \xi_{2,1}} e \lambda x_{1,1} \eta_{1,2} \xi_{2,2} + e^{\lambda \xi_{1,1} - \lambda \xi_{1,2} + \lambda \xi_{2,1}} e \lambda x_{2,2} \eta_{1,2} \xi_{2,2} - e^{\lambda \xi_{2,1}} e \lambda x_{1,1} \eta_{2,2} \xi_{2,2} +$   
 $e^{\lambda \xi_{2,1}} e \lambda x_{2,2} \eta_{2,2} \xi_{2,2} + e^{\lambda \xi_{2,2}} \lambda x_{1,2} \xi_{1,1} \xi_{2,2} + e^{\lambda \xi_{2,1}} e \lambda^2 x_{1,1} \eta_{2,2} \xi_{1,1} \xi_{2,2} -$   
 $e^{\lambda \xi_{2,1}} e \lambda^2 x_{2,2} \eta_{2,2} \xi_{1,1} \xi_{2,2} - 2 e^{\lambda \xi_{1,1} + \lambda \xi_{1,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} e \lambda^2 x_{1,2} \eta_{1,2} \xi_{1,2} \xi_{2,1} -$   
 $2 e^{\lambda \xi_{1,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} e \lambda^2 x_{1,2} \eta_{2,2} \xi_{1,2} \xi_{2,1} + 2 e^{\lambda \xi_{1,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} e \lambda^3 x_{1,2} \eta_{2,2} \xi_{1,1} \xi_{1,2} \xi_{2,1} -$   
 $e^{\lambda \xi_{2,2}} \lambda x_{1,2} \xi_{1,2} \xi_{2,1} - e^{\lambda \xi_{2,1}} e \lambda^2 x_{1,1} \eta_{2,2} \xi_{1,2} \xi_{2,1} + e^{\lambda \xi_{2,1}} e \lambda^2 x_{2,2} \eta_{2,2} \xi_{1,2} \xi_{2,1} -$   
 $2 e^{\lambda \xi_{1,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} e \lambda^3 x_{1,2} \eta_{2,2} \xi_{1,2} \xi_{1,2} \xi_{2,1} + e^{\lambda \xi_{2,2}} \lambda x_{1,2} \xi_{2,1} \xi_{2,1} -$   
 $e^{\lambda \xi_{1,1} - \lambda \xi_{1,2} + \lambda \xi_{2,1} + \lambda \xi_{2,2}} e \lambda^2 x_{1,2} \eta_{1,2} \xi_{2,1}^2 - e^{\lambda \xi_{2,1} + \lambda \xi_{2,2}} e \lambda^2 x_{1,2} \eta_{2,2} \xi_{2,1}^2 +$   
 $e^{\lambda \xi_{2,1} + \lambda \xi_{2,2}} e \lambda^3 x_{1,2} \eta_{2,2} \xi_{1,1} \xi_{2,1}^2 - e^{\lambda \xi_{2,1} + \lambda \xi_{2,2}} e \lambda^3 x_{1,2} \eta_{2,2} \xi_{1,2} \xi_{2,1}^2 +$   
 $e^{\lambda \xi_{1,1} - \lambda \xi_{1,2} + \lambda \xi_{2,1} - \lambda \xi_{2,2}} e y_{1,2} \eta_{1,2} f_{1,1}[\lambda] + e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} e y_{1,2} \eta_{2,2} f_{1,1}[\lambda] -$   
 $e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} e \lambda y_{1,2} \eta_{2,2} \xi_{1,1} f_{1,1}[\lambda] + e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} e \lambda y_{1,2} \eta_{2,2} \xi_{1,2} f_{1,1}[\lambda] +$   
 $e^{\lambda \xi_{1,2} + \lambda \xi_{2,2}} e x_{1,2} \xi_{1,1} f_{1,2}[\lambda] + e^{\lambda \xi_{1,2} + \lambda \xi_{2,2}} e x_{1,2} \xi_{2,1} f_{1,2}[\lambda] -$   
 $e^{\lambda \xi_{1,1} - \lambda \xi_{1,2} + \lambda \xi_{2,1} - \lambda \xi_{2,2}} e y_{1,2} \eta_{1,2} f_{2,2}[\lambda] - e^{\lambda \xi_{1,1} - \lambda \xi_{1,2} + \lambda \xi_{2,1}} e y_{1,2} \eta_{2,2} f_{2,2}[\lambda] +$   
 $e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} e \lambda y_{1,2} \eta_{2,2} \xi_{1,1} f_{2,2}[\lambda] + e^{\lambda \xi_{1,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} e x_{1,2} \xi_{1,2} f_{2,2}[\lambda] +$   
 $e^{\lambda \xi_{1,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} e \lambda x_{1,2} \xi_{1,1} \xi_{1,2} f_{2,2}[\lambda] - e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} e \lambda y_{1,2} \eta_{2,2} \xi_{1,2} f_{2,2}[\lambda] +$   
 $e^{\lambda \xi_{2,2}} e x_{1,2} \xi_{2,2} f_{2,2}[\lambda] + e^{\lambda \xi_{2,2}} e \lambda x_{1,2} \xi_{1,1} \xi_{2,2} f_{2,2}[\lambda] -$   
 $e^{\lambda \xi_{2,2}} e \lambda x_{1,2} \xi_{1,2} \xi_{2,2} f_{2,2}[\lambda] + e^{\lambda \xi_{2,2}} e \lambda x_{1,2} \xi_{2,1} \xi_{2,2} f_{2,2}[\lambda] + e x_{1,1} f_{1,1}'[\lambda] +$   
 $e^{\lambda \xi_{1,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} e \lambda x_{1,2} \xi_{1,2} f_{1,1}'[\lambda] + e^{\lambda \xi_{2,2}} e \lambda x_{1,2} \xi_{2,2} f_{1,1}'[\lambda] +$   
 $e^{\lambda \xi_{1,2} + \lambda \xi_{2,2}} e x_{1,2} f_{1,2}'[\lambda] + x_{2,2} (\xi_{1,2} + \xi_{2,2} + e f_{2,2}'[\lambda]) + e^{\lambda \xi_{1,1} - \lambda \xi_{1,2} + \lambda \xi_{2,1} - \lambda \xi_{2,2}} e y_{1,2} g_{1,2}'[\lambda]$

» eqns

$$\begin{aligned}
& \left\{ -e^{\lambda \xi_{1,1} - \lambda \xi_{1,2} + \lambda \xi_{2,1} - \lambda \xi_{2,2}} \in \eta_{1,2} f_{1,1}[\lambda] - e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} \in \eta_{2,2} f_{1,1}[\lambda] + e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} \in \lambda \eta_{2,2} \xi_{1,1} f_{1,1}[\lambda] - \right. \\
& \quad e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} \in \lambda \eta_{2,2} \xi_{1,2} f_{1,1}[\lambda] + e^{\lambda \xi_{1,1} - \lambda \xi_{1,2} + \lambda \xi_{2,1} - \lambda \xi_{2,2}} \in \eta_{1,2} f_{2,2}[\lambda] + \\
& \quad e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} \in \eta_{2,2} f_{2,2}[\lambda] - e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} \in \lambda \eta_{2,2} \xi_{1,1} f_{2,2}[\lambda] + \\
& \quad e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} \in \lambda \eta_{2,2} \xi_{1,2} f_{2,2}[\lambda] - e^{\lambda \xi_{1,1} - \lambda \xi_{1,2} + \lambda \xi_{2,1} - \lambda \xi_{2,2}} \in g_{1,2}'[\lambda] = 0, \\
& \quad e^{\lambda \xi_{1,1}} \in \lambda \eta_{1,2} \xi_{1,2} + e^{\lambda \xi_{1,2}} \in \lambda \eta_{2,2} \xi_{1,2} - e^{\lambda \xi_{1,2}} \in \lambda^2 \eta_{2,2} \xi_{1,1} \xi_{1,2} + \\
& \quad e^{\lambda \xi_{1,2}} \in \lambda^2 \eta_{2,2} \xi_{1,2} \xi_{1,2} + e^{\lambda \xi_{1,1} - \lambda \xi_{1,2} + \lambda \xi_{2,1}} \in \lambda \eta_{1,2} \xi_{2,1} + e^{\lambda \xi_{2,1}} \in \lambda \eta_{2,2} \xi_{2,1} - \\
& \quad e^{\lambda \xi_{2,1}} \in \lambda^2 \eta_{2,2} \xi_{1,1} \xi_{2,1} + e^{\lambda \xi_{2,1}} \in \lambda^2 \eta_{2,2} \xi_{1,2} \xi_{2,1} - e^{\lambda \xi_{2,1}} \in f_{1,1}'[\lambda] = 0, \\
& \quad e^{\lambda \xi_{1,1} + \lambda \xi_{1,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} \in \lambda^2 \eta_{1,2} \xi_{1,2}^2 + e^{\lambda \xi_{1,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} \in \lambda^2 \eta_{2,2} \xi_{1,2}^2 - \\
& \quad e^{\lambda \xi_{1,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} \in \lambda^3 \eta_{2,2} \xi_{1,1} \xi_{1,2}^2 + e^{\lambda \xi_{1,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} \in \lambda^3 \eta_{2,2} \xi_{1,2}^2 \xi_{1,2} + \\
& \quad 2 e^{\lambda \xi_{1,1} + \lambda \xi_{2,2}} \in \lambda^2 \eta_{1,2} \xi_{1,2} \xi_{2,1} + 2 e^{\lambda \xi_{1,2} + \lambda \xi_{2,2}} \in \lambda^2 \eta_{2,2} \xi_{1,2} \xi_{2,1} - \\
& \quad 2 e^{\lambda \xi_{1,2} + \lambda \xi_{2,2}} \in \lambda^3 \eta_{2,2} \xi_{1,1} \xi_{1,2} \xi_{2,1} + 2 e^{\lambda \xi_{1,2} + \lambda \xi_{2,2}} \in \lambda^3 \eta_{2,2} \xi_{1,2} \xi_{1,2} \xi_{2,1} + \\
& \quad e^{\lambda \xi_{1,1} - \lambda \xi_{1,2} + \lambda \xi_{2,1} + \lambda \xi_{2,2}} \in \lambda^2 \eta_{1,2} \xi_{2,1}^2 + e^{\lambda \xi_{2,1} + \lambda \xi_{2,2}} \in \lambda^2 \eta_{2,2} \xi_{2,1}^2 - e^{\lambda \xi_{2,1} + \lambda \xi_{2,2}} \in \lambda^3 \eta_{2,2} \xi_{1,1} \xi_{2,1}^2 + \\
& \quad e^{\lambda \xi_{2,1} + \lambda \xi_{2,2}} \in \lambda^3 \eta_{2,2} \xi_{1,2} \xi_{2,1}^2 - e^{\lambda \xi_{1,2} + \lambda \xi_{2,2}} \in \xi_{1,1} f_{1,2}[\lambda] - e^{\lambda \xi_{1,2} + \lambda \xi_{2,2}} \in \xi_{2,1} f_{1,2}[\lambda] - \\
& \quad e^{\lambda \xi_{1,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} \in \xi_{1,2} f_{2,2}[\lambda] - e^{\lambda \xi_{1,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} \in \lambda \xi_{1,1} \xi_{1,2} f_{2,2}[\lambda] - e^{\lambda \xi_{2,2}} \in \xi_{2,1} f_{2,2}[\lambda] - \\
& \quad e^{\lambda \xi_{2,2}} \in \lambda \xi_{1,1} \xi_{2,1} f_{2,2}[\lambda] + e^{\lambda \xi_{2,2}} \in \lambda \xi_{1,2} \xi_{2,1} f_{2,2}[\lambda] - e^{\lambda \xi_{2,2}} \in \lambda \xi_{2,1} \xi_{2,1} f_{2,2}[\lambda] - \\
& \quad e^{\lambda \xi_{1,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} \in \lambda \xi_{1,2} f_{1,1}'[\lambda] - e^{\lambda \xi_{2,2}} \in \lambda \xi_{2,1} f_{1,1}'[\lambda] - e^{\lambda \xi_{1,2} + \lambda \xi_{2,2}} \in f_{1,2}'[\lambda] = 0, \\
& \quad -e^{\lambda \xi_{1,1}} \in \lambda \eta_{1,2} \xi_{1,2} - e^{\lambda \xi_{1,2}} \in \lambda \eta_{2,2} \xi_{1,2} + e^{\lambda \xi_{1,2}} \in \lambda^2 \eta_{2,2} \xi_{1,1} \xi_{1,2} - e^{\lambda \xi_{1,2}} \in \lambda^2 \eta_{2,2} \xi_{1,2} \xi_{1,2} - \\
& \quad e^{\lambda \xi_{1,1} - \lambda \xi_{1,2} + \lambda \xi_{2,1}} \in \lambda \eta_{1,2} \xi_{2,1} - e^{\lambda \xi_{2,1}} \in \lambda \eta_{2,2} \xi_{2,1} + e^{\lambda \xi_{2,1}} \in \lambda^2 \eta_{2,2} \xi_{1,1} \xi_{2,1} - \\
& \quad e^{\lambda \xi_{2,1}} \in \lambda^2 \eta_{2,2} \xi_{1,2} \xi_{2,1} - e^{\lambda \xi_{2,1}} \in f_{2,2}'[\lambda] = 0, g_{1,2}[0] = 0, f_{1,1}[0] = 0, f_{1,2}[0] = 0, f_{2,2}[0] = 0 \}
\end{aligned}$$

Out[8]=

$$\begin{aligned}
& \left\{ f_{1,1}[\lambda] \rightarrow \frac{1}{\xi_{1,1}^2 (\xi_{1,1} - \xi_{1,2} + \xi_{2,1})^2} \right. \\
& \quad \eta_{1,2} \left( - \left( (-1 + e^{\lambda \xi_{1,1}}) \xi_{1,2} (\xi_{1,2} - \xi_{2,1})^2 \right) + \xi_{1,1} \xi_{1,2} (\xi_{1,2} - \xi_{2,1}) (-2 + 2 e^{\lambda \xi_{1,1}} + \right. \\
& \quad \left. e^{\lambda \xi_{1,1}} \lambda \xi_{1,2} - e^{\lambda \xi_{1,1}} \lambda \xi_{2,1}) + \lambda \xi_{1,1}^3 (e^{\lambda \xi_{1,1}} \xi_{1,2} + e^{\lambda (\xi_{1,1} - \xi_{1,2} + \xi_{2,1})} \xi_{2,1}) + \right. \\
& \quad \left. \xi_{1,1}^2 (-\xi_{1,2} (-1 + e^{\lambda \xi_{1,1}} + 2 e^{\lambda \xi_{1,1}} \lambda \xi_{1,2} - 2 e^{\lambda \xi_{1,1}} \lambda \xi_{2,1}) - \right. \\
& \quad \left. (-1 + e^{\lambda (\xi_{1,1} - \xi_{1,2} + \xi_{2,1})} + e^{\lambda (\xi_{1,1} - \xi_{1,2} + \xi_{2,1})} \lambda \xi_{1,2} - e^{\lambda (\xi_{1,1} - \xi_{1,2} + \xi_{2,1})} \lambda \xi_{2,1}) \xi_{2,1}) \right) + \\
& \quad \frac{1}{\xi_{1,2}^2 \xi_{2,1}^3} \eta_{2,2} \left( \xi_{1,2} \xi_{1,2} (-1 + e^{\lambda \xi_{1,2}} - e^{\lambda \xi_{1,2}} \lambda \xi_{1,2} + e^{\lambda \xi_{1,2}} \lambda^2 \xi_{1,2}^2) \xi_{2,1}^3 + \xi_{1,2}^3 \xi_{2,1}^2 \right. \\
& \quad \left( \xi_{2,1} (1 - e^{\lambda \xi_{2,1}} + e^{\lambda \xi_{2,1}} \lambda \xi_{2,1}) + \xi_{1,2} (-2 + 2 e^{\lambda \xi_{2,1}} - 2 e^{\lambda \xi_{2,1}} \lambda \xi_{2,1} + e^{\lambda \xi_{2,1}} \lambda^2 \xi_{2,1}^2) \right) \\
& \quad \xi_{2,1} - \xi_{1,1} (\xi_{1,2} (-2 + 2 e^{\lambda \xi_{1,2}} - 2 e^{\lambda \xi_{1,2}} \lambda \xi_{1,2} + e^{\lambda \xi_{1,2}} \lambda^2 \xi_{1,2}^2) \xi_{2,1}^3 + \\
& \quad \xi_{1,2}^3 (-2 + 2 e^{\lambda \xi_{2,1}} - 2 e^{\lambda \xi_{2,1}} \lambda \xi_{2,1} + e^{\lambda \xi_{2,1}} \lambda^2 \xi_{2,1}^2) \xi_{2,1}) \left. \right), \\
& \quad f_{2,2}[\lambda] \rightarrow - \frac{1}{\xi_{1,1}^2 (\xi_{1,1} - \xi_{1,2} + \xi_{2,1})^2} \eta_{1,2} \left( - \left( (-1 + e^{\lambda \xi_{1,1}}) \xi_{1,2} (\xi_{1,2} - \xi_{2,1})^2 \right) + \right. \\
& \quad \xi_{1,1} \xi_{1,2} (\xi_{1,2} - \xi_{2,1}) (-2 + 2 e^{\lambda \xi_{1,1}} + e^{\lambda \xi_{1,1}} \lambda \xi_{1,2} - e^{\lambda \xi_{1,1}} \lambda \xi_{2,1}) + \\
& \quad \lambda \xi_{1,1}^3 (e^{\lambda \xi_{1,1}} \xi_{1,2} + e^{\lambda (\xi_{1,1} - \xi_{1,2} + \xi_{2,1})} \xi_{2,1}) + \\
& \quad \xi_{1,1}^2 (-\xi_{1,2} (-1 + e^{\lambda \xi_{1,1}} + 2 e^{\lambda \xi_{1,1}} \lambda \xi_{1,2} - 2 e^{\lambda \xi_{1,1}} \lambda \xi_{2,1}) - \\
& \quad \left. (-1 + e^{\lambda (\xi_{1,1} - \xi_{1,2} + \xi_{2,1})} + e^{\lambda (\xi_{1,1} - \xi_{1,2} + \xi_{2,1})} \lambda \xi_{1,2} - e^{\lambda (\xi_{1,1} - \xi_{1,2} + \xi_{2,1})} \lambda \xi_{2,1}) \xi_{2,1}) \right) + \\
& \quad \frac{1}{\xi_{1,2}^2 \xi_{2,1}^3} \eta_{2,2} \left( \xi_{1,1} (\xi_{1,2} (-2 + 2 e^{\lambda \xi_{1,2}} - 2 e^{\lambda \xi_{1,2}} \lambda \xi_{1,2} + e^{\lambda \xi_{1,2}} \lambda^2 \xi_{1,2}^2) \xi_{2,1}^3 + \right. \\
& \quad \xi_{1,2}^3 (-2 + 2 e^{\lambda \xi_{2,1}} - 2 e^{\lambda \xi_{2,1}} \lambda \xi_{2,1} + e^{\lambda \xi_{2,1}} \lambda^2 \xi_{2,1}^2) \xi_{2,1}) - \\
& \quad \xi_{1,2} (\xi_{1,2} (-1 + e^{\lambda \xi_{1,2}} - e^{\lambda \xi_{1,2}} \lambda \xi_{1,2} + e^{\lambda \xi_{1,2}} \lambda^2 \xi_{1,2}^2) \xi_{2,1}^3 + \xi_{1,2}^2 (\xi_{2,1} (1 - e^{\lambda \xi_{2,1}} + \\
& \quad \left. e^{\lambda \xi_{2,1}} \lambda \xi_{2,1}) + \xi_{1,2} (-2 + 2 e^{\lambda \xi_{2,1}} - 2 e^{\lambda \xi_{2,1}} \lambda \xi_{2,1} + e^{\lambda \xi_{2,1}} \lambda^2 \xi_{2,1}^2) \xi_{2,1}) \right) \left. \right),
\end{aligned}$$

$$\begin{aligned}
f_{1,2}[\lambda] \rightarrow & \frac{1}{4 \xi_{1,1}^3 (\xi_{1,1} - \xi_{1,2,2} + \xi_{2,1,1})^3} e^{-\lambda (\xi_{1,1} + \xi_{2,1,1})} \left( \frac{1}{(2 \xi_{1,1} - \xi_{1,2,2} + \xi_{2,1,1})^3} \right. \\
& \left( \eta_{1,2} (33 \xi_{1,1}^2 \xi_{1,2}^2 (\xi_{1,2,2} - \xi_{2,1,1})^4 - 9 \xi_{1,1,1} \xi_{1,2}^2 (\xi_{1,2,2} - \xi_{2,1,1})^5 + \xi_{1,2}^2 (\xi_{1,2,2} - \xi_{2,1,1})^6 + \right. \\
& 8 \xi_{1,1}^6 (\xi_{1,2,2} + \xi_{2,1,2})^2 - 12 \xi_{1,1}^5 (\xi_{1,2,2} - \xi_{2,1,1}) (3 \xi_{1,2}^2 + 4 \xi_{1,2} \xi_{2,1,2} + \xi_{2,1,2}^2) + \\
& 6 \xi_{1,1}^4 (\xi_{1,2,2} - \xi_{2,1,1})^2 (11 \xi_{1,2}^2 + 8 \xi_{1,2} \xi_{2,1,2} + \xi_{2,1,2}^2) - \\
& \xi_{1,1}^3 (\xi_{1,2,2} - \xi_{2,1,1})^3 (63 \xi_{1,2}^2 + 16 \xi_{1,2} \xi_{2,1,2} + \xi_{2,1,2}^2) \Big) + \\
& (16 \eta_{2,1,2} \xi_{1,1}^3 (2 \xi_{1,1}^2 - 3 \xi_{1,1} (\xi_{1,2,2} - \xi_{2,1,1}) + (\xi_{1,2,2} - \xi_{2,1,1})^2)^3 \\
& (-\xi_{1,2,2}^2 \xi_{1,2} (\xi_{1,2,2} - 2 \xi_{2,1,1})^4 \xi_{2,1,1}^4 - \xi_{1,2} \xi_{1,2,2}^4 (\xi_{1,2,2} - 2 \xi_{2,1,1})^4 (3 \xi_{1,2,2} - \xi_{2,1,1}) \\
& \xi_{2,1,2} + \xi_{1,2,2}^4 \xi_{2,1,1}^4 (-2 \xi_{1,2,2} + \xi_{2,1,1}) \xi_{2,1,2}^2 + 2 \xi_{1,1}^9 (\xi_{1,2,2} + \xi_{2,1,2})^2 - 3 \xi_{1,1}^8 \\
& (\xi_{1,2,2} + \xi_{2,1,2}) (\xi_{1,2,2} (3 \xi_{1,2,2} - 8 \xi_{2,1,1}) - (2 \xi_{1,2,2} + 3 \xi_{2,1,1}) \xi_{2,1,2}) - 2 \\
& \xi_{1,1,1} (\xi_{1,2,2}^2 (\xi_{1,2,2} - 2 \xi_{2,1,1})^3 \xi_{2,1,1}^3 (2 \xi_{1,2,2}^2 - 7 \xi_{1,2,2} \xi_{2,1,1} + 2 \xi_{2,1,1}^2) - \\
& 2 \xi_{1,2} \xi_{1,2,2}^3 (\xi_{1,2,2} - 2 \xi_{2,1,1})^3 (\xi_{1,2,2}^2 + 4 \xi_{1,2,2} \xi_{2,1,1} - 2 \xi_{2,1,1}^2) \xi_{2,1,2} + \\
& \xi_{1,2,2}^3 \xi_{2,1,1}^3 (4 \xi_{1,2,2}^2 + \xi_{1,2,2} \xi_{2,1,1} - 2 \xi_{2,1,1}^2) \xi_{2,1,2}^2) + 4 \\
& \xi_{1,1,1}^7 (\xi_{1,2,2}^2 (4 \xi_{1,2,2}^2 - 23 \xi_{1,2,2} \xi_{2,1,1} + 31 \xi_{2,1,1}^2) + 2 \xi_{1,2,2} \\
& (-2 \xi_{1,2,2}^2 + \xi_{1,2,2} \xi_{2,1,1} + 13 \xi_{2,1,1}^2) \xi_{2,1,2} + (\xi_{1,2,2}^2 + 7 \xi_{1,2,2} \xi_{2,1,1} + 4 \xi_{2,1,1}^2) \xi_{2,1,2}^2) - \\
& 2 \xi_{1,1,1}^6 (\xi_{1,2,2}^2 (7 \xi_{1,2,2}^3 - 68 \xi_{1,2,2}^2 \xi_{2,1,1} + 199 \xi_{1,2,2} \xi_{2,1,1}^2 - 180 \xi_{2,1,1}^3) - \\
& 2 \xi_{1,2,2} (3 \xi_{1,2,2}^3 - 31 \xi_{1,2,2}^2 \xi_{2,1,1} + 32 \xi_{1,2,2} \xi_{2,1,1}^2 + 38 \xi_{2,1,1}^3) \xi_{2,1,2} + \\
& (2 \xi_{1,2,2}^3 - 11 \xi_{1,2,2}^2 \xi_{2,1,1} - 26 \xi_{1,2,2} \xi_{2,1,1}^2 - 7 \xi_{2,1,1}^3) \xi_{2,1,2}^2) - 6 \xi_{1,1,1}^2 \\
& (\xi_{1,2,2}^2 (\xi_{1,2,2} - 2 \xi_{2,1,1})^2 \xi_{2,1,1}^2 (\xi_{1,2,2}^3 - 8 \xi_{1,2,2}^2 \xi_{2,1,1} + 17 \xi_{1,2,2} \xi_{2,1,1}^2 - 8 \xi_{2,1,1}^3) - \\
& 2 \xi_{1,2,2} \xi_{1,2,2}^2 (\xi_{1,2,2} - 2 \xi_{2,1,1})^2 (\xi_{1,2,2}^3 - 5 \xi_{1,2,2}^2 \xi_{2,1,1} + 2 \xi_{2,1,1}^3) \xi_{2,1,2} + \xi_{1,2,2}^2 \xi_{2,1,1}^2 \\
& (2 \xi_{1,2,2}^3 + 3 \xi_{1,2,2}^2 \xi_{2,1,1} - 2 \xi_{1,2,2} \xi_{2,1,1}^2 - \xi_{2,1,1}^3) \xi_{2,1,2}^2) - 4 \xi_{1,1,1}^3 (\xi_{1,2,2} - \xi_{2,1,1}) \\
& (\xi_{1,2,2}^2 \xi_{2,1,1} (\xi_{1,2,2}^4 - 16 \xi_{1,2,2}^3 \xi_{2,1,1} + 82 \xi_{1,2,2}^2 \xi_{2,1,1}^2 - 170 \xi_{1,2,2} \xi_{2,1,1}^3 + 124 \xi_{2,1,1}^4) + \\
& 2 \xi_{1,2,2} \xi_{1,2,2} (2 \xi_{1,2,2}^4 - \xi_{1,2,2}^3 \xi_{2,1,1} - 28 \xi_{1,2,2}^2 \xi_{2,1,1}^2 + 40 \xi_{1,2,2} \xi_{2,1,1}^3 + 8 \xi_{2,1,1}^4) \xi_{2,1,2} + \\
& \xi_{1,2,2} \xi_{2,1,1} (2 \xi_{1,2,2}^3 + 10 \xi_{1,2,2}^2 \xi_{2,1,1} + 8 \xi_{1,2,2} \xi_{2,1,1}^2 + \xi_{2,1,1}^3) \xi_{2,1,2}^2) + 6 \xi_{1,1,1}^5 \\
& (\xi_{1,2,2}^2 (\xi_{1,2,2}^4 - 16 \xi_{1,2,2}^3 \xi_{2,1,1} + 80 \xi_{1,2,2}^2 \xi_{2,1,1}^2 - 158 \xi_{1,2,2} \xi_{2,1,1}^3 + 107 \xi_{2,1,1}^4) + \\
& 4 \xi_{1,2,2} (\xi_{1,2,2}^4 - \xi_{1,2,2}^3 \xi_{2,1,1} - 11 \xi_{1,2,2}^2 \xi_{2,1,1}^2 + 14 \xi_{1,2,2} \xi_{2,1,1}^3 + 4 \xi_{2,1,1}^4) \xi_{2,1,2} + \\
& (-\xi_{1,2,2}^4 - 2 \xi_{1,2,2}^3 \xi_{2,1,1} + 8 \xi_{1,2,2}^2 \xi_{2,1,1}^2 + 8 \xi_{1,2,2} \xi_{2,1,1}^3 + \xi_{2,1,1}^4) \xi_{2,1,2}^2) + \\
& \xi_{1,1,1}^4 (-\xi_{1,2,2}^2 (\xi_{1,2,2}^5 - 32 \xi_{1,2,2}^4 \xi_{2,1,1} + 268 \xi_{1,2,2}^3 \xi_{2,1,1}^2 - 912 \xi_{1,2,2}^2 \xi_{2,1,1}^3 + \\
& 1353 \xi_{1,2,2} \xi_{2,1,1}^4 - 720 \xi_{2,1,1}^5) - 2 \xi_{1,2,2} (9 \xi_{1,2,2}^5 - 87 \xi_{1,2,2}^4 \xi_{2,1,1} + \\
& 168 \xi_{1,2,2}^3 \xi_{2,1,1}^2 + 28 \xi_{1,2,2}^2 \xi_{2,1,1}^3 - 152 \xi_{1,2,2} \xi_{2,1,1}^4 - 8 \xi_{2,1,1}^5) \xi_{2,1,2} + (-2 \xi_{1,2,2}^5 - \\
& 23 \xi_{1,2,2}^4 \xi_{2,1,1} - 8 \xi_{1,2,2}^3 \xi_{2,1,1}^2 + 52 \xi_{1,2,2}^2 \xi_{2,1,1}^3 + 22 \xi_{1,2,2} \xi_{2,1,1}^4 + \xi_{2,1,1}^5) \xi_{2,1,2}^2) \Big) \Big) / \\
& \left( (\xi_{1,1,1} + \xi_{1,2,2})^4 (\xi_{1,1,1} + \xi_{2,1,1})^4 (\xi_{1,1,1} - \xi_{1,2,2} + 2 \xi_{2,1,1})^4 \right) - \\
& \frac{1}{\xi_{1,2,2}^3 \xi_{2,1,1}^3} \xi_{1,1,1} (\xi_{1,1,1} - \xi_{1,2,2} + \xi_{2,1,1}) \left( e^{\lambda (\xi_{1,1,1} + \xi_{1,2,2})} \xi_{1,1,1} \xi_{1,2}^2 \xi_{2,1,1}^3 \right. \\
& \left( \frac{16 \eta_{2,1,2} \xi_{1,1,1} (2 \xi_{1,1,1} - \xi_{1,2,2}) \xi_{1,2,2}^3 (\xi_{1,1,1} - \xi_{1,2,2} + \xi_{2,1,1})^2}{(\xi_{1,1,1} + \xi_{1,2,2})^4} + \right. \\
& \frac{4 \lambda^3 \eta_{2,1,2} \xi_{1,1,1}^2 (\xi_{1,1,1} - \xi_{1,2,2}) \xi_{1,2,2}^2 (\xi_{1,1,1} - \xi_{1,2,2} + \xi_{2,1,1})^2}{\xi_{1,1,1} + \xi_{1,2,2}} - \\
& \left. \frac{1}{(\xi_{1,1,1} + \xi_{1,2,2})^2} 4 \lambda^2 \eta_{2,1,2} \xi_{1,1,1} \xi_{1,2,2} (2 \xi_{1,1,1}^3 + 3 \xi_{1,1,1}^2 \xi_{1,2,2} - 4 \xi_{1,1,1} \xi_{1,2,2}^2 + \xi_{1,2,2}^3) \right)
\end{aligned}$$

$$\begin{aligned}
& \left( (\xi_{1,1} - \xi_{1,2} + \xi_{2,1})^2 + \frac{1}{(\xi_{1,1} + \xi_{1,2})^3} 4 \lambda \eta_{2,2} \xi_{1,1} (2 \xi_{1,1}^4 + \right. \\
& \quad 5 \xi_{1,1}^3 \xi_{1,2} + 3 \xi_{1,1}^2 \xi_{1,2}^2 - 9 \xi_{1,1} \xi_{1,2}^3 + 3 \xi_{1,2}^4) (\xi_{1,1} - \xi_{1,2} + \xi_{2,1})^2 + \\
& \quad \left. e^{\lambda (\xi_{1,1} - \xi_{1,2})} \eta_{1,2} \xi_{1,2}^3 (1 + 2 \lambda \xi_{1,1} (1 + 2 \lambda \xi_{1,2} - 2 \lambda \xi_{2,1})) \right) + \\
& \frac{1}{\xi_{1,1} - \xi_{1,2} + \xi_{2,1}} e^{\lambda (\xi_{1,1} - \xi_{1,2} + 2 \xi_{2,1})} \xi_{1,1}^2 \xi_{1,2}^3 \\
& \left( \frac{16 \eta_{2,2} \xi_{1,1}^3 (2 \xi_{1,1} - 2 \xi_{1,2} + \xi_{2,1}) (\xi_{1,1} - \xi_{1,2} + \xi_{2,1})^3}{(\xi_{1,1} - \xi_{1,2} + 2 \xi_{2,1})^4} + \right. \\
& \quad \frac{4 \lambda^3 \eta_{2,2} (\xi_{1,1} - \xi_{1,2}) \xi_{1,1}^2 (\xi_{1,1} - \xi_{1,2} + \xi_{2,1})^4}{\xi_{1,1} - \xi_{1,2} + 2 \xi_{2,1}} + e^{\lambda (\xi_{1,1} - \xi_{1,2})} \eta_{1,2} \\
& \quad \xi_{1,1}^3 (1 - 2 \lambda^2 \xi_{1,2}^2 + 2 \lambda \xi_{2,1} - 2 \lambda^2 \xi_{2,1}^2 + 2 \lambda \xi_{1,1} (1 + \lambda \xi_{1,2} - \lambda \xi_{2,1}) + \\
& \quad 2 \lambda \xi_{1,2} (-1 + 2 \lambda \xi_{2,1})) - (4 \lambda^2 \eta_{2,2} \xi_{2,1} (\xi_{1,1} - \xi_{1,2} + \xi_{2,1})^3 (2 \xi_{1,1}^3 - \\
& \quad 2 \xi_{1,2}^3 + 9 \xi_{1,2}^2 \xi_{2,1} - 8 \xi_{1,2} \xi_{2,1}^2 + 2 \xi_{2,1}^3 + \xi_{1,1}^2 (-6 \xi_{1,2} + 9 \xi_{2,1}) + \\
& \quad 2 \xi_{1,1} (3 \xi_{1,2}^2 - 9 \xi_{1,2} \xi_{2,1} + 4 \xi_{2,1}^2)) \Big) / (\xi_{1,1} - \xi_{1,2} + 2 \xi_{2,1})^2 + \\
& \frac{1}{(\xi_{1,1} - \xi_{1,2} + 2 \xi_{2,1})^3} 4 \lambda \eta_{2,2} (\xi_{1,1} - \xi_{1,2} + \xi_{2,1})^3 \\
& \left( 2 \xi_{1,1}^4 + 2 \xi_{1,2}^4 - 13 \xi_{1,2}^3 \xi_{2,1} + 30 \xi_{1,2}^2 \xi_{2,1}^2 - 20 \xi_{1,2} \xi_{2,1}^3 + 4 \xi_{2,1}^4 + \right. \\
& \quad \xi_{1,1}^3 (-8 \xi_{1,2} + 13 \xi_{2,1}) + 3 \xi_{1,1}^2 (4 \xi_{1,2}^2 - 13 \xi_{1,2} \xi_{2,1} + 10 \xi_{2,1}^2) + \\
& \quad \left. \xi_{1,1} (-8 \xi_{1,2}^3 + 39 \xi_{1,2}^2 \xi_{2,1} - 60 \xi_{1,2} \xi_{2,1}^2 + 20 \xi_{2,1}^3) \right) \\
& \xi_{1,2}^2 + 2 e^{\lambda (2 \xi_{1,1} - 2 \xi_{1,2} + \xi_{2,1})} \eta_{1,2} \xi_{1,1}^3 \xi_{1,2}^3 \xi_{2,1}^3 \xi_{2,1} \\
& \left( - \left( 2 e^{\lambda \xi_{1,2}} \xi_{1,2} (4 \lambda^2 \xi_{1,1}^3 + 4 \lambda^2 \xi_{1,1}^2 (-\xi_{1,2} + \xi_{2,1}) + \lambda \xi_{1,1} (\xi_{1,2} - \xi_{2,1})) \right. \right. \\
& \quad \left. \left. (-2 + \lambda \xi_{1,2} - \lambda \xi_{2,1}) + (\xi_{1,2} - \xi_{2,1}) (1 + \lambda \xi_{1,2} - \lambda \xi_{2,1}) \right) \right) / \\
& (2 \xi_{1,1} - \xi_{1,2} + \xi_{2,1})^3 - e^{\lambda \xi_{2,1}} \lambda^2 \xi_{2,1}^2 - \frac{1}{\xi_{1,1}} e^{\lambda \xi_{1,1}} \xi_{1,2} \\
& \left( \eta_{1,2} \xi_{1,2}^3 \xi_{2,1}^3 (2 e^{\lambda \xi_{1,1}} \lambda^2 \xi_{1,1}^4 \xi_{1,2} - e^{\lambda \xi_{1,1}} \xi_{1,2} (\xi_{1,2} - \xi_{2,1})^2 + 2 \lambda \xi_{1,1}^2 \xi_{1,2} \right. \\
& \quad \xi_{1,2} (\xi_{1,2} - \xi_{2,1}) (-4 + 2 e^{\lambda \xi_{1,1}} + e^{\lambda \xi_{1,1}} \lambda \xi_{1,2} - e^{\lambda \xi_{1,1}} \lambda \xi_{2,1}) - 2 \xi_{1,1} \xi_{1,2} \\
& \quad \xi_{1,2} (\xi_{1,2} - \xi_{2,1}) (-e^{\lambda \xi_{1,1}} + (-2 + e^{\lambda \xi_{1,1}}) \lambda \xi_{1,2} - (-2 + e^{\lambda \xi_{1,1}}) \lambda \xi_{2,1}) + \\
& \quad 4 \lambda \xi_{1,1}^3 (\xi_{1,2} + \xi_{2,1})) + 4 \lambda \eta_{2,2} \xi_{1,1}^3 (\xi_{1,1} - \xi_{1,2} + \xi_{2,1})^2 (-\xi_{1,2} \xi_{1,2}^2 \\
& \quad \xi_{2,1}^3 + \xi_{1,2}^3 (-2 \xi_{1,2} + \xi_{2,1}) \xi_{2,1} + 2 \xi_{1,1} (\xi_{1,2} \xi_{2,1}^3 + \xi_{1,2}^3 \xi_{2,1})) \Big) + \\
& \frac{1}{(\xi_{1,1} + \xi_{2,1})^4 (2 \xi_{1,1} - \xi_{1,2} + \xi_{2,1})^3} 4 e^{\lambda \xi_{2,1}} \xi_{2,1}^2 (e^{\lambda \xi_{1,1}} \xi_{1,1}^2 \xi_{1,2} \\
& \left( e^{\lambda (\xi_{1,1} - \xi_{1,2})} \eta_{1,2} \xi_{1,2}^3 \xi_{2,1}^3 (\xi_{1,1} + \xi_{2,1})^4 (4 \xi_{1,1} (\xi_{1,1} - \xi_{1,2} + \xi_{2,1}) + 2 \lambda \right. \\
& \quad (2 \xi_{1,1}^2 - 2 \xi_{1,1} (\xi_{1,2} - \xi_{2,1}) + (\xi_{1,2} - \xi_{2,1})^2) (2 \xi_{1,1} - \xi_{1,2} + \xi_{2,1}) + \\
& \quad \lambda^2 (\xi_{1,2} - \xi_{2,1}) (2 \xi_{1,1} - \xi_{1,2} + \xi_{2,1})^2 (3 \xi_{1,1} - 2 \xi_{1,2} + 2 \xi_{2,1})) + \\
& \quad \eta_{2,2} (\xi_{1,1} - \xi_{1,2} + \xi_{2,1})^2 (2 \xi_{1,1} - \xi_{1,2} + \xi_{2,1})^3 (\lambda \xi_{1,1}^5 \\
& \quad (2 \xi_{2,1}^3 - 2 \lambda \xi_{1,2} \xi_{2,1}^3 + \lambda^2 \xi_{1,2}^2 \xi_{2,1}^3 + \xi_{1,2}^3 (2 - 2 \lambda \xi_{2,1} + \lambda^2 \xi_{2,1}^2)) + \\
& \quad \lambda \xi_{1,1}^4 (8 \xi_{2,1}^4 + \lambda \xi_{1,2}^2 \xi_{2,1}^3 (1 + 4 \lambda \xi_{2,1}) - \xi_{1,2} \xi_{2,1}^3 (1 + 8 \lambda \xi_{2,1}) + \\
& \quad \xi_{1,2}^3 \xi_{2,1} (9 - 9 \lambda \xi_{2,1} + \lambda^2 \xi_{2,1}^2) - \xi_{1,2}^4 (2 - 2 \lambda \xi_{2,1} + \lambda^2 \xi_{2,1}^2)) -
\end{aligned}$$

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$$\begin{aligned}
& \xi_{1,1} \\
& \left( \eta_{1,2} \xi_{2,1}^3 \right. \\
& \quad \left( 1 - \lambda \xi_{1,1} + \lambda \xi_{1,2} - \lambda \xi_{2,1} \right) + \\
& \quad e^{\lambda (-\xi_{1,1} + \xi_{1,2})} \eta_{2,1,2} (\xi_{1,1} - \xi_{1,2} + \xi_{2,1})^2 \\
& \quad \left( \xi_{2,1,1} - \lambda \xi_{2,1,1}^2 + \xi_{1,1,1} (2 - 2\lambda \xi_{2,1,1} + \lambda^2 \xi_{2,1,1}^2) - \right. \\
& \quad \left. \xi_{1,2,2} (2 - 2\lambda \xi_{2,1,1} + \lambda^2 \xi_{2,1,1}^2) \right) \xi_{2,1,2} + \\
& \quad \left( 2 e^{\lambda \xi_{2,1,1}} (\lambda^2 \eta_{1,2} \eta_{2,1,2} (\xi_{1,1} - \xi_{1,2,2}) \xi_{2,1,1}^2 (\xi_{1,1,1}^2 - 2 \xi_{1,1,1} \xi_{1,2,2} + \xi_{1,2,2}^2 - \xi_{2,1,1}^2)^4 - \right. \\
& \quad \lambda \eta_{1,2} \eta_{2,1,2} \xi_{2,1,1} (4 \xi_{1,1,1} - 4 \xi_{1,2,2} + \xi_{2,1,1}) \\
& \quad \left( \xi_{1,1,1}^2 - 2 \xi_{1,1,1} \xi_{1,2,2} + \xi_{1,2,2}^2 - \xi_{2,1,1}^2 \right)^4 + \\
& \quad 2 \eta_{1,2} \eta_{2,1,2} (3 \xi_{1,1,1} - 3 \xi_{1,2,2} + \xi_{2,1,1}) (\xi_{1,1,1}^2 - 2 \xi_{1,1,1} \xi_{1,2,2} + \xi_{1,2,2}^2 - \xi_{2,1,1}^2)^4 - \\
& \quad e^{\lambda (\xi_{1,1,1} - \xi_{1,2,2})} \eta_{1,2}^2 \xi_{2,1,1}^4 (-\xi_{1,1,1} + \xi_{1,2,2} + \xi_{2,1,1})^4 \\
& \quad \left( (\xi_{1,2,2} - \xi_{2,1,1}) (2 + \lambda \xi_{1,2,2} - \lambda \xi_{2,1,1}) + \xi_{1,1,1} (-1 - \lambda \xi_{1,2,2} + \lambda \xi_{2,1,1}) \right) + \\
& \quad e^{\lambda (-\xi_{1,1,1} + \xi_{1,2,2})} \eta_{2,1,2} \xi_{2,1,1} (\xi_{1,1,1} - \xi_{1,2,2} + \xi_{2,1,1})^2 \\
& \quad \left( \lambda^3 \eta_{2,1,2} (\xi_{1,1,1} - \xi_{1,2,2})^2 (\xi_{1,1,1} - \xi_{1,2,2} - \xi_{2,1,1})^3 \xi_{2,1,1}^2 (\xi_{1,1,1} - \xi_{1,2,2} + \xi_{2,1,1})^2 + \right. \\
& \quad 2 \eta_{2,1,2} \xi_{2,1,1}^3 (2 \xi_{1,1,1} - 2 \xi_{1,2,2} + \xi_{2,1,1}) (\xi_{1,1,1} - \xi_{1,2,2} + \xi_{2,1,1})^2 - \\
& \quad \lambda^2 \eta_{2,1,2} (\xi_{1,1,1} - \xi_{1,2,2}) \xi_{2,1,1} (\xi_{1,1,1} - \xi_{1,2,2} + \xi_{2,1,1})^2 (-\xi_{1,1,1} + \xi_{1,2,2} + \xi_{2,1,1})^2 \\
& \quad \left( 2 \xi_{1,1,1}^2 + 2 \xi_{1,2,2}^2 + 3 \xi_{1,2,2} \xi_{2,1,1} - 2 \xi_{2,1,1}^2 - \xi_{1,1,1} (4 \xi_{1,2,2} + 3 \xi_{2,1,1}) \right) + \\
& \quad \lambda \eta_{2,1,2} (\xi_{1,1,1} - \xi_{1,2,2} - \xi_{2,1,1}) (\xi_{1,1,1} - \xi_{1,2,2} + \xi_{2,1,1})^2 (2 \xi_{1,1,1}^4 + 2 \xi_{1,2,2}^4 + 5 \xi_{1,2,2}^3 \xi_{2,1,1} + \\
& \quad 3 \xi_{1,2,2}^2 \xi_{2,1,1}^2 - 5 \xi_{1,2,2} \xi_{2,1,1}^3 + \xi_{2,1,1}^4 - \xi_{1,1,1} (8 \xi_{1,2,2} + 5 \xi_{2,1,1}) + 3 \xi_{1,1,1}^2 (4 \xi_{1,2,2}^2 + \\
& \quad 5 \xi_{1,2,2} \xi_{2,1,1} + \xi_{2,1,1}^2) - \xi_{1,1,1} (8 \xi_{1,2,2}^3 + 15 \xi_{1,2,2}^2 \xi_{2,1,1} + 6 \xi_{1,2,2} \xi_{2,1,1}^2 - 5 \xi_{2,1,1}^3) \left. \right) - \\
& \quad e^{\lambda (\xi_{1,1,1} - \xi_{1,2,2})} \eta_{1,2} (-\xi_{1,1,1} + \xi_{1,2,2} + \xi_{2,1,1})^4 (\xi_{2,1,1}^2 (-2 + \lambda \xi_{2,1,1}) + \xi_{1,2,2} \\
& \quad \xi_{2,1,1} (-2 + \lambda \xi_{2,1,1})^2 - \xi_{1,2,2}^2 (2 - 2\lambda \xi_{2,1,1} + \lambda^2 \xi_{2,1,1}^2) + \xi_{1,1,1} \\
& \quad \left( -\xi_{2,1,1} (3 - 3\lambda \xi_{2,1,1} + \lambda^2 \xi_{2,1,1}^2) + \xi_{1,2,2} (2 - 2\lambda \xi_{2,1,1} + \lambda^2 \xi_{2,1,1}^2) \right) \left. \right) \xi_{2,1,2} \Big) / \\
& \quad \left( \xi_{2,1,1}^4 (\xi_{1,1,1} - \xi_{1,2,2} + \xi_{2,1,1})^4 (-\xi_{1,1,1} + \xi_{1,2,2} + \xi_{2,1,1})^4 \right) - \left( 4 \right. \\
& \quad \eta_{1,2}^2 \\
& \quad \left( 3 \xi_{1,1,1} \xi_{1,2} (\xi_{1,2,2} - \xi_{2,1,1})^2 - \right. \\
& \quad \xi_{1,1,2} (\xi_{1,2,2} - \xi_{2,1,1})^3 + \\
& \quad 3 \xi_{1,1,1}^2 \xi_{1,2} (-\xi_{1,2,2} + \xi_{2,1,1}) + \\
& \quad \left. \left. \xi_{1,1,1}^3 (\xi_{1,2,2} + \xi_{2,1,2}) \right) \right) / \\
& \quad \left( \xi_{1,1,1}^3 (\xi_{1,1,1} - \xi_{1,2,2} + \xi_{2,1,1})^3 \right) + \frac{1}{\xi_{1,2,2}^4 \xi_{2,1,1}^4} \\
& 4 \\
& \eta_{1,2} \\
& \eta_{2,1,2} \\
& \left( \xi_{1,2} \xi_{1,2,2} \xi_{2,1,1}^4 + \right. \\
& \quad \xi_{1,2,2}^4 (3 \xi_{1,2,2} - 2 \xi_{2,1,1}) \xi_{2,1,2} - \\
& \quad \left. 3 \xi_{1,1,1} (\xi_{1,2,2} \xi_{2,1,1}^4 + \xi_{1,2,2}^4 \xi_{2,1,2}) \right) - \\
& \quad \frac{1}{(\xi_{1,1,1} - 2 \xi_{1,2,2})^4 (-\xi_{1,1,1} + \xi_{1,2,2} + \xi_{2,1,1})^4} \\
& 4 \\
& \eta_{2,1,2}^2 \\
& \left( 2 \xi_{1,1,1}^5 (\xi_{1,2,2} + \xi_{2,1,2}) - \right.
\end{aligned}$$

$$\begin{aligned}
& 2 \xi_{1,1}^2 \left( \xi_{1,2} (\xi_{1,2} + \xi_{2,1})^2 (7 \xi_{1,2} + 4 \xi_{2,1}) + 4 \xi_{1,2}^2 (14 \xi_{1,2} - 3 \xi_{2,1}) \xi_{2,2} \right) - \\
& \xi_{1,2} \left( \xi_{1,2} (\xi_{1,2} + \xi_{2,1})^4 + 16 \xi_{1,2}^3 (2 \xi_{1,2} - \xi_{2,1}) \xi_{2,2} \right) + \\
& 2 \xi_{1,1} \left( \xi_{1,2} (\xi_{1,2} + \xi_{2,1})^3 (3 \xi_{1,2} + \xi_{2,1}) + 16 \xi_{1,2}^3 (3 \xi_{1,2} - \xi_{2,1}) \xi_{2,2} \right) + \\
& 4 \xi_{1,1}^3 \left( \xi_{1,2} (4 \xi_{1,2}^2 + 7 \xi_{1,2} \xi_{2,1} + 3 \xi_{2,1}^2) + 2 \xi_{1,2} (8 \xi_{1,2} - \xi_{2,1}) \xi_{2,2} \right) + \\
& \xi_{1,1}^4 (-\xi_{1,2} (9 \xi_{1,2} + 8 \xi_{2,1}) + (-18 \xi_{1,2} + \xi_{2,1}) \xi_{2,2}) + \\
& 2 \lambda \eta_{1,2} \left( - \left( \eta_{1,2} (\xi_{1,2} (\xi_{1,2} - \xi_{2,1})^2 + 2 \xi_{1,1} \xi_{1,2} (-\xi_{1,2} + \xi_{2,1}) + \xi_{1,1}^2 (\xi_{1,2} + \xi_{2,1})) \right) / \right. \\
& \quad \left. \left( \xi_{1,1}^2 (\xi_{1,1} - \xi_{1,2} + \xi_{2,1})^2 \right) + \frac{1}{\xi_{1,2}^3 \xi_{2,1}^3} \right. \\
& \quad \left. \eta_{2,2} (\xi_{1,2} \xi_{1,2} \xi_{2,1}^3 + \xi_{1,2}^3 (2 \xi_{1,2} - \xi_{2,1}) \xi_{2,2} - 2 \xi_{1,1} (\xi_{1,2} \xi_{2,1}^3 + \xi_{1,2}^3 \xi_{2,2})) \right) - \\
& \quad \frac{1}{\xi_{1,1}^2 \xi_{1,2}^4 \xi_{2,1}^3 (\xi_{1,1} - \xi_{1,2} + \xi_{2,1})^2} \\
& \quad 2 \\
& \quad e^{\lambda (-\xi_{1,1} + \xi_{1,2})} \\
& \quad \left( e^{2 \lambda \xi_{1,1} - \lambda \xi_{1,2}} \eta_{1,2}^2 \xi_{1,2} \xi_{1,2}^4 \xi_{2,1}^3 \right. \\
& \quad \left( 2 (\xi_{1,2} - \xi_{2,1}) + \xi_{1,1} (-1 - 2 \lambda \xi_{1,2} + 2 \lambda \xi_{2,1}) \right) - \\
& \quad e^{\lambda \xi_{1,1}} \eta_{1,2} \eta_{2,2} \xi_{1,2} \xi_{2,1}^3 \left( \xi_{1,1}^5 (6 - 4 \lambda \xi_{1,2} + \lambda^2 \xi_{1,2}^2) + \right. \\
& \quad \lambda \xi_{1,2}^4 (\xi_{1,2} - \xi_{2,1})^2 - \lambda \xi_{1,1} \xi_{1,2}^4 (\xi_{1,2} - \xi_{2,1}) (2 + \lambda \xi_{1,2} - \lambda \xi_{2,1}) + \\
& \quad \xi_{1,1}^2 \xi_{1,2} (\xi_{1,2} - \xi_{2,1}) (3 \lambda \xi_{1,2}^2 + 2 \xi_{2,1} - \xi_{1,2} (4 + \lambda \xi_{2,1})) + \\
& \quad \xi_{1,1}^4 (-2 \lambda^2 \xi_{1,2}^3 + 12 \xi_{2,1} + \lambda \xi_{1,2}^2 (9 + 2 \lambda \xi_{2,1}) - 2 \xi_{1,2} (7 + 4 \lambda \xi_{2,1})) + \\
& \quad \xi_{1,1}^3 (2 \lambda^2 \xi_{1,2}^4 + 6 \xi_{2,1}^2 - 4 \xi_{1,2} \xi_{2,1} (5 + \lambda \xi_{2,1}) - \\
& \quad \lambda \xi_{1,2}^3 (9 + 4 \lambda \xi_{2,1}) + \xi_{1,2}^2 (13 + 14 \lambda \xi_{2,1} + \lambda^2 \xi_{2,1}^2)) \left. \right) + \\
& \quad \lambda \eta_{2,2} \xi_{1,2} (\eta_{1,2} \xi_{1,2} \xi_{2,1}^3 (\xi_{1,2} (\xi_{1,2} - \xi_{2,1})^2 + 2 \xi_{1,1} \xi_{1,2} (-\xi_{1,2} + \xi_{2,1}) + \\
& \quad \xi_{1,1}^2 (\xi_{1,2} + \xi_{2,1})) + \eta_{2,2} \xi_{1,1}^2 (\xi_{1,1} - \xi_{1,2} + \xi_{2,1})^2 (-\xi_{1,2} \xi_{1,2} \xi_{2,1}^3 + \\
& \quad \xi_{1,2}^3 (-2 \xi_{1,2} + \xi_{2,1}) \xi_{2,2} + 2 \xi_{1,1} (\xi_{1,2} \xi_{2,1}^3 + \xi_{1,2}^3 \xi_{2,2})) \left. \right) \left. \right\}
\end{aligned}$$

In[\*]:= **SetAlgebra**[gl<sub>2,ε</sub>];

**\$PBWRule**

**MatrixForm**@Table[{b1, b2} → B[b1, b2], {b1, \$Basis}, {b2, \$Basis}]

Out[\*]=

{y<sub>1,2</sub> → 1, x<sub>1,1</sub> → 2, x<sub>1,2</sub> → 3, x<sub>2,2</sub> → 4}

Out[\*]//MatrixForm=

$$\left( \begin{array}{cccc}
\{y_{1,2}, y_{1,2}\} \rightarrow 0 & \{y_{1,2}, x_{1,1}\} \rightarrow y_{1,2} & \{y_{1,2}, x_{1,2}\} \rightarrow -\epsilon x_{1,1} + \epsilon x_{2,2} & \{y_{1,2}, x_{2,2}\} \rightarrow -y_{1,2} \\
\{x_{1,1}, y_{1,2}\} \rightarrow -y_{1,2} & \{x_{1,1}, x_{1,1}\} \rightarrow 0 & \{x_{1,1}, x_{1,2}\} \rightarrow x_{1,2} & \{x_{1,1}, x_{2,2}\} \rightarrow 0 \\
\{x_{1,2}, y_{1,2}\} \rightarrow \epsilon x_{1,1} - \epsilon x_{2,2} & \{x_{1,2}, x_{1,1}\} \rightarrow -x_{1,2} & \{x_{1,2}, x_{1,2}\} \rightarrow 0 & \{x_{1,2}, x_{2,2}\} \rightarrow x_{1,2} \\
\{x_{2,2}, y_{1,2}\} \rightarrow y_{1,2} & \{x_{2,2}, x_{1,1}\} \rightarrow 0 & \{x_{2,2}, x_{1,2}\} \rightarrow -x_{1,2} & \{x_{2,2}, x_{2,2}\} \rightarrow 0
\end{array} \right)$$

```

In[*]:= sol[-1] = 0 $Basis;
Do[
  Block[{ $k = k },
    EchoLabel["$k"] @ $k;
    s0 = EchoLabel["s0"] @ Thread[$Basis -> sol[k - 1]];
    lhs = EchoLabel["lhs"] @ λTangent[
      Join[λ $Basis ($Basis /. {x -> ξ1, y -> η1}), λ $Basis ($Basis /. {x -> ξ2, y -> η2})]
    ];
    rhs = EchoLabel["rhs"] @ Simplify @ λTangent @ Echo[
      $Basis ($Basis /. Echo@{xαβ -> (xαβ /. s0) + εk fαβ[λ], yαβ -> (yαβ /. s0) + εk gαβ[λ]})
    ];
    {s} = DSolve[
      Join[
        (Coefficient[Simplify@CF[lhs - rhs], #] == 0) & /@ $Basis,
        $Basis /. {xαβ -> fαβ[0], yαβ -> gαβ[0]}
      ],
      $Basis /. {xαβ -> fαβ[λ], yαβ -> gαβ[λ]},
      λ
    ];
    sol[k] = EchoLabel["solk"] [
      sol[k - 1] + Simplify[εk ($Basis /. {xαβ -> fαβ[λ], yαβ -> gαβ[λ]} /. s)]
    ],
    {k, 0, $k}
  ];
sol[$k]

```

» \$k 0

» s0 {y<sub>1,2</sub> -> 0, x<sub>1,1</sub> -> 0, x<sub>1,2</sub> -> 0, x<sub>2,2</sub> -> 0}

» lhs 
$$e^{\lambda \xi_{1,1} - \lambda \xi_{1,2} + \lambda \xi_{2,1} - \lambda \xi_{2,2}} y_{1,2} \eta_{1,2} + e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} y_{1,2} \eta_{2,2} + x_{1,1} \xi_{1,1} - e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} \lambda y_{1,2} \eta_{2,2} \xi_{1,1} + e^{\lambda \xi_{1,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} x_{1,2} \xi_{1,2} + e^{\lambda \xi_{1,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} \lambda x_{1,2} \xi_{1,1} \xi_{1,2} + x_{2,2} \xi_{1,2} + e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} \lambda y_{1,2} \eta_{2,2} \xi_{1,2} + x_{1,1} \xi_{2,1} + e^{\lambda \xi_{2,2}} x_{1,2} \xi_{2,2} + e^{\lambda \xi_{2,2}} \lambda x_{1,2} \xi_{1,1} \xi_{2,2} - e^{\lambda \xi_{2,2}} \lambda x_{1,2} \xi_{1,2} \xi_{2,2} + e^{\lambda \xi_{2,2}} \lambda x_{1,2} \xi_{2,1} \xi_{2,2} + x_{2,2} \xi_{2,2}$$

» {x<sub>αβ</sub> -> (x<sub>αβ</sub> /. s0) + ε<sup>k</sup> f<sub>αβ</sub>[λ], y<sub>αβ</sub> -> (y<sub>αβ</sub> /. s0) + ε<sup>k</sup> g<sub>αβ</sub>[λ]}

» {y<sub>1,2</sub> g<sub>1,2</sub>[λ], x<sub>1,1</sub> f<sub>1,1</sub>[λ], x<sub>1,2</sub> f<sub>1,2</sub>[λ], x<sub>2,2</sub> f<sub>2,2</sub>[λ]}

» rhs x<sub>1,1</sub> f<sub>1,1</sub>'[λ] + e<sup>f<sub>2,2</sub>[λ]</sup> x<sub>1,2</sub> (f<sub>1,2</sub>[λ] f<sub>1,1</sub>'[λ] + f<sub>1,2</sub>'[λ]) + x<sub>2,2</sub> f<sub>2,2</sub>'[λ] + e<sup>f<sub>1,1</sub>[λ] - f<sub>2,2</sub>[λ]</sup> y<sub>1,2</sub> g<sub>1,2</sub>'[λ]

» solk {λ (η<sub>1,2</sub> + e<sup>λ (-ξ<sub>1,1</sub> + ξ<sub>1,2</sub>)</sup> η<sub>2,2</sub>), λ (ξ<sub>1,1</sub> + ξ<sub>2,1</sub>), e<sup>-λ ξ<sub>2,1</sub></sup> λ ξ<sub>1,2</sub> + e<sup>-λ ξ<sub>1,2</sub></sup> λ ξ<sub>2,1</sub>, λ (ξ<sub>1,2</sub> + ξ<sub>2,2</sub>)}

» \$k 1

» s0 {y<sub>1,2</sub> -> λ (η<sub>1,2</sub> + e<sup>λ (-ξ<sub>1,1</sub> + ξ<sub>1,2</sub>)</sup> η<sub>2,2</sub>), x<sub>1,1</sub> -> λ (ξ<sub>1,1</sub> + ξ<sub>2,1</sub>), x<sub>1,2</sub> -> e<sup>-λ ξ<sub>2,1</sub></sup> λ ξ<sub>1,2</sub> + e<sup>-λ ξ<sub>1,2</sub></sup> λ ξ<sub>2,1</sub>, x<sub>2,2</sub> -> λ (ξ<sub>1,2</sub> + ξ<sub>2,2</sub>)}

» lhs 
$$e^{\lambda \xi_{1,1} - \lambda \xi_{1,2} + \lambda \xi_{2,1} - \lambda \xi_{2,2}} y_{1,2} \eta_{1,2} + e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} y_{1,2} \eta_{2,2} + x_{1,1} \xi_{1,1} - e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} \lambda y_{1,2} \eta_{2,2} \xi_{1,1} + e^{\lambda \xi_{1,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} x_{1,2} \xi_{1,2} + e^{\lambda \xi_{1,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} \lambda x_{1,2} \xi_{1,1} \xi_{1,2} + x_{2,2} \xi_{1,2} + e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} \lambda y_{1,2} \eta_{2,2} \xi_{1,2} + x_{1,1} \xi_{2,1} + e^{\lambda \xi_{2,2}} x_{1,2} \xi_{2,2} + e^{\lambda \xi_{2,2}} \lambda x_{1,2} \xi_{1,1} \xi_{2,2} - e^{\lambda \xi_{2,2}} \lambda x_{1,2} \xi_{1,2} \xi_{2,2} + e^{\lambda \xi_{2,2}} \lambda x_{1,2} \xi_{2,1} \xi_{2,2} + x_{2,2} \xi_{2,2}$$

$\gg \{x_{\alpha\beta\_} \mapsto (x_{\alpha\beta} / . s0) + \epsilon^k f_{\alpha\beta}[\lambda], y_{\alpha\beta\_} \mapsto (y_{\alpha\beta} / . s0) + \epsilon^k g_{\alpha\beta}[\lambda]\}$   
 $\gg \{y_{1,2} (\lambda (\eta_{1,2} + e^{\lambda (-\xi_{1,1} + \xi_{1,2})} \eta_{2,2}) + \epsilon g_{1,2}[\lambda]), x_{1,1} (\lambda (\xi_{1,1} + \xi_{2,1}) + \epsilon f_{1,1}[\lambda]),$   
 $x_{1,2} (e^{-\lambda \xi_{2,1}} \lambda \xi_{1,2} + e^{-\lambda \xi_{1,2}} \lambda \xi_{2,2} + \epsilon f_{1,2}[\lambda]), x_{2,2} (\lambda (\xi_{1,2} + \xi_{2,2}) + \epsilon f_{2,2}[\lambda])\}$   
 $\gg \text{rhs } e^{-\lambda (3 \xi_{1,2} + 2 \xi_{2,1} + \xi_{2,2})} (e^{\lambda (4 \xi_{1,2} + 2 \xi_{2,1} + 2 \xi_{2,2})} x_{1,2} \xi_{1,2} + e^{\lambda (\xi_{1,1} + 3 \xi_{1,2} + 2 \xi_{2,1} + \xi_{2,2})} \epsilon \lambda x_{2,2} \eta_{1,2} \xi_{1,2} +$   
 $e^{\lambda (4 \xi_{1,2} + 2 \xi_{2,1} + \xi_{2,2})} \epsilon \lambda x_{2,2} \eta_{2,2} \xi_{1,2} + e^{\lambda (4 \xi_{1,2} + \xi_{2,1} + 2 \xi_{2,2})} \lambda x_{1,2} \xi_{1,1} \xi_{1,2} -$   
 $e^{\lambda (4 \xi_{1,2} + 2 \xi_{2,1} + \xi_{2,2})} \epsilon \lambda^2 x_{2,2} \eta_{2,2} \xi_{1,1} \xi_{1,2} - e^{\lambda (\xi_{1,1} + 4 \xi_{1,2} + 2 \xi_{2,1} + 2 \xi_{2,2})} \epsilon \lambda^2 x_{1,2} \eta_{1,2} \xi_{1,2}^2 -$   
 $e^{\lambda (5 \xi_{1,2} + 2 \xi_{2,1} + 2 \xi_{2,2})} \epsilon \lambda^2 x_{1,2} \eta_{2,2} \xi_{1,2}^2 + e^{\lambda (5 \xi_{1,2} + \xi_{2,1} + 2 \xi_{2,2})} \epsilon \lambda^3 x_{1,2} \eta_{2,2} \xi_{1,1} \xi_{1,2}^2 +$   
 $e^{\lambda (3 \xi_{1,2} + 2 \xi_{2,1} + \xi_{2,2})} x_{2,2} \xi_{1,2} + e^{\lambda (4 \xi_{1,2} + 2 \xi_{2,1} + \xi_{2,2})} \epsilon \lambda^2 x_{2,2} \eta_{2,2} \xi_{1,2} \xi_{1,2} -$   
 $e^{\lambda (5 \xi_{1,2} + 2 \xi_{2,1} + 2 \xi_{2,2})} \epsilon \lambda^3 x_{1,2} \eta_{2,2} \xi_{1,2}^2 \xi_{1,2} + e^{\lambda (3 \xi_{1,2} + 2 (\xi_{2,1} + \xi_{2,2}))} x_{1,2} \xi_{2,2} +$   
 $e^{\lambda (\xi_{1,1} + 2 \xi_{1,2} + 3 \xi_{2,1} + \xi_{2,2})} \epsilon \lambda x_{2,2} \eta_{1,2} \xi_{2,2} + e^{\lambda (3 \xi_{1,2} + 3 \xi_{2,1} + \xi_{2,2})} \epsilon \lambda x_{2,2} \eta_{2,2} \xi_{2,2} +$   
 $e^{\lambda (3 \xi_{1,2} + 2 (\xi_{2,1} + \xi_{2,2}))} \lambda x_{1,2} \xi_{1,1} \xi_{2,2} - e^{\lambda (3 \xi_{1,2} + 3 \xi_{2,1} + \xi_{2,2})} \epsilon \lambda^2 x_{2,2} \eta_{2,2} \xi_{1,1} \xi_{2,2} -$   
 $2 e^{\lambda (\xi_{1,1} + 3 \xi_{1,2} + 2 (\xi_{2,1} + \xi_{2,2}))} \epsilon \lambda^2 x_{1,2} \eta_{1,2} \xi_{1,2} \xi_{2,2} - 2 e^{\lambda (2 \xi_{1,2} + \xi_{2,1} + \xi_{2,2})} \epsilon \lambda^2 x_{1,2} \eta_{2,2} \xi_{1,2} \xi_{2,2} +$   
 $2 e^{2\lambda (2 \xi_{1,2} + \xi_{2,1} + \xi_{2,2})} \epsilon \lambda^3 x_{1,2} \eta_{2,2} \xi_{1,1} \xi_{1,2} \xi_{2,2} - e^{\lambda (3 \xi_{1,2} + 2 (\xi_{2,1} + \xi_{2,2}))} \lambda x_{1,2} \xi_{1,2} \xi_{2,2} +$   
 $e^{\lambda (3 \xi_{1,2} + 3 \xi_{2,1} + \xi_{2,2})} \epsilon \lambda^2 x_{2,2} \eta_{2,2} \xi_{1,2} \xi_{2,2} - 2 e^{2\lambda (2 \xi_{1,2} + \xi_{2,1} + \xi_{2,2})} \epsilon \lambda^3 x_{1,2} \eta_{2,2} \xi_{1,2} \xi_{2,2} \xi_{2,2} +$   
 $e^{\lambda (3 \xi_{1,2} + 2 (\xi_{2,1} + \xi_{2,2}))} \lambda x_{1,2} \xi_{2,1} \xi_{2,2} - e^{\lambda (\xi_{1,1} + 2 \xi_{1,2} + 3 \xi_{2,1} + 2 \xi_{2,2})} \epsilon \lambda^2 x_{1,2} \eta_{1,2} \xi_{2,2}^2 -$   
 $e^{\lambda (3 \xi_{1,2} + 3 \xi_{2,1} + 2 \xi_{2,2})} \epsilon \lambda^2 x_{1,2} \eta_{2,2} \xi_{2,2}^2 + e^{\lambda (3 \xi_{1,2} + 3 \xi_{2,1} + 2 \xi_{2,2})} \epsilon \lambda^3 x_{1,2} \eta_{2,2} \xi_{1,1} \xi_{2,2}^2 -$   
 $e^{\lambda (3 \xi_{1,2} + 3 \xi_{2,1} + 2 \xi_{2,2})} \epsilon \lambda^3 x_{1,2} \eta_{2,2} \xi_{1,2} \xi_{2,2}^2 + e^{\lambda (3 \xi_{1,2} + 2 \xi_{2,1} + \xi_{2,2})} x_{2,2} \xi_{2,2} +$   
 $e^{2\lambda (2 \xi_{1,2} + \xi_{2,1} + \xi_{2,2})} \epsilon x_{1,2} \xi_{1,1} f_{1,2}[\lambda] + e^{2\lambda (2 \xi_{1,2} + \xi_{2,1} + \xi_{2,2})} \epsilon x_{1,2} \xi_{2,1} f_{1,2}[\lambda] +$   
 $e^{\lambda (4 \xi_{1,2} + 2 \xi_{2,1} + 2 \xi_{2,2})} \epsilon x_{1,2} \xi_{1,2} f_{2,2}[\lambda] + e^{\lambda (4 \xi_{1,2} + \xi_{2,1} + 2 \xi_{2,2})} \epsilon \lambda x_{1,2} \xi_{1,1} \xi_{1,2} f_{2,2}[\lambda] +$   
 $e^{\lambda (3 \xi_{1,2} + 2 (\xi_{2,1} + \xi_{2,2}))} \epsilon x_{1,2} \xi_{2,2} f_{2,2}[\lambda] + e^{\lambda (3 \xi_{1,2} + 2 (\xi_{2,1} + \xi_{2,2}))} \epsilon \lambda x_{1,2} \xi_{1,1} \xi_{2,2} f_{2,2}[\lambda] -$   
 $e^{\lambda (3 \xi_{1,2} + 2 (\xi_{2,1} + \xi_{2,2}))} \epsilon \lambda x_{1,2} \xi_{1,2} \xi_{2,2} f_{2,2}[\lambda] + e^{\lambda (3 \xi_{1,2} + 2 (\xi_{2,1} + \xi_{2,2}))} \epsilon \lambda x_{1,2} \xi_{2,1} \xi_{2,2} f_{2,2}[\lambda] +$   
 $e^{\lambda (4 \xi_{1,2} + 2 \xi_{2,1} + 2 \xi_{2,2})} \epsilon \lambda x_{1,2} \xi_{1,2} f_{1,1}'[\lambda] + e^{\lambda (3 \xi_{1,2} + 2 (\xi_{2,1} + \xi_{2,2}))} \epsilon \lambda x_{1,2} \xi_{2,1} f_{1,1}'[\lambda] +$   
 $e^{\lambda (2 \xi_{1,2} + 2 \xi_{2,1} + \xi_{2,2})} x_{1,1} (-e^{\lambda \xi_{1,1}} \epsilon \lambda \eta_{1,2} (e^{\lambda \xi_{1,2}} \xi_{1,2} + e^{\lambda \xi_{2,1}} \xi_{2,2}) +$   
 $e^{\lambda \xi_{1,2}} \xi_{1,1} (1 + \epsilon \lambda^2 \eta_{2,2} (e^{\lambda \xi_{1,2}} \xi_{1,2} + e^{\lambda \xi_{2,1}} \xi_{2,2})) +$   
 $e^{\lambda \xi_{1,2}} (\xi_{2,1} - \epsilon \lambda \eta_{2,2} (1 + \lambda \xi_{1,2}) (e^{\lambda \xi_{1,2}} \xi_{1,2} + e^{\lambda \xi_{2,1}} \xi_{2,2}) + \epsilon f_{1,1}'[\lambda])) +$   
 $e^{2\lambda (2 \xi_{1,2} + \xi_{2,1} + \xi_{2,2})} \epsilon x_{1,2} f_{1,2}'[\lambda] + e^{\lambda (3 \xi_{1,2} + 2 \xi_{2,1} + \xi_{2,2})} \epsilon x_{2,2} f_{2,2}'[\lambda] +$   
 $e^{2\lambda \xi_{1,2} + 3 \lambda \xi_{2,1}} y_{1,2} (e^{\lambda \xi_{1,1}} \eta_{1,2} (1 + \epsilon f_{1,1}[\lambda] - \epsilon f_{2,2}[\lambda]) +$   
 $e^{\lambda \xi_{1,2}} \eta_{2,2} (1 - \lambda \xi_{1,1} + \lambda \xi_{1,2}) (1 + \epsilon f_{1,1}[\lambda] - \epsilon f_{2,2}[\lambda]) + e^{\lambda \xi_{1,1}} \epsilon g_{1,2}'[\lambda]))$

Out[\*]=

\$Aborted

Out[\*]=

sol[1]

In[\*]:= Simplify[ $e^{\lambda (-\xi_{1,1} - \xi_{2,1})} \lambda (e^{\lambda \xi_{1,1}} \xi_{1,2} + e^{\lambda (\xi_{1,1} - \xi_{1,2} + \xi_{2,1})} \xi_{2,2})$ ]

Out[\*]=

 $e^{-\lambda \xi_{2,1}} \lambda \xi_{1,2} + e^{-\lambda \xi_{1,2}} \lambda \xi_{2,2}$ 

In[\*]:= \$k

Out[\*]=

1

In[\*]:= sol1 /.  $\lambda \rightarrow 1$ 

Out[\*]=

sol1

```
In[*]:= osol = E_{1,2} \to \{2\} [x_{1,1}[2] \xi_{1,1}[1] + x_{1,1}[2] \xi_{1,1}[2] + e^{-\xi_{1,1}[2]} x_{1,2}[2] \xi_{1,2}[1] + e^{-\xi_{2,2}[1]} x_{1,2}[2] \xi_{1,2}[2] +
e^{-\xi_{1,1}[2]} x_{1,3}[2] \xi_{1,3}[1] + e^{-\xi_{3,3}[1]} x_{1,3}[2] \xi_{1,3}[2] + e^{-\xi_{1,1}[2]} x_{1,4}[2] \xi_{1,4}[1] +
e^{-\xi_{4,4}[1]} x_{1,4}[2] \xi_{1,4}[2] + e^{-\xi_{1,1}[2]} x_{1,5}[2] \xi_{1,5}[1] + e^{-\xi_{5,5}[1]} x_{1,5}[2] \xi_{1,5}[2] +
x_{2,2}[2] \xi_{2,2}[1] + x_{2,2}[2] \xi_{2,2}[2] + e^{-\xi_{2,2}[2]} x_{2,3}[2] \xi_{2,3}[1] - x_{1,3}[2] \xi_{1,2}[2] \xi_{2,3}[1] +
e^{-\xi_{3,3}[1]} x_{2,3}[2] \xi_{2,3}[2] + e^{-\xi_{2,2}[2]} x_{2,4}[2] \xi_{2,4}[1] - x_{1,4}[2] \xi_{1,2}[2] \xi_{2,4}[1] +
e^{-\xi_{4,4}[1]} x_{2,4}[2] \xi_{2,4}[2] + e^{-\xi_{2,2}[2]} x_{2,5}[2] \xi_{2,5}[1] - x_{1,5}[2] \xi_{1,2}[2] \xi_{2,5}[1] +
e^{-\xi_{5,5}[1]} x_{2,5}[2] \xi_{2,5}[2] + x_{3,3}[2] \xi_{3,3}[1] + x_{3,3}[2] \xi_{3,3}[2] + e^{-\xi_{3,3}[2]} x_{3,4}[2] \xi_{3,4}[1] -
x_{1,4}[2] \xi_{1,3}[2] \xi_{3,4}[1] - x_{2,4}[2] \xi_{2,3}[2] \xi_{3,4}[1] + e^{-\xi_{4,4}[1]} x_{3,4}[2] \xi_{3,4}[2] +
e^{-\xi_{3,3}[2]} x_{3,5}[2] \xi_{3,5}[1] - x_{1,5}[2] \xi_{1,3}[2] \xi_{3,5}[1] - x_{2,5}[2] \xi_{2,3}[2] \xi_{3,5}[1] +
e^{-\xi_{5,5}[1]} x_{3,5}[2] \xi_{3,5}[2] + x_{4,4}[2] \xi_{4,4}[1] + x_{4,4}[2] \xi_{4,4}[2] + e^{-\xi_{4,4}[2]} x_{4,5}[2] \xi_{4,5}[1] -
x_{1,5}[2] \xi_{1,4}[2] \xi_{4,5}[1] - x_{2,5}[2] \xi_{2,4}[2] \xi_{4,5}[1] - x_{3,5}[2] \xi_{3,4}[2] \xi_{4,5}[1] +
e^{-\xi_{5,5}[1]} x_{4,5}[2] \xi_{4,5}[2] + x_{5,5}[2] \xi_{5,5}[1] + x_{5,5}[2] \xi_{5,5}[2], 0, 0] [[1]] /. \xi_{i_,i_}[_] \to 0
```

Out[\*]=

```
x_{1,2}[2] \xi_{1,2}[1] + x_{1,2}[2] \xi_{1,2}[2] + x_{1,3}[2] \xi_{1,3}[1] + x_{1,3}[2] \xi_{1,3}[2] +
x_{1,4}[2] \xi_{1,4}[1] + x_{1,4}[2] \xi_{1,4}[2] + x_{1,5}[2] \xi_{1,5}[1] + x_{1,5}[2] \xi_{1,5}[2] + x_{2,3}[2] \xi_{2,3}[1] -
x_{1,3}[2] \xi_{1,2}[2] \xi_{2,3}[1] + x_{2,3}[2] \xi_{2,3}[2] + x_{2,4}[2] \xi_{2,4}[1] - x_{1,4}[2] \xi_{1,2}[2] \xi_{2,4}[1] +
x_{2,4}[2] \xi_{2,4}[2] + x_{2,5}[2] \xi_{2,5}[1] - x_{1,5}[2] \xi_{1,2}[2] \xi_{2,5}[1] + x_{2,5}[2] \xi_{2,5}[2] + x_{3,4}[2] \xi_{3,4}[1] -
x_{1,4}[2] \xi_{1,3}[2] \xi_{3,4}[1] - x_{2,4}[2] \xi_{2,3}[2] \xi_{3,4}[1] + x_{3,4}[2] \xi_{3,4}[2] + x_{3,5}[2] \xi_{3,5}[1] -
x_{1,5}[2] \xi_{1,3}[2] \xi_{3,5}[1] - x_{2,5}[2] \xi_{2,3}[2] \xi_{3,5}[1] + x_{3,5}[2] \xi_{3,5}[2] + x_{4,5}[2] \xi_{4,5}[1] -
x_{1,5}[2] \xi_{1,4}[2] \xi_{4,5}[1] - x_{2,5}[2] \xi_{2,4}[2] \xi_{4,5}[1] - x_{3,5}[2] \xi_{3,4}[2] \xi_{4,5}[1] + x_{4,5}[2] \xi_{4,5}[2]
```

```
In[*]:= Expand@Total[sol1 /. {\xi_{\alpha\beta\_} \to \xi_{\alpha\beta}[1], \eta_{\alpha\beta\_} \to \xi_{\alpha\beta}[2], Rule \to Times, \lambda \to 1, f_{\alpha\beta\_}[_] \to x_{\alpha\beta}[2]}] -
osol
```

Out[\*]=

```
Total[sol1] - x_{1,2}[2] \xi_{1,2}[1] - x_{1,2}[2] \xi_{1,2}[2] - x_{1,3}[2] \xi_{1,3}[1] - x_{1,3}[2] \xi_{1,3}[2] -
x_{1,4}[2] \xi_{1,4}[1] - x_{1,4}[2] \xi_{1,4}[2] - x_{1,5}[2] \xi_{1,5}[1] - x_{1,5}[2] \xi_{1,5}[2] - x_{2,3}[2] \xi_{2,3}[1] +
x_{1,3}[2] \xi_{1,2}[2] \xi_{2,3}[1] - x_{2,3}[2] \xi_{2,3}[2] - x_{2,4}[2] \xi_{2,4}[1] + x_{1,4}[2] \xi_{1,2}[2] \xi_{2,4}[1] -
x_{2,4}[2] \xi_{2,4}[2] - x_{2,5}[2] \xi_{2,5}[1] + x_{1,5}[2] \xi_{1,2}[2] \xi_{2,5}[1] - x_{2,5}[2] \xi_{2,5}[2] - x_{3,4}[2] \xi_{3,4}[1] +
x_{1,4}[2] \xi_{1,3}[2] \xi_{3,4}[1] + x_{2,4}[2] \xi_{2,3}[2] \xi_{3,4}[1] - x_{3,4}[2] \xi_{3,4}[2] - x_{3,5}[2] \xi_{3,5}[1] +
x_{1,5}[2] \xi_{1,3}[2] \xi_{3,5}[1] + x_{2,5}[2] \xi_{2,3}[2] \xi_{3,5}[1] - x_{3,5}[2] \xi_{3,5}[2] - x_{4,5}[2] \xi_{4,5}[1] +
x_{1,5}[2] \xi_{1,4}[2] \xi_{4,5}[1] + x_{2,5}[2] \xi_{2,4}[2] \xi_{4,5}[1] + x_{3,5}[2] \xi_{3,4}[2] \xi_{4,5}[1] - x_{4,5}[2] \xi_{4,5}[2]
```