Pensieve header: The PBW multiplication tensor for $gl_{n,\epsilon}$ using the λ-tangent formalism. Some material from pensieve://Projects/UEA.

```
In[∘]:= SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
        << KnotTheory`
```

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
```

Prolog

```
In[∘]:= BeginPackage["UEA`"];
        Print["UEA` does computations in general universal enveloping
            algebras and PBW algebras. It is in the public domain, available
            at http://drorbn.net/AcademicPensieve/Projects/SolvablePBW/.
            Dror Bar-Natan is committed to support it within
            reason until June 1, 2027. This is version 260601."];
        Print["UEA` implements / extends ",
          Sort@{"**", ∊, $k, CF, SSQ, NilQ, B, m, SetAlgebra, U,
            UB, UProducts, USimp, UU, $Basis, $PBWRule, δ, adPower, adExp},
          "."];
        Begin["`Private`"];
```

```
UEA` does computations in general universal enveloping algebras and PBW algebras. It is in the
  public domain, available at http://drorbn.net/AcademicPensieve/Projects/SolvablePBW/. Dror
  Bar-Natan is committed to support it within reason until June 1, 2027. This is version 260601.
UEA` implements / extends {**, adExp, adPower, B, CF, m, NilQ,
  SetAlgebra, SSQ, U, UB, UProducts, USimp, UU, δ, ∊, $Basis, $k, $PBWRule}.
```

Utilities

```
In[∘]:= δ_{i_,j_} := If[i == j, 1, 0];
```

```
In[∘]:= SSQ[c_. * x_] /; MemberQ[$Basis, x] := ¬ NilQ[x]; (* Semi-Simple Q *)
        NilQ[c_ * x_] /; MemberQ[$Basis, x] := NilQ[x];
        SSQ[x_Plus] := And @@ (SSQ /@ (List @@ x));
        NilQ[x_Plus] := And @@ (NilQ /@ (List @@ x));
```

```
In[∘]:= $k = 1;
```

```
In[∘]:= CF[ℰ_] := Expand[ℰ] /. {∊^k_. /; k > $k → 0};
```

Implementing general universal enveloping algebras

```
In[∘]:= B[0, _] = 0; B[_, 0] = 0;
B[c_ * x_, y_] /; MemberQ[$Basis, x] := CF[c B[x, y]];
B[y_, c_ * x_] /; MemberQ[$Basis, x] := CF[c B[y, x]];
B[x_Plus, y_] := B[#, y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
B[x_, x_] = 0;
B[y_, x_] := CF[-B[x, y]];
```

```
In[∘]:= x_ ≤ y_ := OrderedQ[{x, y} /. $PBWRule]; x_ ≺ y_ := ! OrderedQ[{y, x} /. $PBWRule];
UU_i_ [1] := U_i[];
UU_i_ [x_^p_] := UU_i @@ Table[x, {p}];
UU_i_ [ℰ_] := ℰ /. {
    U[xs__] ⧴ U_i[xs],
    x_ /; MemberQ[$Basis, x] ⧴ U_i[x]
   };
UU_i_ [x_, xs__] := UU_t1[x] UU_t2[xs] // Expand // m_t1,t2→i;
USimp[ℰ_] := Collect[ℰ, Times[U_[__] ..], Expand];
USimp[ℰ_] := Expand[ℰ];
```

```
In[∘]:= m_s__ [0] = 0;
m_s__ [x_Plus] := m_s /@ x;
m_s__ [sd_SeriesData] := MapAt[m_s, sd, {3, All}];
m_i_→j_ [ℰ_] := ℰ /. U_i → U_j;
```

```
In[∘]:= m_i_,j_→k_ [c_. U_i_ [x___] U_j_ []] := c U_k[x];
m_i_,j_→k_ [c_. U_i_ [] U_j_ [y___]] := c U_k[y];
m_i_,j_→k_ [c_. U_i_ [xx___, x_] U_j_ [y_, yy___]] := If[x ≤ y,
   c U_k[xx, x, y, yy],
   ((U_i[xx] (U_j[y, x] + UU_j[B[x, y]]) // Expand // m_i,j→i) U_j[yy] // Expand // m_i,j→k)
     c // USimp
  ];
```

```
In[∘]:= UProducts[{}, 0] = {1}; UProducts[{}, d_Integer] /; d > 0 = {};
UProducts[{i_, is___}, d_Integer] := Sort@
   Flatten@Table[(U_i @@@ Subsets[$Basis, {j}]) u, {j, 0, d}, {u, UProducts[{is}, d - j]}];
```

```
In[∘]:= Supp[ℰ_] := Union@Cases[{ℰ}, U_i_ [___] ⧴ i, ∞];
```

```
In[∘]:= Unprotect[NonCommutativeMultiply];
        NonCommutativeMultiply[x_] := x;
        x_ ⊠ y_ := Module[{is = Supp[x] ⋂ Supp[y], σ, z},
           z = x; Do[z = m_{i→σ@i}[z], {i, is}];
           z = Expand[y z]; Do[z = m_{σ@i,i→i}[z], {i, is}]; z];
        UB[x_, y_] := USimp[x ⊠ y - y ⊠ x];
```

Epilog

```
In[∘]:= End[]; EndPackage[];
```

Predefined Algebras

sl(2)

```
In[∘]:= Print["UEA`SetAlgebra knows \"sl2\"."];
```

UEA`SetAlgebra knows "sl2".

```
In[∘]:= SetAlgebra["sl2"] := (
           Print["In sl2: ⟨e,h,f⟩/([h,e]=2e, [h,f]=-2f, [e,f]=h)."];
           B[h, e] = 2 e; B[h, f] = -2 f; B[e, f] = h;
           $Basis = {e, h, f};
           $PBWRule = {e → 1, h → 2, f → 3};
           NilQ[e] = NilQ[f] = True; NilQ[h] = False;
          );
```

$gl_{n,\epsilon}$

```
In[∘]:= Print["UEA`SetAlgebra knows the ϵ-nilpotent algebra gl_{n,ϵ}."];
```

UEA`SetAlgebra knows the ϵ-nilpotent algebra $gl_{n,\epsilon}$.

In[∘]:=
```
SetAlgebra[gl_{n_,ϵ}] := (
  $Basis = Flatten@{
     Table[y_{α,β}, {β, 2, n}, {α, 1, β - 1}],
     Table[x_{α,β}, {β, 1, n}, {α, 1, β}]
    };
  NilQ[y__] = True; NilQ[x_{α_,β_}] := (α =!= β); (* Nilpotent Q *)
  $PBWRule = Thread[$Basis → Range@Length@$Basis];
  B[x_{i_,j_}, x_{k_,l_}] := δ_{j,k} x_{i,l} - δ_{l,i} x_{k,j};
  B[y_{i_,j_}, y_{k_,l_}] := CF[ϵ δ_{j,k} y_{i,l} - ϵ δ_{l,i} y_{k,j}];
  B[x_{i_,j_}, y_{l_,k_}] := CF[δ_{j,k} x_{i,l} - δ_{l,i} x_{k,j} /. x_{α_,β_} :→ If[α ≤ β, ϵ x_{α,β}, y_{β,α}]];
 );
```

In[∘]:=
```
SetAlgebra[gl_{2,ϵ}];
$PBWRule
MatrixForm@Table[{b1, b2} → B[b1, b2], {b1, $Basis}, {b2, $Basis}]
```

Out[∘]=

$$\{y_{1,2}\to 1,\ x_{1,1}\to 2,\ x_{1,2}\to 3,\ x_{2,2}\to 4\}$$

Out[∘]//MatrixForm=

$$\begin{pmatrix} \{y_{1,2},y_{1,2}\}\to 0 & \{y_{1,2},x_{1,1}\}\to y_{1,2} & \{y_{1,2},x_{1,2}\}\to -\epsilon\, x_{1,1}+\epsilon\, x_{2,2} & \{y_{1,2},x_{2,2}\}\to -y_{1,2} \\ \{x_{1,1},y_{1,2}\}\to -y_{1,2} & \{x_{1,1},x_{1,1}\}\to 0 & \{x_{1,1},x_{1,2}\}\to x_{1,2} & \{x_{1,1},x_{2,2}\}\to 0 \\ \{x_{1,2},y_{1,2}\}\to \epsilon\, x_{1,1}-\epsilon\, x_{2,2} & \{x_{1,2},x_{1,1}\}\to -x_{1,2} & \{x_{1,2},x_{1,2}\}\to 0 & \{x_{1,2},x_{2,2}\}\to x_{1,2} \\ \{x_{2,2},y_{1,2}\}\to y_{1,2} & \{x_{2,2},x_{1,1}\}\to 0 & \{x_{2,2},x_{1,2}\}\to -x_{1,2} & \{x_{2,2},x_{2,2}\}\to 0 \end{pmatrix}$$

In[∘]:=
```
Clear[adPower];
adPower[x_, k_][0] = 0;
adPower[x_, k_][c_ * y_] /; MemberQ[$Basis, y] := CF[c adPower[x, k][y]];
adPower[x_, k_][y_Plus] := adPower[x, k] /@ y;
adPower[_, 0][y_] := y;
adPower[x_, k_][y_] /; MemberQ[$Basis, y] :=
  adPower[x, k][y] = B[adPower[x, k - 1][y], x];
```

In[∘]:=
```
x0 = ($Basis /. {x → ξ0, y → η0}).$Basis
x1 = ($Basis /. {x → ξ1, y → η1}).$Basis
```

Out[∘]=

$$y_{1,2}\,\eta 0_{1,2}+x_{1,1}\,\xi 0_{1,1}+x_{1,2}\,\xi 0_{1,2}+x_{2,2}\,\xi 0_{2,2}$$

Out[∘]=

$$y_{1,2}\,\eta 1_{1,2}+x_{1,1}\,\xi 1_{1,1}+x_{1,2}\,\xi 1_{1,2}+x_{2,2}\,\xi 1_{2,2}$$

In[∘]:=
```
adExp[x_][y_] /; SSQ[x] := Expand@Module[{A, b, c},
   A = Table[Coefficient[B[b, x], c], {c, $Basis}, {b, $Basis}];
   CF[$Basis.Normal[Series[MatrixExp[A], {ϵ, 0, $k}]].
     Table[Coefficient[y, b], {b, $Basis}]]
  ]
```

In[ ]:=
```
SSQ[ξθ1,1 x1,1]
```
Out[ ]=
True

In[ ]:=
```
adExp[ξθ1,1 x1,1][x1]
```
Out[ ]=
$$e^{\xi\theta_{1,1}} y_{1,2}\,\eta 1_{1,2} + x_{1,1}\,\xi 1_{1,1} + e^{-\xi\theta_{1,1}} x_{1,2}\,\xi 1_{1,2} + x_{2,2}\,\xi 1_{2,2}$$

In[ ]:=
```
adExp[x_][y_] /; NilQ[x] := Expand@Module[{k = 0, s = 0},
   While[adPower[x, k][y] =!= 0,
    s += adPower[x, k][y] / k!;
    ++k
   ];
   s
  ]
```

In[ ]:=
```
adExp[ξθ1,2 x1,2][x1]
```
Out[ ]=
$$y_{1,2}\,\eta 1_{1,2} - \epsilon\, x_{1,1}\,\eta 1_{1,2}\,\xi\theta_{1,2} + \epsilon\, x_{2,2}\,\eta 1_{1,2}\,\xi\theta_{1,2} - \epsilon\, x_{1,2}\,\eta 1_{1,2}\,\xi\theta_{1,2}^2 + x_{1,1}\,\xi 1_{1,1} + x_{1,2}\,\xi\theta_{1,2}\,\xi 1_{1,1} + x_{1,2}\,\xi 1_{1,2} + x_{2,2}\,\xi 1_{2,2} - x_{1,2}\,\xi\theta_{1,2}\,\xi 1_{2,2}$$

In[ ]:=
```
λTangent[] = 0;
λTangent[xs___, x_] := adExp[x][λTangent[xs]] + ∂λ x;
λTangent[xs_List] := λTangent @@ xs
```

In[ ]:=
```
$Basis ($Basis /. {x → ξ, y → η})
```
Out[ ]=
$$\{y_{1,2}\,\eta_{1,2},\ x_{1,1}\,\xi_{1,1},\ x_{1,2}\,\xi_{1,2},\ x_{2,2}\,\xi_{2,2}\}$$

In[ ]:=
```
λTangent@(λ $Basis ($Basis /. {x → ξ, y → η}))
```
Out[ ]=
$$e^{\lambda\xi_{1,1}-\lambda\xi_{2,2}} y_{1,2}\,\eta_{1,2} + x_{1,1}\,\xi_{1,1} + e^{\lambda\xi_{2,2}} x_{1,2}\,\xi_{1,2} - e^{\lambda\xi_{1,1}}\,\epsilon\,\lambda\, x_{1,1}\,\eta_{1,2}\,\xi_{1,2} + e^{\lambda\xi_{1,1}}\,\epsilon\,\lambda\, x_{2,2}\,\eta_{1,2}\,\xi_{1,2} + e^{\lambda\xi_{2,2}}\,\lambda\, x_{1,2}\,\xi_{1,1}\,\xi_{1,2} - e^{\lambda\xi_{1,1}+\lambda\xi_{2,2}}\,\epsilon\,\lambda^2\, x_{1,2}\,\eta_{1,2}\,\xi_{1,2}^2 + x_{2,2}\,\xi_{2,2}$$

In[ ]:=
```
$Basis
```
Out[ ]=
$$\{y_{1,2},\ x_{1,1},\ x_{1,2},\ x_{2,2}\}$$

In[◦]:=
```
lhs =
  Join[λ $Basis ($Basis /. {x → ξ1, y → η1}), λ $Basis ($Basis /. {x → ξ2, y → η2})] // λTangent
```

Out[◦]=

$$
\begin{aligned}
&e^{\lambda \xi1_{1,1}-\lambda \xi1_{2,2}+\lambda \xi2_{1,1}-\lambda \xi2_{2,2}} y_{1,2}\, \eta1_{1,2} + e^{\lambda \xi2_{1,1}-\lambda \xi2_{2,2}} y_{1,2}\, \eta2_{1,2} + x_{1,1}\, \xi1_{1,1} - \\
&e^{\lambda \xi2_{1,1}-\lambda \xi2_{2,2}} \lambda\, y_{1,2}\, \eta2_{1,2}\, \xi1_{1,1} + e^{\lambda \xi1_{2,2}-\lambda \xi2_{1,1}+\lambda \xi2_{2,2}} x_{1,2}\, \xi1_{1,2} - e^{\lambda \xi1_{1,1}} \epsilon\, \lambda\, x_{1,1}\, \eta1_{1,2}\, \xi1_{1,2} + \\
&e^{\lambda \xi1_{1,1}} \epsilon\, \lambda\, x_{2,2}\, \eta1_{1,2}\, \xi1_{1,2} + e^{\lambda \xi1_{2,2}} \epsilon\, \lambda\, x_{1,1}\, \eta2_{1,2}\, \xi1_{1,2} - e^{\lambda \xi1_{2,2}} \epsilon\, \lambda\, x_{2,2}\, \eta2_{1,2}\, \xi1_{1,2} + \\
&2\, e^{\lambda \xi1_{1,1}+\lambda \xi2_{1,1}-\lambda \xi2_{2,2}} \epsilon\, \lambda^2\, y_{1,2}\, \eta1_{1,2}\, \eta2_{1,2}\, \xi1_{1,2} - e^{\lambda \xi1_{2,2}+\lambda \xi2_{1,1}-\lambda \xi2_{2,2}} \epsilon\, \lambda^2\, y_{1,2}\, \eta2_{1,2}^2\, \xi1_{1,2} + \\
&e^{\lambda \xi1_{2,2}-\lambda \xi2_{1,1}+\lambda \xi2_{2,2}} \lambda\, x_{1,2}\, \xi1_{1,1}\, \xi1_{1,2} + e^{\lambda \xi1_{2,2}} \epsilon\, \lambda^2\, x_{1,1}\, \eta2_{1,2}\, \xi1_{1,1}\, \xi1_{1,2} - e^{\lambda \xi1_{2,2}} \epsilon\, \lambda^2\, x_{2,2}\, \eta2_{1,2}\, \xi1_{1,1}\, \xi1_{1,2} - \\
&e^{\lambda \xi1_{2,2}+\lambda \xi2_{1,1}-\lambda \xi2_{2,2}} \epsilon\, \lambda^3\, y_{1,2}\, \eta2_{1,2}^2\, \xi1_{1,1}\, \xi1_{1,2} - e^{\lambda \xi1_{1,1}+\lambda \xi1_{2,2}-\lambda \xi2_{1,1}+\lambda \xi2_{2,2}} \epsilon\, \lambda^2\, x_{1,2}\, \eta1_{1,2}\, \xi1_{1,2}^2 + x_{2,2}\, \xi1_{2,2} + \\
&e^{\lambda \xi2_{1,1}-\lambda \xi2_{2,2}} \lambda\, y_{1,2}\, \eta2_{1,2}\, \xi1_{2,2} + x_{1,1}\, \xi2_{1,1} + e^{\lambda \xi2_{2,2}} x_{1,2}\, \xi2_{1,2} - e^{\lambda \xi1_{1,1}-\lambda \xi1_{2,2}+\lambda \xi2_{1,1}} \epsilon\, \lambda\, x_{1,1}\, \eta1_{1,2}\, \xi2_{1,2} + \\
&e^{\lambda \xi1_{1,1}-\lambda \xi1_{2,2}+\lambda \xi2_{1,1}} \epsilon\, \lambda\, x_{2,2}\, \eta1_{1,2}\, \xi2_{1,2} - e^{\lambda \xi2_{1,1}} \epsilon\, \lambda\, x_{1,1}\, \eta2_{1,2}\, \xi2_{1,2} + e^{\lambda \xi2_{1,1}} \epsilon\, \lambda\, x_{2,2}\, \eta2_{1,2}\, \xi2_{1,2} + \\
&e^{\lambda \xi2_{2,2}} \lambda\, x_{1,2}\, \xi1_{1,1}\, \xi2_{1,2} + e^{\lambda \xi2_{1,1}} \epsilon\, \lambda^2\, x_{1,1}\, \eta2_{1,2}\, \xi1_{1,1}\, \xi2_{1,2} - e^{\lambda \xi2_{1,1}} \epsilon\, \lambda^2\, x_{2,2}\, \eta2_{1,2}\, \xi1_{1,1}\, \xi2_{1,2} - \\
&2\, e^{\lambda \xi1_{1,1}+\lambda \xi2_{2,2}} \epsilon\, \lambda^2\, x_{1,2}\, \eta1_{1,2}\, \xi1_{1,2}\, \xi2_{1,2} + 2\, e^{\lambda \xi1_{2,2}+\lambda \xi2_{2,2}} \epsilon\, \lambda^2\, x_{1,2}\, \eta2_{1,2}\, \xi1_{1,2}\, \xi2_{1,2} + \\
&2\, e^{\lambda \xi1_{2,2}+\lambda \xi2_{2,2}} \epsilon\, \lambda^3\, x_{1,2}\, \eta2_{1,2}\, \xi1_{1,1}\, \xi1_{1,2}\, \xi2_{1,2} - e^{\lambda \xi2_{2,2}} \lambda\, x_{1,2}\, \xi1_{2,2}\, \xi2_{1,2} - \\
&e^{\lambda \xi2_{1,1}} \epsilon\, \lambda^2\, x_{1,1}\, \eta2_{1,2}\, \xi1_{2,2}\, \xi2_{1,2} + e^{\lambda \xi2_{1,1}} \epsilon\, \lambda^2\, x_{2,2}\, \eta2_{1,2}\, \xi1_{2,2}\, \xi2_{1,2} + e^{\lambda \xi2_{2,2}} \lambda\, x_{1,2}\, \xi2_{1,1}\, \xi2_{1,2} - \\
&e^{\lambda \xi1_{1,1}-\lambda \xi1_{2,2}+\lambda \xi2_{1,1}+\lambda \xi2_{2,2}} \epsilon\, \lambda^2\, x_{1,2}\, \eta1_{1,2}\, \xi2_{1,2}^2 - e^{\lambda \xi2_{1,1}+\lambda \xi2_{2,2}} \epsilon\, \lambda^2\, x_{1,2}\, \eta2_{1,2}\, \xi2_{1,2}^2 + \\
&e^{\lambda \xi2_{1,1}+\lambda \xi2_{2,2}} \epsilon\, \lambda^3\, x_{1,2}\, \eta2_{1,2}\, \xi1_{1,1}\, \xi2_{1,2}^2 - e^{\lambda \xi2_{1,1}+\lambda \xi2_{2,2}} \epsilon\, \lambda^3\, x_{1,2}\, \eta2_{1,2}\, \xi1_{2,2}\, \xi2_{1,2}^2 + x_{2,2}\, \xi2_{2,2}
\end{aligned}
$$

In[◦]:=
```
rhs = λTangent@
   ($Basis ($Basis /. {x_α_,α_ :→ CF[λ (ξ1_α,α + ξ2_α,α) + ϵ f_α,α[λ]], x_αβ__ → f_αβ[λ], y_αβ__ → g_αβ[λ]}))
```

Out[◦]=

$$
\begin{aligned}
&x_{1,1}\, \xi1_{1,1} + x_{1,1}\, \xi2_{1,1} + e^{\lambda (\xi1_{2,2}+\xi2_{2,2})} x_{1,2}\, \xi1_{1,1}\, f_{1,2}[\lambda] + e^{\lambda (\xi1_{2,2}+\xi2_{2,2})} x_{1,2}\, \xi2_{1,1}\, f_{1,2}[\lambda] + \\
&e^{\lambda (\xi1_{2,2}+\xi2_{2,2})} \epsilon\, x_{1,2}\, \xi1_{1,1}\, f_{1,2}[\lambda]\, f_{2,2}[\lambda] + e^{\lambda (\xi1_{2,2}+\xi2_{2,2})} \epsilon\, x_{1,2}\, \xi2_{1,1}\, f_{1,2}[\lambda]\, f_{2,2}[\lambda] + \\
&\epsilon\, x_{1,1}\, f_{1,1}'[\lambda] + e^{\lambda (\xi1_{2,2}+\xi2_{2,2})} \epsilon\, x_{1,2}\, f_{1,2}[\lambda]\, f_{1,1}'[\lambda] + e^{\lambda (\xi1_{2,2}+\xi2_{2,2})} x_{1,2}\, f_{1,2}'[\lambda] + \\
&e^{\lambda (\xi1_{2,2}+\xi2_{2,2})} \epsilon\, x_{1,2}\, f_{2,2}[\lambda]\, f_{1,2}'[\lambda] + x_{2,2} (\xi1_{2,2} + \xi2_{2,2} + \epsilon\, f_{2,2}'[\lambda]) + \\
&e^{\lambda (\xi1_{1,1}+\xi2_{1,1})-\lambda (\xi1_{2,2}+\xi2_{2,2})} y_{1,2}\, g_{1,2}'[\lambda] + e^{\lambda (\xi1_{1,1}+\xi2_{1,1})-\lambda (\xi1_{2,2}+\xi2_{2,2})} \epsilon\, y_{1,2}\, f_{1,1}[\lambda]\, g_{1,2}'[\lambda] - \\
&e^{\lambda (\xi1_{1,1}+\xi2_{1,1})} \epsilon\, x_{1,1}\, f_{1,2}[\lambda]\, g_{1,2}'[\lambda] + e^{\lambda (\xi1_{1,1}+\xi2_{1,1})} \epsilon\, x_{2,2}\, f_{1,2}[\lambda]\, g_{1,2}'[\lambda] - \\
&e^{\lambda (\xi1_{1,1}+\xi2_{1,1})+\lambda (\xi1_{2,2}+\xi2_{2,2})} \epsilon\, x_{1,2}\, f_{1,2}[\lambda]^2\, g_{1,2}'[\lambda] - e^{\lambda (\xi1_{1,1}+\xi2_{1,1})-\lambda (\xi1_{2,2}+\xi2_{2,2})} \epsilon\, y_{1,2}\, f_{2,2}[\lambda]\, g_{1,2}'[\lambda]
\end{aligned}
$$

In[◦]:=
```
SSBasis = Select[$Basis, SSQ]
```

Out[◦]=

$\{x_{1,1}, x_{2,2}\}$

In[ ]:=
```
eqns = (Coefficient[lhs - rhs, #] == 0) & /@ $Basis
```

Out[ ]=

$$\begin{aligned}
&\{e^{\lambda \xi 1_{1,1}-\lambda \xi 1_{2,2}+\lambda \xi 2_{1,1}-\lambda \xi 2_{2,2}}\, \eta 1_{1,2} + e^{\lambda \xi 2_{1,1}-\lambda \xi 2_{2,2}}\, \eta 2_{1,2} - \\
&\quad e^{\lambda \xi 2_{1,1}-\lambda \xi 2_{2,2}}\, \lambda\, \eta 2_{1,2}\, \xi 1_{1,1} + 2\, e^{\lambda \xi 1_{1,1}+\lambda \xi 2_{1,1}-\lambda \xi 2_{2,2}} \in \lambda^2\, \eta 1_{1,2}\, \eta 2_{1,2}\, \xi 1_{1,2} - \\
&\quad e^{\lambda \xi 1_{2,2}+\lambda \xi 2_{1,1}-\lambda \xi 2_{2,2}} \in \lambda^2\, \eta 2_{1,2}^2\, \xi 1_{1,2} - e^{\lambda \xi 1_{2,2}+\lambda \xi 2_{1,1}-\lambda \xi 2_{2,2}} \in \lambda^3\, \eta 2_{1,2}^2\, \xi 1_{1,1}\, \xi 1_{1,2} + \\
&\quad e^{\lambda \xi 2_{1,1}-\lambda \xi 2_{2,2}}\, \lambda\, \eta 2_{1,2}\, \xi 1_{2,2} - e^{\lambda\, (\xi 1_{1,1}+\xi 2_{1,1})-\lambda\, (\xi 1_{2,2}+\xi 2_{2,2})}\, g_{1,2}{}'[\lambda] - \\
&\quad e^{\lambda\, (\xi 1_{1,1}+\xi 2_{1,1})-\lambda\, (\xi 1_{2,2}+\xi 2_{2,2})} \in f_{1,1}[\lambda]\, g_{1,2}{}'[\lambda] + e^{\lambda\, (\xi 1_{1,1}+\xi 2_{1,1})-\lambda\, (\xi 1_{2,2}+\xi 2_{2,2})} \in f_{2,2}[\lambda]\, g_{1,2}{}'[\lambda] == 0, \\
&-e^{\lambda \xi 1_{1,1}} \in \lambda\, \eta 1_{1,2}\, \xi 1_{1,2} + e^{\lambda \xi 1_{2,2}} \in \lambda\, \eta 2_{1,2}\, \xi 1_{1,2} + e^{\lambda \xi 1_{2,2}} \in \lambda^2\, \eta 2_{1,2}\, \xi 1_{1,1}\, \xi 1_{1,2} - \\
&\quad e^{\lambda \xi 1_{1,1}-\lambda \xi 1_{2,2}+\lambda \xi 2_{1,1}} \in \lambda\, \eta 1_{1,2}\, \xi 2_{1,2} - e^{\lambda \xi 2_{1,1}} \in \lambda\, \eta 2_{1,2}\, \xi 2_{1,2} + e^{\lambda \xi 2_{1,1}} \in \lambda^2\, \eta 2_{1,2}\, \xi 1_{1,1}\, \xi 2_{1,2} - \\
&\quad e^{\lambda \xi 2_{1,1}} \in \lambda^2\, \eta 2_{1,2}\, \xi 1_{2,2}\, \xi 2_{1,2} - \in f_{1,1}{}'[\lambda] + e^{\lambda\, (\xi 1_{1,1}+\xi 2_{1,1})} \in f_{1,2}[\lambda]\, g_{1,2}{}'[\lambda] == 0, \\
&e^{\lambda \xi 1_{2,2}-\lambda \xi 2_{1,1}+\lambda \xi 2_{2,2}}\, \xi 1_{1,2} + e^{\lambda \xi 1_{2,2}-\lambda \xi 2_{1,1}+\lambda \xi 2_{2,2}}\, \lambda\, \xi 1_{1,1}\, \xi 1_{1,2} - e^{\lambda \xi 1_{1,1}+\lambda \xi 1_{2,2}-\lambda \xi 2_{1,1}+\lambda \xi 2_{2,2}} \in \lambda^2\, \eta 1_{1,2}\, \xi 1_{1,2}^2 + \\
&\quad e^{\lambda \xi 2_{2,2}}\, \xi 2_{1,2} + e^{\lambda \xi 2_{2,2}}\, \lambda\, \xi 1_{1,1}\, \xi 2_{1,2} - 2\, e^{\lambda \xi 1_{1,1}+\lambda \xi 2_{2,2}} \in \lambda^2\, \eta 1_{1,2}\, \xi 1_{1,2}\, \xi 2_{1,2} + \\
&\quad 2\, e^{\lambda \xi 1_{2,2}+\lambda \xi 2_{2,2}} \in \lambda^2\, \eta 2_{1,2}\, \xi 1_{1,2}\, \xi 2_{1,2} + 2\, e^{\lambda \xi 1_{2,2}+\lambda \xi 2_{2,2}} \in \lambda^3\, \eta 2_{1,2}\, \xi 1_{1,1}\, \xi 1_{1,2}\, \xi 2_{1,2} - \\
&\quad e^{\lambda \xi 2_{2,2}}\, \lambda\, \xi 1_{2,2}\, \xi 2_{1,2} + e^{\lambda \xi 2_{2,2}}\, \lambda\, \xi 2_{1,1}\, \xi 2_{1,2} - e^{\lambda \xi 1_{1,1}-\lambda \xi 1_{2,2}+\lambda \xi 2_{1,1}+\lambda \xi 2_{2,2}} \in \lambda^2\, \eta 1_{1,2}\, \xi 2_{1,2}^2 - \\
&\quad e^{\lambda \xi 2_{1,1}+\lambda \xi 2_{2,2}} \in \lambda^2\, \eta 2_{1,2}\, \xi 2_{1,2}^2 + e^{\lambda \xi 2_{1,1}+\lambda \xi 2_{2,2}} \in \lambda^3\, \eta 2_{1,2}\, \xi 1_{1,1}\, \xi 2_{1,2}^2 - e^{\lambda \xi 2_{1,1}+\lambda \xi 2_{2,2}} \in \lambda^3\, \eta 2_{1,2}\, \xi 1_{2,2}\, \xi 2_{1,2}^2 - \\
&\quad e^{\lambda\, (\xi 1_{2,2}+\xi 2_{2,2})}\, \xi 1_{1,1}\, f_{1,2}[\lambda] - e^{\lambda\, (\xi 1_{2,2}+\xi 2_{2,2})}\, \xi 2_{1,1}\, f_{1,2}[\lambda] - e^{\lambda\, (\xi 1_{2,2}+\xi 2_{2,2})} \in \xi 1_{1,1}\, f_{1,2}[\lambda]\, f_{2,2}[\lambda] - \\
&\quad e^{\lambda\, (\xi 1_{2,2}+\xi 2_{2,2})} \in \xi 2_{1,1}\, f_{1,2}[\lambda]\, f_{2,2}[\lambda] - e^{\lambda\, (\xi 1_{2,2}+\xi 2_{2,2})} \in f_{1,2}[\lambda]\, f_{1,1}{}'[\lambda] - e^{\lambda\, (\xi 1_{2,2}+\xi 2_{2,2})}\, f_{1,2}{}'[\lambda] - \\
&\quad e^{\lambda\, (\xi 1_{2,2}+\xi 2_{2,2})} \in f_{2,2}[\lambda]\, f_{1,2}{}'[\lambda] + e^{\lambda\, (\xi 1_{1,1}+\xi 2_{1,1})+\lambda\, (\xi 1_{2,2}+\xi 2_{2,2})} \in f_{1,2}[\lambda]^2\, g_{1,2}{}'[\lambda] == 0, \\
&e^{\lambda \xi 1_{1,1}} \in \lambda\, \eta 1_{1,2}\, \xi 1_{1,2} - e^{\lambda \xi 1_{2,2}} \in \lambda\, \eta 2_{1,2}\, \xi 1_{1,2} - e^{\lambda \xi 1_{2,2}} \in \lambda^2\, \eta 2_{1,2}\, \xi 1_{1,1}\, \xi 1_{1,2} + \\
&\quad e^{\lambda \xi 1_{1,1}-\lambda \xi 1_{2,2}+\lambda \xi 2_{1,1}} \in \lambda\, \eta 1_{1,2}\, \xi 2_{1,2} + e^{\lambda \xi 2_{1,1}} \in \lambda\, \eta 2_{1,2}\, \xi 2_{1,2} - e^{\lambda \xi 2_{1,1}} \in \lambda^2\, \eta 2_{1,2}\, \xi 1_{1,1}\, \xi 2_{1,2} + \\
&\quad e^{\lambda \xi 2_{1,1}} \in \lambda^2\, \eta 2_{1,2}\, \xi 1_{2,2}\, \xi 2_{1,2} - \in f_{2,2}{}'[\lambda] - e^{\lambda\, (\xi 1_{1,1}+\xi 2_{1,1})} \in f_{1,2}[\lambda]\, g_{1,2}{}'[\lambda] == 0\}
\end{aligned}$$

In[ ]:=
```
Simplify[eqns]
```

Out[ ]=

$$\begin{aligned}
&\{e^{-\lambda\, (\xi 1_{2,2}-\xi 2_{1,1}+\xi 2_{2,2})} \Big(-e^{2 \lambda \xi 1_{2,2}} \in \lambda^2\, \eta 2_{1,2}^2\, (1 + \lambda\, \xi 1_{1,1})\, \xi 1_{1,2} + e^{\lambda \xi 1_{1,1}}\, \eta 1_{1,2} \left(1 + 2\, e^{\lambda \xi 1_{2,2}} \in \lambda^2\, \eta 2_{1,2}\, \xi 1_{1,2}\right) + \\
&\qquad e^{\lambda \xi 1_{2,2}}\, \eta 2_{1,2}\, (1 - \lambda\, \xi 1_{1,1} + \lambda\, \xi 1_{2,2}) - e^{\lambda \xi 1_{1,1}}\, (1 + \in f_{1,1}[\lambda] - \in f_{2,2}[\lambda])\, g_{1,2}{}'[\lambda]\Big) == 0, \\
&\in \Big(\lambda\, \eta 1_{1,2} \left(e^{\lambda \xi 1_{1,1}}\, \xi 1_{1,2} + e^{\lambda\, (\xi 1_{1,1}-\xi 1_{2,2}+\xi 2_{1,1})}\, \xi 2_{1,2}\right) - \\
&\qquad \lambda\, \eta 2_{1,2} \left(e^{\lambda \xi 1_{2,2}}\, (1 + \lambda\, \xi 1_{1,1})\, \xi 1_{1,2} + e^{\lambda \xi 2_{1,1}}\, (-1 + \lambda\, \xi 1_{1,1} - \lambda\, \xi 1_{2,2})\, \xi 2_{1,2}\right) + \\
&\qquad f_{1,1}{}'[\lambda] - e^{\lambda\, (\xi 1_{1,1}+\xi 2_{1,1})}\, f_{1,2}[\lambda]\, g_{1,2}{}'[\lambda]\Big) == 0, \\
&e^{-\lambda\, (\xi 1_{2,2}+\xi 2_{1,1})} \Big(-e^{\lambda\, (\xi 1_{1,1}+2 \xi 1_{2,2}+\xi 2_{2,2})} \in \lambda^2\, \eta 1_{1,2}\, \xi 1_{1,2}^2 - e^{\lambda\, (\xi 1_{2,2}+\xi 2_{1,1}+\xi 2_{2,2})}\, (-1 - \lambda\, \xi 1_{1,1} + \lambda\, \xi 1_{2,2} - \lambda\, \xi 2_{1,1}) \\
&\qquad\quad \xi 2_{1,2} - e^{\lambda\, (2 \xi 2_{1,1}+\xi 2_{2,2})} \in \lambda^2 \left(e^{\lambda \xi 1_{1,1}}\, \eta 1_{1,2} + e^{\lambda \xi 1_{2,2}}\, \eta 2_{1,2}\, (1 - \lambda\, \xi 1_{1,1} + \lambda\, \xi 1_{2,2})\right) \xi 2_{1,2}^2 + \\
&\qquad e^{\lambda\, (\xi 1_{2,2}+\xi 2_{2,2})}\, \xi 1_{1,2} \Big(-2\, e^{\lambda\, (\xi 1_{1,1}+\xi 2_{1,1})} \in \lambda^2\, \eta 1_{1,2}\, \xi 2_{1,2} + e^{\lambda \xi 1_{2,2}} \left(1 + 2\, e^{\lambda \xi 2_{1,1}} \in \lambda^2\, \eta 2_{1,2}\, \xi 2_{1,2}\right) + \\
&\qquad\qquad e^{\lambda \xi 1_{2,2}}\, \lambda\, \xi 1_{1,1} \left(1 + 2\, e^{\lambda \xi 2_{1,1}} \in \lambda^2\, \eta 2_{1,2}\, \xi 2_{1,2}\right)\Big) - \\
&\qquad e^{\lambda\, (2 \xi 1_{2,2}+\xi 2_{1,1}+\xi 2_{2,2})} \Big(\xi 1_{1,1}\, f_{1,2}[\lambda]\, (1 + \in f_{2,2}[\lambda]) + \xi 2_{1,1}\, f_{1,2}[\lambda]\, (1 + \in f_{2,2}[\lambda]) + \\
&\qquad\qquad \in f_{1,2}[\lambda]\, f_{1,1}{}'[\lambda] + f_{1,2}{}'[\lambda] + \in f_{2,2}[\lambda]\, f_{1,2}{}'[\lambda] - e^{\lambda\, (\xi 1_{1,1}+\xi 2_{1,1})} \in f_{1,2}[\lambda]^2\, g_{1,2}{}'[\lambda]\Big)\Big) == 0, \\
&e^{\lambda\, (\xi 1_{1,1}-\xi 1_{2,2})} \in \lambda\, \eta 1_{1,2} \left(e^{\lambda \xi 1_{2,2}}\, \xi 1_{1,2} + e^{\lambda \xi 2_{1,1}}\, \xi 2_{1,2}\right) == \\
&\quad \in \Big(\lambda\, \eta 2_{1,2} \left(e^{\lambda \xi 1_{2,2}}\, (1 + \lambda\, \xi 1_{1,1})\, \xi 1_{1,2} + e^{\lambda \xi 2_{1,1}}\, (-1 + \lambda\, \xi 1_{1,1} - \lambda\, \xi 1_{2,2})\, \xi 2_{1,2}\right) + \\
&\qquad f_{2,2}{}'[\lambda] + e^{\lambda\, (\xi 1_{1,1}+\xi 2_{1,1})}\, f_{1,2}[\lambda]\, g_{1,2}{}'[\lambda]\Big)\}
\end{aligned}$$

In[ ]:=
```
Length@eqns
```

Out[ ]=

4

In[∘]:=
```
unknowns = $Basis /. {x_αβ__ → f_αβ[λ], y_αβ__ → g_αβ[λ]}
```

Out[∘]=

$$\{g_{1,2}[\lambda],\ f_{1,1}[\lambda],\ f_{1,2}[\lambda],\ f_{2,2}[\lambda]\}$$

In[∘]:=
```
$Basis /. {x_αβ__ :→ f_αβ[0] == 0, y_αβ__ :→ g_αβ[0] == 0}
```

Out[∘]=

$$\{g_{1,2}[0] == 0,\ f_{1,1}[0] == 0,\ f_{1,2}[0] == 0,\ f_{2,2}[0] == 0\}$$

In[∘]:=
```
{sol1} = DSolve[
  Join[
   Simplify[(Coefficient[lhs - rhs, #] == 0) & /@ $Basis],
   $Basis /. {x_αβ__ :→ f_αβ[0] == 0, y_αβ__ :→ g_αβ[0] == 0}
  ],
  $Basis /. {x_αβ__ → f_αβ[λ], y_αβ__ → g_αβ[λ]},
  λ
 ]
```

Set: Lists {sol1} and DSolve[{$e^{-\lambda\,(\text{Subscript}[\ll 3\gg]+\text{Times}[\ll 2\gg]+\text{Subscript}[\ll 3\gg])}$ $(-e^{\text{Times}[\ll 3\gg]}\,\epsilon\,\lambda^2\,\text{Subscript}[\ll 3\gg]^2\,(1+\text{Times}[\ll 2\gg])\,\xi 1_{1,2} + e^{\text{Times}[\ll 2\gg]}\,\eta 1_{1,2}\,(1+\text{Times}[\ll 6\gg]) + e^{\text{Times}[\ll 2\gg]}\,\eta 2_{1,2}\,(1+\text{Times}[\ll 3\gg]+\text{Times}[\ll 2\gg]) - e^{\text{Times}[\ll 2\gg]}\,(1+\text{Times}[\ll 2\gg]+\text{Times}[\ll 3\gg])\,\text{Subscript}[\ll 3\gg]'[\lambda]) == 0$, $\epsilon\,(\lambda\,\eta 1_{1,2}\,(\text{Times}[\ll 2\gg]+\text{Times}[\ll 2\gg]) - \lambda\,\eta 2_{1,2}\,(\text{Times}[\ll 3\gg]+\text{Times}[\ll 3\gg]) + f_{1,1}'[\lambda] - e^{\text{Times}[\ll 2\gg]}\,f_{1,2}[\lambda]\,\text{Subscript}[\ll 3\gg]'[\lambda]) == 0$, $e^{-\lambda\,(\text{Subscript}[\ll 3\gg]+\text{Subscript}[\ll 3\gg])}\,(\ll 1\gg) == 0$, $e^{\lambda\,(\text{Subscript}[\ll 3\gg]+\text{Times}[\ll 2\gg])}\,\epsilon\,\lambda\,\eta 1_{1,2}\,(e^{\text{Times}[\ll 2\gg]}\,\xi 1_{1,2} + e^{\text{Times}[\ll 2\gg]}\,\xi 2_{1,2}) == \epsilon\,(\ll 1\gg)$, $g_{1,2}[0] == 0$, $f_{1,1}[0] == 0$, $f_{1,2}[0] == 0$, $f_{2,2}[0] == 0$}, {$g_{1,2}[\lambda]$, $f_{1,1}[\lambda]$, $f_{\ll 1\gg}[\lambda]$, $f_{2,2}[\lambda]$}, λ] are not the same shape.

Out[∘]=

$$
\begin{aligned}
&\text{DSolve}\Big[\\
&\Big\{e^{-\lambda\,(\xi 1_{2,2}-\xi 2_{1,1}+\xi 2_{2,2})}\,\Big(-e^{2\lambda\,\xi 1_{2,2}}\,\epsilon\,\lambda^2\,\eta 2_{1,2}^2\,(1+\lambda\,\xi 1_{1,1})\,\xi 1_{1,2} + e^{\lambda\,\xi 1_{1,1}}\,\eta 1_{1,2}\,\big(1+2\,e^{\lambda\,\xi 1_{2,2}}\,\epsilon\,\lambda^2\,\eta 2_{1,2}\,\xi 1_{1,2}\big) +\\
&\quad e^{\lambda\,\xi 1_{2,2}}\,\eta 2_{1,2}\,(1-\lambda\,\xi 1_{1,1}+\lambda\,\xi 1_{2,2}) - e^{\lambda\,\xi 1_{1,1}}\,(1+\epsilon\,f_{1,1}[\lambda]-\epsilon\,f_{2,2}[\lambda])\,g_{1,2}'[\lambda]\Big) == 0,\\
&\epsilon\,\Big(\lambda\,\eta 1_{1,2}\,\big(e^{\lambda\,\xi 1_{1,1}}\,\xi 1_{1,2} + e^{\lambda\,(\xi 1_{1,1}-\xi 1_{2,2}+\xi 2_{1,1})}\,\xi 2_{1,2}\big) - \lambda\,\eta 2_{1,2}\,\big(e^{\lambda\,\xi 1_{2,2}}\,(1+\lambda\,\xi 1_{1,1})\,\xi 1_{1,2} +\\
&\quad e^{\lambda\,\xi 2_{1,1}}\,(-1+\lambda\,\xi 1_{1,1}-\lambda\,\xi 1_{2,2})\,\xi 2_{1,2}\big) + f_{1,1}'[\lambda] - e^{\lambda\,(\xi 1_{1,1}+\xi 2_{1,1})}\,f_{1,2}[\lambda]\,g_{1,2}'[\lambda]\Big) == 0,\\
&e^{-\lambda\,(\xi 1_{2,2}+\xi 2_{1,1})}\,\Big(-e^{\lambda\,(\xi 1_{1,1}+2\,\xi 1_{2,2}+\xi 2_{2,2})}\,\epsilon\,\lambda^2\,\eta 1_{1,2}\,\xi 1_{1,2}^2 - e^{\lambda\,(\xi 1_{2,2}+\xi 2_{1,1}+\xi 2_{2,2})}\,(-1-\lambda\,\xi 1_{1,1}+\lambda\,\xi 1_{2,2}-\lambda\,\xi 2_{1,1})\\
&\quad \xi 2_{1,2} - e^{\lambda\,(2\,\xi 2_{1,1}+\xi 2_{2,2})}\,\epsilon\,\lambda^2\,\big(e^{\lambda\,\xi 1_{1,1}}\,\eta 1_{1,2} + e^{\lambda\,\xi 1_{2,2}}\,\eta 2_{1,2}\,(1-\lambda\,\xi 1_{1,1}+\lambda\,\xi 1_{2,2})\big)\,\xi 2_{1,2}^2 +\\
&\quad e^{\lambda\,(\xi 1_{2,2}+\xi 2_{2,2})}\,\xi 1_{1,2}\,\big(-2\,e^{\lambda\,(\xi 1_{1,1}+\xi 2_{1,1})}\,\epsilon\,\lambda^2\,\eta 1_{1,2}\,\xi 2_{1,2} + e^{\lambda\,\xi 1_{2,2}}\,\big(1+2\,e^{\lambda\,\xi 2_{1,1}}\,\epsilon\,\lambda^2\,\eta 2_{1,2}\,\xi 2_{1,2}\big) +\\
&\quad\quad e^{\lambda\,\xi 1_{2,2}}\,\lambda\,\xi 1_{1,1}\,\big(1+2\,e^{\lambda\,\xi 2_{1,1}}\,\epsilon\,\lambda^2\,\eta 2_{1,2}\,\xi 2_{1,2}\big)\big) -\\
&\quad e^{\lambda\,(2\,\xi 1_{2,2}+\xi 2_{1,1}+\xi 2_{2,2})}\,\big(\xi 1_{1,1}\,f_{1,2}[\lambda]\,(1+\epsilon\,f_{2,2}[\lambda]) + \xi 2_{1,1}\,f_{1,2}[\lambda]\,(1+\epsilon\,f_{2,2}[\lambda]) +\\
&\quad\quad \epsilon\,f_{1,2}[\lambda]\,f_{1,1}'[\lambda] + f_{1,2}'[\lambda] + \epsilon\,f_{2,2}[\lambda]\,f_{1,2}'[\lambda] - e^{\lambda\,(\xi 1_{1,1}+\xi 2_{1,1})}\,\epsilon\,f_{1,2}[\lambda]^2\,g_{1,2}'[\lambda]\big)\Big) == 0,\\
&e^{\lambda\,(\xi 1_{1,1}-\xi 1_{2,2})}\,\epsilon\,\lambda\,\eta 1_{1,2}\,\big(e^{\lambda\,\xi 1_{2,2}}\,\xi 1_{1,2} + e^{\lambda\,\xi 2_{1,1}}\,\xi 2_{1,2}\big) ==\\
&\ \epsilon\,\Big(\lambda\,\eta 2_{1,2}\,\big(e^{\lambda\,\xi 1_{2,2}}\,(1+\lambda\,\xi 1_{1,1})\,\xi 1_{1,2} + e^{\lambda\,\xi 2_{1,1}}\,(-1+\lambda\,\xi 1_{1,1}-\lambda\,\xi 1_{2,2})\,\xi 2_{1,2}\big) +\\
&\quad f_{2,2}'[\lambda] + e^{\lambda\,(\xi 1_{1,1}+\xi 2_{1,1})}\,f_{1,2}[\lambda]\,g_{1,2}'[\lambda]\Big),\\
&g_{1,2}[0] == 0,\ f_{1,1}[0] == 0,\ f_{1,2}[0] == 0,\ f_{2,2}[0] == 0\Big\},\ \{g_{1,2}[\lambda],\\
&f_{1,1}[\lambda],\\
&f_{1,2}[\lambda],\\
&f_{2,2}[\lambda]\},\ \lambda\Big]
\end{aligned}
$$

In[∘]:= $k

Out[∘]=

1

In[∘]:= $\text{sol1} \;/.\; \lambda \to 1$

Out[∘]=

sol1

In[∘]:=
$$\begin{aligned}\text{osol} = \mathbb{E}_{\{1,2\}\to\{2\}}\Big[&x_{1,1}[2]\,\xi_{1,1}[1] + x_{1,1}[2]\,\xi_{1,1}[2] + e^{-\xi_{1,1}[2]}\,x_{1,2}[2]\,\xi_{1,2}[1] + e^{-\xi_{2,2}[1]}\,x_{1,2}[2]\,\xi_{1,2}[2] +\\
&e^{-\xi_{1,1}[2]}\,x_{1,3}[2]\,\xi_{1,3}[1] + e^{-\xi_{3,3}[1]}\,x_{1,3}[2]\,\xi_{1,3}[2] + e^{-\xi_{1,1}[2]}\,x_{1,4}[2]\,\xi_{1,4}[1] +\\
&e^{-\xi_{4,4}[1]}\,x_{1,4}[2]\,\xi_{1,4}[2] + e^{-\xi_{1,1}[2]}\,x_{1,5}[2]\,\xi_{1,5}[1] + e^{-\xi_{5,5}[1]}\,x_{1,5}[2]\,\xi_{1,5}[2] +\\
&x_{2,2}[2]\,\xi_{2,2}[1] + x_{2,2}[2]\,\xi_{2,2}[2] + e^{-\xi_{2,2}[2]}\,x_{2,3}[2]\,\xi_{2,3}[1] - x_{1,3}[2]\,\xi_{1,2}[2]\,\xi_{2,3}[1] +\\
&e^{-\xi_{3,3}[1]}\,x_{2,3}[2]\,\xi_{2,3}[2] + e^{-\xi_{2,2}[2]}\,x_{2,4}[2]\,\xi_{2,4}[1] - x_{1,4}[2]\,\xi_{1,2}[2]\,\xi_{2,4}[1] +\\
&e^{-\xi_{4,4}[1]}\,x_{2,4}[2]\,\xi_{2,4}[2] + e^{-\xi_{2,2}[2]}\,x_{2,5}[2]\,\xi_{2,5}[1] - x_{1,5}[2]\,\xi_{1,2}[2]\,\xi_{2,5}[1] +\\
&e^{-\xi_{5,5}[1]}\,x_{2,5}[2]\,\xi_{2,5}[2] + x_{3,3}[2]\,\xi_{3,3}[1] + x_{3,3}[2]\,\xi_{3,3}[2] + e^{-\xi_{3,3}[2]}\,x_{3,4}[2]\,\xi_{3,4}[1] -\\
&x_{1,4}[2]\,\xi_{1,3}[2]\,\xi_{3,4}[1] - x_{2,4}[2]\,\xi_{2,3}[2]\,\xi_{3,4}[1] + e^{-\xi_{4,4}[1]}\,x_{3,4}[2]\,\xi_{3,4}[2] +\\
&e^{-\xi_{3,3}[2]}\,x_{3,5}[2]\,\xi_{3,5}[1] - x_{1,5}[2]\,\xi_{1,3}[2]\,\xi_{3,5}[1] - x_{2,5}[2]\,\xi_{2,3}[2]\,\xi_{3,5}[1] +\\
&e^{-\xi_{5,5}[1]}\,x_{3,5}[2]\,\xi_{3,5}[2] + x_{4,4}[2]\,\xi_{4,4}[1] + x_{4,4}[2]\,\xi_{4,4}[2] + e^{-\xi_{4,4}[2]}\,x_{4,5}[2]\,\xi_{4,5}[1] -\\
&x_{1,5}[2]\,\xi_{1,4}[2]\,\xi_{4,5}[1] - x_{2,5}[2]\,\xi_{2,4}[2]\,\xi_{4,5}[1] - x_{3,5}[2]\,\xi_{3,4}[2]\,\xi_{4,5}[1] +\\
&e^{-\xi_{5,5}[1]}\,x_{4,5}[2]\,\xi_{4,5}[2] + x_{5,5}[2]\,\xi_{5,5}[1] + x_{5,5}[2]\,\xi_{5,5}[2],\ 0,\ 0\Big][\![1]\!] \;/.\; \xi_{i_,i_}[_] \to 0\end{aligned}$$

Out[∘]=
$$\begin{aligned}&x_{1,2}[2]\,\xi_{1,2}[1] + x_{1,2}[2]\,\xi_{1,2}[2] + x_{1,3}[2]\,\xi_{1,3}[1] + x_{1,3}[2]\,\xi_{1,3}[2] +\\
&x_{1,4}[2]\,\xi_{1,4}[1] + x_{1,4}[2]\,\xi_{1,4}[2] + x_{1,5}[2]\,\xi_{1,5}[1] + x_{1,5}[2]\,\xi_{1,5}[2] + x_{2,3}[2]\,\xi_{2,3}[1] -\\
&x_{1,3}[2]\,\xi_{1,2}[2]\,\xi_{2,3}[1] + x_{2,3}[2]\,\xi_{2,3}[2] + x_{2,4}[2]\,\xi_{2,4}[1] - x_{1,4}[2]\,\xi_{1,2}[2]\,\xi_{2,4}[1] +\\
&x_{2,4}[2]\,\xi_{2,4}[2] + x_{2,5}[2]\,\xi_{2,5}[1] - x_{1,5}[2]\,\xi_{1,2}[2]\,\xi_{2,5}[1] + x_{2,5}[2]\,\xi_{2,5}[2] + x_{3,4}[2]\,\xi_{3,4}[1] -\\
&x_{1,4}[2]\,\xi_{1,3}[2]\,\xi_{3,4}[1] - x_{2,4}[2]\,\xi_{2,3}[2]\,\xi_{3,4}[1] + x_{3,4}[2]\,\xi_{3,4}[2] + x_{3,5}[2]\,\xi_{3,5}[1] -\\
&x_{1,5}[2]\,\xi_{1,3}[2]\,\xi_{3,5}[1] - x_{2,5}[2]\,\xi_{2,3}[2]\,\xi_{3,5}[1] + x_{3,5}[2]\,\xi_{3,5}[2] + x_{4,5}[2]\,\xi_{4,5}[1] -\\
&x_{1,5}[2]\,\xi_{1,4}[2]\,\xi_{4,5}[1] - x_{2,5}[2]\,\xi_{2,4}[2]\,\xi_{4,5}[1] - x_{3,5}[2]\,\xi_{3,4}[2]\,\xi_{4,5}[1] + x_{4,5}[2]\,\xi_{4,5}[2]\end{aligned}$$

In[∘]:=
$$\text{Expand@Total}[\text{sol1} \;/.\; \{\xi_{\alpha\beta\_\_} \mapsto \xi_{\alpha\beta}[1],\ \eta_{\alpha\beta\_\_} \mapsto \xi_{\alpha\beta}[2]\ ,\ \text{Rule} \to \text{Times},\ \lambda \to 1,\ f_{\alpha\beta\_\_}[\_] \mapsto x_{\alpha\beta}[2]\}] - \text{osol}$$

Out[∘]=
$$\begin{aligned}&\text{Total}[\text{sol1}] - x_{1,2}[2]\,\xi_{1,2}[1] - x_{1,2}[2]\,\xi_{1,2}[2] - x_{1,3}[2]\,\xi_{1,3}[1] - x_{1,3}[2]\,\xi_{1,3}[2] -\\
&x_{1,4}[2]\,\xi_{1,4}[1] - x_{1,4}[2]\,\xi_{1,4}[2] - x_{1,5}[2]\,\xi_{1,5}[1] - x_{1,5}[2]\,\xi_{1,5}[2] - x_{2,3}[2]\,\xi_{2,3}[1] +\\
&x_{1,3}[2]\,\xi_{1,2}[2]\,\xi_{2,3}[1] - x_{2,3}[2]\,\xi_{2,3}[2] - x_{2,4}[2]\,\xi_{2,4}[1] + x_{1,4}[2]\,\xi_{1,2}[2]\,\xi_{2,4}[1] -\\
&x_{2,4}[2]\,\xi_{2,4}[2] - x_{2,5}[2]\,\xi_{2,5}[1] + x_{1,5}[2]\,\xi_{1,2}[2]\,\xi_{2,5}[1] - x_{2,5}[2]\,\xi_{2,5}[2] - x_{3,4}[2]\,\xi_{3,4}[1] +\\
&x_{1,4}[2]\,\xi_{1,3}[2]\,\xi_{3,4}[1] + x_{2,4}[2]\,\xi_{2,3}[2]\,\xi_{3,4}[1] - x_{3,4}[2]\,\xi_{3,4}[2] - x_{3,5}[2]\,\xi_{3,5}[1] +\\
&x_{1,5}[2]\,\xi_{1,3}[2]\,\xi_{3,5}[1] + x_{2,5}[2]\,\xi_{2,3}[2]\,\xi_{3,5}[1] - x_{3,5}[2]\,\xi_{3,5}[2] - x_{4,5}[2]\,\xi_{4,5}[1] +\\
&x_{1,5}[2]\,\xi_{1,4}[2]\,\xi_{4,5}[1] + x_{2,5}[2]\,\xi_{2,4}[2]\,\xi_{4,5}[1] + x_{3,5}[2]\,\xi_{3,4}[2]\,\xi_{4,5}[1] - x_{4,5}[2]\,\xi_{4,5}[2]\end{aligned}$$