

Pensieve header: The PBW multiplication tensor for $\mathfrak{gl}_n$ using the λ -tangent formalism. Some material from pensieve://Projects/UEA.

```
In[ ]:= SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
```

Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at <http://katlas.org/wiki/KnotTheory>.

Prolog

```
In[ ]:= BeginPackage["UEA`"];
Print["UEA` does computations in general universal enveloping
algebras and PBW algebras. It is in the public domain, available
at http://drorbn.net/AcademicPensieve/Projects/SolvablePBW/.
Dror Bar-Natan is committed to support it within
reason until June 1, 2027. This is version 260601."];
Print["UEA` implements / extends ",
Sort@{"**", B, m, SetAlgebra, U, UB, UProducts, USimp, UU, $Basis, $PBWRule,  $\delta$ },
"."];
Begin["`Private`"];
```

UEA` does computations in general universal enveloping algebras and PBW algebras. It is in the public domain, available at <http://drorbn.net/AcademicPensieve/Projects/SolvablePBW/>. Dror Bar-Natan is committed to support it within reason until June 1, 2027. This is version 260601.
UEA` implements / extends $\{**, B, m, \text{SetAlgebra}, U, UB, \text{UProducts}, \text{USimp}, UU, \delta, \$Basis, \$PBWRule\}$.

Utilities

```
In[ ]:=  $\delta_{i,j}$  := If[i == j, 1, 0];

SSQ[x_] :=  $\neg$  NilQ[x]; (* Semi-Simple Q *)
```

Implementing general universal enveloping algebras

```
In[ ]:= B[0, _] = 0; B[_ , 0] = 0;
B[c_ * x_, y_] /; MemberQ[$Basis, x] := Expand[c B[x, y]];
B[y_, c_ * x_] /; MemberQ[$Basis, x] := Expand[c B[y, x]];
B[x_Plus, y_] := B[#, y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
B[x_, x_] = 0;
B[y_, x_] := Expand[-B[x, y]];
```

```
In[*]:= x_ ≤ y_ := OrderedQ[{x, y} /. $PBWRule]; x_ < y_ := ! OrderedQ[{y, x} /. $PBWRule];
UU_i_[1] := U_i[];
UU_i_[x_p_] := UU_i_@@Table[x, {p}];
UU_i_[ε_] := ε /. {
  U[xs__] => U_i[xs],
  x_ /; MemberQ[$Basis, x] => U_i[x]
};
UU_i_[x_, xs__] := UU_t1[x] UU_t2[xs] // Expand // m_t1,t2->i;
USimp[ε_] := Collect[ε, Times[U[___] ..], Expand];
USimp[ε_] := Expand[ε];
```

```
In[*]:= m_s__[0] = 0;
m_s__[x_Plus] := m_s_/@x;
m_s__[sd_SeriesData] := MapAt[m_s, sd, {3, All}];
m_i->j_[ε_] := ε /. U_i->U_j;
```

```
In[*]:= m_i,j->k_[c_. U_i_[x__] U_j[]] := c U_k[x];
m_i,j->k_[c_. U_i[] U_j_[y__]] := c U_k[y];
m_i,j->k_[c_. U_i_[xx__, x_] U_j_[y_, yy__]] := If[x ≤ y,
  c U_k[xx, x, y, yy],
  ((U_i[xx] (U_j[y, x] + UU_j[B[x, y]]) // Expand // m_i,j->i) U_j[yy] // Expand // m_i,j->k)
  c // USimp
];
```

```
In[*]:= UProducts[{}, 0] = {1}; UProducts[{}, d_Integer] /; d > 0 = {};
UProducts[{i_, is__}, d_Integer] := Sort@
  Flatten@Table[(U_i@@@Subsets[$Basis, {j}]) u, {j, 0, d}, {u, UProducts[{is, d-j}]}];
```

```
In[*]:= Supp[ε_] := Union@Cases[{ε}, U_i[___] => i, ∞];
```

```
In[*]:= Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ⋈ y_ := Module[{is = Supp[x] ∩ Supp[y], σ, z},
  z = x; Do[z = m_i->σ@i[z], {i, is}];
  z = Expand[y z]; Do[z = m_σ@i,i->i[z], {i, is}]; z];
UB[x_, y_] := USimp[x ⋈ y - y ⋈ x];
```

Epilog

```
In[*]:= End[]; EndPackage[];
```

Predefined Algebras

sl(2)

```
In[*]:= Print["UEA`SetAlgebra knows \"sl2\"."];
```

UEA`SetAlgebra knows "sl2".

```
SetAlgebra["sl2"] := (
  Print["In sl2: <e,h,f> / ([h,e]=2e, [h,f]=-2f, [e,f]=h)."];
  B[h, e] = 2 e; B[h, f] = -2 f; B[e, f] = h;
  $Basis = {e, h, f};
  $PBWRule = {e → 1, h → 2, f → 3};
  NilQ[e] = NilQ[f] = True; NilQ[h] = False;
);
```

gl_{n,ε}

```
In[*]:= Print["UEA`SetAlgebra knows the ε-nilpotent algebra gln,ε."];
```

UEA`SetAlgebra knows the ε-nilpotent algebra gl_{n,ε}.

```
SetAlgebra[gln,ε] := (
  $Basis = Flatten@{
    Table[yα,β, {β, 2, n}, {α, 1, β - 1}],
    Table[xα,β, {β, 1, n}, {α, 1, β}]
  };
  NilQ[y__] = True; NilQ[xα,β] = (α != β); (* Nilpotent Q *)
  $PBWRule = Thread[$Basis → Range@Length@$Basis];
  B[xi,j, xk,l] := δj,k xi,l - δl,i xk,j;
  B[yi,j, yk,l] := ε δj,k yi,l - ε δl,i yk,j;
  B[xi,j, yl,k] := δj,k xi,l - δl,i xk,j /. xα,β → If[α ≤ β, ε xα,β, yα,β];
);
```

```

In[*]:= SetAlgebra[gl3,ε];
$PBWRule
MatrixForm@Table[{b1, b2} → B[b1, b2], {b1, $Basis}, {b2, $Basis}]

Out[*]=
{y1,2 → 1, y1,3 → 2, y2,3 → 3, x1,1 → 4, x1,2 → 5, x2,2 → 6, x1,3 → 7, x2,3 → 8, x3,3 → 9}

Out[*]//MatrixForm=

$$\left( \begin{array}{cccc} \{y_{1,2}, y_{1,2}\} \rightarrow 0 & \{y_{1,2}, y_{1,3}\} \rightarrow 0 & \{y_{1,2}, y_{2,3}\} \rightarrow \epsilon y_{1,3} & \{y_{1,2}, x_{1,1}\} \rightarrow y_{1,3} \\ \{y_{1,3}, y_{1,2}\} \rightarrow 0 & \{y_{1,3}, y_{1,3}\} \rightarrow 0 & \{y_{1,3}, y_{2,3}\} \rightarrow 0 & \{y_{1,3}, x_{1,1}\} \rightarrow y_{2,3} \\ \{y_{2,3}, y_{1,2}\} \rightarrow -\epsilon y_{1,3} & \{y_{2,3}, y_{1,3}\} \rightarrow 0 & \{y_{2,3}, y_{2,3}\} \rightarrow 0 & \{y_{2,3}, x_{1,1}\} \rightarrow y_{2,3} \\ \{x_{1,1}, y_{1,2}\} \rightarrow -y_{2,1} & \{x_{1,1}, y_{1,3}\} \rightarrow -y_{3,1} & \{x_{1,1}, y_{2,3}\} \rightarrow 0 & \{x_{1,1}, x_{1,1}\} \rightarrow y_{2,3} \\ \{x_{1,2}, y_{1,2}\} \rightarrow \epsilon x_{1,1} - \epsilon x_{2,2} & \{x_{1,2}, y_{1,3}\} \rightarrow -y_{3,2} & \{x_{1,2}, y_{2,3}\} \rightarrow 0 & \{x_{1,2}, x_{1,1}\} \rightarrow -y_{3,2} \\ \{x_{2,2}, y_{1,2}\} \rightarrow y_{2,1} & \{x_{2,2}, y_{1,3}\} \rightarrow 0 & \{x_{2,2}, y_{2,3}\} \rightarrow -y_{3,2} & \{x_{2,2}, x_{1,1}\} \rightarrow -y_{3,2} \\ \{x_{1,3}, y_{1,2}\} \rightarrow -\epsilon x_{2,3} & \{x_{1,3}, y_{1,3}\} \rightarrow \epsilon x_{1,1} - \epsilon x_{3,3} & \{x_{1,3}, y_{2,3}\} \rightarrow \epsilon x_{1,2} & \{x_{1,3}, x_{1,1}\} \rightarrow -y_{3,2} \\ \{x_{2,3}, y_{1,2}\} \rightarrow 0 & \{x_{2,3}, y_{1,3}\} \rightarrow y_{2,1} & \{x_{2,3}, y_{2,3}\} \rightarrow \epsilon x_{2,2} - \epsilon x_{3,3} & \{x_{2,3}, x_{1,1}\} \rightarrow -y_{3,2} \\ \{x_{3,3}, y_{1,2}\} \rightarrow 0 & \{x_{3,3}, y_{1,3}\} \rightarrow y_{3,1} & \{x_{3,3}, y_{2,3}\} \rightarrow y_{3,2} & \{x_{3,3}, x_{1,1}\} \rightarrow -y_{3,2} \end{array} \right)$$


In[*]:= $k = 2;
x0 = ($Basis /. {x → ξ0, y → η0}).$Basis + 0[ε]$k+1
x1 = ($Basis /. {x → ξ1, y → η1}).$Basis + 0[ε]$k+1

Out[*]=
(y1,2 η0,2 + y1,3 η0,3 + y2,3 η0,3 + x1,1 ξ0,1 + x1,2 ξ0,2 + x1,3 ξ0,3 + x2,2 ξ0,2 + x2,3 ξ0,3 + x3,3 ξ0,3) + 0[ε]3

Out[*]=
(y1,2 η1,2 + y1,3 η1,3 + y2,3 η1,3 + x1,1 ξ1,1 + x1,2 ξ1,2 + x1,3 ξ1,3 + x2,2 ξ1,2 + x2,3 ξ1,3 + x3,3 ξ1,3) + 0[ε]3

In[*]:= A = Table[Coefficient[B[b, Normal@x0], c] + 0[ε]$k+1, {c, $Basis}, {b, $Basis}]

Out[*]=
{ {0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3 },
  {η0,2ε + 0[ε]3, 0[ε]3, -η0,2ε + 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3 },
  {0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3 },
  {-ξ0,2ε + 0[ε]3, -ξ0,3ε + 0[ε]3, 0[ε]3, 0[ε]3, η0,2ε + 0[ε]3, 0[ε]3, η0,3ε + 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3 },
  {(-ξ0,1 + ξ0,2) + 0[ε]3, -ξ0,2 + 0[ε]3, η0,2ε + 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3 },
  {ξ0,2ε + 0[ε]3, 0[ε]3, -ξ0,3ε + 0[ε]3, 0[ε]3, -η0,2ε + 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3 },
  {0[ε]3, η0,2ε + 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3 },
  {ξ0,2ε + 0[ε]3, 0[ε]3, (-ξ0,1 + ξ0,3) + 0[ε]3, -ξ0,2 + 0[ε]3, -ξ0,3 + 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3 },
  {ξ0,3ε + 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3 },
  {(-ξ0,2 + ξ0,3) + 0[ε]3, -ξ0,3 + 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3 },
  {ξ0,2ε + 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3, 0[ε]3 } }

```

In[*]:= **A // Normal // MatrixForm**

Out[*]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \in \eta_{0,2,3} & 0 & -\in \eta_{0,1,2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\in \xi_{0,1,2} & -\in \xi_{0,1,3} & 0 & 0 & \in \eta_{0,1,2} & 0 & \in \eta_{0,1,3} & 0 & 0 \\ 0 & 0 & -\in \xi_{0,1,3} & \xi_{0,1,2} & -\xi_{0,1,1} + \xi_{0,2,2} & -\xi_{0,1,2} & \in \eta_{0,2,3} & 0 & 0 \\ \in \xi_{0,1,2} & 0 & -\in \xi_{0,2,3} & 0 & -\in \eta_{0,1,2} & 0 & 0 & \in \eta_{0,2,3} & 0 \\ 0 & 0 & 0 & \xi_{0,1,3} & \xi_{0,2,3} & 0 & -\xi_{0,1,1} + \xi_{0,3,3} & -\xi_{0,1,2} & -\xi_{0,1,3} \\ \in \xi_{0,1,3} & 0 & 0 & 0 & 0 & \xi_{0,2,3} & -\in \eta_{0,1,2} & -\xi_{0,2,2} + \xi_{0,3,3} & -\xi_{0,2,3} \\ 0 & \in \xi_{0,1,3} & \in \xi_{0,2,3} & 0 & 0 & 0 & -\in \eta_{0,1,3} & -\in \eta_{0,2,3} & 0 \end{pmatrix}$$

In[*]:= **MatrixExp[(A // Normal) /. $\epsilon \rightarrow 0$] // Simplify**

Out[*]=

$$\begin{aligned} & \{ \{1, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 1, 0, 0, 0, 0, 0, 0, 0\}, \\ & \{0, 0, 1, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 1, 0, 0, 0, 0, 0\}, \\ & \left\{ 0, 0, 0, \frac{(1 - e^{-\xi_{0,1,1} + \xi_{0,2,2}}) \xi_{0,1,2}}{\xi_{0,1,1} - \xi_{0,2,2}}, e^{-\xi_{0,1,1} + \xi_{0,2,2}}, -\frac{(1 - e^{-\xi_{0,1,1} + \xi_{0,2,2}}) \xi_{0,1,2}}{\xi_{0,1,1} - \xi_{0,2,2}}, 0, 0, 0 \right\}, \\ & \{0, 0, 0, 0, 0, 1, 0, 0, 0\}, \left\{ 0, 0, 0, \right. \\ & \left(e^{-\xi_{0,1,1}} \left(\xi_{0,1,1} \left(- \left((e^{\xi_{0,2,2}} - e^{\xi_{0,3,3}}) \xi_{0,1,2} \xi_{0,2,3} \right) + (e^{\xi_{0,1,1}} - e^{\xi_{0,3,3}}) \xi_{0,1,3} (\xi_{0,2,2} - \xi_{0,3,3}) \right) - (e^{\xi_{0,1,1}} - e^{\xi_{0,3,3}}) \right. \right. \\ & \left. \left. \xi_{0,1,3} \xi_{0,2,2} (\xi_{0,2,2} - \xi_{0,3,3}) + \xi_{0,1,2} \xi_{0,2,3} \left((e^{\xi_{0,1,1}} - e^{\xi_{0,3,3}}) \xi_{0,2,2} - (e^{\xi_{0,1,1}} - e^{\xi_{0,2,2}}) \xi_{0,3,3} \right) \right) \right) / \\ & \left. \left((\xi_{0,1,1} - \xi_{0,2,2}) (\xi_{0,1,1} - \xi_{0,3,3}) (\xi_{0,2,2} - \xi_{0,3,3}) \right), \frac{e^{-\xi_{0,1,1}} (e^{\xi_{0,2,2}} - e^{\xi_{0,3,3}}) \xi_{0,2,3}}{\xi_{0,2,2} - \xi_{0,3,3}}, \right. \\ & \left. \frac{e^{-\xi_{0,1,1} - \xi_{0,2,2}} (-e^{\xi_{0,1,1}} + e^{\xi_{0,2,2}}) (e^{\xi_{0,2,2}} - e^{\xi_{0,3,3}}) \xi_{0,1,2} \xi_{0,2,3}}{(\xi_{0,1,1} - \xi_{0,2,2}) (\xi_{0,2,2} - \xi_{0,3,3})}, e^{-\xi_{0,1,1} + \xi_{0,3,3}}, \right. \\ & \left. \frac{e^{\xi_{0,3,3}} (e^{-\xi_{0,1,1}} - e^{-\xi_{0,2,2}}) \xi_{0,1,2}}{\xi_{0,1,1} - \xi_{0,2,2}}, \frac{1}{(\xi_{0,1,1} - \xi_{0,2,2}) (\xi_{0,1,1} - \xi_{0,3,3}) (\xi_{0,2,2} - \xi_{0,3,3})} \right. \\ & \left. e^{-\xi_{0,1,1} - \xi_{0,2,2}} \left(\xi_{0,1,1} (e^{\xi_{0,1,1}} (e^{\xi_{0,2,2}} - e^{\xi_{0,3,3}}) \xi_{0,1,2} \xi_{0,2,3} - e^{\xi_{0,2,2}} (e^{\xi_{0,1,1}} - e^{\xi_{0,3,3}}) \xi_{0,1,3} (\xi_{0,2,2} - \xi_{0,3,3})) + \right. \right. \\ & \left. e^{\xi_{0,2,2}} (e^{\xi_{0,1,1}} - e^{\xi_{0,3,3}}) \xi_{0,1,3} \xi_{0,2,2} (\xi_{0,2,2} - \xi_{0,3,3}) - \right. \\ & \left. \left. \xi_{0,1,2} \xi_{0,2,3} (e^{\xi_{0,2,2}} (e^{\xi_{0,1,1}} - e^{\xi_{0,3,3}}) \xi_{0,2,2} - e^{\xi_{0,3,3}} (e^{\xi_{0,1,1}} - e^{\xi_{0,2,2}}) \xi_{0,3,3}) \right) \right\}, \\ & \left\{ 0, 0, 0, 0, 0, \frac{(1 - e^{-\xi_{0,2,2} + \xi_{0,3,3}}) \xi_{0,2,3}}{\xi_{0,2,2} - \xi_{0,3,3}}, 0, e^{-\xi_{0,2,2} + \xi_{0,3,3}}, -\frac{(1 - e^{-\xi_{0,2,2} + \xi_{0,3,3}}) \xi_{0,2,3}}{\xi_{0,2,2} - \xi_{0,3,3}} \right\}, \\ & \{0, 0, 0, 0, 0, 0, 0, 0, 1\} \} \end{aligned}$$

In[*]:= **MatrixExp[A // Normal]**

Out[*]=

\$Aborted

In[*]:= **MatrixExp**[A]

Eigenvalues: Unable to find all roots of the characteristic polynomial.

Out[*]=

MatrixExp[{ {0[ε]³, 0[ε]³, 0[ε]³, 0[ε]³, 0[ε]³, 0[ε]³, 0[ε]³, 0[ε]³, 0[ε]³},
 {η_{02,3}ε + 0[ε]³, 0[ε]³, -η_{01,2}ε + 0[ε]³, 0[ε]³, 0[ε]³, 0[ε]³, 0[ε]³, 0[ε]³, 0[ε]³},
 {0[ε]³, 0[ε]³, 0[ε]³, 0[ε]³, 0[ε]³, 0[ε]³, 0[ε]³, 0[ε]³, 0[ε]³},
 {-ξ_{01,2}ε + 0[ε]³, -ξ_{01,3}ε + 0[ε]³, 0[ε]³, 0[ε]³, η_{01,2}ε + 0[ε]³, 0[ε]³,
 η_{01,3}ε + 0[ε]³, 0[ε]³, 0[ε]³}, {0[ε]³, 0[ε]³, -ξ_{01,3}ε + 0[ε]³, ξ_{01,2} + 0[ε]³,
 (-ξ_{01,1} + ξ_{02,2}) + 0[ε]³, -ξ_{01,2} + 0[ε]³, η_{02,3}ε + 0[ε]³, 0[ε]³, 0[ε]³},
 {ξ_{01,2}ε + 0[ε]³, 0[ε]³, -ξ_{02,3}ε + 0[ε]³, 0[ε]³, -η_{01,2}ε + 0[ε]³, 0[ε]³,
 0[ε]³, η_{02,3}ε + 0[ε]³, 0[ε]³}, {0[ε]³, 0[ε]³, 0[ε]³, ξ_{01,3} + 0[ε]³,
 ξ_{02,3} + 0[ε]³, 0[ε]³, (-ξ_{01,1} + ξ_{03,3}) + 0[ε]³, -ξ_{01,2} + 0[ε]³, -ξ_{01,3} + 0[ε]³},
 {ξ_{01,3}ε + 0[ε]³, 0[ε]³, 0[ε]³, 0[ε]³, 0[ε]³, ξ_{02,3} + 0[ε]³, -η_{01,2}ε + 0[ε]³,
 (-ξ_{02,2} + ξ_{03,3}) + 0[ε]³, -ξ_{02,3} + 0[ε]³}, {0[ε]³, ξ_{01,3}ε + 0[ε]³, ξ_{02,3}ε + 0[ε]³,
 0[ε]³, 0[ε]³, 0[ε]³, -η_{01,3}ε + 0[ε]³, -η_{02,3}ε + 0[ε]³, 0[ε]³}}]

In[*]:= **adExp**[x_][y_] := **Expand@Module**[{A, b, c},
 A = **Table**[**Coefficient**[B[b, x], c], {c, \$Basis}, {b, \$Basis}];
 \$Basis.**MatrixExp**[A].**Table**[**Coefficient**[y, b], {b, \$Basis}]
]

In[*]:= **adExp**[x0][x1]

Out[*]=

x_{1,2} η_{1,2} + x_{1,3} η_{1,3} + x_{1,4} η_{1,4} + x_{1,5} η_{1,5} + x_{2,3} η_{2,3} + x_{2,4} η_{2,4} + x_{2,5} η_{2,5} + x_{3,4} η_{3,4} + x_{3,5} η_{3,5} + x_{4,5} η_{4,5} -
 x_{1,3} η_{2,3} ξ_{1,2} - x_{1,4} η_{2,4} ξ_{1,2} - x_{1,5} η_{2,5} ξ_{1,2} - x_{1,4} η_{3,4} ξ_{1,3} - x_{1,5} η_{3,5} ξ_{1,3} - x_{1,5} η_{4,5} ξ_{1,4} +
 x_{1,3} η_{1,2} ξ_{2,3} - x_{2,4} η_{3,4} ξ_{2,3} - x_{2,5} η_{3,5} ξ_{2,3} + $\frac{1}{2}$ x_{1,4} η_{3,4} ξ_{1,2} ξ_{2,3} + $\frac{1}{2}$ x_{1,5} η_{3,5} ξ_{1,2} ξ_{2,3} +
 x_{1,4} η_{1,2} ξ_{2,4} - x_{2,5} η_{4,5} ξ_{2,4} + $\frac{1}{2}$ x_{1,5} η_{4,5} ξ_{1,2} ξ_{2,4} + x_{1,5} η_{1,2} ξ_{2,5} + x_{1,4} η_{1,3} ξ_{3,4} +
 x_{2,4} η_{2,3} ξ_{3,4} - x_{3,5} η_{4,5} ξ_{3,4} - x_{1,4} η_{2,3} ξ_{1,2} ξ_{3,4} + $\frac{1}{2}$ x_{1,5} η_{4,5} ξ_{1,3} ξ_{3,4} + $\frac{1}{2}$ x_{1,4} η_{1,2} ξ_{2,3} ξ_{3,4} +
 $\frac{1}{2}$ x_{2,5} η_{4,5} ξ_{2,3} ξ_{3,4} - $\frac{1}{6}$ x_{1,5} η_{4,5} ξ_{1,2} ξ_{2,3} ξ_{3,4} + x_{1,5} η_{1,3} ξ_{3,5} + x_{2,5} η_{2,3} ξ_{3,5} - x_{1,5} η_{2,3} ξ_{1,2} ξ_{3,5} +
 $\frac{1}{2}$ x_{1,5} η_{1,2} ξ_{2,3} ξ_{3,5} + x_{1,5} η_{1,4} ξ_{4,5} + x_{2,5} η_{2,4} ξ_{4,5} + x_{3,5} η_{3,4} ξ_{4,5} - x_{1,5} η_{2,4} ξ_{1,2} ξ_{4,5} -
 x_{1,5} η_{3,4} ξ_{1,3} ξ_{4,5} - x_{2,5} η_{3,4} ξ_{2,3} ξ_{4,5} + $\frac{1}{2}$ x_{1,5} η_{3,4} ξ_{1,2} ξ_{2,3} ξ_{4,5} + $\frac{1}{2}$ x_{1,5} η_{1,2} ξ_{2,4} ξ_{4,5} +
 $\frac{1}{2}$ x_{1,5} η_{1,3} ξ_{3,4} ξ_{4,5} + $\frac{1}{2}$ x_{2,5} η_{2,3} ξ_{3,4} ξ_{4,5} - $\frac{1}{2}$ x_{1,5} η_{2,3} ξ_{1,2} ξ_{3,4} ξ_{4,5} + $\frac{1}{6}$ x_{1,5} η_{1,2} ξ_{2,3} ξ_{3,4} ξ_{4,5}

In[*]:= **Coefficient**[adExp[x0][y0], x1,5]

Out[*]=

$$\begin{aligned} & \eta_{1,5} - \eta_{2,5} \xi_{1,2} - \eta_{3,5} \xi_{1,3} - \eta_{4,5} \xi_{1,4} + \frac{1}{2} \eta_{3,5} \xi_{1,2} \xi_{2,3} + \frac{1}{2} \eta_{4,5} \xi_{1,2} \xi_{2,4} + \eta_{1,2} \xi_{2,5} + \frac{1}{2} \eta_{4,5} \xi_{1,3} \xi_{3,4} - \\ & \frac{1}{6} \eta_{4,5} \xi_{1,2} \xi_{2,3} \xi_{3,4} + \eta_{1,3} \xi_{3,5} - \eta_{2,3} \xi_{1,2} \xi_{3,5} + \frac{1}{2} \eta_{1,2} \xi_{2,3} \xi_{3,5} + \eta_{1,4} \xi_{4,5} - \eta_{2,4} \xi_{1,2} \xi_{4,5} - \eta_{3,4} \xi_{1,3} \xi_{4,5} + \\ & \frac{1}{2} \eta_{3,4} \xi_{1,2} \xi_{2,3} \xi_{4,5} + \frac{1}{2} \eta_{1,2} \xi_{2,4} \xi_{4,5} + \frac{1}{2} \eta_{1,3} \xi_{3,4} \xi_{4,5} - \frac{1}{2} \eta_{2,3} \xi_{1,2} \xi_{3,4} \xi_{4,5} + \frac{1}{6} \eta_{1,2} \xi_{2,3} \xi_{3,4} \xi_{4,5} \end{aligned}$$

In[*]:= **λTangent**[] = 0;

λTangent[xs___, x_] := adExp[x][λTangent[xs]] + ∂_λx;

λTangent[xs_List] := λTangent@@xs

In[*]:= **\$Basis** (\$Basis /. x → ξ)

Out[*]=

$$\{x_{1,2} \xi_{1,2}, x_{1,3} \xi_{1,3}, x_{2,3} \xi_{2,3}, x_{1,4} \xi_{1,4}, x_{2,4} \xi_{2,4}, x_{3,4} \xi_{3,4}, x_{1,5} \xi_{1,5}, x_{2,5} \xi_{2,5}, x_{3,5} \xi_{3,5}, x_{4,5} \xi_{4,5}\}$$

In[*]:= **λTangent**@(λ\$Basis (\$Basis /. x → ξ))

Out[*]=

$$\begin{aligned} & x_{1,2} \xi_{1,2} + x_{1,3} \xi_{1,3} + x_{1,4} \xi_{1,4} + x_{1,5} \xi_{1,5} + x_{2,3} \xi_{2,3} + \lambda x_{1,3} \xi_{1,2} \xi_{2,3} + \\ & x_{2,4} \xi_{2,4} + \lambda x_{1,4} \xi_{1,2} \xi_{2,4} + x_{2,5} \xi_{2,5} + \lambda x_{1,5} \xi_{1,2} \xi_{2,5} + x_{3,4} \xi_{3,4} + \lambda x_{1,4} \xi_{1,3} \xi_{3,4} + \\ & \lambda x_{2,4} \xi_{2,3} \xi_{3,4} + \lambda^2 x_{1,4} \xi_{1,2} \xi_{2,3} \xi_{3,4} + x_{3,5} \xi_{3,5} + \lambda x_{1,5} \xi_{1,3} \xi_{3,5} + \lambda x_{2,5} \xi_{2,3} \xi_{3,5} + \\ & \lambda^2 x_{1,5} \xi_{1,2} \xi_{2,3} \xi_{3,5} + x_{4,5} \xi_{4,5} + \lambda x_{1,5} \xi_{1,4} \xi_{4,5} + \lambda x_{2,5} \xi_{2,4} \xi_{4,5} + \lambda^2 x_{1,5} \xi_{1,2} \xi_{2,4} \xi_{4,5} + \\ & \lambda x_{3,5} \xi_{3,4} \xi_{4,5} + \lambda^2 x_{1,5} \xi_{1,3} \xi_{3,4} \xi_{4,5} + \lambda^2 x_{2,5} \xi_{2,3} \xi_{3,4} \xi_{4,5} + \lambda^3 x_{1,5} \xi_{1,2} \xi_{2,3} \xi_{3,4} \xi_{4,5} \end{aligned}$$

In[*]:= **lhs** = Join[λ\$Basis (\$Basis /. x → ξ), λ\$Basis (\$Basis /. x → η)] // λTangent

Out[*]=

$$\begin{aligned} & x_{1,2} \eta_{1,2} + x_{1,3} \eta_{1,3} + x_{1,4} \eta_{1,4} + x_{1,5} \eta_{1,5} + x_{2,3} \eta_{2,3} + \lambda x_{1,3} \eta_{1,2} \eta_{2,3} + x_{2,4} \eta_{2,4} + \lambda x_{1,4} \eta_{1,2} \eta_{2,4} + \\ & x_{2,5} \eta_{2,5} + \lambda x_{1,5} \eta_{1,2} \eta_{2,5} + x_{3,4} \eta_{3,4} + \lambda x_{1,4} \eta_{1,3} \eta_{3,4} + \lambda x_{2,4} \eta_{2,3} \eta_{3,4} + \lambda^2 x_{1,4} \eta_{1,2} \eta_{2,3} \eta_{3,4} + \\ & x_{3,5} \eta_{3,5} + \lambda x_{1,5} \eta_{1,3} \eta_{3,5} + \lambda x_{2,5} \eta_{2,3} \eta_{3,5} + \lambda^2 x_{1,5} \eta_{1,2} \eta_{2,3} \eta_{3,5} + x_{4,5} \eta_{4,5} + \lambda x_{1,5} \eta_{1,4} \eta_{4,5} + \\ & \lambda x_{2,5} \eta_{2,4} \eta_{4,5} + \lambda^2 x_{1,5} \eta_{1,2} \eta_{2,4} \eta_{4,5} + \lambda x_{3,5} \eta_{3,4} \eta_{4,5} + \lambda^2 x_{1,5} \eta_{1,3} \eta_{3,4} \eta_{4,5} + \lambda^2 x_{2,5} \eta_{2,3} \eta_{3,4} \eta_{4,5} + \\ & \lambda^3 x_{1,5} \eta_{1,2} \eta_{2,3} \eta_{3,4} \eta_{4,5} + x_{1,2} \xi_{1,2} + \lambda x_{1,3} \xi_{1,2} \xi_{2,3} + \lambda x_{1,4} \xi_{1,2} \xi_{2,4} + \lambda x_{1,5} \xi_{1,2} \xi_{2,5} + \\ & \lambda^2 x_{1,4} \xi_{1,2} \xi_{2,3} \xi_{3,4} + \lambda^2 x_{1,5} \xi_{1,2} \xi_{2,3} \xi_{3,5} + \lambda^2 x_{1,5} \xi_{1,2} \xi_{2,4} \xi_{4,5} + \lambda^3 x_{1,5} \xi_{1,2} \xi_{2,3} \xi_{3,4} \xi_{4,5} + \\ & x_{1,3} \xi_{1,3} + \lambda x_{1,4} \xi_{1,3} \xi_{3,4} + \lambda x_{1,5} \xi_{1,3} \xi_{3,5} + \lambda^2 x_{1,5} \xi_{1,3} \xi_{3,4} \xi_{4,5} + x_{1,4} \xi_{1,4} + \lambda x_{1,5} \xi_{1,4} \xi_{4,5} + \\ & x_{1,5} \xi_{1,5} + x_{2,3} \xi_{2,3} - \lambda x_{1,3} \xi_{1,2} \xi_{2,3} + \lambda x_{2,4} \xi_{2,3} \xi_{3,4} - \lambda^2 x_{1,4} \xi_{1,2} \xi_{2,3} \xi_{3,4} + \lambda x_{2,5} \xi_{2,3} \xi_{3,5} - \\ & \lambda^2 x_{1,5} \xi_{1,2} \xi_{2,3} \xi_{3,5} + \lambda^2 x_{2,5} \xi_{2,3} \xi_{3,4} \xi_{4,5} - \lambda^3 x_{1,5} \xi_{1,2} \xi_{2,3} \xi_{3,4} \xi_{4,5} + \lambda x_{1,3} \xi_{1,2} \xi_{2,3} + \\ & \lambda^2 x_{1,4} \xi_{1,2} \xi_{2,3} \xi_{3,4} + \lambda^2 x_{1,5} \xi_{1,2} \xi_{2,3} \xi_{3,5} + \lambda^3 x_{1,5} \xi_{1,2} \xi_{2,3} \xi_{3,4} \xi_{4,5} + x_{2,4} \xi_{2,4} - \lambda x_{1,4} \xi_{1,2} \xi_{2,4} + \\ & \lambda x_{2,5} \xi_{2,4} \xi_{4,5} - \lambda^2 x_{1,5} \xi_{1,2} \xi_{2,4} \xi_{4,5} + \lambda x_{1,4} \xi_{1,2} \xi_{2,4} \xi_{4,5} + \lambda^2 x_{1,5} \xi_{1,2} \xi_{2,4} \xi_{4,5} + x_{2,5} \xi_{2,5} - \\ & \lambda x_{1,5} \xi_{1,2} \xi_{2,5} + \lambda x_{1,5} \xi_{1,2} \xi_{2,5} + x_{3,4} \xi_{3,4} - \lambda x_{1,4} \xi_{1,3} \xi_{3,4} - \lambda x_{2,4} \xi_{2,3} \xi_{3,4} + \lambda x_{3,5} \xi_{3,4} \xi_{4,5} - \\ & \lambda^2 x_{1,5} \xi_{1,3} \xi_{3,4} \xi_{4,5} - \lambda^2 x_{2,5} \xi_{2,3} \xi_{3,4} \xi_{4,5} + \lambda x_{1,4} \xi_{1,3} \xi_{3,4} + \lambda^2 x_{1,5} \xi_{1,3} \xi_{3,4} \xi_{4,5} + \lambda x_{2,4} \xi_{2,3} \xi_{3,4} - \\ & \lambda^2 x_{1,4} \xi_{1,2} \xi_{2,3} \xi_{3,4} + \lambda^2 x_{2,5} \xi_{2,3} \xi_{3,4} \xi_{4,5} - \lambda^3 x_{1,5} \xi_{1,2} \xi_{2,3} \xi_{3,4} \xi_{4,5} + \lambda^2 x_{1,4} \xi_{1,2} \xi_{2,3} \xi_{3,4} + \\ & \lambda^3 x_{1,5} \xi_{1,2} \xi_{2,3} \xi_{3,4} \xi_{4,5} + x_{3,5} \xi_{3,5} - \lambda x_{1,5} \xi_{1,3} \xi_{3,5} - \lambda x_{2,5} \xi_{2,3} \xi_{3,5} + \lambda x_{1,5} \xi_{1,3} \xi_{3,5} + \\ & \lambda x_{2,5} \xi_{2,3} \xi_{3,5} - \lambda^2 x_{1,5} \xi_{1,2} \xi_{2,3} \xi_{3,5} + \lambda^2 x_{1,5} \xi_{1,2} \xi_{2,3} \xi_{3,5} + x_{4,5} \xi_{4,5} - \lambda x_{1,5} \xi_{1,4} \xi_{4,5} - \\ & \lambda x_{2,5} \xi_{2,4} \xi_{4,5} - \lambda x_{3,5} \xi_{3,4} \xi_{4,5} + \lambda x_{1,5} \xi_{1,4} \xi_{4,5} + \lambda x_{2,5} \xi_{2,4} \xi_{4,5} - \lambda^2 x_{1,5} \xi_{1,2} \xi_{2,4} \xi_{4,5} + \\ & \lambda^2 x_{1,5} \xi_{1,2} \xi_{2,4} \xi_{4,5} + \lambda x_{3,5} \xi_{3,4} \xi_{4,5} - \lambda^2 x_{1,5} \xi_{1,3} \xi_{3,4} \xi_{4,5} - \lambda^2 x_{2,5} \xi_{2,3} \xi_{3,4} \xi_{4,5} + \\ & \lambda^2 x_{1,5} \xi_{1,3} \xi_{3,4} \xi_{4,5} + \lambda^2 x_{2,5} \xi_{2,3} \xi_{3,4} \xi_{4,5} - \lambda^3 x_{1,5} \xi_{1,2} \xi_{2,3} \xi_{3,4} \xi_{4,5} + \lambda^3 x_{1,5} \xi_{1,2} \xi_{2,3} \xi_{3,4} \xi_{4,5} \end{aligned}$$

In[*]:= rhs = λ Tangent@(\$Basis (\$Basis /. $x_{\alpha\beta} \rightarrow f_{\alpha\beta}[\lambda]$))

Out[*]=

$$\begin{aligned} & x_{1,2} f_{1,2}'[\lambda] + x_{1,3} f_{2,3}[\lambda] f_{1,2}'[\lambda] + x_{1,4} f_{2,4}[\lambda] f_{1,2}'[\lambda] + x_{1,5} f_{2,5}[\lambda] f_{1,2}'[\lambda] + \\ & x_{1,4} f_{2,3}[\lambda] f_{3,4}[\lambda] f_{1,2}'[\lambda] + x_{1,5} f_{2,3}[\lambda] f_{3,5}[\lambda] f_{1,2}'[\lambda] + x_{1,5} f_{2,4}[\lambda] f_{4,5}[\lambda] f_{1,2}'[\lambda] + \\ & x_{1,5} f_{2,3}[\lambda] f_{3,4}[\lambda] f_{4,5}[\lambda] f_{1,2}'[\lambda] + x_{1,3} f_{1,3}'[\lambda] + x_{1,4} f_{3,4}[\lambda] f_{1,3}'[\lambda] + x_{1,5} f_{3,5}[\lambda] f_{1,3}'[\lambda] + \\ & x_{1,5} f_{3,4}[\lambda] f_{4,5}[\lambda] f_{1,3}'[\lambda] + x_{1,4} f_{1,4}'[\lambda] + x_{1,5} f_{4,5}[\lambda] f_{1,4}'[\lambda] + x_{1,5} f_{1,5}'[\lambda] + x_{2,3} f_{2,3}'[\lambda] + \\ & x_{2,4} f_{3,4}[\lambda] f_{2,3}'[\lambda] + x_{2,5} f_{3,5}[\lambda] f_{2,3}'[\lambda] + x_{2,5} f_{3,4}[\lambda] f_{4,5}[\lambda] f_{2,3}'[\lambda] + x_{2,4} f_{2,4}'[\lambda] + \\ & x_{2,5} f_{4,5}[\lambda] f_{2,4}'[\lambda] + x_{2,5} f_{2,5}'[\lambda] + x_{3,4} f_{3,4}'[\lambda] + x_{3,5} f_{4,5}[\lambda] f_{3,4}'[\lambda] + x_{3,5} f_{3,5}'[\lambda] + x_{4,5} f_{4,5}'[\lambda] \end{aligned}$$

In[*]:= (Coefficient[lhs - rhs, #] == 0) & /@ \$Basis

Out[*]=

$$\begin{aligned} & \{ \eta_{1,2} + \xi_{1,2} - f_{1,2}'[\lambda] = 0, \\ & \eta_{1,3} + \lambda \eta_{1,2} \eta_{2,3} + \lambda \eta_{2,3} \xi_{1,2} + \xi_{1,3} - \lambda \eta_{1,2} \xi_{2,3} + \lambda \xi_{1,2} \xi_{2,3} - f_{2,3}[\lambda] f_{1,2}'[\lambda] - f_{1,3}'[\lambda] = 0, \\ & \eta_{2,3} + \xi_{2,3} - f_{2,3}'[\lambda] = 0, \\ & \eta_{1,4} + \lambda \eta_{1,2} \eta_{2,4} + \lambda \eta_{1,3} \eta_{3,4} + \lambda^2 \eta_{1,2} \eta_{2,3} \eta_{3,4} + \lambda \eta_{2,4} \xi_{1,2} + \lambda^2 \eta_{2,3} \eta_{3,4} \xi_{1,2} + \lambda \eta_{3,4} \xi_{1,3} + \xi_{1,4} - \\ & \lambda^2 \eta_{1,2} \eta_{3,4} \xi_{2,3} + \lambda^2 \eta_{3,4} \xi_{1,2} \xi_{2,3} - \lambda \eta_{1,2} \xi_{2,4} + \lambda \xi_{1,2} \xi_{2,4} - \lambda \eta_{1,3} \xi_{3,4} + \lambda \xi_{1,3} \xi_{3,4} - \lambda^2 \eta_{1,2} \xi_{2,3} \xi_{3,4} + \\ & \lambda^2 \xi_{1,2} \xi_{2,3} \xi_{3,4} - f_{2,4}[\lambda] f_{1,2}'[\lambda] - f_{2,3}[\lambda] f_{3,4}[\lambda] f_{1,2}'[\lambda] - f_{3,4}[\lambda] f_{1,3}'[\lambda] - f_{1,4}'[\lambda] = 0, \\ & \eta_{2,4} + \lambda \eta_{2,3} \eta_{3,4} + \lambda \eta_{3,4} \xi_{2,3} + \xi_{2,4} - \lambda \eta_{2,3} \xi_{3,4} + \lambda \xi_{2,3} \xi_{3,4} - f_{3,4}[\lambda] f_{2,3}'[\lambda] - f_{2,4}'[\lambda] = 0, \\ & \eta_{3,4} + \xi_{3,4} - f_{3,4}'[\lambda] = 0, \\ & \eta_{1,5} + \lambda \eta_{1,2} \eta_{2,5} + \lambda \eta_{1,3} \eta_{3,5} + \lambda^2 \eta_{1,2} \eta_{2,3} \eta_{3,5} + \lambda \eta_{1,4} \eta_{4,5} + \lambda^2 \eta_{1,2} \eta_{2,4} \eta_{4,5} + \lambda^2 \eta_{1,3} \eta_{3,4} \eta_{4,5} + \\ & \lambda^3 \eta_{1,2} \eta_{2,3} \eta_{3,4} \eta_{4,5} + \lambda \eta_{2,5} \xi_{1,2} + \lambda^2 \eta_{2,3} \eta_{3,5} \xi_{1,2} + \lambda^2 \eta_{2,4} \eta_{4,5} \xi_{1,2} + \lambda^3 \eta_{2,3} \eta_{3,4} \eta_{4,5} \xi_{1,2} + \\ & \lambda \eta_{3,5} \xi_{1,3} + \lambda^2 \eta_{3,4} \eta_{4,5} \xi_{1,3} + \lambda \eta_{4,5} \xi_{1,4} + \xi_{1,5} - \lambda^2 \eta_{1,2} \eta_{3,5} \xi_{2,3} - \lambda^3 \eta_{1,2} \eta_{3,4} \eta_{4,5} \xi_{2,3} + \\ & \lambda^2 \eta_{3,5} \xi_{1,2} \xi_{2,3} + \lambda^3 \eta_{3,4} \eta_{4,5} \xi_{1,2} \xi_{2,3} - \lambda^2 \eta_{1,2} \eta_{4,5} \xi_{2,4} + \lambda^2 \eta_{4,5} \xi_{1,2} \xi_{2,4} - \\ & \lambda \eta_{1,2} \xi_{2,5} + \lambda \xi_{1,2} \xi_{2,5} - \lambda^2 \eta_{1,3} \eta_{4,5} \xi_{3,4} + \lambda^2 \eta_{4,5} \xi_{1,3} \xi_{3,4} - \lambda^3 \eta_{1,2} \eta_{4,5} \xi_{2,3} \xi_{3,4} + \\ & \lambda^3 \eta_{4,5} \xi_{1,2} \xi_{2,3} \xi_{3,4} - \lambda \eta_{1,3} \xi_{3,5} + \lambda \xi_{1,3} \xi_{3,5} - \lambda^2 \eta_{1,2} \xi_{2,3} \xi_{3,5} + \lambda^2 \xi_{1,2} \xi_{2,3} \xi_{3,5} - \\ & \lambda \eta_{1,4} \xi_{4,5} + \lambda \xi_{1,4} \xi_{4,5} - \lambda^2 \eta_{1,2} \xi_{2,4} \xi_{4,5} + \lambda^2 \xi_{1,2} \xi_{2,4} \xi_{4,5} - \lambda^2 \eta_{1,3} \xi_{3,4} \xi_{4,5} + \\ & \lambda^2 \xi_{1,3} \xi_{3,4} \xi_{4,5} - \lambda^3 \eta_{1,2} \xi_{2,3} \xi_{3,4} \xi_{4,5} + \lambda^3 \xi_{1,2} \xi_{2,3} \xi_{3,4} \xi_{4,5} - f_{2,5}[\lambda] f_{1,2}'[\lambda] - \\ & f_{2,3}[\lambda] f_{3,5}[\lambda] f_{1,2}'[\lambda] - f_{2,4}[\lambda] f_{4,5}[\lambda] f_{1,2}'[\lambda] - f_{2,3}[\lambda] f_{3,4}[\lambda] f_{4,5}[\lambda] f_{1,2}'[\lambda] - \\ & f_{3,5}[\lambda] f_{1,3}'[\lambda] - f_{3,4}[\lambda] f_{4,5}[\lambda] f_{1,3}'[\lambda] - f_{4,5}[\lambda] f_{1,4}'[\lambda] - f_{1,5}'[\lambda] = 0, \\ & \eta_{2,5} + \lambda \eta_{2,3} \eta_{3,5} + \lambda \eta_{2,4} \eta_{4,5} + \lambda^2 \eta_{2,3} \eta_{3,4} \eta_{4,5} + \lambda \eta_{3,5} \xi_{2,3} + \lambda^2 \eta_{3,4} \eta_{4,5} \xi_{2,3} + \lambda \eta_{4,5} \xi_{2,4} + \xi_{2,5} - \\ & \lambda^2 \eta_{2,3} \eta_{4,5} \xi_{3,4} + \lambda^2 \eta_{4,5} \xi_{2,3} \xi_{3,4} - \lambda \eta_{2,3} \xi_{3,5} + \lambda \xi_{2,3} \xi_{3,5} - \lambda \eta_{2,4} \xi_{4,5} + \lambda \xi_{2,4} \xi_{4,5} - \lambda^2 \eta_{2,3} \xi_{3,4} \xi_{4,5} + \\ & \lambda^2 \xi_{2,3} \xi_{3,4} \xi_{4,5} - f_{3,5}[\lambda] f_{2,3}'[\lambda] - f_{3,4}[\lambda] f_{4,5}[\lambda] f_{2,3}'[\lambda] - f_{4,5}[\lambda] f_{2,4}'[\lambda] - f_{2,5}'[\lambda] = 0, \\ & \eta_{3,5} + \lambda \eta_{3,4} \eta_{4,5} + \lambda \eta_{4,5} \xi_{3,4} + \xi_{3,5} - \lambda \eta_{3,4} \xi_{4,5} + \lambda \xi_{3,4} \xi_{4,5} - f_{4,5}[\lambda] f_{3,4}'[\lambda] - f_{3,5}'[\lambda] = 0, \\ & \eta_{4,5} + \xi_{4,5} - f_{4,5}'[\lambda] = 0 \} \end{aligned}$$


```

In[*]:= {sol} = DSolve[
  Join[
    (Coefficient[lhs - rhs, #] == 0) & /@ $Basis,
    $Basis /. xαβ__ => fαβ[0] == 0
  ],
  $Basis /. xαβ__ => fαβ[λ],
  λ
]

Out[*]=
{ {f1,2[λ] → λ η1,2 + λ ξ1,2, f1,3[λ] → λ η1,3 + λ ξ1,3 - λ2 η1,2 ξ2,3,
  f2,3[λ] → λ η2,3 + λ ξ2,3, f1,4[λ] → -λ (-η1,4 - ξ1,4 + λ η1,2 ξ2,4 + λ η1,3 ξ3,4),
  f2,4[λ] → -λ (-η2,4 - ξ2,4 + λ η2,3 ξ3,4), f3,4[λ] → λ (η3,4 + ξ3,4), f4,5[λ] → λ η4,5 + λ ξ4,5,
  f3,5[λ] → λ η3,5 + λ ξ3,5 - λ2 η3,4 ξ4,5, f2,5[λ] → λ η2,5 + λ ξ2,5 - λ2 η2,3 ξ3,5 - λ2 η2,4 ξ4,5,
  f1,5[λ] → λ η1,5 + λ ξ1,5 - λ2 η1,2 ξ2,5 - λ2 η1,3 ξ3,5 - λ2 η1,4 ξ4,5} }

In[*]:= sol /. λ → 1

Out[*]=
{f1,2[1] → η1,2 + ξ1,2, f1,3[1] → η1,3 + ξ1,3 - η1,2 ξ2,3, f2,3[1] → η2,3 + ξ2,3,
  f1,4[1] → η1,4 + ξ1,4 - η1,2 ξ2,4 - η1,3 ξ3,4, f2,4[1] → η2,4 + ξ2,4 - η2,3 ξ3,4,
  f3,4[1] → η3,4 + ξ3,4, f4,5[1] → η4,5 + ξ4,5, f3,5[1] → η3,5 + ξ3,5 - η3,4 ξ4,5,
  f2,5[1] → η2,5 + ξ2,5 - η2,3 ξ3,5 - η2,4 ξ4,5, f1,5[1] → η1,5 + ξ1,5 - η1,2 ξ2,5 - η1,3 ξ3,5 - η1,4 ξ4,5}

In[*]:= osol = E{1,2}→{2} [x1,1[2] ξ1,1[1] + x1,1[2] ξ1,1[2] + e-ξ1,1[2] x1,2[2] ξ1,2[1] + e-ξ2,2[1] x1,2[2] ξ1,2[2] +
  e-ξ1,1[2] x1,3[2] ξ1,3[1] + e-ξ3,3[1] x1,3[2] ξ1,3[2] + e-ξ1,1[2] x1,4[2] ξ1,4[1] +
  e-ξ4,4[1] x1,4[2] ξ1,4[2] + e-ξ1,1[2] x1,5[2] ξ1,5[1] + e-ξ5,5[1] x1,5[2] ξ1,5[2] +
  x2,2[2] ξ2,2[1] + x2,2[2] ξ2,2[2] + e-ξ2,2[2] x2,3[2] ξ2,3[1] - x1,3[2] ξ1,2[2] ξ2,3[1] +
  e-ξ3,3[1] x2,3[2] ξ2,3[2] + e-ξ2,2[2] x2,4[2] ξ2,4[1] - x1,4[2] ξ1,2[2] ξ2,4[1] +
  e-ξ4,4[1] x2,4[2] ξ2,4[2] + e-ξ2,2[2] x2,5[2] ξ2,5[1] - x1,5[2] ξ1,2[2] ξ2,5[1] +
  e-ξ5,5[1] x2,5[2] ξ2,5[2] + x3,3[2] ξ3,3[1] + x3,3[2] ξ3,3[2] + e-ξ3,3[2] x3,4[2] ξ3,4[1] -
  x1,4[2] ξ1,3[2] ξ3,4[1] - x2,4[2] ξ2,3[2] ξ3,4[1] + e-ξ4,4[1] x3,4[2] ξ3,4[2] +
  e-ξ3,3[2] x3,5[2] ξ3,5[1] - x1,5[2] ξ1,3[2] ξ3,5[1] - x2,5[2] ξ2,3[2] ξ3,5[1] +
  e-ξ5,5[1] x3,5[2] ξ3,5[2] + x4,4[2] ξ4,4[1] + x4,4[2] ξ4,4[2] + e-ξ4,4[2] x4,5[2] ξ4,5[1] -
  x1,5[2] ξ1,4[2] ξ4,5[1] - x2,5[2] ξ2,4[2] ξ4,5[1] - x3,5[2] ξ3,4[2] ξ4,5[1] +
  e-ξ5,5[1] x4,5[2] ξ4,5[2] + x5,5[2] ξ5,5[1] + x5,5[2] ξ5,5[2], 0, 0] [[1]] /. ξi_,i_[_] → 0

Out[*]=
x1,2[2] ξ1,2[1] + x1,2[2] ξ1,2[2] + x1,3[2] ξ1,3[1] + x1,3[2] ξ1,3[2] +
  x1,4[2] ξ1,4[1] + x1,4[2] ξ1,4[2] + x1,5[2] ξ1,5[1] + x1,5[2] ξ1,5[2] + x2,3[2] ξ2,3[1] -
  x1,3[2] ξ1,2[2] ξ2,3[1] + x2,3[2] ξ2,3[2] + x2,4[2] ξ2,4[1] - x1,4[2] ξ1,2[2] ξ2,4[1] +
  x2,4[2] ξ2,4[2] + x2,5[2] ξ2,5[1] - x1,5[2] ξ1,2[2] ξ2,5[1] + x2,5[2] ξ2,5[2] + x3,4[2] ξ3,4[1] -
  x1,4[2] ξ1,3[2] ξ3,4[1] - x2,4[2] ξ2,3[2] ξ3,4[1] + x3,4[2] ξ3,4[2] + x3,5[2] ξ3,5[1] -
  x1,5[2] ξ1,3[2] ξ3,5[1] - x2,5[2] ξ2,3[2] ξ3,5[1] + x3,5[2] ξ3,5[2] + x4,5[2] ξ4,5[1] -
  x1,5[2] ξ1,4[2] ξ4,5[1] - x2,5[2] ξ2,4[2] ξ4,5[1] - x3,5[2] ξ3,4[2] ξ4,5[1] + x4,5[2] ξ4,5[2]

In[*]:= Expand@Total[sol /. {ξαβ__ => ξαβ[1], ηαβ__ => ξαβ[2], Rule → Times, λ → 1, fαβ[_] => xαβ[2]}] -
  osol

Out[*]=
0

```