

Pensieve header: The Alexander polynomial using an  $S \times F$  matrix.

```
In[*]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\Signatures"];
Open["KnotTheory"];
```

```
In[*]:= A1[K_] := Module[{spd, x, a, s = 0, c, cs, A, is, M, es},
  spd = (Times @@ PD[K]) /. x_X => If[PositiveQ@x, Xp@@x, Xm@@x];
  cs = spd /. {
    Xp[i_, j_, k_, l_] => a[j, ++s, i] a[k, ++s, -j] a[-l, ++s, -k] a[-i, ++s, l],
    Xm[i_, j_, k_, l_] => a[-j, ++s, i] a[k, ++s, j] a[l, ++s, -k] a[-i, ++s, -l]
  } /. a[i_, x_, j_] a[j_, y_, k_] => a[i, x, y, k] /. a[i_, x_, j_] => a[x];
  A = Table[0, {Length[cs]}, {Length[cs]}];
  Do[is = Position[cs, #][[1, 1]] & /@ {4 i - {3, 2, 1, 0}};

  A[[is, is]] += If[Head[spd[[i]]] === Xp,  $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 2 & 0 & -1 \end{pmatrix}$ ,  $\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix}$ ],
```

```
In[*]:= Table[Simplify[A1[K] / Alexander[K][w]], {K, AllKnots[{3, 8}]}]
```

**KnotTheory:** Loading precomputed data in PD4Knots`.

```
Out[*]= {3 w, 4 w, 5 w^2, 3 w, 4 w, 8 w^2, 9 w^2, 7 w^3, 3 w, 5 w^2, 3 w, 5 w^2, 8 w^2, 9 w^2, 4 w, 12 w^3, 4 w, 8 w^2, 12 w^3,
  8 w^2, 15 w^3, 9 w^2, 16 w^3, 15 w^3, 8 w^2, 12 w^2, 9 w^2, 8 w^2, 5 w^2, 15 w^3, 16 w^3, 16 w^3, 15 w^3, 5 w^2, 5 w^2}
```

```
In[*]:= Union@Table[Simplify[A1[K] / Alexander[K][w]], {K, AllKnots[{3, 12}]}]
```

**KnotTheory:** Loading precomputed data in DTCode4KnotsTo11`.

**KnotTheory:** The GaussCode to PD conversion was written by Siddarth Sankaran at the University of Toronto in the summer of 2005.

\*\*\*

**KnotTheory:** Loading precomputed data in KnotTheory/12A.dts.

**KnotTheory:** Loading precomputed data in KnotTheory/12N.dts.

**General:** Further output of KnotTheory::loading will be suppressed during this calculation. **i**

```
Out[*]= {3 w, 4 w, 5 w^2, 8 w^2, 9 w^2, 12 w^2, 7 w^3, 12 w^3, 15 w^3, 16 w^3, 20 w^3,
  24 w^3, 27 w^3, 32 w^3, 9 w^4, 16 w^4, 21 w^4, 24 w^4, 25 w^4, 28 w^4, 36 w^4, 40 w^4, 45 w^4,
  48 w^4, 52 w^4, 56 w^4, 60 w^4, 64 w^4, 68 w^4, 72 w^4, 76 w^4, 80 w^4, 84 w^4, 88 w^4, 92 w^4, 96 w^4, 100 w^4,
  104 w^4, 108 w^4, 112 w^4, 116 w^4, 120 w^4, 124 w^4, 128 w^4, 132 w^4, 136 w^4, 140 w^4, 144 w^4, 148 w^4, 152 w^4, 156 w^4, 160 w^4,
  164 w^4, 168 w^4, 172 w^4, 176 w^4, 180 w^4, 184 w^4, 188 w^4, 192 w^4, 196 w^4, 200 w^4, 204 w^4, 208 w^4, 212 w^4, 216 w^4, 220 w^4, 224 w^4, 228 w^4, 232 w^4, 236 w^4, 240 w^4,
  244 w^4, 248 w^4, 252 w^4, 256 w^4, 260 w^4, 264 w^4, 268 w^4, 272 w^4, 276 w^4, 280 w^4, 284 w^4, 288 w^4, 292 w^4, 296 w^4, 300 w^4, 304 w^4, 308 w^4, 312 w^4, 316 w^4, 320 w^4,
  324 w^4, 328 w^4, 332 w^4, 336 w^4, 340 w^4, 344 w^4, 348 w^4, 352 w^4, 356 w^4, 360 w^4, 364 w^4, 368 w^4, 372 w^4, 376 w^4, 380 w^4, 384 w^4, 388 w^4, 392 w^4, 396 w^4, 400 w^4,
  404 w^4, 408 w^4, 412 w^4, 416 w^4, 420 w^4, 424 w^4, 428 w^4, 432 w^4, 436 w^4, 440 w^4, 444 w^4, 448 w^4, 452 w^4, 456 w^4, 460 w^4, 464 w^4, 468 w^4, 472 w^4, 476 w^4, 480 w^4,
  484 w^4, 488 w^4, 492 w^4, 496 w^4, 500 w^4, 504 w^4, 508 w^4, 512 w^4, 516 w^4, 520 w^4, 524 w^4, 528 w^4, 532 w^4, 536 w^4, 540 w^4, 544 w^4, 548 w^4, 552 w^4, 556 w^4, 560 w^4,
  564 w^4, 568 w^4, 572 w^4, 576 w^4, 580 w^4, 584 w^4, 588 w^4, 592 w^4, 596 w^4, 600 w^4, 604 w^4, 608 w^4, 612 w^4, 616 w^4, 620 w^4, 624 w^4, 628 w^4, 632 w^4, 636 w^4, 640 w^4,
  644 w^4, 648 w^4, 652 w^4, 656 w^4, 660 w^4, 664 w^4, 668 w^4, 672 w^4, 676 w^4, 680 w^4, 684 w^4, 688 w^4, 692 w^4, 696 w^4, 700 w^4, 704 w^4, 708 w^4, 712 w^4, 716 w^4, 720 w^4,
  724 w^4, 728 w^4, 732 w^4, 736 w^4, 740 w^4, 744 w^4, 748 w^4, 752 w^4, 756 w^4, 760 w^4, 764 w^4, 768 w^4, 772 w^4, 776 w^4, 780 w^4, 784 w^4, 788 w^4, 792 w^4, 796 w^4, 800 w^4,
  804 w^4, 808 w^4, 812 w^4, 816 w^4, 820 w^4, 824 w^4, 828 w^4, 832 w^4, 836 w^4, 840 w^4, 844 w^4, 848 w^4, 852 w^4, 856 w^4, 860 w^4, 864 w^4, 868 w^4, 872 w^4, 876 w^4, 880 w^4,
  884 w^4, 888 w^4, 892 w^4, 896 w^4, 900 w^4, 904 w^4, 908 w^4, 912 w^4, 916 w^4, 920 w^4, 924 w^4, 928 w^4, 932 w^4, 936 w^4, 940 w^4, 944 w^4, 948 w^4, 952 w^4, 956 w^4, 960 w^4,
  964 w^4, 968 w^4, 972 w^4, 976 w^4, 980 w^4, 984 w^4, 988 w^4, 992 w^4, 996 w^4, 1000 w^4}
```

```
In[*]:= MatrixForm@Table[
  {Knot[A1[K_]] Alexander[K1[K_]] Simplify[A1[K_]/Alexander[K1[K_]]} / {K AllKnots[3, 7]}]
Out[*]//MatrixForm=

$$\begin{pmatrix} \text{Knot}[3, 1] & 3(1 - \omega + \omega^2) & -1 + \frac{1}{\omega} + \omega & 3\omega \\ \text{Knot}[4, 1] & -4(1 - 3\omega + \omega^2) & 3 - \frac{1}{\omega} - \omega & 4\omega \\ \text{Knot}[5, 1] & 5(1 - \omega + \omega^2 - \omega^3 + \omega^4) & 1 + \frac{1}{\omega^2} - \frac{1}{\omega} - \omega + \omega^2 & 5\omega^2 \\ \text{Knot}[5, 2] & 6 - 9\omega + 6\omega^2 & -3 + \frac{2}{\omega} + 2\omega & 3\omega \\ \text{Knot}[6, 1] & -8 + 20\omega - 8\omega^2 & 5 - \frac{2}{\omega} - 2\omega & 4\omega \\ \text{Knot}[6, 2] & -8(1 - 3\omega + 3\omega^2 - 3\omega^3 + \omega^4) & -3 - \frac{1}{\omega^2} + \frac{3}{\omega} + 3\omega - \omega^2 & 8\omega^2 \\ \text{Knot}[6, 3] & 9(1 - 3\omega + 5\omega^2 - 3\omega^3 + \omega^4) & 5 + \frac{1}{\omega^2} - \frac{3}{\omega} - 3\omega + \omega^2 & 9\omega^2 \\ \text{Knot}[7, 1] & 7(1 - \omega + \omega^2 - \omega^3 + \omega^4 - \omega^5 + \omega^6) & -1 + \frac{1}{\omega^3} - \frac{1}{\omega^2} + \frac{1}{\omega} + \omega - \omega^2 + \omega^3 & 7\omega^3 \\ \text{Knot}[7, 2] & 9 - 15\omega + 9\omega^2 & -5 + \frac{3}{\omega} + 3\omega & 3\omega \\ \text{Knot}[7, 3] & 5(2 - 3\omega + 3\omega^2 - 3\omega^3 + 2\omega^4) & 3 + \frac{2}{\omega^2} - \frac{3}{\omega} - 3\omega + 2\omega^2 & 5\omega^2 \end{pmatrix}$$

```

```
In[*]:= MatrixSignature[A_] := Total[Sign[Select[Eigenvalues[A], Abs[#] > 10^-12 &]]];
MatrixSignature[A_] := Total[Sign[Eigenvalues[A]]];
```

```
In[*]:= A2[K_] := Module[{spd, x, a, s = 0, c, cs, A, is, M, es},
  spd = (Times@@PD[K]) /. x_X => If[PositiveQ@x, Xp@@x, Xm@@x];
  cs = spd /. {
    Xp[i_, j_, k_, l_] => a[j, ++s, i] a[k, ++s, -j] a[-l, ++s, -k] a[-i, ++s, l],
    Xm[i_, j_, k_, l_] => a[-j, ++s, i] a[k, ++s, j] a[l, ++s, -k] a[-i, ++s, -l]
  } /. a[i_, x_, j_] a[j_, y_, k_] => a[i, x, y, k] /. a[i_, x_, j_] => a[x];
  A = Table[0, {Length[cs]}, {Length[cs]}];
  Do[is = Position[cs, #][[1, 1]] & /@ (4 i - {3, 2, 1, 0});
  A[[is, is]] += If[Head[spd[[i]]] === Xp,

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 2 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix}],
  {i, Length[spd]}];
  MatrixForm[M = Factor[(\omega A - A^T) / 2]]];$$

```

```
In[*]:= MatrixForm@Union@Table[A2[K], {K, AllKnots[{3, 8}]}]
Out[*]//MatrixForm=
```

3	2	2	2	3	-3	3	$\omega$	Knot[3, 1]
4	2	0	0	5	0	4	$\omega$	Knot[4, 1]
5	2	2	2	7	-5	3	$\omega$	Knot[5, 2]
5	4	4	4	5	-5	5	$\omega^2$	Knot[5, 1]
6	2	0	0	9	-2	4	$\omega$	Knot[6, 1]
6	4	0	0	13	0	9	$\omega^2$	Knot[6, 3]
6	4	2	2	11	-2	8	$\omega^2$	Knot[6, 2]
7	2	-2	-2	15	7	3	$\omega$	Knot[7, 4]
7	2	2	2	11	-7	3	$\omega$	Knot[7, 2]
7	4	-4	-4	13	7	5	$\omega^2$	Knot[7, 3]
7	4	0	0	21	1	9	$\omega^2$	Knot[7, 7]
7	4	2	2	19	-3	8	$\omega^2$	Knot[7, 6]
7	4	4	4	17	-7	5	$\omega^2$	Knot[7, 5]
7	6	6	6	7	-7	7	$\omega^3$	Knot[7, 1]
8	2	0	0	13	-4	4	$\omega$	Knot[8, 1]
8	2	0	0	17	0	4	$\omega$	Knot[8, 3]
8	4	0	0	9	-2	5	$\omega^2$	Knot[8, 20]
8	4	0	0	25	2	9	$\omega^2$	Knot[8, 8]
8	4	0	0	29	0	12	$\omega^2$	Knot[8, 12]
8	4	0	0	29	2	9	$\omega^2$	Knot[8, 13]
8	4	2	2	15	-4	5	$\omega^2$	Knot[8, 21]
8	4	2	2	19	0	8	$\omega^2$	Knot[8, 4]
8	4	2	2	23	-4	8	$\omega^2$	Knot[8, 6]
8	4	2	2	27	-4	8	$\omega^2$	Knot[8, 11]
8	4	2	2	31	-4	8	$\omega^2$	Knot[8, 14]

From the ICERM-2305 handout:  $t = 1 - \omega$ ;  $r = t + t^*$ ;  $v = \text{Re}[\omega]$ ;  $u = \text{Re}[\omega^{1/2}]$

$$\text{In[*]} := \{A1p, A1m\} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 2 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix} \right\};$$

$$\begin{pmatrix} -r & -t & 2t & t^* \\ -t^* & r & t^* & r \end{pmatrix}, \begin{pmatrix} r & -t & -2t^* & t^* \\ -t^* & r & t^* & r \end{pmatrix},$$

```
In[*]:= MatrixForm[lhs = RotateRight[(1 - \omega) A1p + (1 - \omega^*) A1p^T, {1, 1}]]
MatrixForm[rhs = Tlp /. {t -> 1 - \omega, r -> t + t^*}]
```

$$\text{Out[*]} // \text{MatrixForm} = \begin{pmatrix} -2 + \omega + \text{Conjugate}[\omega] & -1 + \omega & 2(1 - \omega) & 1 - \text{Conjugate}[\omega] \\ -1 + \text{Conjugate}[\omega] & 0 & 1 - \text{Conjugate}[\omega] & 0 \\ 2(1 - \text{Conjugate}[\omega]) & 1 - \omega & -2 + \omega + \text{Conjugate}[\omega] & -1 + \text{Conjugate}[\omega] \end{pmatrix}$$

$$\text{Out[*]} // \text{MatrixForm} = \begin{pmatrix} -2 + \omega + \text{Conjugate}[\omega] & -1 + \omega & 2(1 - \omega) & 1 - \text{Conjugate}[\omega] \\ -1 + \text{Conjugate}[\omega] & 0 & 1 - \text{Conjugate}[\omega] & 0 \\ 2(1 - \text{Conjugate}[\omega]) & 1 - \omega & -2 + \omega + \text{Conjugate}[\omega] & -1 + \text{Conjugate}[\omega] \end{pmatrix}$$

```
Out[*]= True
```



```

In[*]:= A4[K_] := Module[{u, v, XingsByArmpits, bends, faces, p, A, is, es},
  v = (ω + ω*) / 2; u = √(v + 1) / 2;
  XingsByArmpits = List@@PD[K] /.
    x : X[i_, j_, k_, l_] => If[PositiveQ[x], X, [-i, j, k, -l], X, [-j, k, l, -i]];
  bends = Times@@XingsByArmpits /. _[X][a_, b_, c_, d_] => pa,-d pb,-a pc,-b pd,-c;
  faces = bends /. px_,y_ py_,z_ => px,y,z;
  A = Table[0, Length@faces, Length@faces];
  Do[is = Position[faces, #][[1, 1]] & /@ List@@x;
    A[[is, is]] += If[Head[x] === X,
      
$$\begin{pmatrix} v & u & 1 & u \\ u & 1 & u & 1 \\ 1 & u & v & u \\ u & 1 & u & 1 \end{pmatrix}, - \begin{pmatrix} v & u & 1 & u \\ u & 1 & u & 1 \\ 1 & u & v & u \\ u & 1 & u & 1 \end{pmatrix}],
    -
  ];
  tab = Table[K → Factor@FullSimplify[{A4[K],  $\frac{A4[K]}{\dots}$ }], {K, AllKnots[{3, 10}]}];
  Union@Exponent[tab[[All, 2, 2]], ω]
Out[*]= {1, 2, 3}
  tab11 =
    Table[K → Factor@FullSimplify[{A4[K],  $\frac{A4[K]}{\dots}$ }], {K, AllKnots[{11, 11}]}];
  Union@Exponent[tab11[[All, 2, 2]], ω]
Out[*]= {1, 2, 3, 4}
  Cases[tab11, p /; Exponent[p[[2, 2]], ω] == 4]
Out[*]= {Knot[11, Alternating, 121] →
  
$$\left\{ - \frac{(1 - 9\omega + 29\omega^2 - 41\omega^3 + 29\omega^4 - 9\omega^5 + \omega^6)^2 (1 + 6\omega + 22\omega^2 + 59\omega^3 + 126\omega^4 + 59\omega^5 + 22\omega^6 + 6\omega^7 + \omega^8)}{\omega^{10}} \right.$$

  
$$1 - 6\omega + 22\omega^2 - 59\omega^3 + 126\omega^4 - 59\omega^5 + 22\omega^6 - 6\omega^7 + \omega^8$$$$

```

```

In[*]:= A5[K_] := Module[{u, v, XingsByArmpits, bends, faces, p, A, is, es, keeps},
  v = (ω + ω*) / 2; u = √(v + 1) / 2;
  XingsByArmpits = List@@PD[K] /.
    x : X[i_, j_, k_, l_] => If[PositiveQ[x], X, [-i, j, k, -l], X, [-j, k, l, -i]];
  bends = Times@@XingsByArmpits /. _[X][a_, b_, c_, d_] => pa,-d pb,-a pc,-b pd,-c;
  faces = bends //. px_,y_ py_,z_ => px,y,z;
  A = Table[0, Length@faces, Length@faces];
  Do[is = Position[faces, #] [[1, 1]] & /@ List@@x;
    A[[is, is]] += If[Head[x] === X,
      
$$\begin{pmatrix} v & u & 1 & u \\ u & 1 & u & 1 \\ 1 & u & v & u \\ u & 1 & u & 1 \end{pmatrix}, -\begin{pmatrix} v & u & 1 & u \\ u & 1 & u & 1 \\ 1 & u & v & u \\ u & 1 & u & 1 \end{pmatrix}],
    {x, XingsByArmpits}];
  tab = Table[K -> Factor@FullSimplify[{A5[K],  $\frac{A5[K]}{A5[K]}$ }], {K, AllKnots[{3, 10}]}];$$

```

```

In[*]:= tab[[1 ;; 10]] // Column
Out[*]=
Knot[3, 1] -> { - (1-ω+ω²)² / ω², -1 }
Knot[4, 1] -> { (1-3ω+ω²)² (1+ω+ω²)² / ω⁴, (1+ω+ω²)² / ω² }
Knot[5, 1] -> { - (1-ω+ω²-ω³+ω⁴)² / ω⁴, -1 }
Knot[5, 2] -> { - (2-3ω+2ω²)² / ω², -1 }
Knot[6, 1] -> { (-2+ω)² (-1+2ω)² (1+ω+ω²)² / ω⁴, (1+ω+ω²)² / ω² }
Knot[6, 2] -> { (1+ω)² (1-3ω+3ω²-3ω³+ω⁴)² / ω⁵, (1+ω)² / ω }
Knot[6, 3] -> { - (1-3ω+5ω²-3ω³+ω⁴)² / ω⁴, -1 }

```

```

In[*]:= Union@tab[[All, 2, 2]]
Out[*]=
{-1, 1, - (1+ω)² / ω, (1+ω)² / ω, - (1+ω)² (1+ω²)² / ω³, (1+ω)² (1+ω²)² / ω³, - (1+ω+ω²)² / ω, (1+ω+ω²)² / ω}

```

```

In[*]:= tab11 =
  Table[K -> Factor@FullSimplify[{A5[K],  $\frac{A5[K]}{A5[K]}$ }], {K, AllKnots[{11, 11}]}];

```

```

In[*]:= Union@tab11[[All, 2, 2]]
Out[*]=
{-1, 1, - (1+ω)² / ω, (1+ω)² / ω, - (1+ω)² (1+ω²)² / ω³,
  (1+ω)² (1+ω²)² / ω³, (1+ω+ω²)² / ω, (1+ω+ω²)² / ω,
  (1+ω+ω²)² / ω, (1+ω+ω²)² / ω, (1+ω+ω²+ω³+ω⁴)² / ω⁴}

```

```
In[*]:= Monitor[
  tab12 =
  Table[K -> Factor@FullSimplify[{{A5[K],  $\frac{A5[K]}{Alexander[K][\omega]^2}$ }}, {K, AllKnots[{12, 12}}]],
```

```
In[*]:= Union@tab12[[All, 2, 2]]
Out[*]=
{-1, 1, - $\frac{(1+\omega)^2}{\omega}$ ,  $\frac{(1+\omega)^2}{\omega}$ , - $\frac{(1+\omega)^2(1+\omega^2)^2}{\omega^3}$ ,  $\frac{(1+\omega)^2(1+\omega^2)^2}{\omega^3}$ ,
 $(1+\omega+\omega^2)^2$ ,  $(1+\omega+\omega^2)^2$ ,  $(1+\omega+\omega^2+\omega^3+\omega^4)^2$ ,  $(1+\omega+\omega^2+\omega^3+\omega^4)^2$ ,
```

```
In[*]:= A6[K_] := Module[{u, v, XingsByArmpits, bends, faces, p, A, is, es, rad, throws, keeps},
  v = (\omega + \omega*) / 2; u = \sqrt{(v + 1) / 2};
  XingsByArmpits = List@@PD[K] /.
  x : X[i_, j_, k_, l_] => If[PositiveQ[x], X+[-i, j, k, -l], X[-j, k, l, -i]];
  bends = Times@@XingsByArmpits /. _[X][a_, b_, c_, d_] => p_{a,-d} p_{b,-a} p_{c,-b} p_{d,-c};
  faces = bends /. p_{x_,y_} p_{y_,z_} => p_{x,y,z};
  A = Table[0, Length@faces, Length@faces];
  Do[is = Position[faces, #][[1, 1]] & /@ List@@@x;
  A[[is, is]] += If[Head[x] == X+,
  (

$$\begin{pmatrix} v & u & 1 & u \\ u & 1 & u & 1 \\ 1 & u & v & u \\ u & 1 & u & 1 \end{pmatrix}, - \begin{pmatrix} v & u & 1 & u \\ u & 1 & u & 1 \\ 1 & u & v & u \\ u & 1 & u & 1 \end{pmatrix} ]],
  {x, XingsByArmpits}];
  M = Factor[A /. \omega* -> \omega^{-1}];$$

```

```
In[*]:= Column@Table[K -> Factor@FullSimplify[{{A6[K],  $\frac{A6[K]}{Alexander[K][\omega]^2}$ }}, {K, AllKnots[{3, 7}}]]]
Out[*]=
```

$$\begin{aligned} \text{Knot}[3, 1] &\rightarrow \left\{ -\frac{(1+\omega)^2(1-\omega+\omega^2)^2}{\omega^3}, -\frac{(1+\omega)^2}{\omega} \right\} \\ \text{Knot}[4, 1] &\rightarrow \left\{ \frac{(1-3\omega+\omega^2)^2}{\omega^2}, 1 \right\} \\ \text{Knot}[5, 1] &\rightarrow \left\{ -\frac{(1+\omega)^2(1-\omega+\omega^2-\omega^3+\omega^4)^2}{\omega^5}, -\frac{(1+\omega)^2}{\omega} \right\} \\ \text{Knot}[5, 2] &\rightarrow \left\{ -\frac{(2-3\omega+2\omega^2)^2}{\omega^2}, -1 \right\} \\ \text{Knot}[6, 1] &\rightarrow \left\{ \frac{(-2+\omega)^2(-1+2\omega)^2}{\omega^2}, 1 \right\} \\ \text{Knot}[6, 2] &\rightarrow \left\{ \frac{(1+\omega)^2(1-3\omega+3\omega^2-3\omega^3+\omega^4)^2}{\omega^5}, \frac{(1+\omega)^2}{\omega} \right\} \\ \text{Knot}[6, 3] &\rightarrow \left\{ -\frac{(1-3\omega+5\omega^2-3\omega^3+\omega^4)^2}{\omega^4}, -1 \right\} \\ \text{Knot}[7, 1] &\rightarrow \left\{ -\frac{(1+\omega)^2(1-\omega+\omega^2-\omega^3+\omega^4-\omega^5+\omega^6)^2}{\omega^7}, -\frac{(1+\omega)^2}{\omega} \right\} \\ \text{Knot}[7, 2] &\rightarrow \left\{ -\frac{(3-5\omega+3\omega^2)^2}{\omega^2}, -1 \right\} \\ \text{Knot}[7, 3] &\rightarrow \left\{ \frac{(1+\omega)^2(2-3\omega+3\omega^2-3\omega^3+2\omega^4)^2}{\omega^5}, \frac{(1+\omega)^2}{\omega} \right\} \end{aligned}$$

```

In[ ]:= A7[K_] := Module[{u, v, XingsByArmpits, bends, faces, p, A, is, es, rad, throws, keeps},
  v = (ω + ω*) / 2; u = √((v + 1) / 2);
  XingsByArmpits = List@@PD[K] /.
    x : X[i_, j_, k_, l_] => If[PositiveQ[x], X,[-i, j, k, -l], X,[-j, k, l, -i]];
  bends = Times@@XingsByArmpits /. _[X][a_, b_, c_, d_] => pa,-d pb,-a pc,-b pd,-c;
  faces = bends /. px-,y-,z- => px,y,z;
  A = Table[0, Length@faces, Length@faces];
  Do[is = Position[faces, #][[1, 1]] & /@ List@@x;
    A[[is, is]] += If[Head[x] === X,
      (

$$\begin{pmatrix} v & u & 1 & u \\ u & 1 & u & 1 \\ 1 & u & v & u \\ u & 1 & u & 1 \end{pmatrix}, - \begin{pmatrix} v & u & 1 & u \\ u & 1 & u & 1 \\ 1 & u & v & u \\ u & 1 & u & 1 \end{pmatrix} \right],
      {x, XingsByArmpits}];
  M = Factor[A /. ω* -> ω-1];
  throws = RandomChoice[Subsets[Range@Length@M, {2}]]];$$

```

Out[ ]:=

$$\begin{aligned}
 \text{Knot}[3, 1] &\rightarrow \left\{ -\frac{(1-\omega+\omega^2)^2}{\omega^2}, -1 \right\} \\
 \text{Knot}[4, 1] &\rightarrow \left\{ \frac{(1-3\omega+\omega^2)^2(1+\omega+\omega^2)^2}{\omega^4}, \frac{(1+\omega+\omega^2)^2}{\omega^2} \right\} \\
 \text{Knot}[5, 1] &\rightarrow \{0, 0\} \\
 \text{Knot}[5, 2] &\rightarrow \left\{ -\frac{(1+\omega)^2(2-3\omega+2\omega^2)^2}{\omega^3}, -\frac{(1+\omega)^2}{\omega} \right\} \\
 \text{Knot}[6, 1] &\rightarrow \left\{ \frac{(-2+\omega)^2(-1+2\omega)^2}{\omega^2}, 1 \right\} \\
 \text{Knot}[6, 2] &\rightarrow \left\{ \frac{(1-3\omega+3\omega^2-3\omega^3+\omega^4)^2}{\omega^4}, 1 \right\} \\
 \text{Knot}[6, 3] &\rightarrow \left\{ -\frac{(1-3\omega+5\omega^2-3\omega^3+\omega^4)^2}{\omega^4}, -1 \right\} \\
 \text{Knot}[7, 1] &\rightarrow \left\{ -\frac{(1-\omega+\omega^2-\omega^3+\omega^4-\omega^5+\omega^6)^2}{\omega^6}, -1 \right\} \\
 \text{Knot}[7, 2] &\rightarrow \left\{ -\frac{(3-5\omega+3\omega^2)^2}{\omega^2}, -1 \right\} \\
 \text{Knot}[7, 3] &\rightarrow \left\{ \frac{(2-3\omega+3\omega^2-3\omega^3+2\omega^4)^2}{\omega^4}, 1 \right\}
 \end{aligned}$$



```
In[ ]:= A8[K_] := Module[{u, v, XingsByArmpits, bends, faces, p, A, is, es, rad, throws, keeps},
  v = (ω + ω*) / 2; u = √(v + 1) / 2;
  XingsByArmpits = List @@ PD[K] /.
    x : X[i_, j_, k_, l_] => If[PositiveQ[x], X, [-i, j, k, -l], X, [-j, k, l, -i]];
  bends = Times @@ XingsByArmpits /. _[X][a_, b_, c_, d_] => Pa,-d Pb,-a Pc,-b Pd,-c;
  faces = bends /. pX_,y_ pY_,z_ => pX,y,z;
  A = Table[0, Length@faces, Length@faces];
  Do[is = Position[faces, #][[1, 1]] & /@ List @@ x;
    A[[is, is]] += If[Head[x] === X,
      (
        (v u 1 u)
        (u 1 u 1)
        (1 u v u)
        (u 1 u 1)
      ), -
      (
        (v u 1 u)
        (u 1 u 1)
        (1 u v u)
        (u 1 u 1)
      )
    ], {x, XingsByArmpits}];
  M = Factor[A /. ω* -> ω^-1];
  throws = Position[faces, #][[1, 1]] & /@ List @@ XingsByArmpits[[1, 1] :: 2] ;
```

Out[ ]:=

$$\begin{aligned} \text{Knot}[3, 1] &\rightarrow \left\{ -\frac{(1-\omega+\omega^2)^2}{\omega^2}, -1 \right\} \\ \text{Knot}[4, 1] &\rightarrow \left\{ \frac{(1-3\omega+\omega^2)^2}{\omega^2}, 1 \right\} \\ \text{Knot}[5, 1] &\rightarrow \left\{ -\frac{(1-\omega+\omega^2-\omega^3+\omega^4)^2}{\omega^4}, -1 \right\} \\ \text{Knot}[5, 2] &\rightarrow \left\{ -\frac{(2-3\omega+2\omega^2)^2}{\omega^2}, -1 \right\} \\ \text{Knot}[6, 1] &\rightarrow \left\{ \frac{(-2+\omega)^2(-1+2\omega)^2}{\omega^2}, 1 \right\} \\ \text{Knot}[6, 2] &\rightarrow \left\{ \frac{(1-3\omega+3\omega^2-3\omega^3+\omega^4)^2}{\omega^4}, 1 \right\} \\ \text{Knot}[6, 3] &\rightarrow \left\{ -\frac{(1-3\omega+5\omega^2-3\omega^3+\omega^4)^2}{\omega^4}, -1 \right\} \\ \text{Knot}[7, 1] &\rightarrow \left\{ -\frac{(1-\omega+\omega^2-\omega^3+\omega^4-\omega^5+\omega^6)^2}{\omega^6}, -1 \right\} \\ \text{Knot}[7, 2] &\rightarrow \left\{ -\frac{(3-5\omega+3\omega^2)^2}{\omega^2}, -1 \right\} \\ \text{Knot}[7, 3] &\rightarrow \left\{ \frac{(2-3\omega+3\omega^2-3\omega^3+2\omega^4)^2}{\omega^4}, 1 \right\} \end{aligned}$$

```
In[ ]:= tab =
```

```
Table[K -> Factor@FullSimplify[{a8 = A8[K],  $\frac{a8}{\omega^2}$ }], {K, AllKnots[{3, 11]}]};
```

```
In[ ]:= Union@tab[[All, 2, 2]]
```

Out[ ]:=

$$\{-1, 1\}$$

```

In[*]:= MatrixSignature[A_] := Total[Sign[Select[Eigenvalues[A], Abs[#] > 10-12 &]]];
Writhe[K_] := Sum[If[PositiveQ[x], 1, -1], {x, List@@PD@K}];
S8[K_, ω1_] :=
Module[{u, v, XingsByArmpits, bends, faces, p, A, is, es, rad, throws, keeps},
v = (ω + ω*) / 2; u = √(v + 1) / 2;
XingsByArmpits = List@@PD[K] /.
x : X[i_, j_, k_, l_] => If[PositiveQ[x], X_[-i, j, k, -l], X_-j, k, l, -i]];
bends = Times@@XingsByArmpits /. _[X][a_, b_, c_, d_] => pa,-d pb,-a pc,-b pd,-c;
faces = bends //. px_,y_ py_,z_ => px,y,z;
A = Table[0, Length@faces, Length@faces];
Do[is = Position[faces, #][[1, 1]] & /@ List@@x;
A[[is, is]] += If[Head[x] === X_.,

$$\begin{pmatrix} v & u & 1 & u \\ u & 1 & u & 1 \\ 1 & u & v & u \\ u & 1 & u & 1 \end{pmatrix}, - \begin{pmatrix} v & u & 1 & u \\ u & 1 & u & 1 \\ 1 & u & v & u \\ u & 1 & u & 1 \end{pmatrix}],
{x, XingsByArmpits}];$$

```

