Pensieve header: The Alexander polynomial using an $F\times F$ matrix.

```
In[•]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\Signatures"];
Once[<<KnotTheory`]
```

```
In[•]:= A1[K_] := Module[{spd, x, a, s = 0, c, cs, A, is, M, es},
    spd = (Times @@ PD[K]) /. x_X :> If[PositiveQ@x, Xp @@ x, Xm @@ x];
    cs = spd /. {
        Xp[i_, j_, k_, l_] :> a[j, ++s, i] a[k, ++s, -j] a[-l, ++s, -k] a[-i, ++s, l],
        Xm[i_, j_, k_, l_] :> a[-j, ++s, i] a[k, ++s, j] a[l, ++s, -k] a[-i, ++s, -l]
      } //. a[i_, x__, j_] a[j_, y__, k_] :> a[i, x, y, k] /. a[i_, x__, j_] :> a[x];
    A = Table[0, {Length[cs]}, {Length[cs]}];
    Do[is = Position[cs, #][[1, 1]] & /@ (4 i - {3, 2, 1, 0});

     A[[is, is]] += If[Head[spd[[i]]] === Xp, ( 0  0  0  0 ), ( 1 -1  0  0 )],
                                               ( 1 -1  0  0 )  ( 0  0  0  0 )
                                               ( 0 -1  0  1 )  (-2  1  1  0 )
                                               (-1  2  0 -1 )  ( 1  0 -1  0 )
```

```
In[•]:= Table[Simplify[A1[K] / Alexander[K][ω]], {K, AllKnots[{3, 8}]}]
```

KnotTheory: Loading precomputed data in PD4Knots`.

Out[•]=

$\{3\,\omega, 4\,\omega, 5\,\omega^2, 3\,\omega, 4\,\omega, 8\,\omega^2, 9\,\omega^2, 7\,\omega^3, 3\,\omega, 5\,\omega^2, 3\,\omega, 5\,\omega^2, 8\,\omega^2, 9\,\omega^2, 4\,\omega, 12\,\omega^3, 4\,\omega, 8\,\omega^2, 12\,\omega^3,$
$8\,\omega^2, 15\,\omega^3, 9\,\omega^2, 16\,\omega^3, 15\,\omega^3, 8\,\omega^2, 12\,\omega^2, 9\,\omega^2, 8\,\omega^2, 5\,\omega^2, 15\,\omega^3, 16\,\omega^3, 16\,\omega^3, 15\,\omega^3, 5\,\omega^2, 5\,\omega^2\}$

```
In[•]:= Union@Table[Simplify[A1[K] / Alexander[K][ω]], {K, AllKnots[{3, 12}]}]
```

KnotTheory: Loading precomputed data in DTCode4KnotsTo11`.

KnotTheory: The GaussCode to PD conversion was written by Siddarth Sankaran at the University of Toronto in the summer of 2005.

KnotTheory: Loading precomputed data in KnotTheory/12A.dts.

KnotTheory: Loading precomputed data in KnotTheory/12N.dts.

General: Further output of KnotTheory::loading will be suppressed during this calculation.

Out[•]=

$\{3\,\omega, 4\,\omega, 5\,\omega^2, 8\,\omega^2, 9\,\omega^2, 12\,\omega^2, 7\,\omega^3, 12\,\omega^3, 15\,\omega^3, 16\,\omega^3, 20\,\omega^3,$
$24\,\omega^3, 27\,\omega^3, 32\,\omega^3, 9\,\omega^4, 16\,\omega^4, 21\,\omega^4, 24\,\omega^4, 25\,\omega^4, 28\,\omega^4, 36\,\omega^4, 40\,\omega^4, 45\,\omega^4,$

In[∘]:=
```
MatrixForm@Table[
  {K, A1[K], Alexander[K][ω], Simplify[A1[K] / Alexander[K][ω]]}, {K, AllKnots[{3, 7}]}]]
```

Out[∘]//MatrixForm=

Knot[3, 1]	$3\left(1-\omega+\omega^2\right)$	$-1+\frac{1}{\omega}+\omega$	$3\,\omega$
Knot[4, 1]	$-4\left(1-3\,\omega+\omega^2\right)$	$3-\frac{1}{\omega}-\omega$	$4\,\omega$
Knot[5, 1]	$5\left(1-\omega+\omega^2-\omega^3+\omega^4\right)$	$1+\frac{1}{\omega^2}-\frac{1}{\omega}-\omega+\omega^2$	$5\,\omega^2$
Knot[5, 2]	$6-9\,\omega+6\,\omega^2$	$-3+\frac{2}{\omega}+2\,\omega$	$3\,\omega$
Knot[6, 1]	$-8+20\,\omega-8\,\omega^2$	$5-\frac{2}{\omega}-2\,\omega$	$4\,\omega$
Knot[6, 2]	$-8\left(1-3\,\omega+3\,\omega^2-3\,\omega^3+\omega^4\right)$	$-3-\frac{1}{\omega^2}+\frac{3}{\omega}+3\,\omega-\omega^2$	$8\,\omega^2$
Knot[6, 3]	$9\left(1-3\,\omega+5\,\omega^2-3\,\omega^3+\omega^4\right)$	$5+\frac{1}{\omega^2}-\frac{3}{\omega}-3\,\omega+\omega^2$	$9\,\omega^2$
Knot[7, 1]	$7\left(1-\omega+\omega^2-\omega^3+\omega^4-\omega^5+\omega^6\right)$	$-1+\frac{1}{\omega^3}-\frac{1}{\omega^2}+\frac{1}{\omega}+\omega-\omega^2+\omega^3$	$7\,\omega^3$
Knot[7, 2]	$9-15\,\omega+9\,\omega^2$	$-5+\frac{3}{\omega}+3\,\omega$	$3\,\omega$
Knot[7, 3]	$5\left(2-3\,\omega+3\,\omega^2-3\,\omega^3+2\,\omega^4\right)$	$3+\frac{2}{\omega^2}-\frac{3}{\omega}-3\,\omega+2\,\omega^2$	$5\,\omega^2$

In[∘]:=
```
MatrixSignature[A_] := Total[Sign[Select[Eigenvalues[A], Abs[#] > 10^-12 &]]];
MatrixSignature[A_] := Total[Sign[Eigenvalues[A]]];
```

In[∘]:=
```
A2[K_] := Module[{spd, x, a, s = 0, c, cs, A, is, M, es},
  spd = (Times @@ PD[K]) /. x_X :> If[PositiveQ@x, Xp @@ x, Xm @@ x];
  cs = spd /. {
      Xp[i_, j_, k_, l_] :> a[j, ++s, i] a[k, ++s, -j] a[-l, ++s, -k] a[-i, ++s, l],
      Xm[i_, j_, k_, l_] :> a[-j, ++s, i] a[k, ++s, j] a[l, ++s, -k] a[-i, ++s, -l]
     } //. a[i_, x__, j_] a[j_, y__, k_] :> a[i, x, y, k] /. a[i_, x__, j_] :> a[x];
  A = Table[0, {Length[cs]}, {Length[cs]}];
  Do[is = Position[cs, #][[1, 1]] & /@ (4 i - {3, 2, 1, 0});
   A[[is, is]] += If[Head[spd[[i]]] === Xp, (0  0  0  0
                                             1 -1  0  0
                                             0 -1  0  1
                                            -1  2  0 -1), (1 -1  0  0
                                                           0  0  0  0
                                                          -2  1  1  0
                                                           1  0 -1  0)],
   {i, Length[spd]}];
  MatrixForm[M = Factor[(ω A - A^T) / 2]];
```

In[ ]:=
```
MatrixForm@Union@Table[A2[K], {K, AllKnots[{3, 8}]}]
```
Out[ ]//MatrixForm=

3	2	2	2	3	-3	3ω	Knot[3, 1]
4	2	0	0	5	0	4ω	Knot[4, 1]
5	2	2	2	7	-5	3ω	Knot[5, 2]
5	4	4	4	5	-5	$5\omega^2$	Knot[5, 1]
6	2	0	0	9	-2	4ω	Knot[6, 1]
6	4	0	0	13	0	$9\omega^2$	Knot[6, 3]
6	4	2	2	11	-2	$8\omega^2$	Knot[6, 2]
7	2	-2	-2	15	7	3ω	Knot[7, 4]
7	2	2	2	11	-7	3ω	Knot[7, 2]
7	4	-4	-4	13	7	$5\omega^2$	Knot[7, 3]
7	4	0	0	21	1	$9\omega^2$	Knot[7, 7]
7	4	2	2	19	-3	$8\omega^2$	Knot[7, 6]
7	4	4	4	17	-7	$5\omega^2$	Knot[7, 5]
7	6	6	6	7	-7	$7\omega^3$	Knot[7, 1]
8	2	0	0	13	-4	4ω	Knot[8, 1]
8	2	0	0	17	0	4ω	Knot[8, 3]
8	4	0	0	9	-2	$5\omega^2$	Knot[8, 20]
8	4	0	0	25	2	$9\omega^2$	Knot[8, 8]
8	4	0	0	29	0	$12\omega^2$	Knot[8, 12]
8	4	0	0	29	2	$9\omega^2$	Knot[8, 13]
8	4	2	2	15	-4	$5\omega^2$	Knot[8, 21]
8	4	2	2	19	0	$8\omega^2$	Knot[8, 4]
8	4	2	2	23	-4	$8\omega^2$	Knot[8, 6]
8	4	2	2	27	-4	$8\omega^2$	Knot[8, 11]
8	4	2	2	31	-4	$8\omega^2$	Knot[8, 14]
8	4	4	4	33	8	$5\omega^2$	Knot[8, 15]

From the ICERM-2305 handout: $t = 1 - \omega$; r=t+t*; v=Re[ω]; u = Re[$\omega^{1/2}$]

In[ ]:=
$$\{A1p, A1m\} = \left\{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 2 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix}\right\};$$

$$\left\{\begin{pmatrix} -r & -t & 2t & t^* \\ -t^* & 0 & t^* & 0 \end{pmatrix}, \begin{pmatrix} r & -t & -2t^* & t^* \\ -t^* & 0 & t^* & 0 \end{pmatrix}\right\}$$

In[ ]:=
```
MatrixForm[lhs = RotateRight[(1 - ω) A1p + (1 - ω*) A1pᵀ, {1, 1}]]
MatrixForm[rhs = TLp //. {t → 1 - ω, r → t + t*}]
```
Out[ ]//MatrixForm=
$$\begin{pmatrix} -2+\omega+\text{Conjugate}[\omega] & -1+\omega & 2(1-\omega) & 1-\text{Conjugate}[\omega] \\ -1+\text{Conjugate}[\omega] & 0 & 1-\text{Conjugate}[\omega] & 0 \\ 2(1-\text{Conjugate}[\omega]) & 1-\omega & -2+\omega+\text{Conjugate}[\omega] & -1+\text{Conjugate}[\omega] \end{pmatrix}$$

Out[ ]//MatrixForm=
$$\begin{pmatrix} -2+\omega+\text{Conjugate}[\omega] & -1+\omega & 2(1-\omega) & 1-\text{Conjugate}[\omega] \\ -1+\text{Conjugate}[\omega] & 0 & 1-\text{Conjugate}[\omega] & 0 \\ 2(1-\text{Conjugate}[\omega]) & 1-\omega & -2+\omega+\text{Conjugate}[\omega] & -1+\text{Conjugate}[\omega] \end{pmatrix}$$

Out[ ]=
True

In[∘]:=
```
MatrixForm[lhs = (1 - ω) A1m + (1 - ω*) A1mᵀ]
MatrixForm[rhs = TLm //. {t → 1 - ω, r → t + t*}]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 2-\omega-\text{Conjugate}[\omega] & -1+\omega & -2\,(1-\text{Conjugate}[\omega]) & 1-\text{Conjugate}[\omega] \\ -1+\text{Conjugate}[\omega] & 0 & 1-\text{Conjugate}[\omega] & 0 \\ -2\,(1-\omega) & 1-\omega & 2-\omega-\text{Conjugate}[\omega] & -1+\text{Conjugate}[\omega] \end{pmatrix}$$

Out[∘]//MatrixForm=

$$\begin{pmatrix} 2-\omega-\text{Conjugate}[\omega] & -1+\omega & -2\,(1-\text{Conjugate}[\omega]) & 1-\text{Conjugate}[\omega] \\ -1+\text{Conjugate}[\omega] & 0 & 1-\text{Conjugate}[\omega] & 0 \\ -2\,(1-\omega) & 1-\omega & 2-\omega-\text{Conjugate}[\omega] & -1+\text{Conjugate}[\omega] \end{pmatrix}$$

Out[∘]=
True

Barring mistakes, the following is equivalent to A1, but with methodology as in SideBySide.nb:

In[∘]:=
```
A3[K_] := Module[{t, r, XingsByArmpits, bends, faces, p, A, is, es, keeps},
  t = 1 - ω; r = t + t*;
  XingsByArmpits = List @@ PD[K] /.
    x : X[i_, j_, k_, l_] :→ If[PositiveQ[x], X₊[-i, j, k, -l], X₋[-j, k, l, -i]];
  bends = Times @@ XingsByArmpits /. _[X][a_, b_, c_, d_] :→ p_{a,-d} p_{b,-a} p_{c,-b} p_{d,-c};
  faces = bends //. p_{x__,y_} p_{y_,z__} :→ p_{x,y,z};
  A = Table[0, Length@faces, Length@faces];
  Do[is = Position[faces, #][[1, 1]] & /@ List @@ x;
   A[[is, is]] += If[Head[x] === X₊,
     {{-r, -t, 2t, t*}, {-t*, 0, t*, 0}, {2t*, t, -r, -t*}, {t, 0, -t, 0}},
     {{r, -t, -2t*, t*}, {-t*, 0, t*, 0}, {-2t, t, r, -t*}, {t, 0, -t, 0}}],
   {x, XingsByArmpits}];
```

Out[∘]=
```
{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
```

In[∘]:=
```
Table[Simplify[A3[K] / Alexander[K][ω]], {K, AllKnots[{3, 10}]}]
```

Out[∘]=
```
{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
```

And here's the Kas version:

```
In[◦]:= A4[K_] := Module[{u, v, XingsByArmpits, bends, faces, p, A, is, es},
    v = (ω + ω*) / 2; u = √((v + 1) / 2);
    XingsByArmpits = List @@ PD[K] /.
      x : X[i_, j_, k_, l_] :> If[PositiveQ[x], X₊[-i, j, k, -l], X₋[-j, k, l, -i]];
    bends = Times @@ XingsByArmpits /. _[X][a_, b_, c_, d_] :> p[a,-d] p[b,-a] p[c,-b] p[d,-c];
    faces = bends //. p[x__,y_] p[y_,z__] :> p[x,y,z];
    A = Table[0, Length@faces, Length@faces];
    Do[is = Position[faces, #][[1, 1]] & /@ List @@ x;
     A[[is, is]] += If[Head[x] === X₊,
       ( v u 1 u )    ( v u 1 u )
       ( u 1 u 1 ), - ( u 1 u 1 )],
       ( 1 u v u )    ( 1 u v u )
       ( u 1 u 1 )    ( u 1 u 1 )

   tab = Table[K → Factor@FullSimplify[{A4[K], A4[K] / ...}], {K, AllKnots[{3, 10}]}];

In[◦]:= Union@Exponent[tab[[All, 2, 2]], ω]
Out[◦]=
{1, 2, 3}

   tab11 =
     Table[K → Factor@FullSimplify[{A4[K], A4[K] / ...}], {K, AllKnots[{11, 11}]}];

In[◦]:= Union@Exponent[tab11[[All, 2, 2]], ω]
Out[◦]=
{1, 2, 3, 4}

In[◦]:= Cases[tab11, p_ /; Exponent[p[[2, 2]], ω] == 4]
Out[◦]=
```

$\{\text{Knot}[11, \text{Alternating}, 121] \to$

$\left\{-\frac{\left(1-9\omega+29\omega^2-41\omega^3+29\omega^4-9\omega^5+\omega^6\right)^2\left(1+6\omega+22\omega^2+59\omega^3+126\omega^4+59\omega^5+22\omega^6+6\omega^7+\omega^8\right)}{\omega^{10}}\right.$

$1+6\omega+22\omega^2+59\omega^3+126\omega^4+59\omega^5+22\omega^6+6\omega^7+\omega^8$

In[∘]:=
```
A5[K_] := Module[{u, v, XingsByArmpits, bends, faces, p, A, is, es, keeps},
  v = (ω + ω*) / 2; u = √((v + 1) / 2);
  XingsByArmpits = List @@ PD[K] /.
    x : X[i_, j_, k_, l_] :> If[PositiveQ[x], X₊[-i, j, k, -l], X₋[-j, k, l, -i]];
  bends = Times @@ XingsByArmpits /. _[X][a_, b_, c_, d_] :> p_{a,-d} p_{b,-a} p_{c,-b} p_{d,-c};
  faces = bends //. p_{x__,y_} p_{y_,z__} :> p_{x,y,z};
  A = Table[0, Length@faces, Length@faces];
  Do[is = Position[faces, #][[1, 1]] & /@ List @@ x;
   A[[is, is]] += If[Head[x] === X₊,
      (v u 1 u
       u 1 u 1
       1 u v u
       u 1 u 1), -(v u 1 u
                   u 1 u 1
                   1 u v u
                   u 1 u 1)],
   {x, XingsByArmpits}];
tab = Table[K → Factor@FullSimplify[{A5[K], A5[K] / Alexander[K][...]^2}], {K, AllKnots[{3, 10}]}];
```

In[∘]:= `tab[[1 ;; 10]] // Column`

Out[∘]=

Knot[3, 1] → $\left\{-\frac{(1-\omega+\omega^2)^2}{\omega^2}, -1\right\}$

Knot[4, 1] → $\left\{\frac{(1-3\,\omega+\omega^2)^2\,(1+\omega+\omega^2)^2}{\omega^4}, \frac{(1+\omega+\omega^2)^2}{\omega^2}\right\}$

Knot[5, 1] → $\left\{-\frac{(1-\omega+\omega^2-\omega^3+\omega^4)^2}{\omega^4}, -1\right\}$

Knot[5, 2] → $\left\{-\frac{(2-3\,\omega+2\,\omega^2)^2}{\omega^2}, -1\right\}$

Knot[6, 1] → $\left\{\frac{(-2+\omega)^2\,(-1+2\,\omega)^2\,(1+\omega+\omega^2)^2}{\omega^4}, \frac{(1+\omega+\omega^2)^2}{\omega^2}\right\}$

Knot[6, 2] → $\left\{\frac{(1+\omega)^2\,(1-3\,\omega+3\,\omega^2-3\,\omega^3+\omega^4)^2}{\omega^5}, \frac{(1+\omega)^2}{\omega}\right\}$

Knot[6, 3] → $\left\{-\frac{(1-3\,\omega+5\,\omega^2-3\,\omega^3+\omega^4)^2}{\omega^4}, -1\right\}$

In[∘]:= `Union@tab[[All, 2, 2]]`

Out[∘]=

$\left\{-1, 1, -\frac{(1+\omega)^2}{...}, \frac{(1+\omega)^2}{...}, -\frac{(1+\omega)^2\,(1+\omega^2)^2}{...}, \frac{(1+\omega)^2\,(1+\omega^2)^2}{...}, -\frac{(1+\omega+\omega^2)^2}{...}, \frac{(1+\omega+\omega^2)^2}{...}\right\}$

In[∘]:=
```
tab11 =
  Table[K → Factor@FullSimplify[{A5[K], A5[K] / ...}], {K, AllKnots[{11, 11}]}];
```

In[∘]:= `Union@tab11[[All, 2, 2]]`

Out[∘]=

$\left\{-1, 1, -\frac{(1+\omega)^2}{\omega}, \frac{(1+\omega)^2}{\omega}, -\frac{(1+\omega)^2\,(1+\omega^2)^2}{\omega^3},\right.$

$(1+\omega)^2\,(1+\omega^2)^2 \quad (1+\omega+\omega^2)^2 \quad (1+\omega+\omega^2)^2 \quad (1+\omega+\omega^2+\omega^3+\omega^4)^2,$

In[∘]:=
```
Monitor[
  tab12 =
   Table[K → Factor@FullSimplify[{A5[K], A5[K] / Alexander[K][ω]^2}], {K, AllKnots[{12, 12}]}],
```

In[∘]:=
```
Union@tab12[[All, 2, 2]]
```

Out[∘]=

$$\left\{-1,\ 1,\ -\frac{(1+\omega)^2}{\omega},\ \frac{(1+\omega)^2}{\omega},\ -\frac{(1+\omega)^2\left(1+\omega^2\right)^2}{\omega^3},\ \frac{(1+\omega)^2\left(1+\omega^2\right)^2}{\omega^3},\right.$$
$$\left(1+\omega+\omega^2\right)^2 \quad \left(1+\omega+\omega^2\right)^2 \quad \left(1+\omega+\omega^2+\omega^3+\omega^4\right)^2 \quad \left(1+\omega+\omega^2+\omega^3+\omega^4\right)^2,$$

In[∘]:=
```
A6[K_] := Module[{u, v, XingsByArmpits, bends, faces, p, A, is, es, rad, throws, keeps},
   v = (ω + ω*) / 2; u = Sqrt[(v + 1) / 2];
   XingsByArmpits = List @@ PD[K] /.
     x : X[i_, j_, k_, l_] :→ If[PositiveQ[x], X₊[-i, j, k, -l], X₋[-j, k, l, -i]];
   bends = Times @@ XingsByArmpits /. _[X][a_, b_, c_, d_] :→ p_{a,-d} p_{b,-a} p_{c,-b} p_{d,-c};
   faces = bends //. p_{x__,y_} p_{y_,z__} :→ p_{x,y,z};
   A = Table[0, Length@faces, Length@faces];
   Do[is = Position[faces, #][[1, 1]] & /@ List @@ x;
    A[[is, is]] += If[Head[x] === X₊,
       ( v u 1 u
         u 1 u 1
         1 u v u
         u 1 u 1 ), -( v u 1 u
                       u 1 u 1
                       1 u v u
                       u 1 u 1 )],
    {x, XingsByArmpits}];
   M = Factor[A /. ω* → ω^-1];
```

In[∘]:=
```
Column@Table[K → Factor@FullSimplify[{A6[K], A6[K] / Alexander[K][ω]^2}], {K, AllKnots[{3, 7}]}]
```

Out[∘]=

$$\text{Knot}[3,1] \to \left\{-\frac{(1+\omega)^2\left(1-\omega+\omega^2\right)^2}{\omega^3},\ -\frac{(1+\omega)^2}{\omega}\right\}$$

$$\text{Knot}[4,1] \to \left\{\frac{\left(1-3\,\omega+\omega^2\right)^2}{\omega^2},\ 1\right\}$$

$$\text{Knot}[5,1] \to \left\{-\frac{(1+\omega)^2\left(1-\omega+\omega^2-\omega^3+\omega^4\right)^2}{\omega^5},\ -\frac{(1+\omega)^2}{\omega}\right\}$$

$$\text{Knot}[5,2] \to \left\{-\frac{\left(2-3\,\omega+2\,\omega^2\right)^2}{\omega^2},\ -1\right\}$$

$$\text{Knot}[6,1] \to \left\{\frac{(-2+\omega)^2\,(-1+2\,\omega)^2}{\omega^2},\ 1\right\}$$

$$\text{Knot}[6,2] \to \left\{\frac{(1+\omega)^2\left(1-3\,\omega+3\,\omega^2-3\,\omega^3+\omega^4\right)^2}{\omega^5},\ \frac{(1+\omega)^2}{\omega}\right\}$$

$$\text{Knot}[6,3] \to \left\{-\frac{\left(1-3\,\omega+5\,\omega^2-3\,\omega^3+\omega^4\right)^2}{\omega^4},\ -1\right\}$$

$$\text{Knot}[7,1] \to \left\{-\frac{(1+\omega)^2\left(1-\omega+\omega^2-\omega^3+\omega^4-\omega^5+\omega^6\right)^2}{\omega^7},\ -\frac{(1+\omega)^2}{\omega}\right\}$$

$$\text{Knot}[7,2] \to \left\{-\frac{\left(3-5\,\omega+3\,\omega^2\right)^2}{\omega^2},\ -1\right\}$$

$$\text{Knot}[7,3] \to \left\{\frac{(1+\omega)^2\left(2-3\,\omega+3\,\omega^2-3\,\omega^3+2\,\omega^4\right)^2}{\omega^5},\ \frac{(1+\omega)^2}{\omega}\right\}$$

In[∘]:=
```
A7[K_] := Module[{u, v, XingsByArmpits, bends, faces, p, A, is, es, rad, throws, keeps},
  v = (ω + ω*) / 2; u = √((v + 1) / 2);
  XingsByArmpits = List @@ PD[K] /.
    x : X[i_, j_, k_, l_] :> If[PositiveQ[x], X₊[-i, j, k, -l], X₋[-j, k, l, -i]];
  bends = Times @@ XingsByArmpits /. _[X][a_, b_, c_, d_] :> p_{a,-d} p_{b,-a} p_{c,-b} p_{d,-c};
  faces = bends //. p_{x__,y_} p_{y_,z__} :> p_{x,y,z};
  A = Table[0, Length@faces, Length@faces];
  Do[is = Position[faces, #][[1, 1]] & /@ List @@ x;
   A[[is, is]] += If[Head[x] === X₊,
     {{v, u, 1, u}, {u, 1, u, 1}, {1, u, v, u}, {u, 1, u, 1}}, -{{v, u, 1, u}, {u, 1, u, 1}, {1, u, v, u}, {u, 1, u, 1}}],
   {x, XingsByArmpits}];
  M = Factor[A /. ω* → ω^-1];
  throws = RandomChoice[Subsets[Range@Length@M, {2}]];
```

Out[∘]=

$\text{Knot}[3, 1] \to \left\{-\frac{(1-\omega+\omega^2)^2}{\omega^2}, -1\right\}$

$\text{Knot}[4, 1] \to \left\{\frac{(1-3\omega+\omega^2)^2 (1+\omega+\omega^2)^2}{\omega^4}, \frac{(1+\omega+\omega^2)^2}{\omega^2}\right\}$

$\text{Knot}[5, 1] \to \{0, 0\}$

$\text{Knot}[5, 2] \to \left\{-\frac{(1+\omega)^2 (2-3\omega+2\omega^2)^2}{\omega^3}, -\frac{(1+\omega)^2}{\omega}\right\}$

$\text{Knot}[6, 1] \to \left\{\frac{(-2+\omega)^2 (-1+2\omega)^2}{\omega^2}, 1\right\}$

$\text{Knot}[6, 2] \to \left\{\frac{(1-3\omega+3\omega^2-3\omega^3+\omega^4)^2}{\omega^4}, 1\right\}$

$\text{Knot}[6, 3] \to \left\{-\frac{(1-3\omega+5\omega^2-3\omega^3+\omega^4)^2}{\omega^4}, -1\right\}$

$\text{Knot}[7, 1] \to \left\{-\frac{(1-\omega+\omega^2-\omega^3+\omega^4-\omega^5+\omega^6)^2}{\omega^6}, -1\right\}$

$\text{Knot}[7, 2] \to \left\{-\frac{(3-5\omega+3\omega^2)^2}{\omega^2}, -1\right\}$

$\text{Knot}[7, 3] \to \left\{\frac{(2-3\omega+3\omega^2-3\omega^3+2\omega^4)^2}{\omega^4}, 1\right\}$

In[∘]:=
```
A8[K_] := Module[{u, v, XingsByArmpits, bends, faces, p, A, is, es, rad, throws, keeps},
   v = (ω + ω*) / 2; u = √((v + 1) / 2);
   XingsByArmpits = List @@ PD[K] /.
     x : X[i_, j_, k_, l_] :→ If[PositiveQ[x], X+[-i, j, k, -l], X-[-j, k, l, -i]];
   bends = Times @@ XingsByArmpits /. _[X][a_, b_, c_, d_] :→ p[a,-d] p[b,-a] p[c,-b] p[d,-c];
   faces = bends //. p[x__,y_] p[y_,z__] :→ p[x,y,z];
   A = Table[0, Length@faces, Length@faces];
   Do[is = Position[faces, #][[1, 1]] & /@ List @@ x;
    A[[is, is]] += If[Head[x] === X+,
      {{v, u, 1, u}, {u, 1, u, 1}, {1, u, v, u}, {u, 1, u, 1}}, -{{v, u, 1, u}, {u, 1, u, 1}, {1, u, v, u}, {u, 1, u, 1}}],
    {x, XingsByArmpits}];
   M = Factor[A /. ω* → ω^-1];
   throws = Position[faces, #][[1, 1]] & /@ List @@ XingsByArmpits[[1, 1 ;; 2]];
```

Out[∘]=

Knot[3, 1] → $\left\{-\frac{(1-\omega+\omega^2)^2}{\omega^2}, -1\right\}$

Knot[4, 1] → $\left\{\frac{(1-3\,\omega+\omega^2)^2}{\omega^2}, 1\right\}$

Knot[5, 1] → $\left\{-\frac{(1-\omega+\omega^2-\omega^3+\omega^4)^2}{\omega^4}, -1\right\}$

Knot[5, 2] → $\left\{-\frac{(2-3\,\omega+2\,\omega^2)^2}{\omega^2}, -1\right\}$

Knot[6, 1] → $\left\{\frac{(-2+\omega)^2\,(-1+2\,\omega)^2}{\omega^2}, 1\right\}$

Knot[6, 2] → $\left\{\frac{(1-3\,\omega+3\,\omega^2-3\,\omega^3+\omega^4)^2}{\omega^4}, 1\right\}$

Knot[6, 3] → $\left\{-\frac{(1-3\,\omega+5\,\omega^2-3\,\omega^3+\omega^4)^2}{\omega^4}, -1\right\}$

Knot[7, 1] → $\left\{-\frac{(1-\omega+\omega^2-\omega^3+\omega^4-\omega^5+\omega^6)^2}{\omega^6}, -1\right\}$

Knot[7, 2] → $\left\{-\frac{(3-5\,\omega+3\,\omega^2)^2}{\omega^2}, -1\right\}$

Knot[7, 3] → $\left\{\frac{(2-3\,\omega+3\,\omega^2-3\,\omega^3+2\,\omega^4)^2}{\omega^4}, 1\right\}$

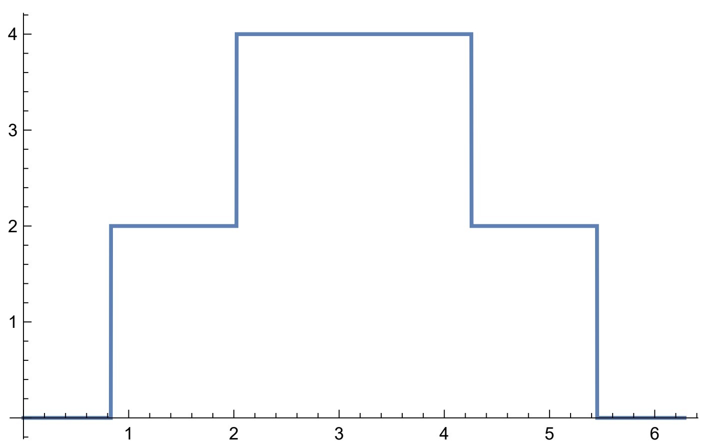
In[∘]:=
```
tab =
  Table[K → Factor@FullSimplify[{a8 = A8[K], a8/───}], {K, AllKnots[{3, 11}]}];
```

In[∘]:=
```
Union@tab[[All, 2, 2]]
```

Out[∘]=
```
{-1, 1}
```

In[∘]:=
```
MatrixSignature[A_] := Total[Sign[Select[Eigenvalues[A], Abs[#] > 10^-12 &]]];
Writhe[K_] := Sum[If[PositiveQ[x], 1, -1], {x, List @@ PD@K}];
S8[K_, ω1_] :=
  Module[{u, v, XingsByArmpits, bends, faces, p, A, is, es, rad, throws, keeps},
   v = (ω + ω*) / 2; u = Sqrt[(v + 1) / 2];
   XingsByArmpits = List @@ PD[K] /.
     x : X[i_, j_, k_, l_] :→ If[PositiveQ[x], X₊[-i, j, k, -l], X₋[-j, k, l, -i]];
   bends = Times @@ XingsByArmpits /. _[X][a_, b_, c_, d_] :→ p_{a,-d} p_{b,-a} p_{c,-b} p_{d,-c};
   faces = bends //. p_{x__,y_} p_{y_,z__} :→ p_{x,y,z};
   A = Table[0, Length@faces, Length@faces];
   Do[is = Position[faces, #][[1, 1]] & /@ List @@ x;
    A[[is, is]] += If[Head[x] === X₊,
      ( v  u  1  u
        u  1  u  1
        1  u  v  u
        u  1  u  1 ), -( v  u  1  u
                        u  1  u  1
                        1  u  v  u
                        u  1  u  1 )],
    {x, XingsByArmpits}];
```

Out[∘]=



Out[∘]=