Pensieve header: Analyzing Reidemeister moves by hand.

Integration

As in Talks/Beijing-2407

In[∘]:=
```
CF[ω_. ℰ_𝔼, specs___] := CF[ω, specs] CF[#, specs] & /@ ℰ;
CF[ℰ_List] := CF /@ ℰ;
CF[q_, vp_, h_] := Module[{H}, Expand@Collect[q, vp, H] /. H → h];
CF[q_, vp_] := CF[q, vp, Factor];
CF[q_] := CF[q, ϵ | (p | x | π | ξ)_ | (p | x | π | ξ)₊ | (p | x | π | ξ)₋];
```

In[∘]:=
```
𝔼 /: 𝔼[A_] 𝔼[B_] := 𝔼[A + B];
```

In[∘]:=
```
$π = Identity;
```

In[∘]:=
```
Unprotect[Integrate];
∫ω_. 𝔼[L_] ⅆ(vs_List) := Module[{n, L0, Q, Δ, G, Z0, Z, λ, DZ, DDZ, FZ, a, b},
   n = Length@vs; L0 = L /. ϵ → 0;
   Q = Table[(-∂_{vs⟦a⟧,vs⟦b⟧} L0) /. Thread[vs → 0] /. (p | x)__ → 0, {a, n}, {b, n}];
   If[(Δ = Det[Q]) == 0, Return@"Degenerate Q!"];
   Z = Z0 = CF@$π[L + vs.Q.vs / 2]; G = Inverse[Q];
   FixedPoint[(DZ = Table[∂_v Z, {v, vs}];
      DDZ = Table[∂_u DZ, {u, vs}];
      FZ = Sum[G⟦a, b⟧ (DDZ⟦a, b⟧ + DZ⟦a⟧ DZ⟦b⟧), {a, n}, {b, n}] / 2;
      Z = CF[Z0 + ∫_0^λ $π[FZ] ⅆλ]) &, Z];
   PowerExpand@Factor[ω Δ^(-1/2)] 𝔼[CF[Z /. λ → 1 /. Thread[vs → 0]]]];
Protect[Integrate];
```

In[∘]:=
```
∫𝔼[- μ x^2 / 2 + ⅈ ξ x] ⅆ{x}
```

Out[∘]=

$$\frac{\mathbb{E}\left[-\frac{\xi^2}{2\mu}\right]}{\sqrt{\mu}}$$

In[∘]:=
```
FofG = ∫𝔼[- μ (x - a)^2 / 2 + ⅈ ξ x] ⅆ{x}
```

Out[∘]=

$$\frac{\mathbb{E}\left[\frac{i\,(2\,a\,\mu+i\,\xi)\,\xi}{2\,\mu}\right]}{\sqrt{\mu}}$$

In[∘]:=
```
∫FofG 𝔼[-ⅈ ξ x] ⅆ{ξ}
```
Out[∘]=

$$\mathbb{E}\left[-\frac{1}{2}\,(a-x)^2\,\mu\right]$$

In[∘]:=
```
L = -1/2 {x1, x2}.(a b; b c).{x1, x2} + {ξ1, ξ2}.{x1, x2};
Z12 = ∫𝔼[L] ⅆ{x1, x2}
```
Out[∘]=

$$\frac{\mathbb{E}\left[\frac{c\,\xi_1^2}{2\,(-b^2+a\,c)}+\frac{b\,\xi_1\,\xi_2}{b^2-a\,c}+\frac{a\,\xi_2^2}{2\,(-b^2+a\,c)}\right]}{\sqrt{-b^2+a\,c}}$$

In[∘]:=
```
{Z1 = ∫𝔼[L] ⅆ{x1}, Z12 == ∫Z1 ⅆ{x2}}
```
Out[∘]=

$$\left\{\frac{\mathbb{E}\left[-\frac{(-b^2+a\,c)\,x_2^2}{2\,a}-\frac{b\,x_2\,\xi_1}{a}+\frac{\xi_1^2}{2\,a}+x_2\,\xi_2\right]}{\sqrt{a}},\ \text{True}\right\}$$

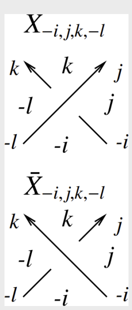
In[∘]:=
```
$π = Normal[# + O[ϵ]^13] &; ∫𝔼[-ϕ^2/2 + ϵ ϕ^3/6] ⅆ{ϕ}
```
Out[∘]=

$$\mathbb{E}\left[\frac{5\,\epsilon^2}{24}+\frac{5\,\epsilon^4}{16}+\frac{1105\,\epsilon^6}{1152}+\frac{565\,\epsilon^8}{128}+\frac{82\,825\,\epsilon^{10}}{3072}+\frac{19\,675\,\epsilon^{12}}{96}\right]$$

The Quadratic

Matrices from SSAlexander4Tangles.nb

In[∘]:=



$$\left(\begin{array}{l} M = \begin{pmatrix} -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix} \\ \overline{M} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix} \end{array}\right);$$

In[∘]:=
```
F[i_, j_, k_, l_] := (1 + t^2) p[i] x[i]/t - (1 + t^2) p[k] x[k]/t - p[j] (x[i]/t - x[k]/t) - p[l] (t x[i] - t x[k]);
Q[i_, j_, k_, l_] := {p[i], p[j], p[k], p[l]}.(t M - t^-1 M^T).{x[i], x[j], x[k], x[l]} + F[i, j, k, l];
Q̄[i_, j_, k_, l_] := {p[i], p[j], p[k], p[l]}.(t M̄ - t^-1 M̄^T).{x[i], x[j], x[k], x[l]} + F[i, j, k, l];
```

In[∘]:= **CF[$Q_{i,j,k,l}$]**

Out[∘]=

$$\frac{2\,p_i\,x_i}{t} - \frac{2\,p_k\,x_i}{t} - t\,p_i\,x_j + t\,p_k\,x_j + 2\,t\,p_i\,x_k - 2\,t\,p_k\,x_k - \frac{p_i\,x_l}{t} + \frac{p_k\,x_l}{t}$$

In[∘]:= **{CF[$Q_{i,j,k,l}$ /. x_ → x], CF[$Q_{i,j,k,l}$ /. p_ → p]}**

Out[∘]=

$$\left\{\frac{(1+t^2)\,x\,p_i}{t} - \frac{(1+t^2)\,x\,p_k}{t},\ 0\right\}$$

In[∘]:= **Collect[$Q_{i,j,k,l}$, p_, Simplify]**

Out[∘]=

$$-\frac{p_i\,(-2\,x_i + t^2\,x_j - 2\,t^2\,x_k + x_l)}{t} + \frac{p_k\,(-2\,x_i + t^2\,x_j - 2\,t^2\,x_k + x_l)}{t}$$

In[∘]:= **Collect[$Q_{i,j,k,l}$, x_, Simplify]**

Out[∘]=

$$\frac{2\,(p_i - p_k)\,x_i}{t} + t\,(-p_i + p_k)\,x_j + 2\,t\,(p_i - p_k)\,x_k + \frac{(-p_i + p_k)\,x_l}{t}$$

In[∘]:= **CF[$Q_{i,j,k,l}$ /. {p_i → (p_+ + p_-) / 2, p_k → (p_+ − p_-) / 2, x_i → (x_+ + x_-) / 2, x_k → (x_+ − x_-) / 2}]**

Out[∘]=

$$-\frac{(-1+t)\,(1+t)\,p_-\,x_-}{t} + \frac{(1+t^2)\,p_-\,x_+}{t} - t\,p_-\,x_j - \frac{p_-\,x_l}{t}$$

In[∘]:= **CF[$\overline{Q}_{i,j,k,l}$ /. {p_i → (p_+ + p_-) / 2, p_k → (p_+ − p_-) / 2, x_i → (x_+ + x_-) / 2, x_k → (x_+ − x_-) / 2}]**

Out[∘]=

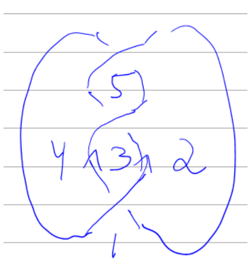
$$\frac{(-1+t)\,(1+t)\,p_-\,x_-}{t} + \frac{(1+t^2)\,p_-\,x_+}{t} - t\,p_-\,x_j - \frac{p_-\,x_l}{t}$$

In[∘]:= **CF[$Q_{i,j,k,l}$ − $\overline{Q}_{i,j,k,l}$ /. {p_i → (p_+ + p_-) / 2, p_k → (p_+ − p_-) / 2, x_i → (x_+ + x_-) / 2, x_k → (x_+ − x_-) / 2}]**

Out[∘]=

$$-\frac{2\,(-1+t)\,(1+t)\,p_-\,x_-}{t}$$

The Trefoil



In[∘]:= **L = CF[$Q_{1,2,3,4}$ + $Q_{3,2,5,4}$ + $Q_{5,2,1,4}$]**

Out[∘]=

$$-\frac{2(-1+t)(1+t)p_1x_1}{t}-\frac{2p_3x_1}{t}+2tp_5x_1+2tp_1x_3-\frac{2(-1+t)(1+t)p_3x_3}{t}-\frac{2p_5x_3}{t}-\frac{2p_1x_5}{t}+2tp_3x_5-\frac{2(-1+t)(1+t)p_5x_5}{t}$$

In[∘]:= **{CF[(∂_{p_1}L) + (∂_{p_3}L) + (∂_{p_5}L)], CF[(∂_{x_1}L) + (∂_{x_3}L) + (∂_{x_5}L)]}**

Out[∘]=

{0, 0}

In[∘]:= **$\int \mathbb{E}$[L] d{p_1, p_3, x_1, x_3}**

Out[∘]=

$$\frac{t^2\,\mathbb{E}[0]}{4\left(1-t^2+t^4\right)}$$

In[∘]:= **CF[L, (p | x)$_5$]**

Out[∘]=

$$\frac{2p_5\left(t^2x_1-x_3\right)}{t}-\frac{2\left(-p_1x_1+t^2p_1x_1+p_3x_1-t^2p_1x_3-p_3x_3+t^2p_3x_3\right)}{t}+\frac{2\left(-p_1+t^2p_3\right)x_5}{t}-\frac{2(-1+t)(1+t)p_5x_5}{t}$$

In[∘]:= **CF[L /. {p_5 → p_+ - p_1 - p_3, x_5 → x_+ - x_1 - x_3}]**

Out[∘]=

$$-\frac{2(-1+t)(1+t)p_+x_+}{t}+\frac{2\left(-2+t^2\right)x_+p_1}{t}+\frac{2\left(-1+2t^2\right)x_+p_3}{t}+\frac{2\left(-1+2t^2\right)p_+x_1}{t}-\frac{6(-1+t)(1+t)p_1x_1}{t}-6tp_3x_1+\frac{2\left(-2+t^2\right)p_+x_3}{t}+\frac{6p_1x_3}{t}-\frac{6(-1+t)(1+t)p_3x_3}{t}$$

In[∘]:= **CF[L /. {p_5 → p_+ - p_1 - p_3, x_5 → x_+ - x_1 - x_3, p_3 → p_0 - p_1, x_3 → x_0 - x_1}]**

Out[∘]=

$$-\frac{2(-1+t)(1+t)p_+x_+}{t}+2tx_+p_0-\frac{4x_+p_1}{t}+\frac{2(-1+t)(1+t)x_+p_3}{t}-\frac{2p_+x_0}{t}-\frac{2(-1+t)(1+t)p_0x_0}{t}+4tp_1x_0+\frac{2p_3x_0}{t}+4tp_+x_1-\frac{4p_0x_1}{t}-\frac{8(-1+t)(1+t)p_1x_1}{t}-4tp_3x_1+\frac{2(-1+t)(1+t)p_+x_3}{t}-2tp_0x_3+\frac{4p_1x_3}{t}-\frac{2(-1+t)(1+t)p_3x_3}{t}$$

In[∘]:= **$\int \mathbb{E}$[L] d{p_1, p_5, x_1, x_5}**

Out[∘]=

$$\frac{t^2\,\mathbb{E}[0]}{4\left(1-t^2+t^4\right)}$$

In[∘]:= CF[Q_{1,2,3,4} + Q_{3,2,5,4} + Q_{5,2,1,4} /. {p_1 → -p_3 - p_5, x_1 → -x_3 - x_5}]

Out[∘]=

$$-\frac{6\ (-1+t)\ (1+t)\ p_3\ x_3}{t} - 6\ t\ p_5\ x_3 + \frac{6\ p_3\ x_5}{t} - \frac{6\ (-1+t)\ (1+t)\ p_5\ x_5}{t}$$

Re-sectioning

In[∘]:= {CF[Q_{i,j,k,l} /. (p | x)_i → 0], CF[Q_{i,j,k,l} /. (p | x)_k → 0]}

Out[∘]=

$$\left\{t\ p_k\ x_j - 2\ t\ p_k\ x_k + \frac{p_k\ x_l}{t},\ \frac{2\ p_i\ x_i}{t} - t\ p_i\ x_j - \frac{p_i\ x_l}{t}\right\}$$

In[∘]:= {lhs, rhs} = {∫𝔼[Q_{i,j,k,l} /. (p | x)_i → 0] ⅆ{p_k, x_k}, ∫𝔼[Q_{i,j,k,l} /. (p | x)_k → 0] ⅆ{p_i, x_i}}

Out[∘]=

$$\left\{-\frac{i\ \mathbb{E}[0]}{2\ t},\ -\frac{1}{2}\ i\ t\ \mathbb{E}[0]\right\}$$

In[∘]:= lhs == rhs

Out[∘]=

$$-\frac{i\ \mathbb{E}[0]}{2\ t} == -\frac{1}{2}\ i\ t\ \mathbb{E}[0]$$

In[∘]:= f = ξ_j x_j + π_j p_j + ξ_l x_l + π_l p_l;

{lhs, rhs} =

{∫𝔼[(Q_{i,j,k,l} + f) /. (p | x)_i → 0] ⅆ{p_k, x_k}, ∫𝔼[(Q_{i,j,k,l} + f) /. (p | x)_k → 0] ⅆ{p_i, x_i}}

lhs == rhs

Out[∘]=

$$\left\{-\frac{i\ \mathbb{E}[p_j\ \pi_j + p_l\ \pi_l + x_j\ \xi_j + x_l\ \xi_l]}{2\ t},\ -\frac{1}{2}\ i\ t\ \mathbb{E}[p_j\ \pi_j + p_l\ \pi_l + x_j\ \xi_j + x_l\ \xi_l]\right\}$$

Out[∘]=

$$-\frac{i\ \mathbb{E}[p_j\ \pi_j + p_l\ \pi_l + x_j\ \xi_j + x_l\ \xi_l]}{2\ t} == -\frac{1}{2}\ i\ t\ \mathbb{E}[p_j\ \pi_j + p_l\ \pi_l + x_j\ \xi_j + x_l\ \xi_l]$$

In[∘]:=
```
f = ξj xj + πj pj + ξl xl + πl pl + π (pi - pk) + ξ (xi - xk);
{lhs, rhs} =
  {∫𝔼[(Qi,j,k,l + f) /. (p | x)i → 0] ⅆ{pk, xk}, ∫𝔼[(Qi,j,k,l + f) /. (p | x)k → 0] ⅆ{pi, xi}}
lhs == rhs
```

Out[∘]=

$$\left\{-\frac{i\,\mathbb{E}\left[\frac{\pi\xi}{2t}+p_j\pi_j+p_l\pi_l-\frac{\xi x_j}{2}-\frac{\xi x_l}{2t^2}+x_j\xi_j+x_l\xi_l\right]}{2t},\right.$$
$$\left.-\frac{1}{2}\,i\,t\,\mathbb{E}\left[-\frac{1}{2}\pi t\xi+p_j\pi_j+p_l\pi_l+\frac{1}{2}t^2\xi x_j+\frac{\xi x_l}{2}+x_j\xi_j+x_l\xi_l\right]\right\}$$

Out[∘]=

$$-\frac{i\,\mathbb{E}\left[\frac{\pi\xi}{2t}+p_j\pi_j+p_l\pi_l-\frac{\xi x_j}{2}-\frac{\xi x_l}{2t^2}+x_j\xi_j+x_l\xi_l\right]}{2t}==$$
$$-\frac{1}{2}\,i\,t\,\mathbb{E}\left[-\frac{1}{2}\pi t\xi+p_j\pi_j+p_l\pi_l+\frac{1}{2}t^2\xi x_j+\frac{\xi x_l}{2}+x_j\xi_j+x_l\xi_l\right]$$

In[∘]:=
```
f = ξj xj + πj pj + ξl xl + πl pl + π (pi - pk) + ξ (xi - xk);
{lhs, rhs} =
  {∫𝔼[(Q̄i,j,k,l + f) /. (p | x)i → 0] ⅆ{pk, xk}, ∫𝔼[(Q̄i,j,k,l + f) /. (p | x)k → 0] ⅆ{pi, xi}}
lhs == rhs
```

Out[∘]=

$$\left\{-\frac{1}{2}\,i\,t\,\mathbb{E}\left[\frac{\pi t\xi}{2}+p_j\pi_j+p_l\pi_l-\frac{1}{2}t^2\xi x_j-\frac{\xi x_l}{2}+x_j\xi_j+x_l\xi_l\right],\right.$$
$$\left.-\frac{i\,\mathbb{E}\left[-\frac{\pi\xi}{2t}+p_j\pi_j+p_l\pi_l+\frac{\xi x_j}{2}+\frac{\xi x_l}{2t^2}+x_j\xi_j+x_l\xi_l\right]}{2t}\right\}$$

Out[∘]=

$$-\frac{1}{2}\,i\,t\,\mathbb{E}\left[\frac{\pi t\xi}{2}+p_j\pi_j+p_l\pi_l-\frac{1}{2}t^2\xi x_j-\frac{\xi x_l}{2}+x_j\xi_j+x_l\xi_l\right]==$$
$$-\frac{i\,\mathbb{E}\left[-\frac{\pi\xi}{2t}+p_j\pi_j+p_l\pi_l+\frac{\xi x_j}{2}+\frac{\xi x_l}{2t^2}+x_j\xi_j+x_l\xi_l\right]}{2t}$$

R2b

In[∘]:= $Q_{1,2,3,4} + \overline{Q}_{3,2,5,4}$ // CF

Out[∘]=

$$\frac{2 p_1 x_1}{t} - \frac{2 p_3 x_1}{t} - t p_1 x_2 + t p_5 x_2 + 2 t p_1 x_3 - 2 t p_5 x_3 - \frac{p_1 x_4}{t} + \frac{p_5 x_4}{t} + \frac{2 p_3 x_5}{t} - \frac{2 p_5 x_5}{t}$$

In[∘]:= CF[$Q_{1,2,3,4} + \overline{Q}_{3,2,5,4}$, $(p \mid x)_3$]

Out[∘]=

$$2 t (p_1 - p_5) x_3 - \frac{2 p_3 (x_1 - x_5)}{t} - \frac{-2 p_1 x_1 + t^2 p_1 x_2 - t^2 p_5 x_2 + p_1 x_4 - p_5 x_4 + 2 p_5 x_5}{t}$$

In[∘]:= Expand[$Q_{1,2,3,4} + \overline{Q}_{3,2,5,4}$ /. {$p_5 \to p_1$, $x_5 \to x_1$}]

Out[∘]=

0

In[∘]:= CF[$Q_{1,2,3,4} + \overline{Q}_{3,2,5,4}$ /. $(p \mid x)_3 \to 0$]

Out[∘]=

$$\frac{2 p_1 x_1}{t} - t p_1 x_2 + t p_5 x_2 - \frac{p_1 x_4}{t} + \frac{p_5 x_4}{t} - \frac{2 p_5 x_5}{t}$$

In[∘]:= CF[$Q_{1,2,3,4} + \overline{Q}_{3,2,5,4}$ /. $(p \mid x)_3 \to 0$ /. p_ → p]

Out[∘]=

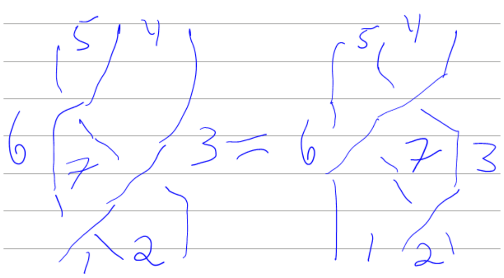
$$\frac{2 p x_1}{t} - \frac{2 p x_5}{t}$$

In[∘]:= CF[$Q_{1,2,3,4} + \overline{Q}_{3,2,5,4}$ /. p_ → p]

Out[∘]=

0

R3



In[∘]:= lhs = CF[$Q_{1,2,7,6} + Q_{2,3,4,7} + Q_{7,4,5,6}$];
rhs = CF[$Q_{2,3,7,1} + Q_{1,7,5,6} + Q_{7,3,4,5}$]

Out[∘]=

$$\frac{2 p_1 x_1}{t} - \frac{p_2 x_1}{t} - \frac{2 p_5 x_1}{t} + \frac{p_7 x_1}{t} + \frac{2 p_2 x_2}{t} - \frac{2 p_7 x_2}{t} - t p_2 x_3 + t p_4 x_3 - 2 t p_4 x_4 + 2 t p_7 x_4 + 2 t p_1 x_5 + \frac{p_4 x_5}{t} - 2 t p_5 x_5 - \frac{p_7 x_5}{t} - \frac{p_1 x_6}{t} + \frac{p_5 x_6}{t} - t p_1 x_7 + 2 t p_2 x_7 - \frac{2 p_4 x_7}{t} + t p_5 x_7 - \frac{2 (-1 + t) (1 + t) p_7 x_7}{t}$$

In[∘]:= `Simplify[lhs == rhs]`

Out[∘]=

$$\frac{1}{t}\left(-3 p_7 x_1 - t^2 p_1 x_2 - 2 p_4 x_2 + 2 p_7 x_2 + t^2 p_7 x_2 - 3 t^2 p_7 x_4 - 2 t^2 p_1 x_5 - p_4 x_5 + p_7 x_5 + 2 t^2 p_7 x_5 + 3 t^2 p_1 x_7 + 3 p_4 x_7 + p_5 \left(2 x_1 + t^2 x_4 - \left(2 + t^2\right) x_7\right) + p_2 \left(x_1 + 2 t^2 x_4 - \left(1 + 2 t^2\right) x_7\right)\right) == 0$$

In[∘]:= `CF[lhs, (p | x)₇]`

Out[∘]=

$$\frac{p_7 \left(-2 x_1 + t^2 x_2 - t^2 x_4 + 2 t^2 x_5\right)}{t} - \frac{1}{t}\left(-2 p_1 x_1 + t^2 p_1 x_2 - 2 p_2 x_2 + 2 p_4 x_2 + t^2 p_2 x_3 - t^2 p_4 x_3 - 2 t^2 p_2 x_4 + 2 t^2 p_4 x_4 - t^2 p_5 x_4 + 2 t^2 p_5 x_5 + p_1 x_6 - p_5 x_6\right) + \frac{\left(2 t^2 p_1 - p_2 + p_4 - 2 p_5\right) x_7}{t} - \frac{2 (-1 + t) (1 + t) p_7 x_7}{t}$$

In[∘]:= `∫𝔼[lhs] ⅆ{p₇, x₇}`

Out[∘]=

$$-\frac{1}{2\left(-1 + t^2\right)} \, \mathbb{i} \, t \, \mathbb{E}\Big[-\frac{2 p_1 x_1}{(-1 + t) t (1 + t)} + \frac{p_2 x_1}{(-1 + t) t (1 + t)} - \frac{p_4 x_1}{(-1 + t) t (1 + t)} + \frac{2 p_5 x_1}{(-1 + t) t (1 + t)} + \frac{t p_1 x_2}{(-1 + t) (1 + t)} + \frac{\left(-4 + 3 t^2\right) p_2 x_2}{2 (-1 + t) t (1 + t)} - \frac{\left(-4 + 3 t^2\right) p_4 x_2}{2 (-1 + t) t (1 + t)} - \frac{t p_5 x_2}{(-1 + t) (1 + t)} - t p_2 x_3 + t p_4 x_3 - \frac{t^3 p_1 x_4}{(-1 + t) (1 + t)} + \frac{t \left(-3 + 4 t^2\right) p_2 x_4}{2 (-1 + t) (1 + t)} - \frac{t \left(-3 + 4 t^2\right) p_4 x_4}{2 (-1 + t) (1 + t)} + \frac{t^3 p_5 x_4}{(-1 + t) (1 + t)} + \frac{2 t^3 p_1 x_5}{(-1 + t) (1 + t)} - \frac{t p_2 x_5}{(-1 + t) (1 + t)} + \frac{t p_4 x_5}{(-1 + t) (1 + t)} - \frac{2 t^3 p_5 x_5}{(-1 + t) (1 + t)} - \frac{p_1 x_6}{t} + \frac{p_5 x_6}{t}\Big]$$

In[∘]:= `CF[rhs, (p | x)₇]`

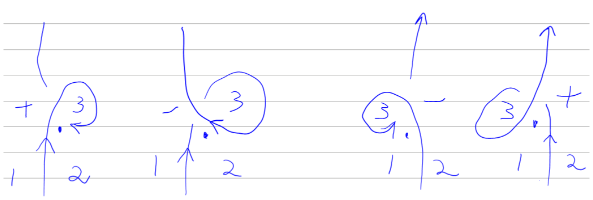
Out[∘]=

$$\frac{p_7 \left(x_1 - 2 x_2 + 2 t^2 x_4 - x_5\right)}{t} + \frac{1}{t}\left(2 p_1 x_1 - p_2 x_1 - 2 p_5 x_1 + 2 p_2 x_2 - t^2 p_2 x_3 + t^2 p_4 x_3 - 2 t^2 p_4 x_4 + 2 t^2 p_1 x_5 + p_4 x_5 - 2 t^2 p_5 x_5 - p_1 x_6 + p_5 x_6\right) - \frac{\left(t^2 p_1 - 2 t^2 p_2 + 2 p_4 - t^2 p_5\right) x_7}{t} - \frac{2 (-1 + t) (1 + t) p_7 x_7}{t}$$

In[•]:=
$$\left(\int \mathbb{E}[\texttt{rhs}]\, d\{p_7, x_7\}\right) == \left(\int \mathbb{E}[\texttt{lhs}]\, d\{p_7, x_7\}\right)$$

Out[•]=
$$-\frac{1}{2\left(-1+t^2\right)}\, i\, t\, \mathbb{E}\Big[\frac{\left(-4+3t^2\right) p_1 x_1}{2(-1+t)\, t\, (1+t)} + \frac{p_2 x_1}{(-1+t)\, t\, (1+t)} - \frac{p_4 x_1}{(-1+t)\, t\, (1+t)} - \frac{\left(-4+3t^2\right) p_5 x_1}{2(-1+t)\, t\, (1+t)} + \frac{t\, p_1 x_2}{(-1+t)(1+t)} - \frac{2 p_2 x_2}{(-1+t)\, t\, (1+t)} + \frac{2 p_4 x_2}{(-1+t)\, t\, (1+t)} - \frac{t\, p_5 x_2}{(-1+t)(1+t)} - t\, p_2 x_3 + t\, p_4 x_3 - \frac{t^3 p_1 x_4}{(-1+t)(1+t)} + \frac{2t^3 p_2 x_4}{(-1+t)(1+t)} - \frac{2t^3 p_4 x_4}{(-1+t)(1+t)} + \frac{t^3 p_5 x_4}{(-1+t)(1+t)} + \frac{t\left(-3+4t^2\right) p_1 x_5}{2(-1+t)(1+t)} - \frac{t\, p_2 x_5}{(-1+t)(1+t)} + \frac{t\, p_4 x_5}{(-1+t)(1+t)} - \frac{t\left(-3+4t^2\right) p_5 x_5}{2(-1+t)(1+t)} - \frac{p_1 x_6}{t} + \frac{p_5 x_6}{t}\Big] ==$$
$$-\frac{1}{2\left(-1+t^2\right)}\, i\, t\, \mathbb{E}\Big[-\frac{2 p_1 x_1}{(-1+t)\, t\, (1+t)} + \frac{p_2 x_1}{(-1+t)\, t\, (1+t)} - \frac{p_4 x_1}{(-1+t)\, t\, (1+t)} + \frac{2 p_5 x_1}{(-1+t)\, t\, (1+t)} + \frac{t\, p_1 x_2}{(-1+t)(1+t)} + \frac{\left(-4+3t^2\right) p_2 x_2}{2(-1+t)\, t\, (1+t)} - \frac{\left(-4+3t^2\right) p_4 x_2}{2(-1+t)\, t\, (1+t)} - \frac{t\, p_5 x_2}{(-1+t)(1+t)} - t\, p_2 x_3 + t\, p_4 x_3 - \frac{t^3 p_1 x_4}{(-1+t)(1+t)} + \frac{t\left(-3+4t^2\right) p_2 x_4}{2(-1+t)(1+t)} - \frac{t\left(-3+4t^2\right) p_4 x_4}{2(-1+t)(1+t)} + \frac{t^3 p_5 x_4}{(-1+t)(1+t)} + \frac{2t^3 p_1 x_5}{(-1+t)(1+t)} - \frac{t\, p_2 x_5}{(-1+t)(1+t)} + \frac{t\, p_4 x_5}{(-1+t)(1+t)} - \frac{2t^3 p_5 x_5}{(-1+t)(1+t)} - \frac{p_1 x_6}{t} + \frac{p_5 x_6}{t}\Big]$$

R1s

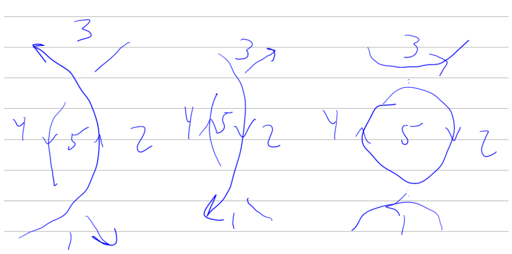


In[•]:=
$$\texttt{R1s} = \texttt{Simplify}@\left\{Q_{2,3,2,1},\ \overline{Q}_{2,3,2,1},\ \overline{Q}_{1,2,1,3},\ Q_{1,2,1,3}\right\}$$

Out[•]=

{0, 0, 0, 0}

R2C$_+$ and R2C$_-$



In[∘]:=
$$\{\text{Collect}[\overline{Q}_{4,1,2,5}+Q_{2,3,4,5},\ t,\ \text{Simplify}],\ \text{Collect}[\overline{Q}_{2,5,4,1}+Q_{4,5,2,3},\ t,\ \text{Simplify}]\}$$

Out[∘]=
$$\left\{t\,(p_2-p_4)\,(x_1-x_3),\ -\frac{(p_2-p_4)\,(x_1-x_3)}{t}\right\}$$

In[∘]:=
$$\{\text{Collect}[Q_{4,1,2,5}+\overline{Q}_{2,3,4,5},\ t,\ \text{Simplify}],\ \text{Collect}[Q_{2,5,4,1}+\overline{Q}_{4,5,2,3},\ t,\ \text{Simplify}]\}$$

Out[∘]=
$$\left\{t\,(p_2-p_4)\,(x_1-x_3),\ -\frac{(p_2-p_4)\,(x_1-x_3)}{t}\right\}$$

In[∘]:=
$$\mathbb{E}\left[\overline{Q}_{4,1,2,5}+Q_{2,3,4,5}+i\left(\sum_{k=1}^{4}(\pi_k\,p_k+\xi_k\,x_k)\right)\right]$$

Out[∘]=
$$\mathbb{E}\Big[(t\,p_2-t\,p_4)\,x_1+\left(-\frac{p_1}{t}+\left(-\frac{1}{t}+t\right)p_2+\frac{2\,p_4}{t}-t\,p_5\right)x_2+\left(\left(\frac{1}{t}-t\right)p_2+\frac{p_3}{t}-\frac{2\,p_4}{t}+t\,p_5\right)x_2+$$
$$(-t\,p_2+t\,p_4)\,x_3+\left(2\,t\,p_2-\frac{p_3}{t}+\left(\frac{1}{t}-t\right)p_4-t\,p_5\right)x_4+\left(\frac{p_1}{t}-2\,t\,p_2+\left(-\frac{1}{t}+t\right)p_4+t\,p_5\right)x_4-$$
$$p_3\left(\frac{x_2}{t}-\frac{x_4}{t}\right)-p_1\left(-\frac{x_2}{t}+\frac{x_4}{t}\right)-p_5\,(t\,x_2-t\,x_4)-p_5\,(-t\,x_2+t\,x_4)+\left(\frac{p_2}{t}-\frac{p_4}{t}\right)x_5+$$
$$\left(-\frac{p_2}{t}+\frac{p_4}{t}\right)x_5+i\,(p_1\,\pi_1+p_2\,\pi_2+p_3\,\pi_3+p_4\,\pi_4+x_1\,\xi_1+x_2\,\xi_2+x_3\,\xi_3+x_4\,\xi_4)\Big]$$

In[∘]:=
```
px_i_ := Sequence[p_i, x_i];
```

In[∘]:=
$$\int\mathbb{E}\left[\overline{Q}_{4,1,2,5}+Q_{2,3,4,5}+i\left(\sum_{k=1}^{4}(\pi_k\,p_k+\xi_k\,x_k)\right)\right]d\{px_4\}$$

Out[∘]=
Degenerate Q!

In[∘]:=
$$\text{CF}\left[\mathbb{E}\left[\overline{Q}_{4,1,2,5}+Q_{2,3,4,5}+i\left(\sum_{k=1}^{4}(\pi_k\,p_k+\xi_k\,x_k)\right)\right],\ (p\mid x)_4\right]$$

Out[∘]=
$$\mathbb{E}[i\,p_1\,\pi_1+i\,p_2\,\pi_2+i\,p_3\,\pi_3+t\,p_2\,x_1-t\,p_2\,x_3+p_4\,(i\,\pi_4-t\,x_1+t\,x_3)+i\,x_1\,\xi_1+i\,x_2\,\xi_2+i\,x_3\,\xi_3+i\,x_4\,\xi_4]$$

In[∘]:=
$$\int\mathbb{E}\left[\overline{Q}_{4,1,2,5}+Q_{2,3,4,5}+i\left(\sum_{k=1}^{4}(\pi_k\,p_k+\xi_k\,x_k)\right)\right]d\{px_3,\ px_4\}$$

Out[∘]=
Degenerate Q!

In[∘]:=
$$\left(\int\mathbb{E}\left[\overline{Q}_{4,1,2,5}+Q_{2,3,4,5}+i\left(\sum_{k=1}^{4}(\pi_k\,p_k+\xi_k\,x_k)\right)\right]d\{px_3,\ px_4\}\right)\ /.$$
$$\{\pi_1\to-\pi_3,\ \xi_1\to-\xi_3,\ \pi_2\to-\pi_4,\ \xi_2\to-\xi_4\}$$

Out[∘]=
Degenerate Q!

In[∘]:=
$$\text{CF}[\overline{Q}_{4,1,2,5}+Q_{2,3,4,5}+i\,(\pi_v\,(p_1-p_3)+\xi_v\,(x_1-x_3)+\pi_h\,(p_2-p_4)+\xi_h\,(x_2-x_4))]$$

Out[∘]=
$$i\,p_2\,\pi_h-i\,p_4\,\pi_h+i\,p_1\,\pi_v-i\,p_3\,\pi_v+t\,p_2\,x_1-t\,p_4\,x_1-t\,p_2\,x_3+t\,p_4\,x_3+i\,x_2\,\xi_h-i\,x_4\,\xi_h+i\,x_1\,\xi_v-i\,x_3\,\xi_v$$

In[ ]:= $\int \mathbb{E}\left[\overline{Q}_{4,1,2,5} + Q_{2,3,4,5} + i\,(\pi v\,(p_1 - p_3) + \xi v\,(x_1 - x_3) + \pi h\,(p_2 - p_4) + \xi h\,(x_2 - x_4))\right] d\{px_3, px_4\}$

Out[ ]= Degenerate Q!

The potentials

For the crossings:

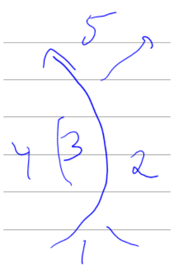
In[ ]:= $L = Q_{i,j,k,l};\ CF@\{(\partial_{p_i} L) + (\partial_{p_k} L),\ (\partial_{x_i} L) + (\partial_{x_k} L),\ \partial_{p_j} L,\ \partial_{x_j} L,\ \partial_{p_l} L,\ \partial_{x_l} L\}$

Out[ ]= $\left\{0,\ \frac{2\left(1+t^2\right) p_i}{t} - \frac{2\left(1+t^2\right) p_k}{t},\ 0,\ -t\,p_i + t\,p_k,\ 0,\ -\frac{p_i}{t} + \frac{p_k}{t}\right\}$

In[ ]:= $L = \overline{Q}_{i,j,k,l};\ CF@\{(\partial_{p_i} L) + (\partial_{p_k} L),\ (\partial_{x_i} L) + (\partial_{x_k} L),\ \partial_{p_j} L,\ \partial_{x_j} L,\ \partial_{p_l} L,\ \partial_{x_l} L\}$

Out[ ]= $\left\{0,\ \frac{2\left(1+t^2\right) p_i}{t} - \frac{2\left(1+t^2\right) p_k}{t},\ 0,\ -t\,p_i + t\,p_k,\ 0,\ -\frac{p_i}{t} + \frac{p_k}{t}\right\}$

For R2b:



In[ ]:= $L = Q_{1,2,3,4} + \overline{Q}_{3,2,5,4};$
$CF@\{(\partial_{p_1} L) + (\partial_{p_3} L) + (\partial_{p_5} L),\ (\partial_{x_1} L) + (\partial_{x_3} L) + (\partial_{x_5} L),\ \partial_{p_2} L,\ \partial_{x_2} L,\ \partial_{p_4} L,\ \partial_{x_4} L\}$

Out[ ]= $\left\{0,\ \frac{2\left(1+t^2\right) p_1}{t} - \frac{2\left(1+t^2\right) p_5}{t},\ 0,\ -t\,p_1 + t\,p_5,\ 0,\ -\frac{p_1}{t} + \frac{p_5}{t}\right\}$

In[ ]:= $L = Q_{1,2,3,4} + \overline{Q}_{3,2,5,4}$ /. $(p\,|\,x)_3 \to 0;$
$CF@\{(\partial_{p_1} L) + (\partial_{p_5} L),\ (\partial_{x_1} L) + (\partial_{x_5} L),\ \partial_{p_2} L,\ \partial_{x_2} L,\ \partial_{p_4} L,\ \partial_{x_4} L\}$

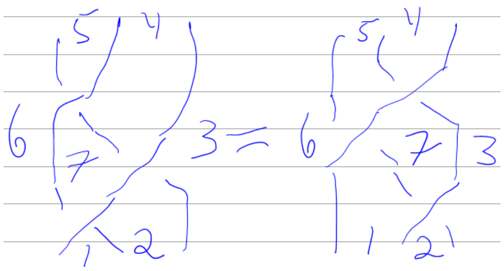
Out[ ]= $\left\{\frac{2\,x_1}{t} - \frac{2\,x_5}{t},\ \frac{2\,p_1}{t} - \frac{2\,p_5}{t},\ 0,\ -t\,p_1 + t\,p_5,\ 0,\ -\frac{p_1}{t} + \frac{p_5}{t}\right\}$

For σ_1^2:

In[ ]:= $L = Q_{1,2,3,4} + Q_{3,2,5,4};$
$CF@\{(\partial_{p_1} L) + (\partial_{p_3} L) + (\partial_{p_5} L),\ (\partial_{x_1} L) + (\partial_{x_3} L) + (\partial_{x_5} L),\ \partial_{p_2} L,\ \partial_{x_2} L,\ \partial_{p_4} L,\ \partial_{x_4} L\}$

Out[ ]= $\left\{0,\ \frac{2\left(1+t^2\right) p_1}{t} - \frac{2\left(1+t^2\right) p_5}{t},\ 0,\ -t\,p_1 + t\,p_5,\ 0,\ -\frac{p_1}{t} + \frac{p_5}{t}\right\}$

For R3:



In[∘]:=
```
lhs = CF[Q_{1,2,7,6} + Q_{2,3,4,7} + Q_{7,4,5,6}];
rhs = CF[Q_{2,3,7,1} + Q_{1,7,5,6} + Q_{7,3,4,5}];
L = lhs; CF@{(∂_{p_1} L) + (∂_{p_7} L) + (∂_{p_5} L), (∂_{x_1} L) + (∂_{x_7} L) + (∂_{x_5} L),
  (∂_{p_2} L) + (∂_{p_4} L), (∂_{x_2} L) + (∂_{x_4} L), ∂_{p_3} L, ∂_{x_3} L, ∂_{p_6} L, ∂_{x_6} L}
L = rhs; CF@{(∂_{p_1} L) + (∂_{p_5} L), (∂_{x_1} L) + (∂_{x_5} L),
  (∂_{p_2} L) + (∂_{p_7} L) + (∂_{p_4} L), (∂_{x_2} L) + (∂_{x_7} L) + (∂_{x_4} L), ∂_{p_3} L, ∂_{x_3} L, ∂_{p_6} L, ∂_{x_6} L}
L = Cases[(∫𝔼[lhs] ⅆ{p_7, x_7}), 𝔼[L_] :→ L][[1]];
CF@{(∂_{p_1} L) + (∂_{p_5} L), (∂_{x_1} L) + (∂_{x_5} L), (∂_{p_2} L) + (∂_{p_4} L), (∂_{x_2} L) + (∂_{x_4} L), ∂_{p_3} L, ∂_{x_3} L, ∂_{p_6} L, ∂_{x_6} L}
L = Cases[(∫𝔼[rhs] ⅆ{p_7, x_7}), 𝔼[L_] :→ L][[1]];
CF@{(∂_{p_1} L) + (∂_{p_5} L), (∂_{x_1} L) + (∂_{x_5} L), (∂_{p_2} L) + (∂_{p_4} L), (∂_{x_2} L) + (∂_{x_4} L), ∂_{p_3} L, ∂_{x_3} L, ∂_{p_6} L, ∂_{x_6} L}
```

Out[∘]=

$$\left\{0, \frac{2\left(1+t^2\right)p_1}{t} - \frac{p_2}{t} + \frac{p_4}{t} - \frac{2\left(1+t^2\right)p_5}{t}, 0, -t\,p_1 + \frac{2\left(1+t^2\right)p_2}{t} - \frac{2\left(1+t^2\right)p_4}{t} + t\,p_5, 0, -t\,p_2 + t\,p_4, 0, -\frac{p_1}{t} + \frac{p_5}{t}\right\}$$

Out[∘]=

$$\left\{0, \frac{2\left(1+t^2\right)p_1}{t} - \frac{p_2}{t} + \frac{p_4}{t} - \frac{2\left(1+t^2\right)p_5}{t}, 0, -t\,p_1 + \frac{2\left(1+t^2\right)p_2}{t} - \frac{2\left(1+t^2\right)p_4}{t} + t\,p_5, 0, -t\,p_2 + t\,p_4, 0, -\frac{p_1}{t} + \frac{p_5}{t}\right\}$$

Out[∘]=

$$\left\{0, \frac{2\left(1+t^2\right)p_1}{t} - \frac{p_2}{t} + \frac{p_4}{t} - \frac{2\left(1+t^2\right)p_5}{t}, 0, -t\,p_1 + \frac{2\left(1+t^2\right)p_2}{t} - \frac{2\left(1+t^2\right)p_4}{t} + t\,p_5, 0, -t\,p_2 + t\,p_4, 0, -\frac{p_1}{t} + \frac{p_5}{t}\right\}$$

Out[∘]=

$$\left\{0, \frac{2\left(1+t^2\right)p_1}{t} - \frac{p_2}{t} + \frac{p_4}{t} - \frac{2\left(1+t^2\right)p_5}{t}, 0, -t\,p_1 + \frac{2\left(1+t^2\right)p_2}{t} - \frac{2\left(1+t^2\right)p_4}{t} + t\,p_5, 0, -t\,p_2 + t\,p_4, 0, -\frac{p_1}{t} + \frac{p_5}{t}\right\}$$

R3 with flipped middle crossing:

In[•]:=

```
lhs = CF[Q_{1,2,7,6} + Q̄_{2,3,4,7} + Q_{7,4,5,6}];
rhs = CF[Q_{2,3,7,1} + Q̄_{1,7,5,6} + Q_{7,3,4,5}];
L = lhs; CF@{(∂_{p1} L) + (∂_{p7} L) + (∂_{p5} L), (∂_{x1} L) + (∂_{x7} L) + (∂_{x5} L),
   (∂_{p2} L) + (∂_{p4} L), (∂_{x2} L) + (∂_{x4} L), ∂_{p3} L, ∂_{x3} L, ∂_{p6} L, ∂_{x6} L}
L = rhs; CF@{(∂_{p1} L) + (∂_{p5} L), (∂_{x1} L) + (∂_{x5} L),
   (∂_{p2} L) + (∂_{p7} L) + (∂_{p4} L), (∂_{x2} L) + (∂_{x7} L) + (∂_{x4} L), ∂_{p3} L, ∂_{x3} L, ∂_{p6} L, ∂_{x6} L}
L = Cases[(∫𝔼[lhs] d{p7, x7}), 𝔼[L_] :> L][[1]];
CF@{(∂_{p1} L) + (∂_{p5} L), (∂_{x1} L) + (∂_{x5} L), (∂_{p2} L) + (∂_{p4} L), (∂_{x2} L) + (∂_{x4} L), ∂_{p3} L, ∂_{x3} L, ∂_{p6} L, ∂_{x6} L}
L = Cases[(∫𝔼[rhs] d{p7, x7}), 𝔼[L_] :> L][[1]];
CF@{(∂_{p1} L) + (∂_{p5} L), (∂_{x1} L) + (∂_{x5} L), (∂_{p2} L) + (∂_{p4} L), (∂_{x2} L) + (∂_{x4} L), ∂_{p3} L, ∂_{x3} L, ∂_{p6} L, ∂_{x6} L}
```

Out[•]=

$$\left\{0, \frac{2\left(1+t^2\right)p_1}{t} - \frac{p_2}{t} + \frac{p_4}{t} - \frac{2\left(1+t^2\right)p_5}{t}, 0, -t\,p_1 + \frac{2\left(1+t^2\right)p_2}{t} - \frac{2\left(1+t^2\right)p_4}{t} + t\,p_5, 0, -t\,p_2 + t\,p_4, 0, -\frac{p_1}{t} + \frac{p_5}{t}\right\}$$

Out[•]=

$$\left\{0, \frac{2\left(1+t^2\right)p_1}{t} - \frac{p_2}{t} + \frac{p_4}{t} - \frac{2\left(1+t^2\right)p_5}{t}, 0, -t\,p_1 + \frac{2\left(1+t^2\right)p_2}{t} - \frac{2\left(1+t^2\right)p_4}{t} + t\,p_5, 0, -t\,p_2 + t\,p_4, 0, -\frac{p_1}{t} + \frac{p_5}{t}\right\}$$

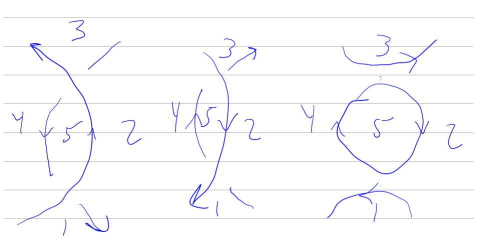
Out[•]=

$$\left\{0, \frac{2\left(1+t^2\right)p_1}{t} - \frac{p_2}{t} + \frac{p_4}{t} - \frac{2\left(1+t^2\right)p_5}{t}, 0, -t\,p_1 + \frac{2\left(1+t^2\right)p_2}{t} - \frac{2\left(1+t^2\right)p_4}{t} + t\,p_5, 0, -t\,p_2 + t\,p_4, 0, -\frac{p_1}{t} + \frac{p_5}{t}\right\}$$

Out[•]=

$$\left\{0, \frac{2\left(1+t^2\right)p_1}{t} - \frac{p_2}{t} + \frac{p_4}{t} - \frac{2\left(1+t^2\right)p_5}{t}, 0, -t\,p_1 + \frac{2\left(1+t^2\right)p_2}{t} - \frac{2\left(1+t^2\right)p_4}{t} + t\,p_5, 0, -t\,p_2 + t\,p_4, 0, -\frac{p_1}{t} + \frac{p_5}{t}\right\}$$

For R2C$_+$:



In[◦]:=

```
L = CF[Q̄_{4,1,2,5} + Q_{2,3,4,5}];
CF@{(∂_{p_2} L) + (∂_{p_4} L), (∂_{x_2} L) + (∂_{x_4} L), ∂_{p_1} L, ∂_{x_1} L, ∂_{p_3} L, ∂_{x_3} L}
```

Out[◦]=

$\{0, 0, 0, t\, p_2 - t\, p_4, 0, -t\, p_2 + t\, p_4\}$

R2C_+ with one crossing flipped:

In[◦]:=

```
L = CF[Q_{4,1,2,5} + Q_{2,3,4,5}];
CF@{(∂_{p_2} L) + (∂_{p_4} L), (∂_{x_2} L) + (∂_{x_4} L), ∂_{p_1} L, ∂_{x_1} L, ∂_{p_3} L, ∂_{x_3} L}
```

Out[◦]=

$\{0, 0, 0, t\, p_2 - t\, p_4, 0, -t\, p_2 + t\, p_4\}$

In[◦]:=

```
Factor[Q_{4,1,2,5} - Q̄_{4,1,2,5}]
```

Out[◦]=

$$-\frac{2\,(-1+t)\,(1+t)\,(p_2-p_4)\,(x_2-x_4)}{t}$$