Pensieve header: Analyzing Reidemeister moves by hand.

Integration

As in Talks/Beijing-2407

In[∘]:=
```
CF[ω_. ℰ_𝔼] := CF[ω] CF /@ ℰ;
CF[ℰ_List] := CF /@ ℰ;
CF[q_] :=
  Expand@Collect[q, ϵ | (p | x | π | ξ)_ | (p | x | π | ξ)₊ | (p | x | π | ξ)₋, F] /. F → Factor;
```

In[∘]:=
```
𝔼 /: 𝔼[A_] 𝔼[B_] := 𝔼[A + B];
```

In[∘]:=
```
(ω1_. 𝔼[L1_]) ≡ (ω2_. 𝔼[L2_]) := Simplify[(ω1 == ω2) ∧ (L1 == L2)]
```

In[∘]:=
```
$π = Identity;
```

In[∘]:=
```
Unprotect[Integrate];
∫ω_. 𝔼[L_] ⅆ(vs_List) := Module[{n, L0, Q, Δ, G, Z0, Z, λ, DZ, DDZ, FZ, a, b},
   n = Length@vs; L0 = L /. ϵ → 0;
   Q = Table[(-∂_{vs⟦a⟧,vs⟦b⟧} L0) /. Thread[vs → 0] /. (p | x)__ → 0, {a, n}, {b, n}];
   If[(Δ = Det[Q]) == 0, Return@"Degenerate Q!"];
   Z = Z0 = CF@$π[L + vs.Q.vs / 2]; G = Inverse[Q];
   FixedPoint[(DZ = Table[∂_v Z, {v, vs}];
      DDZ = Table[∂_u DZ, {u, vs}];
      FZ = Sum[G⟦a, b⟧ (DDZ⟦a, b⟧ + DZ⟦a⟧ DZ⟦b⟧), {a, n}, {b, n}] / 2;
      Z = CF[Z0 + ∫_0^λ $π[FZ] ⅆλ]) &, Z];
   PowerExpand@Factor[ω Δ^(-1/2)] 𝔼[CF[Z /. λ → 1 /. Thread[vs → 0]]]];
Protect[Integrate];
```

In[∘]:= $\int \mathbb{E}\left[-\mu\, x^2/2 + i\,\xi\, x\right] ⅆ\{x\}$

Out[∘]=

$$\frac{\mathbb{E}\left[-\frac{\xi^2}{2\mu}\right]}{\sqrt{\mu}}$$

In[∘]:= FofG = $\int \mathbb{E}\left[-\mu\,(x-a)^2/2 + i\,\xi\, x\right] ⅆ\{x\}$

Out[∘]=

$$\frac{\mathbb{E}\left[\frac{i\,(2\,a\,\mu + i\,\xi)\,\xi}{2\,\mu}\right]}{\sqrt{\mu}}$$

In[∘]:= $\int \mathbf{FofG}\ \mathbb{E}[-\mathrm{i}\ \xi\ x]\ d\{\xi\}$

Out[∘]=

$$\mathbb{E}\left[-\frac{1}{2}\ (a-x)^2\ \mu\right]$$

In[∘]:= $\mathbf{L} = -\frac{1}{2}\ \{x_1, x_2\}.\begin{pmatrix} a & b \\ b & c \end{pmatrix}.\{x_1, x_2\} + \{\xi_1, \xi_2\}.\{x_1, x_2\};$

$\mathbf{Z12} = \int \mathbb{E}[\mathbf{L}]\ d\{x_1, x_2\}$

Out[∘]=

$$\frac{\mathbb{E}\left[\frac{c\ \xi_1^2}{2\ (-b^2+a\ c)} + \frac{b\ \xi_1\ \xi_2}{b^2-a\ c} + \frac{a\ \xi_2^2}{2\ (-b^2+a\ c)}\right]}{\sqrt{-b^2+a\ c}}$$

In[∘]:= $\left\{\mathbf{Z1} = \int \mathbb{E}[\mathbf{L}]\ d\{x_1\},\ \mathbf{Z12} == \int \mathbf{Z1}\ d\{x_2\}\right\}$

Out[∘]=

$$\left\{\frac{\mathbb{E}\left[-\frac{(-b^2+a\ c)\ x_2^2}{2\ a} - \frac{b\ x_2\ \xi_1}{a} + \frac{\xi_1^2}{2\ a} + x_2\ \xi_2\right]}{\sqrt{a}},\ \mathrm{True}\right\}$$

In[∘]:= $\$\pi = \mathbf{Normal}\left[\# + \mathbf{O}[\epsilon]^{13}\right]\ \&;\ \int \mathbb{E}\left[-\phi^2/2 + \epsilon\ \phi^3/6\right]\ d\{\phi\}$

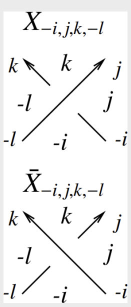
Out[∘]=

$$\mathbb{E}\left[\frac{5\ \epsilon^2}{24} + \frac{5\ \epsilon^4}{16} + \frac{1105\ \epsilon^6}{1152} + \frac{565\ \epsilon^8}{128} + \frac{82\,825\ \epsilon^{10}}{3072} + \frac{19\,675\ \epsilon^{12}}{96}\right]$$

The Quadratic

Matrices from SSAlexander4Tangles.nb

In[∘]:=



$$\left(\begin{array}{l} Q_{i\_,j\_,k\_,l\_} := \frac{(-1+t)\ (1+t)\ p\_\ x\_}{t}\ /.\ \{p\_ \to p_i - p_k,\ x\_ \to x_i - x_k\}; \\ \overline{Q}_{i\_,j\_,k\_,l\_} := -\frac{(-1+t)\ (1+t)\ p\_\ x\_}{t}\ /.\ \{p\_ \to p_i - p_k,\ x\_ \to x_i - x_k\}; \end{array}\right);$$

In[∘]:= $\mathbf{CF}[Q_{i,j,k,l}]$

Out[∘]=

$$\frac{(-1+t)\ (1+t)\ p_i\ x_i}{t} - \frac{(-1+t)\ (1+t)\ p_k\ x_i}{t} - \frac{(-1+t)\ (1+t)\ p_i\ x_k}{t} + \frac{(-1+t)\ (1+t)\ p_k\ x_k}{t}$$

In[∘]:= $\mathbf{Collect}[Q_{i,j,k,l},\ p\_,\ \mathbf{Simplify}]$

Out[∘]=

$$\frac{(-1+t)\ (1+t)\ p_i\ (x_i - x_k)}{t} - \frac{(-1+t)\ (1+t)\ p_k\ (x_i - x_k)}{t}$$

In[∘]:= **Collect[$Q_{i,j,k,l}$, x_, Simplify]**

Out[∘]=

$$\frac{(-1+t)\,(1+t)\,(p_i-p_k)\,x_i}{t}-\frac{(-1+t)\,(1+t)\,(p_i-p_k)\,x_k}{t}$$

In[∘]:= **CF[$Q_{i,j,k,l}$ /. {p_i → (p_+ + p_-) / 2, p_k → (p_+ - p_-) / 2, x_i → (x_+ + x_-) / 2, x_k → (x_+ - x_-) / 2}]**

Out[∘]=

$$\frac{(-1+t)\,(1+t)\,p_-\,x_-}{t}$$

In[∘]:= **CF[$\overline{Q}_{i,j,k,l}$ /. {p_i → (p_+ + p_-) / 2, p_k → (p_+ - p_-) / 2, x_i → (x_+ + x_-) / 2, x_k → (x_+ - x_-) / 2}]**

Out[∘]=

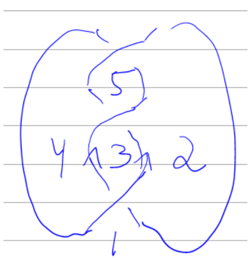
$$-\frac{(-1+t)\,(1+t)\,p_-\,x_-}{t}$$

In[∘]:= **CF[$Q_{i,j,k,l}$ - $\overline{Q}_{i,j,k,l}$ /. {p_i → (p_+ + p_-) / 2, p_k → (p_+ - p_-) / 2, x_i → (x_+ + x_-) / 2, x_k → (x_+ - x_-) / 2}]**

Out[∘]=

$$\frac{2\,(-1+t)\,(1+t)\,p_-\,x_-}{t}$$

The Trefoil



In[∘]:= **L = CF[$Q_{1,2,3,4}$ + $Q_{3,2,5,4}$ + $Q_{5,2,1,4}$]**

Out[∘]=

$$\frac{2\,(-1+t)\,(1+t)\,p_1\,x_1}{t}-\frac{(-1+t)\,(1+t)\,p_3\,x_1}{t}-\frac{(-1+t)\,(1+t)\,p_5\,x_1}{t}-\frac{(-1+t)\,(1+t)\,p_1\,x_3}{t}+\frac{2\,(-1+t)\,(1+t)\,p_3\,x_3}{t}-\frac{(-1+t)\,(1+t)\,p_5\,x_3}{t}-\frac{(-1+t)\,(1+t)\,p_1\,x_5}{t}-\frac{(-1+t)\,(1+t)\,p_3\,x_5}{t}+\frac{2\,(-1+t)\,(1+t)\,p_5\,x_5}{t}$$

In[∘]:= **{CF[(∂_{p_1}L) + (∂_{p_3}L) + (∂_{p_5}L)], CF[(∂_{x_1}L) + (∂_{x_3}L) + (∂_{x_5}L)]}**

Out[∘]=

{0, 0}

In[∘]:= **$\int$𝔼[L] ⅆ{p_1, p_3, x_1, x_3}**

Out[∘]=

$$\frac{t^2\,\mathbb{E}[0]}{3\left(-1+t^2\right)^2}$$

In[∘]:= $\int \mathbb{E}[L]\, d\{p_1, p_5, x_1, x_5\}$

Out[∘]=

$$\frac{t^2\,\mathbb{E}[0]}{3\left(-1+t^2\right)^2}$$

In[∘]:= $CF[Q_{1,2,3,4} + Q_{3,2,5,4} + Q_{5,2,1,4}\ /.\ \{p_1 \to -p_3 - p_5,\ x_1 \to -x_3 - x_5\}]$

Out[∘]=

$$\frac{6\,(-1+t)\,(1+t)\,p_3\,x_3}{t} + \frac{3\,(-1+t)\,(1+t)\,p_5\,x_3}{t} + \frac{3\,(-1+t)\,(1+t)\,p_3\,x_5}{t} + \frac{6\,(-1+t)\,(1+t)\,p_5\,x_5}{t}$$

Re-sectioning

In[∘]:= $\{CF[Q_{i,j,k,l}\ /.\ (p\mid x)_i \to 0],\ CF[Q_{i,j,k,l}\ /.\ (p\mid x)_k \to 0]\}$

Out[∘]=

$$\left\{\frac{(-1+t)\,(1+t)\,p_k\,x_k}{t},\ \frac{(-1+t)\,(1+t)\,p_i\,x_i}{t}\right\}$$

In[∘]:= $\{lhs, rhs\} = \left\{\int \mathbb{E}[Q_{i,j,k,l}\ /.\ (p\mid x)_i \to 0]\, d\{p_k, x_k\},\ \int \mathbb{E}[Q_{i,j,k,l}\ /.\ (p\mid x)_k \to 0]\, d\{p_i, x_i\}\right\}$

Out[∘]=

$$\left\{-\frac{i\,t\,\mathbb{E}[0]}{-1+t^2},\ -\frac{i\,t\,\mathbb{E}[0]}{-1+t^2}\right\}$$

In[∘]:= $lhs == rhs$

Out[∘]=

True

In[∘]:= $f = \xi_j\, x_j + \pi_j\, p_j + \xi_l\, x_l + \pi_l\, p_l;$

$\{lhs, rhs\} =$

$\left\{\int \mathbb{E}[(Q_{i,j,k,l} + f)\ /.\ (p\mid x)_i \to 0]\, d\{p_k, x_k\},\ \int \mathbb{E}[(Q_{i,j,k,l} + f)\ /.\ (p\mid x)_k \to 0]\, d\{p_i, x_i\}\right\}$

$lhs == rhs$

Out[∘]=

$$\left\{-\frac{i\,t\,\mathbb{E}[p_j\,\pi_j + p_l\,\pi_l + x_j\,\xi_j + x_l\,\xi_l]}{-1+t^2},\ -\frac{i\,t\,\mathbb{E}[p_j\,\pi_j + p_l\,\pi_l + x_j\,\xi_j + x_l\,\xi_l]}{-1+t^2}\right\}$$

Out[∘]=

True

In[∘]:= $f = \xi_j\, x_j + \pi_j\, p_j + \xi_l\, x_l + \pi_l\, p_l + \pi\,(p_i - p_k) + \xi\,(x_i - x_k);$

$\{lhs, rhs\} =$

$\left\{\int \mathbb{E}[(Q_{i,j,k,l} + f)\ /.\ (p\mid x)_i \to 0]\, d\{p_k, x_k\},\ \int \mathbb{E}[(Q_{i,j,k,l} + f)\ /.\ (p\mid x)_k \to 0]\, d\{p_i, x_i\}\right\}$

$lhs == rhs$

Out[∘]=

$$\left\{-\frac{i\,t\,\mathbb{E}\left[-\frac{\pi\,t\,\xi}{(-1+t)\,(1+t)} + p_j\,\pi_j + p_l\,\pi_l + x_j\,\xi_j + x_l\,\xi_l\right]}{-1+t^2},\ -\frac{i\,t\,\mathbb{E}\left[-\frac{\pi\,t\,\xi}{(-1+t)\,(1+t)} + p_j\,\pi_j + p_l\,\pi_l + x_j\,\xi_j + x_l\,\xi_l\right]}{-1+t^2}\right\}$$

Out[∘]=

True

In[∘]:=
```
f = ξj xj + πj pj + ξl xl + πl pl + π (pi - pk) + ξ (xi - xk);
{lhs, rhs} =
  {∫𝔼[(Q̄i,j,k,l + f) /. (p | x)i → 0] ⅆ{pk, xk}, ∫𝔼[(Q̄i,j,k,l + f) /. (p | x)k → 0] ⅆ{pi, xi}}
lhs == rhs
```

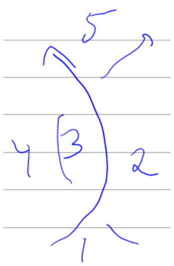
Out[∘]=

$$\left\{-\frac{\mathbb{i}\, t\, \mathbb{E}\left[\frac{\pi\, t\, \xi}{(-1+t)\,(1+t)} + p_j\, \pi_j + p_l\, \pi_l + x_j\, \xi_j + x_l\, \xi_l\right]}{-1+t^2},\ -\frac{\mathbb{i}\, t\, \mathbb{E}\left[\frac{\pi\, t\, \xi}{(-1+t)\,(1+t)} + p_j\, \pi_j + p_l\, \pi_l + x_j\, \xi_j + x_l\, \xi_l\right]}{-1+t^2}\right\}$$

Out[∘]=

True

R2b



In[∘]:=
```
Q1,2,3,4 + Q̄3,2,5,4 // Expand
```

Out[∘]=

$$-\frac{p_1\, x_1}{t} + t\, p_1\, x_1 + \frac{p_3\, x_1}{t} - t\, p_3\, x_1 + \frac{p_1\, x_3}{t} - t\, p_1\, x_3 - \frac{p_5\, x_3}{t} + t\, p_5\, x_3 - \frac{p_3\, x_5}{t} + t\, p_3\, x_5 + \frac{p_5\, x_5}{t} - t\, p_5\, x_5$$

In[∘]:=
```
Expand@Collect[Q1,2,3,4 + Q̄3,2,5,4, (p | x)3, F] /. F → Factor
```

Out[∘]=

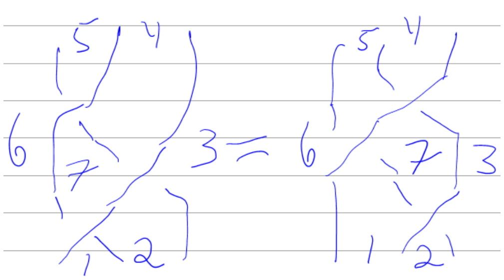
$$-\frac{(-1+t)\,(1+t)\,(p_1 - p_5)\,x_3}{t} - \frac{(-1+t)\,(1+t)\,p_3\,(x_1 - x_5)}{t} + \frac{(-1+t)\,(1+t)\,(p_1\, x_1 - p_5\, x_5)}{t}$$

In[∘]:=
```
Expand[Q1,2,3,4 + Q̄3,2,5,4 /. {p5 → p1, x5 → x1}]
```

Out[∘]=

0

R3



In[∘]:=
```
lhs = Simplify[Q1,2,7,6 + Q2,3,4,7 + Q7,4,5,6];
rhs = Simplify[Q2,3,7,1 + Q1,7,5,6 + Q7,3,4,5]
```

Out[∘]=

$$\frac{(-1+t)(1+t)((p_1-p_5)(x_1-x_5)+(p_2-p_7)(x_2-x_7)+(p_4-p_7)(x_4-x_7))}{t}$$

In[∘]:=
```
Simplify[lhs == rhs]
```

Out[∘]=

$$\frac{1}{t}\left(-1+t^2\right)(p_4 x_2+p_2 x_4-p_1 x_5+p_7(x_1-x_2-x_4+x_5)+p_1 x_7-p_2 x_7-p_4 x_7+p_5(-x_1+x_7)) == 0$$

In[∘]:=
```
Expand@Collect[lhs, (p | x)7, F] /. F → Simplify
```

Out[∘]=

$$-\frac{(-1+t)(1+t)p_7(x_1+x_5)}{t}+\frac{(-1+t)(1+t)(p_1 x_1+(p_2-p_4)(x_2-x_4)+p_5 x_5)}{t}-\frac{(-1+t)(1+t)(p_1+p_5)x_7}{t}+\left(-\frac{2}{t}+2t\right)p_7 x_7$$

In[∘]:=
```
∫𝔼[lhs] ⅆ{p7, x7}
```

Out[∘]=

$$-\frac{1}{2\left(-1+t^2\right)}\, i\, t\, \mathbb{E}\left[\frac{(-1+t)(1+t)p_1 x_1}{2t}-\frac{(-1+t)(1+t)p_5 x_1}{2t}+\frac{(-1+t)(1+t)p_2 x_2}{t}-\frac{(-1+t)(1+t)p_4 x_2}{t}-\frac{(-1+t)(1+t)p_2 x_4}{t}+\frac{(-1+t)(1+t)p_4 x_4}{t}-\frac{(-1+t)(1+t)p_1 x_5}{2t}+\frac{(-1+t)(1+t)p_5 x_5}{2t}\right]$$

In[∘]:=
```
Expand@Collect[rhs, (p | x)7, F] /. F → Simplify
```

Out[∘]=

$$-\frac{(-1+t)(1+t)p_7(x_2+x_4)}{t}+\frac{(-1+t)(1+t)(p_2 x_2+p_4 x_4+(p_1-p_5)(x_1-x_5))}{t}-\frac{(-1+t)(1+t)(p_2+p_4)x_7}{t}+\left(-\frac{2}{t}+2t\right)p_7 x_7$$

In[∘]:=
```
Simplify[((∫𝔼[rhs] ⅆ{p7, x7}) == (∫𝔼[lhs] ⅆ{p7, x7}))]
```

Out[∘]=

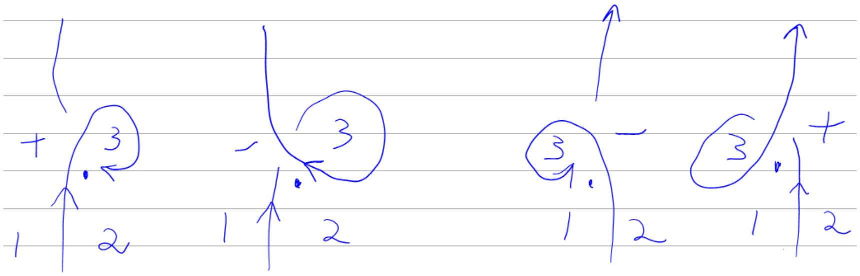
$$\frac{1}{-1+t^2}\,t\left(\mathbb{E}\left[\frac{\left(-1+t^2\right)((p_2-p_4)(x_2-x_4)+2p_1(x_1-x_5)-2p_5(x_1-x_5))}{2t}\right]-\mathbb{E}\left[\frac{\left(-1+t^2\right)(2(p_2-p_4)(x_2-x_4)+p_1(x_1-x_5)+p_5(-x_1+x_5))}{2t}\right]\right) == 0$$

In[∘]:=
```
((∫𝔼[rhs] ⅆ{p7, x7}) ≡ (∫𝔼[lhs] ⅆ{p7, x7}))
```

Out[∘]=

$$\frac{\left(-1+t^2\right)((p_2-p_4)(x_2-x_4)+p_5(x_1-x_5)+p_1(-x_1+x_5))}{t} == 0$$

R1s

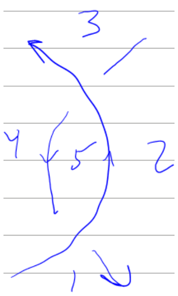


In[∘]:= **R1s = Simplify@$\{Q_{2,3,2,1}, \overline{Q}_{2,3,2,1}, \overline{Q}_{1,2,1,3}, Q_{1,2,1,3}\}$**

Out[∘]=

{0, 0, 0, 0}

R2C$_+$



In[∘]:= **lhs = Collect[$\overline{Q}_{4,1,2,5} + Q_{2,3,4,5}$, t, Simplify]**

Out[∘]=

0

R2C$_-$

In[∘]:= **lhs = Collect[$\overline{Q}_{2,5,4,1} + Q_{4,5,2,3}$, t, Simplify]**

Out[∘]=

0