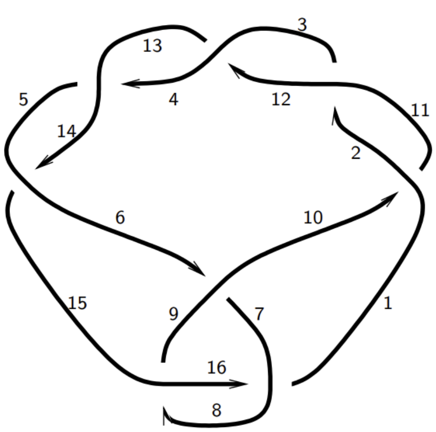
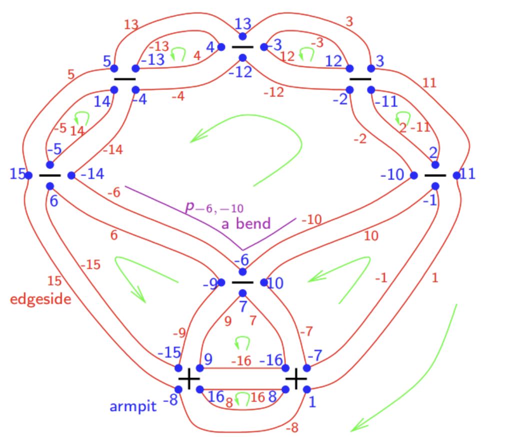
Pensieve header: The Alexander polynomial using a Seifert formula derived from an $F\times F$ matrix. Continues AlexanderAsFxF.nb of pensieve://Projects/Signatures.

In[∘]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\SeifertBox"];
Once[<< KnotTheory`]
```

In[∘]:=
```
PD[epd_EPD] := PD @@ epd /. {X_{i_,j_} :> X[j, i + 1, j + 1, i], X̄_{i_,j_} :> X[j, i, j + 1, i + 1]}
```

In[∘]:=
```
Rot[{X_, φ_}] := {X, φ};
Rot[pd_PD] := Module[{n, xs, x, rots, Xp, Xm, front = {1}, k},
   n = Length@pd; rots = Table[0, {2 n + 1}];
   xs = Cases[pd, x_X :> { Xp[x[[4]], x[[1]]]   PositiveQ@x
                          { Xm[x[[2]], x[[1]]]   True        ];
   For[k = 1, k ≤ 2 n, ++k,
    If[FreeQ[front, -k],
     front = Flatten@Replace[front, k → (xs /. {
              Xp[k, l_] | Xm[l_, k] :> {l + 1, k + 1, -l},
              Xp[l_, k] | Xm[k, l_] :> (++rots[[l]]; {-l, k + 1, l + 1}),
              _Xp | _Xm :> {}
             }), {1}],
     Cases[front, k | -k] /. {k, -k} :> --rots[[k]];
    ]
   ];
   {xs /. {Xp[i_, j_] :> {+1, i, j}, Xm[i_, j_] :> {-1, i, j}}, rots} ];
Rot[K_] := Rot[PD[K]];
```





$PD[X[10,1,11,2], X[2,11,3,12], \ldots]$
(diagram from Talks/CMS-2112)

$\{X_{-}[-1,11,2,-10], X_{-}[-11,3,12,-2], \ldots\}$

In[◦]:=

```
A1[K_] := Module[
  {ω$, XingsByArmpits, bends, faces, p, A$, is, Cs, φ, SeifertCycles, sc$, drops, keeps},
  XingsByArmpits = List @@ PD[K] /.
    x : X[i_, j_, k_, l_] :> If[PositiveQ[x], X+[-i, j, k, -l], X-[-j, k, l, -i]];
  bends = Times @@ XingsByArmpits /. _[X][a_, b_, c_, d_] :> p_{a,-d} p_{b,-a} p_{c,-b} p_{d,-c};
  faces = bends //. p_{x__,y_} p_{y_,z__} :> p_{x,y,z};
  A = Table[0, Length@faces, Length@faces];
  Do[is = Position[faces, #][[1, 1]] & /@ List @@ x;
   A[[is, is]] += If[Head[x] === X+,
```

$$\begin{pmatrix} \frac{(-1+\omega)\ (1+\omega)}{2\,\omega} & \frac{\omega}{2} & -\omega & \frac{1}{2\,\omega} \\ -\frac{1}{2\,\omega} & 0 & \frac{1}{2\,\omega} & 0 \\ \frac{1}{\omega} & -\frac{\omega}{2} & \frac{(-1+\omega)\ (1+\omega)}{2\,\omega} & -\frac{1}{2\,\omega} \\ -\frac{\omega}{2} & 0 & \frac{\omega}{2} & 0 \end{pmatrix}, \begin{pmatrix} -\frac{(-1+\omega)\ (1+\omega)}{2\,\omega} & \frac{\omega}{2} & -\frac{1}{\omega} & \frac{1}{2\,\omega} \\ -\frac{1}{2\,\omega} & 0 & \frac{1}{2\,\omega} & 0 \\ \omega & -\frac{\omega}{2} & -\frac{(-1+\omega)\ (1+\omega)}{2\,\omega} & -\frac{1}{2\,\omega} \\ -\frac{\omega}{2} & 0 & \frac{\omega}{2} & 0 \end{pmatrix}\Bigg],$$

```
   {x, XingsByArmpits}];
  {Cs, φ} = Rot[PD@K]; φ[[-1]] = -1;
  SeifertCycles = Times @@ (Cs /. {1 | -1, i_, j_} :> sc_{i,j+1} sc_{j,i+1}) //.
    sc_{i__,j_} sc_{j_,k__} :> sc_{i,j,k} /. sc_{i_,j___,i_} :> sc_{i,j};
  drops = {-1} ⋃ Cases[SeifertCycles, sc_{is__} :> -Total@φ[[{is}]] Min[{is}]];
  keeps = Complement[Range@Length@A, FirstPosition[#][faces][[1]] & /@ drops];
  Factor@Det[A[[keeps, keeps]] /. ω → T^(1/2)]
 ]
Table[Simplify[A1[K] / Alexander[K][T]], {K, AllKnots[{3, 8}]}]
```

Out[◦]=

{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}

In[◦]:=

```
A1[Knot[3, 1]]
```

Out[◦]=

$$\frac{1 - T + T^2}{T}$$

In[◦]:=

```
A // MatrixForm
```

Out[◦]//MatrixForm=

$$\begin{pmatrix} -\frac{(-1+\omega)\ (1+\omega)}{\omega} & -\frac{1}{\omega} & \omega & 0 & 0 \\ \omega & -\frac{(-1+\omega)\ (1+\omega)}{\omega} & -\frac{1}{\omega} & 0 & 0 \\ -\frac{1}{\omega} & \omega & -\frac{(-1+\omega)\ (1+\omega)}{\omega} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In[∘]:=

```
A2[K_] := Module[
  {ω$, XingsByArmpits, bends, Faces$, p, A$, is, Cs, φ, SeifertCycles, sc$, drops, keeps$},
  XingsByArmpits = List @@ PD[K] /.
    x : X[i_, j_, k_, l_] ⧴ If[PositiveQ[x], X_+[-i, j, k, -l], X_-[-j, k, l, -i]];
  bends = Times @@ XingsByArmpits /. _[X][a_, b_, c_, d_] ⧴ p_{a,-d} p_{b,-a} p_{c,-b} p_{d,-c};
  Faces = bends //. p_{x__,y_} p_{y_,z__} ⧴ p_{x,y,z};
  {Cs, φ} = Rot[PD@K]; φ⟦-1⟧ = -1;
  SeifertCycles = Times @@ (Cs /. {1 | -1, i_, j_} ⧴ sc_{i,j+1} sc_{j,i+1}) //.
     sc_{i__,j_} sc_{j_,k__} ⧴ sc_{i,j,k} /. sc_{i_,j___,i_} ⧴ sc_{i,j};
  Walls = Cases[SeifertCycles, sc_{is__} ⧴ -Total@φ⟦{is}⟧ {is}];
  A = Table[0, Length@Faces + Length@SeifertCycles, Length@Faces + Length@SeifertCycles];
  Do[is = Position[Faces, #]⟦1, 1⟧ & /@ List @@ x;
   A⟦is, is⟧ += If[Head[x] === X_+,
```

$$\left(\begin{array}{cccc} \frac{(-1+\omega)\ (1+\omega)}{2\,\omega} & \frac{\omega}{2} & -\omega & \frac{1}{2\,\omega} \\ -\frac{1}{2\,\omega} & 0 & \frac{1}{2\,\omega} & 0 \\ \frac{1}{\omega} & -\frac{\omega}{2} & \frac{(-1+\omega)\ (1+\omega)}{2\,\omega} & -\frac{1}{2\,\omega} \\ -\frac{\omega}{2} & 0 & \frac{\omega}{2} & 0 \end{array}\right), \left(\begin{array}{cccc} -\frac{(-1+\omega)\ (1+\omega)}{2\,\omega} & \frac{\omega}{2} & -\frac{1}{\omega} & \frac{1}{2\,\omega} \\ -\frac{1}{2\,\omega} & 0 & \frac{1}{2\,\omega} & 0 \\ \omega & -\frac{\omega}{2} & -\frac{(-1+\omega)\ (1+\omega)}{2\,\omega} & -\frac{1}{2\,\omega} \\ -\frac{\omega}{2} & 0 & \frac{\omega}{2} & 0 \end{array}\right)],$$

```
   {x, XingsByArmpits}];
  Do[
   j = FirstPosition[f /. 2 Length[Cs] + 1 → 1][Faces]⟦1⟧;
   A⟦Length[Faces] + w, j⟧ = 1;
   A⟦j, Length[Faces] + w⟧ = -1,
   {w, Length@Walls}, {f, Walls⟦w⟧}
  ];
  keeps = DeleteCases[Range@Length@A, FirstPosition[-1][Faces]⟦1⟧];
  Factor@Det[A⟦keeps, keeps⟧ /. ω → T^(1/2)]
 ]
Table[Simplify[A2[K] / Alexander[K][T]], {K, AllKnots[{3, 8}]}]
```

Out[∘]=

{9, 16, 25, 4, 16, 64, 9, 49, 4, 4, 4, 16, 16, 81, 16, 144,
16, 64, 144, 64, 9, 9, 256, 9, 64, 36, 9, 36, 9, 9, 256, 16, 9, 9, 9}

In[∘]:=

```
A2[Knot[3, 1]]
```

Out[∘]=

$$\frac{9\left(1-T+T^2\right)}{T}$$

In[∘]:=

```
Faces
```

Out[∘]=

$p\$45201_{-6,3,-6}\ p\$45201_{-4,1,-4}\ p\$45201_{-2,5,-2}\ p\$45201_{-1,-5,-3,-1}\ p\$45201_{6,2,4,6}$

In[∘]:= **Walls**

Out[∘]=

{{6, 4, 2}, {1, 5, 3, 7}}

In[∘]:= **A // MatrixForm**

Out[∘]//MatrixForm=

$$\begin{pmatrix} -\frac{(-1+\omega)\ (1+\omega)}{\omega} & -\frac{1}{\omega} & \omega & 0 & 0 & 0 & -1 \\ \omega & -\frac{(-1+\omega)\ (1+\omega)}{\omega} & -\frac{1}{\omega} & 0 & 0 & 0 & -1 \\ -\frac{1}{\omega} & \omega & -\frac{(-1+\omega)\ (1+\omega)}{\omega} & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In[∘]:= **keeps**

Out[∘]=

{1, 2, 3, 5, 6, 7}