Pensieve header: Full testing of all functions.

Startup

In[•]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\SL2Portfolio2"];
<< KnotTheory`;
<< "../Profile/Profile.m";
<< "Engine-With-w.m";
<< "Objects-With-w.m";
<< "KT.m";
BeginProfile[];
PopupWindow[Button["Show Profile Monitor"],
 Dynamic[PrintProfile[], UpdateInterval → 3, TrackedSymbols → {}]]
```

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.

This is Profile.m of http://www.drorbn.net/AcademicPensieve/Projects/Profile/.

This version: April 2020. Original version: July 1994.
```

» Warning: On Sep 4 2019 I swapped the operations
$\in$ and η. Some incompatibilities may arise in older notebooks.

Out[•]=

Show Profile Monitor

In[•]:=
```
$k = 2; (*ℏ=γ=1;*)
```

Utilities

In[•]:=
```
HL[ℰ_] := Style[ℰ, Background → If[TrueQ@ℰ, , ]];
```

Testing

In[•]:=
```
Block[{$k = 1},
  {am → am_{i,j→k}, bm → bm_{i,j→k}, dm → dm_{i,j→k}, R → R_{i,j}, R̄ → R̄_{i,j}, P → P_{i,j}, aS → aS_i,
   āS → āS_i, bS → bS_i, b̄S → b̄S_i, dS → dS_i, aΔ → aΔ_{i→j,k}, bΔ → bΔ_{i→j,k}, dΔ → dΔ_{i→j,k},
   C → C_i, C̄ → C̄_i, Kink → Kink_i, K̄ink → K̄ink_i, b2t → b2t_i, t2b → t2b_i, km → km_{i,j→k}
  }] // Column
```

Out[•]=

$$\mathtt{am}\to\mathbb{E}_{\{i,j\}\to\{k\}}\left[a_k\,\alpha_i+a_k\,\alpha_j,\ \frac{x_k\,\xi_i}{\mathcal{A}_j}+x_k\,\xi_j,\ 1\right]$$

$$\mathtt{bm}\to\mathbb{E}_{\{i,j\}\to\{k\}}\left[b_k\,\beta_i+b_k\,\beta_j,\ y_k\,\eta_i+y_k\,\eta_j,\ 1-y_k\,\beta_i\,\eta_j\,\epsilon+0[\epsilon]^2\right]$$

$$\mathtt{dm}\to\mathbb{E}_{\{i,j\}\to\{k\}}\left[a_k\,\alpha_i+a_k\,\alpha_j+b_k\,\beta_i+b_k\,\beta_j,\ y_k\,\eta_i+\frac{y_k\,\eta_j}{\mathcal{A}_i}+\frac{x_k\,\xi_i}{\mathcal{A}_j}+\frac{(1-B_k)\,\eta_j\,\xi_i}{\hbar}+x_k\,\xi_j,\right.$$
$$\left.1+\left(-\frac{y_k\,\beta_i\,\eta_j}{\mathcal{A}_i}-\frac{x_k\,\beta_j\,\xi_i}{\mathcal{A}_j}+a_k\,B_k\,\eta_j\,\xi_i+\frac{\gamma\,\hbar\,x_k\,y_k\,\eta_j\,\xi_i}{\mathcal{A}_i\,\mathcal{A}_j}+\frac{(\gamma-3\,\gamma\,B_k)\,y_k\,\eta_j^2\,\xi_i}{2\,\mathcal{A}_i}+\frac{(\gamma-3\,\gamma\,B_k)\,x_k\,\eta_j\,\xi_i^2}{2\,\mathcal{A}_j}+\frac{\left(\gamma-4\,\gamma\,B_k+3\,\gamma\,B_k^2\right)\eta_j^2\,\xi_i^2}{4\,\hbar}\right)\epsilon+\right.$$
$$\left.0[\epsilon]^2\right]$$

$$\mathtt{R}\to\mathbb{E}_{\{\}\to\{i,j\}}\left[\hbar\,a_j\,b_i,\ \hbar\,x_j\,y_i,\ 1-\frac{1}{4}\left(\gamma\,\hbar^3\,x_j^2\,y_i^2\right)\epsilon+0[\epsilon]^2\right]$$

$$\overline{\mathtt{R}}\to\mathbb{E}_{\{\}\to\{i,j\}}\left[-\hbar\,a_j\,b_i,\ -\frac{\hbar\,x_j\,y_i}{B_i},\ 1+\left(-\frac{\hbar^2\,a_j\,x_j\,y_i}{B_i}-\frac{3\,\gamma\,\hbar^3\,x_j^2\,y_i^2}{4\,B_i^2}\right)\epsilon+0[\epsilon]^2\right]$$

$$\mathtt{P}\to\mathbb{E}_{\{i,j\}\to\{\}}\left[\frac{\alpha_j\,\beta_i}{\hbar},\ \frac{\eta_i\,\xi_j}{\hbar},\ 1+\frac{\gamma\,\eta_i^2\,\xi_j^2\,\epsilon}{4\,\hbar}+0[\epsilon]^2\right]$$

$$\mathtt{aS}\to\mathbb{E}_{\{i\}\to\{i\}}\left[-a_i\,\alpha_i,\ -x_i\,\mathcal{A}_i\,\xi_i,\ 1+\left(-\hbar\,a_i\,x_i\,\mathcal{A}_i\,\xi_i-\frac{1}{2}\,\gamma\,\hbar\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2\right)\epsilon+0[\epsilon]^2\right]$$

$$\overline{\mathtt{aS}}\to\mathbb{E}_{\{i\}\to\{i\}}\left[-a_i\,\alpha_i,\ -x_i\,\mathcal{A}_i\,\xi_i,\ 1+\left(\gamma\,\hbar\,x_i\,\mathcal{A}_i\,\xi_i-\hbar\,a_i\,x_i\,\mathcal{A}_i\,\xi_i-\frac{1}{2}\,\gamma\,\hbar\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2\right)\epsilon+0[\epsilon]^2\right]$$

$$\mathtt{bS}\to\mathbb{E}_{\{i\}\to\{i\}}\left[-b_i\,\beta_i,\ -\frac{y_i\,\eta_i}{B_i},\ 1+\left(-\frac{y_i\,\beta_i\,\eta_i}{B_i}-\frac{\gamma\,\hbar\,y_i^2\,\eta_i^2}{2\,B_i^2}\right)\epsilon+0[\epsilon]^2\right]$$

$$\overline{\mathtt{bS}}\to\mathbb{E}_{\{i\}\to\{i\}}\left[-b_i\,\beta_i,\ -\frac{y_i\,\eta_i}{B_i},\ 1+\left(\frac{\gamma\,\hbar\,y_i\,\eta_i}{B_i}-\frac{y_i\,\beta_i\,\eta_i}{B_i}-\frac{\gamma\,\hbar\,y_i^2\,\eta_i^2}{2\,B_i^2}\right)\epsilon+0[\epsilon]^2\right]$$

$$\mathtt{dS}\to\mathbb{E}_{\{i\}\to\{i\}}\left[-a_i\,\alpha_i-b_i\,\beta_i,\ -\frac{y_i\,\mathcal{A}_i\,\eta_i}{B_i}-x_i\,\mathcal{A}_i\,\xi_i+\frac{(\mathcal{A}_i-B_i\,\mathcal{A}_i)\,\eta_i\,\xi_i}{\hbar\,B_i},\right.$$
$$1+\left(\frac{\gamma\,\hbar\,y_i\,\mathcal{A}_i\,\eta_i}{B_i}-\frac{y_i\,\mathcal{A}_i\,\beta_i\,\eta_i}{B_i}-\frac{\gamma\,\hbar\,y_i^2\,\mathcal{A}_i^2\,\eta_i^2}{2\,B_i^2}-\hbar\,a_i\,x_i\,\mathcal{A}_i\,\xi_i-x_i\,\mathcal{A}_i\,\beta_i\,\xi_i+\frac{a_i\,\mathcal{A}_i\,\eta_i\,\xi_i}{B_i}-\right.$$
$$\frac{\gamma\,\hbar\,x_i\,y_i\,\mathcal{A}_i^2\,\eta_i\,\xi_i}{B_i}+\frac{(-\gamma\,\mathcal{A}_i+\gamma\,B_i\,\mathcal{A}_i)\,\eta_i\,\xi_i}{B_i}+\frac{(\mathcal{A}_i-B_i\,\mathcal{A}_i)\,\beta_i\,\eta_i\,\xi_i}{\hbar\,B_i}+\frac{y_i\left(3\,\gamma\,\mathcal{A}_i^2-\gamma\,B_i\,\mathcal{A}_i^2\right)\eta_i^2\,\xi_i}{2\,B_i^2}-$$
$$\left.\left.\frac{1}{2}\,\gamma\,\hbar\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2+\frac{x_i\left(3\,\gamma\,\mathcal{A}_i^2-\gamma\,B_i\,\mathcal{A}_i^2\right)\eta_i\,\xi_i^2}{2\,B_i}+\frac{\left(-3\,\gamma\,\mathcal{A}_i^2+4\,\gamma\,B_i\,\mathcal{A}_i^2-\gamma\,B_i^2\,\mathcal{A}_i^2\right)\eta_i^2\,\xi_i^2}{4\,\hbar\,B_i^2}\right)\epsilon+0[\epsilon]^2\right]$$

$$\mathtt{a\Delta}\to\mathbb{E}_{\{i\}\to\{j,k\}}\left[a_j\,\alpha_i+a_k\,\alpha_i,\ x_j\,\xi_i+x_k\,\xi_i,\ 1+\left(-\hbar\,a_j\,x_k\,\xi_i+\frac{1}{2}\,\gamma\,\hbar\,x_j\,x_k\,\xi_i^2\right)\epsilon+0[\epsilon]^2\right]$$

$$\mathtt{b\Delta}\to\mathbb{E}_{\{i\}\to\{j,k\}}\left[b_j\,\beta_i+b_k\,\beta_i,\ B_k\,y_j\,\eta_i+y_k\,\eta_i,\ 1+\frac{1}{2}\,\gamma\,\hbar\,B_k\,y_j\,y_k\,\eta_i^2\,\epsilon+0[\epsilon]^2\right]$$

$$\mathtt{d\Delta}\to\mathbb{E}_{\{i\}\to\{j,k\}}\left[a_j\,\alpha_i+a_k\,\alpha_i+b_j\,\beta_i+b_k\,\beta_i,\ y_j\,\eta_i+B_j\,y_k\,\eta_i+x_j\,\xi_i+x_k\,\xi_i,\right.$$
$$\left.1+\left(\frac{1}{2}\,\gamma\,\hbar\,B_j\,y_j\,y_k\,\eta_i^2-\hbar\,a_j\,x_k\,\xi_i+\frac{1}{2}\,\gamma\,\hbar\,x_j\,x_k\,\xi_i^2\right)\epsilon+0[\epsilon]^2\right]$$

$$\mathtt{C}\to\mathbb{E}_{\{\}\to\{i\}}\left[0,\ 0,\ \sqrt{B_i}-\frac{1}{2}\left(\hbar\,a_i\,\sqrt{B_i}\right)\epsilon+0[\epsilon]^2\right]$$

$$\overline{\mathtt{C}}\to\mathbb{E}_{\{\}\to\{i\}}\left[0,\ 0,\ \frac{1}{\sqrt{B_i}}+\frac{\hbar\,a_i\,\epsilon}{2\,\sqrt{B_i}}+0[\epsilon]^2\right]$$

$$\mathtt{Kink}\to\mathbb{E}_{\{\}\to\{i\}}\left[\hbar\,a_i\,b_i,\ \hbar\,x_i\,y_i,\ \frac{1}{\sqrt{B_i}}+\left(\frac{\hbar\,a_i}{2\,\sqrt{B_i}}-\frac{\gamma\,\hbar^3\,x_i^2\,y_i^2}{4\,\sqrt{B_i}}\right)\epsilon+0[\epsilon]^2\right]$$

$$\overline{\mathtt{Kink}}\to\mathbb{E}_{\{\}\to\{i\}}\left[-\hbar\,a_i\,b_i,\ -\frac{\hbar\,x_i\,y_i}{B_i},\ \sqrt{B_i}+\left(-\frac{1}{2}\,\hbar\,a_i\,\sqrt{B_i}-\frac{\hbar^2\,a_i\,x_i\,y_i}{\sqrt{B_i}}-\frac{3\,\gamma\,\hbar^3\,x_i^2\,y_i^2}{4\,B_i^{3/2}}\right)\epsilon+0[\epsilon]^2\right]$$

$$\mathtt{b2t}\to\mathbb{E}_{\{i\}\to\{i\}}\left[a_i\,\alpha_i+\frac{t_i\,\beta_i}{\gamma},\ y_i\,\eta_i+x_i\,\xi_i,\ 1+\frac{a_i\,\beta_i\,\epsilon}{\gamma}+0[\epsilon]^2\right]$$

$$\mathtt{t2b}\to\mathbb{E}_{\{i\}\to\{i\}}\left[a_i\,\alpha_i+\gamma\,b_i\,\tau_i,\ y_i\,\eta_i+x_i\,\xi_i,\ 1-a_i\,\tau_i\,\epsilon+0[\epsilon]^2\right]$$

$$\mathtt{km}\to\mathbb{E}_{\{i,j\}\to\{k\}}\left[a_k\,\alpha_i+a_k\,\alpha_j,\ y_k\,\eta_i+\frac{y_k\,\eta_j}{\mathcal{A}_i}+\frac{x_k\,\xi_i}{\mathcal{A}_j}+\frac{(1-T)\,\eta_j\,\xi_i}{\hbar}+x_k\,\xi_j,\right.$$
$$\left.1+\left(2\,T\,a_k\,\eta_j\,\xi_i+\frac{\gamma\,\hbar\,x_k\,y_k\,\eta_j\,\xi_i}{\mathcal{A}_i\,\mathcal{A}_j}+\frac{(\gamma-3\,T\,\gamma)\,y_k\,\eta_j^2\,\xi_i}{2\,\mathcal{A}_i}+\frac{(\gamma-3\,T\,\gamma)\,x_k\,\eta_j\,\xi_i^2}{2\,\mathcal{A}_j}+\frac{\left(\gamma-4\,T\,\gamma+3\,T^2\,\gamma\right)\eta_j^2\,\xi_i^2}{4\,\hbar}\right)\epsilon+0[\epsilon]^2\right]$$

Check that on the generators this agrees with our conventions in the handout:

```
In[•]:= Timing@
 {{"[a,x]" → ((𝔼{}→{1,2}[0, 0, a2 x1] // am1,2→1)[3] - (𝔼{}→{1,2}[0, 0, a1 x2] // am1,2→1)[3]),
    "[b,y]" → ((𝔼{}→{1,2}[0, 0, y2 b1] // bm1,2→1)[3] - (𝔼{}→{1,2}[0, 0, y1 b2] // bm1,2→1)[3])} /.
   z_1 → z,
  {"Δ[y]" → Last[𝔼{}→{1}[0, 0, y1] ~B1~ bΔ1→1,2],
   "Δ[b]" → Last[𝔼{}→{1}[0, 0, b1] ~B1~ bΔ1→1,2],
   "Δ[a]" → Last[𝔼{}→{1}[0, 0, a1] ~B1~ aΔ1→1,2],
   "Δ[x]" → Last[𝔼{}→{1}[0, 0, x1] ~B1~ aΔ1→1,2]},
  {
    "S(a)" → ((𝔼{}→{1}[0, 0, a1] ~B1~ aS1)[3]),
    "S(x)" → ((𝔼{}→{1}[0, 0, x1] ~B1~ aS1)[3]),
    "S(b)" → ((𝔼{}→{1}[0, 0, b1] ~B1~ bS1)[3]),
    "S(y)" → ((𝔼{}→{1}[0, 0, y1] ~B1~ bS1)[3])
   } /. z_1 → z}
```

Out[•]=

$$\Big\{1.48438,\ \Big\{\{[a,x] \to -x\gamma,\ [b,y] \to -y\epsilon + O[\epsilon]^3\},\ \Big\{\Delta[y] \to (B_2 y_1 + y_2) + O[\epsilon]^3,\ \Delta[b] \to (b_1 + b_2) + O[\epsilon]^3,$$
$$\Delta[a] \to (a_1 + a_2) + O[\epsilon]^3,\ \Delta[x] \to (x_1 + x_2) - \hbar\, a_1 x_2 \epsilon + \frac{1}{2}\hbar^2 a_1^2 x_2 \epsilon^2 + O[\epsilon]^3\Big\},\ \Big\{S(a) \to -a + O[\epsilon]^3,$$
$$S(x) \to -x - a x \hbar \epsilon - \frac{1}{2}\left(a^2 x \hbar^2\right)\epsilon^2 + O[\epsilon]^3,\ S(b) \to -b + O[\epsilon]^3,\ S(y) \to -\frac{y}{B} + O[\epsilon]^3\Big\}\Big\}\Big\}$$

Hopf algebra axioms on both sides separately.

Associativity of am and bm:

```
In[•]:= Timing@Block[{$k = 3},
  HL /@ {(am1,2→1 // am1,3→1) ≡ (am2,3→2 // am1,2→1), (bm1,2→1 // bm1,3→1) ≡ (bm2,3→2 // bm1,2→1)}
 ]
```

Out[•]=

```
{0.25, {True, True}}
```

R and *P* are inverses:

```
In[•]:= Timing@Block[{$k = 3}, {Ri,j, Pi,k, HL[(Ri,j // Pi,k) ≡ aσk→j]}]
```

Out[•]=

$$\Big\{0.5,\ \Big\{\mathbb{E}_{\{\}\to\{i,j\}}\Big[\hbar\, a_j b_i,\ \hbar\, x_j y_i,\ 1 - \frac{1}{4}\left(\gamma\hbar^3 x_j^2 y_i^2\right)\epsilon + \left(\frac{1}{9}\gamma^2\hbar^5 x_j^3 y_i^3 + \frac{1}{32}\gamma^2\hbar^6 x_j^4 y_i^4\right)\epsilon^2 +$$
$$\left(\frac{1}{48}\gamma^3\hbar^5 x_j^2 y_i^2 - \frac{1}{16}\gamma^3\hbar^7 x_j^4 y_i^4 - \frac{1}{36}\gamma^3\hbar^8 x_j^5 y_i^5 - \frac{1}{384}\gamma^3\hbar^9 x_j^6 y_i^6\right)\epsilon^3 + O[\epsilon]^4\Big],$$
$$\mathbb{E}_{\{i,k\}\to\{\}}\Big[\frac{\alpha_k \beta_i}{\hbar},\ \frac{\eta_i \xi_k}{\hbar},\ 1 + \frac{\gamma \eta_i^2 \xi_k^2 \epsilon}{4\hbar} + \frac{\left(36\gamma^2\hbar^2\eta_i^2\xi_k^2 + 40\gamma^2\hbar\,\eta_i^3\xi_k^3 + 9\gamma^2\eta_i^4\xi_k^4\right)\epsilon^2}{288\hbar^2} +$$
$$\left(\frac{1}{24}\gamma^3\hbar\,\eta_i^2\xi_k^2 + \frac{1}{6}\gamma^3\eta_i^3\xi_k^3 + \frac{13\gamma^3\eta_i^4\xi_k^4}{96\hbar} + \frac{5\gamma^3\eta_i^5\xi_k^5}{144\hbar^2} + \frac{\gamma^3\eta_i^6\xi_k^6}{384\hbar^3}\right)\epsilon^3 + O[\epsilon]^4\Big],\ \text{True}\Big\}\Big\}$$

as and $\overline{aS}$ are inverses, bs and $\overline{bS}$ are inverses:

```
In[•]:= Timing[HL /@ {(aS̄1 // aS1) ≡ aσ1→1, (bS̄1 // bS1) ≡ bσ1→1}]
Out[•]=
{0.390625, {True, True}}
```

(co)-associativity on both sides

```
In[•]:= Timing[HL /@ {(aΔ1→1,2 // aΔ2→2,3) ≡ (aΔ1→1,3 // aΔ1→1,2), (bΔ1→1,2 // bΔ2→2,3) ≡ (bΔ1→1,3 // bΔ1→1,2),
    (am1,2→1 // am1,3→1) ≡ (am2,3→2 // am1,2→1), (bm1,2→1 // bm1,3→1) ≡ (bm2,3→2 // bm1,2→1)}]
Out[•]=
{0.375, {True, True, True, True}}
```

Δ is an algebra morphism

```
In[•]:= Timing[HL /@ {(am1,2→1 // aΔ1→1,2) ≡ ((aΔ1→1,3 aΔ2→2,4) // (am3,4→2 am1,2→1)),
    (bm1,2→1 // bΔ1→1,2) ≡ ((bΔ1→1,3 bΔ2→2,4) // (bm3,4→2 bm1,2→1))}]
Out[•]=
{0.40625, {True, True}}
```

An explicit formula for aS_i

```
In[•]:= Timing@Block[{$k = 4}, HL[aSi ≡ (𝔼{i}→{i,j}[-αi aj, -ξi xi,
        Sum[Expand[e^(ξi xi) (-ħ γ ϵ)^k / (2^k k!) Nest[Expand[xi^2 ∂{xi,2} #] &, e^(-ξi e^(ħ ϵ ai) xi), k]], {k, 0, $k}]]$k //
      ami,j→i)]]
Out[•]=
{3.5, True}
```

S is convolution inverse of id

```
In[•]:= Timing[HL[# ≡ sϵ1 sη1] & /@ {
    (aΔ1→1,2 ~ B1 ~ aS1) ~ B1,2 ~ am1,2→1, (aΔ1→1,2 ~ B2 ~ aS2) ~ B1,2 ~ am1,2→1,
    (bΔ1→1,2 ~ B1 ~ bS1) ~ B1,2 ~ bm1,2→1, (bΔ1→1,2 ~ B2 ~ bS2) ~ B1,2 ~ bm1,2→1}]
Out[•]=
{0.484375, {True, True, True, True}}
```

But not with the opposite product:

```
In[•]:= Timing[Short[# ≡ sε1 sη1] & /@ {
    (aΔ1→1,2 ~ B1 ~ aS1) ~ B1,2 ~ am2,1→1, (aΔ1→1,2 ~ B2 ~ aS2) ~ B1,2 ~ am2,1→1,
    (bΔ1→1,2 ~ B1 ~ bS1) ~ B1,2 ~ bm2,1→1, (bΔ1→1,2 ~ B2 ~ bS2) ~ B1,2 ~ bm2,1→1}]
```

Out[•]=

$$\left\{0.59375, \left\{\frac{1}{2}\left(-2\gamma\in\hbar\, x_1\, \mathcal{A}_1\, \xi_1+\gamma^2\in^2\hbar^2\, x_1\, \mathcal{A}_1\, \xi_1-2\ \text{<<6>>}\ \text{<<1>>}+2\gamma^2\in^2\hbar^2\, x_1^2\, \mathcal{A}_1^2\, \xi_1^2\right)==0,\right.\right.$$
$$\frac{1}{2}\left(-2\gamma\in\hbar\, x_1\, \xi_1-\gamma^2\in^2\hbar^2\, x_1\, \xi_1+2\gamma^2\in^2\hbar^2\, x_1^2\, \xi_1^2\right)==0,$$
$$\frac{1}{2}\left(-2\gamma\in\hbar\, y_1\, \eta_1-\gamma^2\in^2\hbar^2\, y_1\, \eta_1+2\gamma^2\in^2\hbar^2\, y_1^2\, \eta_1^2\right)==0,$$
$$\left.\left.\frac{-2\gamma\in\hbar\, B_1\, y_1\, \eta_1+\text{<<1>>}\ \text{<<1>>}\ \text{<<1>>}+2\gamma^2\ \text{<<3>>}\ \eta_1^2}{2\, B_1^2}==0\right\}\right\}$$

S is an algebra anti-(co)morphism

```
In[•]:= Timing[HL /@ {am1,2→1 ~ B1 ~ aS1 ≡ (aS1 aS2) ~ B1,2 ~ am2,1→1, bm1,2→1 ~ B1 ~ bS1 ≡ (bS1 bS2) ~ B1,2 ~ bm2,1→1,
    aS1 ~ B1 ~ aΔ1→1,2 ≡ aΔ1→2,1 ~ B1,2 ~ (aS1 aS2), bS1 ~ B1 ~ bΔ1→1,2 ≡ bΔ1→2,1 ~ B1,2 ~ (bS1 bS2)}]
```

Out[•]=

{0.734375, {True, True, True, True}}

Pairing axioms

```
In[•]:= Timing[HL /@ {(bm1,2→1 E{3}→{3}[α3 a3, ξ3 x3, 1]) ~ B1,3 ~ P1,3 ≡
     (E{1}→{1}[β1 b1, η1 y1, 1] E{2}→{2}[β2 b2, η2 y2, 1] aΔ3→4,5) ~ B1,4 ~ P1,4 ~ B2,5 ~ P2,5,
    (bΔ1→1,2 E{3}→{3}[α3 a3, ξ3 x3, 1] E{4}→{4}[α4 a4, ξ4 x4, 1]) ~ B1,3 ~ P1,3 ~ B2,4 ~ P2,4 ≡
     (E{1}→{1}[β1 b1, η1 y1, 1] am3,4→3) ~ B1,3 ~ P1,3 }]
```

Out[•]=

{0.328125, {True, True}}

```
In[•]:= Timing[HL /@ {((bS1 aσ2→2) // P1,2) ≡ ((bσ1→1 aS2) // P1,2),
    (bS1 aσ2→2) ~ B1,2 ~ P1,2 ≡ (bσ1→1 aS2) ~ B1,2 ~ P1,2}]
```

Out[•]=

{0.3125, {True, True}}

Tests for the double.

Check the double formulas on the generators agree with SL2Portfolio.pdf:

```
In[•]:= Timing@{{
          "[a,y]" → ((𝔼{}→{1,2}[0, 0, y2 a1] ~B1,2~ dm1,2→1)[3] - (𝔼{}→{1,2}[0, 0, y1 a2] ~B1,2~ dm1,2→1)[3
          "[b,x]"  →
           ((𝔼{}→{1,2}[0, 0, x2 b1] ~B1,2~ dm1,2→1)[3] - (𝔼{}→{1,2}[0, 0, x1 b2] ~B1,2~ dm1,2→1)[3]),
          "xy-qyx" → ((𝔼{}→{1,2}[0, 0, x1 y2] ~B1,2~ dm1,2→1)[3] -
              (1 + ϵ) (𝔼{}→{1,2}[0, 0, y1 x2] ~B1,2~ dm1,2→1)[3])
        } /. {z_1 → z} // Expand // Factor,
       {
         "Δ(a)" → ((𝔼{}→{1}[0, 0, a1] ~B1~ dΔ1→1,2)[3]),
         "Δ(x)" → ((𝔼{}→{1}[0, 0, x1] ~B1~ dΔ1→1,2)[3]),
         "Δ(b)" → ((𝔼{}→{1}[0, 0, b1] ~B1~ dΔ1→1,2)[3]),
         "Δ(y)" → ((𝔼{}→{1}[0, 0, y1] ~B1~ dΔ1→1,2)[3])
        } // Simplify,
       {
          "S(a)" → ((𝔼{}→{1}[0, 0, a1] ~B1~ dS1)[3]),
          "S(x)" → ((𝔼{}→{1}[0, 0, x1] ~B1~ dS1)[3]),
          "S(b)" → ((𝔼{}→{1}[0, 0, b1] ~B1~ dS1)[3]),
          "S(y)" → ((𝔼{}→{1}[0, 0, y1] ~B1~ dS1)[3])
         } /. {z_1 → z} // Simplify
       }
```

Out[•]=

$$\left\{4.125,\ \left\{\left\{[a,y] \to -y\,\gamma + O[\epsilon]^3,\ [b,x] \to x\,\epsilon + O[\epsilon]^3,\right.\right.\right.$$

$$xy\text{-}qyx \to \frac{1-B}{\hbar} + (a\,B - x\,y + x\,y\,\gamma\,\hbar)\,\epsilon + \left(-\frac{1}{2}\,a^2\,B\,\hbar + \frac{1}{2}\,x\,y\,\gamma^2\,\hbar^2\right)\epsilon^2 + O[\epsilon]^3\Big\},$$

$$\Big\{\Delta(a) \to (a_1 + a_2) + O[\epsilon]^3,\ \Delta(x) \to (x_1 + x_2) - \hbar\,a_1\,x_2\,\epsilon + \frac{1}{2}\,\hbar^2\,a_1^2\,x_2\,\epsilon^2 + O[\epsilon]^3,$$

$$\Delta(b) \to (b_1 + b_2) + O[\epsilon]^3,\ \Delta(y) \to (y_1 + B_1\,y_2) + O[\epsilon]^3\Big\},$$

$$\Big\{S(a) \to -a + O[\epsilon]^3,\ S(x) \to -x - a\,x\,\hbar\,\epsilon - \frac{1}{2}\left(a^2\,x\,\hbar^2\right)\epsilon^2 + O[\epsilon]^3,$$

$$\left.\left.\left.S(b) \to -b + O[\epsilon]^3,\ S(y) \to -\frac{y}{B} + \frac{y\,\gamma\,\hbar\,\epsilon}{B} - \frac{\left(y\,\gamma^2\,\hbar^2\right)\epsilon^2}{2\,B} + O[\epsilon]^3\right\}\right\}\right\}$$

```
In[•]:= {HL[(𝔼{1,2}→{1,2}[α1 a1 + α2 a2, ξ1 x1 + ξ2 x2, 1] // dm1,2→1) ≡ am1,2→1],
         HL[(𝔼{1,2}→{1,2}[β1 b1 + β2 b2, η1 y1 + η2 y2, 1] // dm1,2→1) ≡ bm1,2→1]}
```

Out[•]=

```
{True, True}
```

(co)-associativity

```
In[•]:= Timing[HL /@ {(dΔ1→1,2 // dΔ2→2,3) ≡ (dΔ1→1,3 // dΔ1→1,2), (dm1,2→1 // dm1,3→1) ≡ (dm2,3→2 // dm1,2→1)}]
```

Out[•]=

```
{2.0625, {True, True}}
```

Δ is an algebra morphism

```
In[•]:= Timing@HL[(dm_{1,2→1} // dΔ_{1→1,2}) ≡ ((dΔ_{1→1,3} dΔ_{2→2,4}) // (dm_{3,4→2} dm_{1,2→1}))]
Out[•]=
{2.07813, True}
```

dS and $\overline{\mathrm{dS}}$ are inverses:

```
In[•]:= Timing@HL[(dS̄_1 // dS_1) ≡ dσ_{1→1}]
Out[•]=
{1.60938, True}
```

S_2 inverts R, but not S_1:

```
In[•]:= Timing@{(R_{1,2} // dS_1) ≡ R̄_{1,2}, HL[(R_{1,2} // dS_2) ≡ R̄_{1,2}]}
Out[•]=
{0.421875, {1/(4 B_1^3) (4 γ ϵ^2 ħ^2 B_1^2 x_2 y_1 - 2 γ^2 ϵ^2 ħ^3 B_1^2 x_2 y_1 + 4 γ ϵ^2 ħ^3 a_2 B_1^2 x_2 y_1 +
      8 γ^2 ϵ^2 ħ^4 B_1 x_2^2 y_1^2 - 4 γ ϵ^2 ħ^4 a_2 B_1 x_2^2 y_1^2 - 3 γ^2 ϵ^2 ħ^5 x_2^3 y_1^3) == 0, True}}
```

dS is convolution inverse of id

```
In[•]:= Timing[HL[# ≡ dϵ_1 dη_1] & /@ {(dΔ_{1→1,2} // dS_1) // dm_{1,2→1}, (dΔ_{1→1,2} // dS_2) // dm_{1,2→1}}]
Out[•]=
{3.625, {True, True}}
```

dS is a (co)-algebra anti-morphism

```
In[•]:= Timing[HL /@
  Expand /@ {(dm_{1,2→1} // dS_1) ≡ ((dS_1 dS_2) // dm_{2,1→1}), (dS_1 // dΔ_{1→1,2}) ≡ (dΔ_{1→2,1} // (dS_1 dS_2))}]
Out[•]=
{7.26563, {True, True}}
```

Quasi-triangular axiom 1:

```
In[•]:= Timing[
 HL /@ {(R_{1,3} // dΔ_{1→1,2}) ≡ ((R_{1,4} R_{2,3}) // dm_{3,4→3}), (R_{1,2} // dΔ_{2→2,3}) ≡ ((R_{1,2} R_{4,3}) // dm_{1,4→1})}]
Out[•]=
{0.453125, {True, True}}
```

Quasi-triangular axiom 2:

```
In[•]:= Timing@HL[((dΔ_{1→1,2} R_{3,4}) // (dm_{1,3→1} dm_{2,4→2})) ≡ ((R_{1,2} dΔ_{1→3,4}) // (dm_{1,4→1} dm_{2,3→2}))]
Out[•]=
{1.8125, True}
```

The Drinfel'd element inverse property, $(u_1\,\overline{u}_2)$ // $\mathrm{dm}_{1,2\to1} \equiv \mathrm{d}\epsilon_i$:

```
In[•]:= Timing@HL[((R_{1,2} ~ B_1 ~ dS_1 ~ B_{1,2} ~ dm_{2,1→i}) (R_{1,2} ~ B_2 ~ dS_2 ~ B_2 ~ dS_2 ~ B_{1,2} ~ dm_{2,1→j})) ~ B_{i,j} ~ dm_{i,j→i} ≡ dη_i]
Out[•]=
{1.51563, True}
```

The ribbon element v satisfies $v^2 = S(u)\,u$. The spinner C=uv^{-1}. It is convenient to compute $z = S(u)\,u^{-1}$ which is something easy.

```
In[•]:= Timing@Block[{$k = 2},
  (((R_{1,2} ~ B_1 ~ dS_1 ~ B_{1,2} ~ dm_{2,1→i}) ~ B_i ~ dS_i) (R_{1,2} ~ B_2 ~ dS_2 ~ B_2 ~ dS_2 ~ B_{1,2} ~ dm_{2,1→j})) ~ B_{i,j} ~ dm_{i,j→i}]
```

```
Out[•]= {2.0625, 𝔼_{}→{i}[0, 0, 1/B_i + ħ a_i ∈/B_i + ħ^2 a_i^2 ∈^2/(2 B_i) + O[∈]^3]}
```

```
In[•]:= Timing@Block[{$k = 2}, HL /@ {((C_i C̄_j) // dm_{i,j→i}) ≡ dη_i, ((C̄_i C̄_j) // dm_{i,j→i}) ≡
      (((R_{1,2} // dS_1 // dm_{2,1→i}) // dS_i) (R_{1,2} // dS_2 // dS_2 // dm_{2,1→j}) // dm_{i,j→i})}]
```

```
Out[•]= {2.3125, {True, True}}
```

```
In[•]:= Timing@Block[{$k = 2}, HL /@ {(C_i C̄_j) ~ B_{i,j} ~ dm_{i,j→i} ≡ 𝔼_{}→{i}[0, 0, 1], (C̄_i C̄_j) ~ B_{i,j} ~ dm_{i,j→i} ≡
      (((R_{1,2} ~ B_1 ~ dS_1 ~ B_{1,2} ~ dm_{2,1→i}) ~ B_i ~ dS_i) (R_{1,2} ~ B_2 ~ dS_2 ~ B_2 ~ dS_2 ~ B_{1,2} ~ dm_{2,1→j})) ~ B_{i,j} ~ dm_{i,j→i}}]
```

```
Out[•]= {2.29688, {True, True}}
```

Reidemeister 2:

```
In[•]:= Timing[HL[# ≡ dη_1 dη_2] & /@ {(R̄_{1,2} R_{3,4}) // (dm_{1,3→1} dm_{2,4→2}), (R_{1,2} R̄_{3,4}) // (dm_{1,3→1} dm_{2,4→2})}]
```

```
Out[•]= {1.32813, {True, True}}
```

Cyclic Reidemeister 2:

```
In[•]:= Timing@HL[((R_{1,4} R̄_{5,2} C̄_3) // dm_{2,4→2} // dm_{1,3→1} // dm_{1,5→1}) ≡ C̄_1 dη_2]
```

```
Out[•]= {0.90625, True}
```

Reidemeister 3:

```
In[•]:= Timing@HL[(R_{1,2} R_{6,3} R_{4,5} // dm_{1,6→1} dm_{2,4→2} dm_{3,5→3}) ≡ (R_{2,3} R_{1,4} R_{5,6} // dm_{1,5→1} dm_{2,6→2} dm_{3,4→3})]
```

```
Out[•]= {2.92188, True}
```

Relations between the four kinks:

```
In[•]:= Timing[HL /@ {Kink_i ≡ ((R_{3,1} C_2) // dm_{1,2→1} // dm_{1,3→i}),
    Kink̄_j ≡ ((R̄_{3,1} C̄_2) // dm_{1,2→1} // dm_{1,3→j}), ((Kink_i Kink̄_j) // dm_{i,j→1}) ≡ dη_1}]
```

```
Out[•]= {2.67188, {True, True, True}}
```

The Trefoil

```
In[•]:= Timing@Block[{$k = 1},
  Z31 = R_{1,5} R_{6,2} R_{3,7} C̄_4 Kink̄_8 Kink̄_9 Kink̄_10;
  Do[Z31 = Z31 // dm_{1,r→1}, {r, 2, 10}];
  {Simplify /@ Z31, Simplify /@ (Z31 ~B_1~ b2t_1 /. T_1 → T)}]
```

Out[•]=

$$\left\{2.20313,\ \left\{\mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ \frac{B_1}{1-B_1+B_1^2}-\frac{\hbar\,B_1\left(-a_1\left(-1+B_1-B_1^3+B_1^4\right)+\gamma\left(B_1-2B_1^2-2B_1^4+2\hbar\,x_1\,y_1+B_1^3\,(3+2\hbar\,x_1\,y_1)\right)\right)\epsilon}{\left(1-B_1+B_1^2\right)^3}+\right.\right.$$

$$\left.\left.O[\epsilon]^2\right],\ \mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ \frac{T}{1-T+T^2}+\right.\right.$$

$$\left.\left.\frac{T\hbar\left(T\left(-1+2T-3T^2+2T^3\right)\gamma+2\left(-1+T-T^3+T^4\right)a_1-2\left(1+T^3\right)\gamma\hbar\,x_1\,y_1\right)\epsilon}{\left(1-T+T^2\right)^3}+O[\epsilon]^2\right]\right\}\right\}$$

b2t, t2b, knot tensors.

```
In[•]:= HL[(b2t_i // t2b_i) ≡ dσ_{i→i}]
```

Out[•]=

True

```
In[•]:= t2b_i // b2t_i
```

Out[•]=

$$\mathbb{E}_{\{i\}\to\{i\}}\left[a_i\,\alpha_i+t_i\,\tau_i,\ y_i\,\eta_i+x_i\,\xi_i,\ 1+O[\epsilon]^3\right]$$

Reidemeister 2:

```
In[•]:= Timing[HL[# ≡ dη_1 dη_2] & /@ {(kR̄_{1,2} kR_{3,4}) // (km_{1,3→1} km_{2,4→2}), (kR_{1,2} kR̄_{3,4}) // (km_{1,3→1} km_{2,4→2})}]
```

Out[•]=

{2.21875, {True, True}}

Cyclic Reidemeister 2:

```
In[•]:= Timing@HL[((kR_{1,4} kR̄_{5,2} kC̄_3) // km_{2,4→2} // km_{1,3→1} // km_{1,5→1}) ≡ kC̄_1 dη_2]
```

Out[•]=

{1.01563, True}

Reidemeister 3:

```
In[•]:= Timing@HL[(kR_{1,2} kR_{4,3} kR_{5,6} // km_{1,4→1} // km_{2,5→2} // km_{3,6→3}) ≡
    (kR_{1,6} kR_{2,3} kR_{4,5} // km_{1,4→1} // km_{2,5→2} // km_{3,6→3})]
```

Out[•]=

{1.5625, True}

Relations between the four kinks:

```
In[•]:= Timing[HL /@ {kKink_i ≡ ((kR_{3,1} kC_2) // km_{1,2→1} // km_{1,3→i}),
    kKink̄_j ≡ ((kR̄_{3,1} kC̄_2) // km_{1,2→1} // km_{1,3→j}), ((kKink_i kKink̄_j) // km_{i,j→1}) ≡ dη_1}]
```

Out[•]=

{1.51563, {True, True, True}}

The Trefoil

```
In[•]:= Timing@Block[{$k = 1},
  Z31 = kR_{1,5} kR_{6,2} kR_{3,7} cC_4 cKink_8 cKink_9 cKink_10;
  Do[Z31 = Z31 // km_{1,r→1}, {r, 2, 10}];
  Simplify /@ Z31]
```

```
Out[•]= {1.48438, 𝔼_{}→{1}[0, 0, T/(1 - T + T^2) + (T ħ (T (-1 + 2 T - 3 T^2 + 2 T^3) γ + 2 (-1 + T - T^3 + T^4) a_1 - 2 (1 + T^3) γ ħ x_1 y_1) ∈)/(1 - T + T^2)^3 + O[∈]^2]}
```

```
In[•]:= Timing@Block[{$k = 1},
  Z31 = kR_{1,5} kR_{6,2} kR_{3,7} cC_4 cKink_8 cKink_9 cKink_10;
  Do[Z31 = Z31 ~ B_{1,r} ~ km_{1,r→1}, {r, 2, 10}];
  Simplify /@ Z31]
```

```
Out[•]= {1.23438, 𝔼_{}→{1}[0, 0, T/(1 - T + T^2) + (T ħ (T (-1 + 2 T - 3 T^2 + 2 T^3) γ + 2 (-1 + T - T^3 + T^4) a_1 - 2 (1 + T^3) γ ħ x_1 y_1) ∈)/(1 - T + T^2)^3 + O[∈]^2]}
```

CU

Associativity of CU:

```
In[•]:= Timing@Block[{$k = 3}, HL[(cm_{1,2→1} // cm_{1,3→1}) ≡ (cm_{2,3→2} // cm_{1,2→1})]]
```

```
Out[•]= {6.0625, True}
```

Associativity, co-associativity, and Δ is an algebra morphism:

```
In[•]:= Timing@Block[{$k = 3}, HL /@ {(cm_{1,2→1} // cm_{1,3→1}) ≡ (cm_{2,3→2} // cm_{1,2→1}),
    (cΔ_{1→1,2} // cΔ_{2→2,3}) ≡ (cΔ_{1→1,3} // cΔ_{1→1,2}),
    (cm_{1,2→1} // cΔ_{1→1,2}) ≡ ((cΔ_{1→1,3} cΔ_{2→2,4}) // (cm_{3,4→2} cm_{1,2→1}))}]
```

```
Out[•]= {7.57813, {True, True, True}}
```

S is convolution inverse of id:

```
In[•]:= Timing@Block[{$k = 3}, HL[# ≡ cε_1 cη_1] & /@ {
    (cΔ_{1→1,2} ~ B_1 ~ cS_1) ~ B_{1,2} ~ cm_{1,2→1}, (cΔ_{1→1,2} ~ B_2 ~ cS_2) ~ B_{1,2} ~ cm_{1,2→1}}]
```

```
Out[•]= {3.375, {True, True}}
```

S is an algebra anti-(co)morphism

```
In[•]:= Timing@Block[{$k = 3},
          HL /@ {cm_{1,2→1} ~ B_1 ~ cS_1 ≡ (cS_1 cS_2) ~ B_{1,2} ~ cm_{2,1→1},  cS_1 ~ B_1 ~ cΔ_{1→1,2} ≡ cΔ_{1→2,1} ~ B_{1,2} ~ (cS_1 cS_2)}]
Out[•]=
        {11.4375, {True, True}}
```

Classical is the $\hbar \to 0$ limit of quantum:

```
In[•]:= ClassicalLimit[f_] := Normal@Series[Normal[f] /. U2l, {ħ, 0, 0}] //. l2U;
        (*ClassicalLimit[f_]:=Limit[Normal[f]/.U2l,ħ→0]//.l2U;*)
        Timing[HL /@ Simplify /@
            {cm_{1,2→3} ≡ ClassicalLimit /@ dm_{1,2→3},
             (cΔ_{1→2,3} /. τ_1 → 0) ≡ ClassicalLimit /@ dΔ_{1→2,3}, cS_1 ≡ ClassicalLimit /@ dS_1}]
Out[•]=
        {3.64063, {True, True, True}}
```

```
In[•]:= EndProfile[];

In[•]:= PrintProfile[]
Out[•]=
        ProfileRoot is root. Profiled time: 90.793
          (   210)   1.047/ 74.511 above B
          (    59)   2.659/ 16.109 above Boot
          (   270)   0.079/  0.173 above CF
        CCF: called 67319 times, time in 21.671/36.418
          ( 67319)  21.671/ 36.418 under CF
          ( 67319)  10.948/ 14.747 above Together
        Zip: called 3168 times, time in 20.285/114.069
          (   294)   2.314/ 16.128 under LZip
          (   294)   1.725/  8.084 under QZip
          (  2580)  16.246/ 89.857 under Zip
          (  3168)   3.927/  3.927 above Collect
          (  2580)  16.246/ 89.857 above Zip
        CF: called 13068 times, time in 17.881/54.299
          (  1433)   1.003/  1.844 under EEQ
          (   179)   0.205/  0.344 under Boot
          (  1416)   7.565/ 26.166 under LZip
          (   270)   0.079/  0.173 under ProfileRoot
          (  9770)   9.029/ 25.772 under QZip
          ( 67319)  21.671/ 36.418 above CCF
        Together: called 67319 times, time in 10.948/14.747
          ( 67319)  10.948/ 14.747 under CCF
          ( 67319)   3.799/  3.799 above Exp
        LZip: called 294 times, time in 4.084/49.226
          (   294)   4.084/ 49.226 under B
          (  1433)   1.004/  2.848 above EEQ
          (  1416)   7.565/ 26.166 above CF
          (   294)   2.314/ 16.128 above Zip
        Collect: called 3168 times, time in 3.927/3.927
          (  3168)   3.927/  3.927 under Zip
```

```
Boot: called 83 times, time in 3.894/23.5
  (    24)   1.235/  7.391 under Boot
  (    59)   2.659/ 16.109 under ProfileRoot
  (    84)   0.250/ 11.871 above B
  (    24)   1.235/  7.391 above Boot
  (   179)   0.205/  0.344 above CF
Exp: called 67319 times, time in 3.799/3.799
  ( 67319)   3.799/  3.799 under Together
QZip: called 294 times, time in 2.003/35.859
  (   294)   2.003/ 35.859 under B
  (  9770)   9.029/ 25.772 above CF
  (   294)   1.725/  8.084 above Zip
B: called 294 times, time in 1.297/86.382
  (    84)   0.250/ 11.871 under Boot
  (   210)   1.047/ 74.511 under ProfileRoot
  (   294)   4.084/ 49.226 above LZip
  (   294)   2.003/ 35.859 above QZip
EEQ: called 1433 times, time in 1.004/2.848
  (  1433)   1.004/  2.848 under LZip
  (  1433)   1.003/  1.844 above CF
```