

IMPLEMENTING THE CLASSICAL AND QUANTUM sl_2 PORTFOLIO OF OPERATIONS

DROR BAR-NATAN AND ROLAND VAN DER VEEN

ABSTRACT. Building up from some new or lightly used theoretical tools, especially “solvable approximation” and “Gaussian differential operators”, we give a clean and efficient computer implementation of the quantum sl_2 portfolio of operations. Beyond the theoretical interest and the satisfaction that one obtains when complicated formulas come to life, become specific, and check, we explain (and implement and prove) why our results are valuable in knot theory.

We mean business! Page [??](#) displays a program which is a complete implementation of the quantum sl_2 portfolio of operations. Page [??](#) displays a variant of that program tailored to efficiently compute the “Rozansky-Overbay” invariants. Appendix [A](#) contains a tabulation of some of these invariants on knots with up to 10 crossings. Much more is at [ωεβ/:=http://drorbn.net/SL2PO/](http://drorbn.net/SL2PO/).

CONTENTS

1. Introduction	3
1.1. A Quick Reminder of Algebras and R-matrices	3
1.2. An Expansion of the Abstract	4
1.3. Plan of the paper.	4
1.4. Acknowledgement	4
2. Rotational Virtual Tangles	5
3. Meta-Hopf Algebras and the Drinfel’d Double Construction	7
3.1. Meta-monoids vs. monoid objects in a monoidal category	8
4. The Category of Perturbed Gaussian Differential Operators	9
5. Solvable Approximations of Semi-Simple Lie Algebras	11
6. Odds and Ends	13
Appendix A. Tables	13
References	15

computations below

Date: First edition Not Yet, 2018, this edition Aug. 9, 2019.

2010 Mathematics Subject Classification. 57M25.

Key words and phrases. knots, tangles, knot polynomials, Lie algebras, Lie bialgebras, quantization, 2-parameter quantum groups, Drinfel’d double.

This work was partially supported by NSERC grant RGPIN-2018-04350.

1. INTRODUCTION

In Section [1.1](#) of the introduction we briefly and schematically recall how certain algebras lead to knot invariants, only so as to explain what exactly it is that we aim to implement and why. Section [1.2](#) of the introduction is the abstract of this paper, expanded from one paragraph to a few pages. Section [1.3](#) of the introduction is an introduction to the rest of the paper — a summary of what happens in it, and in what order.

1.1. A Quick Reminder of Algebras and R-matrices. A “Hopf Algebra” is a vector space $(U, +, \cdot)$ (over \mathbb{Q} , for simplicity) along with a number of further operations: a “product” $m: U \otimes U \rightarrow U$, a “coproduct” $\Delta: U \rightarrow U \otimes U$, an “antipode” $S: U \rightarrow U$, a “unit” $\eta: \mathbb{Q} \rightarrow U$ and a “counit” $\epsilon: U \rightarrow \mathbb{Q}$ (which of course are required to satisfy some axioms). If U is also equipped with a “braiding” $R \in U \otimes U$ and a “cuap element” $C \in U$ and these satisfy a few further axioms, then U is a “ribbon Hopf algebra”. It is sometimes (but not always) useful to add to the mix a “pairing element” $P \in U^* \otimes U^*$, which is dual to the element R .

Ribbon Hopf algebras are immensely useful in low dimensional topology, as they lead to knot and tangle invariants which are well-behaved under “strand stitching”, “strand doubling”, “strand reversal”, and a few lesser operations. See e.g. [Oh, Section 4.2] and our quick summary in [Aside 1.1](#) and in [Aside ??](#).

Yet from the perspective of topology, the algebras U that one uses seem like great wastelands with a few pearls hidden within. From the perspective of [Aside 1.1](#) and [Aside ??](#)

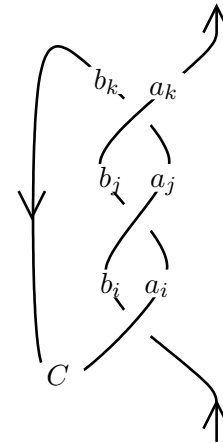
Draw K as a long knot in the plane so that at each crossing the two crossing strands are flowing up, and so that the two ends of K are flowing up.

Put a copy of $R = \sum a_i \otimes b_i$ on every positive crossing of K with the “ a ” side on the over-strand and the “ b ” side on the under-strand, labeling these a ’s and b ’s with distinct indices i, j, k, \dots (similarly put copies of $R^{-1} = \sum a'_i \otimes b'_i$ on the negative crossings; these are absent in our example). Put a copy of $C^{\pm 1}$ on every cuap where the tangent to the knot is pointing to the right (meaning, a C on every such cup and a C^{-1} on every such cap).

Form an expression $z(K)$ in U by multiplying all the a, b, C letters as they are seen when traveling along K and then summing over all the indices, as shown.

If R and C satisfy some conditions dictated by the standard Reidemeister moves of knot theory, the resulting $z(K)$ is a knot invariant.

Abstractly, $z(K)$ is obtained by tensoring together several copies of $R^{\pm 1} \in U^{\otimes 2}$ and $C^{\pm 1} \in U$ to get an intermediate result $z_0 \in U^{\otimes S}$, where S is a finite set with two elements for each crossing of K and one element for each right-pointing cuap. We then multiply the different tensor factors in z_0 in an order dictated by K to get an output in a single copy of U .



$$z(K) = \sum_{i,j,k} b_i a_j b_k C a_i b_j a_k$$

ASIDE 1.1. The standard methodology on an example knot.

the vector space structure of U is completely irrelevant as the operations of addition (+) and multiplication by a scalar (\cdot) are never used. All that matters are those elements (the “pearls”) within tensor powers $U^{\otimes S}$ of U that can be written using the “generators” R and C , using tensor products $U^{\otimes S_1} \times U^{\otimes S_2} \rightarrow U^{\otimes (S_1 \sqcup S_2)}$, and using the multiplication m (extended to tensor powers) *yet without using + and \cdot* ¹.

MORE.

1.2. An Expansion of the Abstract. MORE.

1.3. Plan of the paper. MORE.

1.4. Acknowledgement. MORE.

¹It does not matter whether or not Δ , S , η , and ϵ are used for the generation of the “pearls”, as the axioms of a ribbon Hopf algebra imply that anything that can be generated with them can also be generated without them.

2. ROTATIONAL VIRTUAL TANGLES

sec:RVK

Sections [2-5](#) of this paper can be regarded as independent paperlets which can be read in any order.

MORE.

3. META-HOPF ALGEBRAS AND THE DRINFEL'D DOUBLE CONSTRUCTION

Sections 2–5 of this paper can be regarded as independent paperlets which can be read in any order.

A Hopf algebra is a vector space U with some operations which satisfy some axioms (see Aside 3.1). These axioms, labeled (1)–(9) in the aside, never directly mention the vector space structure of U ; that structure is only used for the formation of the spaces $U^{\otimes n}$ on which the operations are defined and in which the axioms are stated. But this calls for a generalization — why not replace $U^{\otimes n}$ with sets U_n that do not need to be vector spaces, and replace operations such as $Id \otimes \cdots \otimes m \otimes \cdots \otimes Id: U^{\otimes n} \rightarrow U^{\otimes(n-1)}$ with arbitrary maps $m_n: U_n \rightarrow U_{n-1}$ in such a manner that the axioms (1)–(9) would still make sense?

A bit of further reflection² leads one to realize that m should be replaced with a family of operations m_k^{ij} which generalize “multiply the content of the i th tensor factor with the content of the j th tensor factor putting the result as a k th tensor factor”, and that in fact, the restriction that the labels of the tensor factors would be natural numbers is a (minor) handicap. Hence we come to the following convention and definition:

Convention 3.1. If S , A , and B are finite sets, $A \subset S$, and $(S \setminus A) \cap B = \emptyset$ we write $S \setminus A \cup B$ for $(S \setminus A) \sqcup B$. In fact, whenever we write $S \setminus A \cup B$ we automatically add the assumptions that $A \subset S$ and $(S \setminus A) \cap B = \emptyset$, even if this is not explicitly stated. In this context we often suppress braces and commas when referring to sets with a small number of elements, and automatically assume that these elements are distinct. Hence for example $S \setminus ij \cup kl = (S \setminus \{i, j\}) \sqcup \{k, l\}$, and the assumptions $i \neq j$, $k \neq l$, $\{i, j\} \subset S$, and $(S \setminus \{i, j\}) \cap \{k, l\} = \emptyset$ are silently made. Finally, if A or B are omitted from the notation, the omitted set is assumed to be the empty set and all further conventions still apply.

Definition 3.2. A meta-Hopf algebra (in the category of sets) is an assignment $U: S \mapsto U_S$ that assigns a (possibly big) set U_S to every finite set S (we ignore the easily-resolved issues

²A fuller but longer explanation is at [KBH, Section 10.3].

If U is a vector space over \mathbb{Q} (or another field) we form $U^{\otimes n} = U \otimes \cdots \otimes U$ (n times), and in particular $U^{\otimes 0} = \mathbb{Q}$, $U^{\otimes 1} = U$, $U^{\otimes 2} = U \otimes U$, etc. We identify $U^{\otimes 1} \cong U^{\otimes 1} \otimes U^{\otimes 0} \cong U^{\otimes 0} \otimes U^{\otimes 1}$. With these conventions, a Hopf Algebra is a vector space U endowed with maps $m: U^{\otimes 2} \rightarrow U^{\otimes 1}$, $\Delta: U^{\otimes 1} \rightarrow U^{\otimes 2}$, $\eta: U^{\otimes 0} \rightarrow U^{\otimes 1}$, $\epsilon: U^{\otimes 1} \rightarrow U^{\otimes 0}$, and an invertible $S: U^{\otimes 1} \rightarrow U^{\otimes 1}$ such that:

- | | |
|--|---|
| (1) $(m \otimes Id) // m = (Id \otimes m) // m.$ | (6) $\eta = (\eta \otimes \eta) // m.$ |
| (2) $(\eta \otimes Id) // m = (Id \otimes \eta) // m = Id.$ | (7) $\epsilon = \Delta // (\epsilon \otimes \epsilon).$ |
| (3) $\Delta // (\Delta \otimes Id) = \Delta // (Id \otimes \Delta).$ | (8) $\Delta // (S \otimes Id) // m = \Delta // (Id \otimes S) // m = \epsilon // \eta.$ |
| (4) $\Delta // (\epsilon \otimes Id) = \Delta // (Id \otimes \epsilon) = Id.$ | |
| (5) $m // \Delta = (\Delta \otimes \delta) // (Id \otimes \sigma \otimes Id) // (m \otimes m),$
where $\sigma: U_2 \rightarrow U_2$ is the transposition. | |

Note that we are not assuming $m = \sigma // m$ (“commutativity”) or $\Delta = \Delta // \sigma$ (“co-commutativity”).

ASIDE 3.1. Ordinary Hopf Algebras.

that come with the likes of “the set of all finite sets”), along with the following families of operations and axioms:

Most Interesting. For any finite S , operations $m_k^{ij}: U_S \rightarrow U_{S \setminus ij \cup k}$ called “meta-multiplications” or “stitchings”, $\Delta_{jk}^i: U_S \rightarrow U_{S \setminus i \cup jk}$ “meta-comultiplications” or “doublings”, $\eta_i: U_S \rightarrow U_{S \cup i}$ “meta-units”, $\epsilon^i: U_S \rightarrow U_{S \setminus i}$ “meta-counits”, and $S_i: U_S \rightarrow U_S$ “meta-antipodes”, satisfying the following axioms (compare with [Aside 3.1](#)):

- | | |
|---|--|
| (1) $m_i^{ij} // m_i^{ik} = m_j^{jk} // m_i^{ij}$. | (5) $m_k^{ij} // \Delta_{ij}^k = \Delta_{kl}^i // \Delta_{mn}^j // m_i^{km} // m_j^{ln}$. |
| (2) $\eta_i // m_j^{ij} = \eta_j // m_i^{ij} = Id$. | (6) $\eta_k = \eta_i // \eta_j // m_k^{ij}$. |
| (3) $\Delta_{ik}^i // \Delta_{ij}^i = \Delta_{ij}^i // \Delta_{jk}^j$. | (7) $\epsilon^k = \Delta_{ij}^k // \epsilon_i // \epsilon_j$. |
| (4) $\Delta_{ij}^j // \epsilon^i = \Delta_{ij}^i // \epsilon^j = Id$. | (8) $\Delta_{jk}^i // S_j // m_i^{jk} = \Delta_{jk}^i // S_k // m_i^{jk} = \epsilon^i // \eta_i$. |

Note that we are not assuming $m_k^{ij} = m_k^{ji}$ (“commutativity”) or $\Delta_{jk}^i = \Delta_{kj}^i$ (“cocommutativity”).

MORE.

3.1. Meta-monoids vs. monoid objectss in a monoidal category. MORE.

4. THE CATEGORY OF PERTURBED GAUSSIAN DIFFERENTIAL OPERATORS

Sections [2-5](#) of this paper can be regarded as independent paperlets which can be read in any order.

MORE.

5. SOLVABLE APPROXIMATIONS OF SEMI-SIMPLE LIE ALGEBRAS

Sections [2-5](#) of this paper can be regarded as independent paperlets which can be read in any order.

MORE.

ec:SolvApp

[se:seBVKolvApp](#)

6. ODDS AND ENDS

MORE.

APPENDIX A. TABLES

MORE.

app:Tables

REFERENCES

- [BN] D. Bar-Natan, *Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant*, Acta Mathematica Vietnamica **40-2** (2015) 271–329, [arXiv:1308.1721](#). See pp. 7.
- [IW] E. Inonu and E. P. Wigner, *On the Contraction of Groups and their Representations*, Proc. Nat. Acad. Sci. **39** (1953) 510–524. See pp. .
- [Ka] L. H. Kauffman, *Rotational Virtual Knots and Quantum Link Invariants*, [arXiv:1509.00578](#). See pp. .
- [Oh] T. Ohtsuki, *Quantum Invariants*, Series on Knots and Everything **29**, World Scientific 2002. See pp. 3.
- [Ov] A. Overbay, *Perturbative Expansion of the Colored Jones Polynomial*, University of North Carolina PhD thesis, [ωεβ/Ov](#). See pp. .
- [Ro1] L. Rozansky, *A contribution of the trivial flat connection to the Jones polynomial and Witten's invariant of 3d manifolds, I*, Comm. Math. Phys. **175-2** (1996) 275–296, [arXiv:hep-th/9401061](#). See pp. .
- [Ro2] L. Rozansky, *The Universal R-Matrix, Burau Representation and the Melvin-Morton Expansion of the Colored Jones Polynomial*, Adv. Math. **134-1** (1998) 1–31, [arXiv:q-alg/9604005](#). See pp. .
- [Ro3] L. Rozansky, *A Universal $U(1)$ -RCC Invariant of Links and Rationality Conjecture*, [arXiv:math/0201139](#). See pp. .
- [Wo] *Wolfram Language & System Documentation Center*, [ωεβ/Wolf](#). See pp. .



DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, TORONTO
ONTARIO M5S 2E4, CANADA

Email address: drorbn@math.toronto.edu

URL: <http://www.math.toronto.edu/drorbn>

MATHEMATISCH INSTITUUT, UNIVERSITEIT LEIDEN, NIELS BOHRWEG
1, 2333 CA LEIDEN, THE NETHERLANDS

Email address: roland.mathematics@gmail.com

URL: <http://www.rolandvdv.nl/>