Oregon Handout as of July 30, 2011 (2)

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page 2: "Examples & interpretations" ?

The Pure Virtual Braid Group is Quadratic[1]

Dror Bar–Natan and Peter Lee in Oregon, August 2011
http://www.math.toronto.edu/~drorbn/Talks/Oregon-1108/
Foots & refs on PDF version, page 3.

Let K be a unital algebra over a field $\mathbb{F}$ with $\operatorname{char}\mathbb{F}=0$, and let $I\subset K$ be an "augmentation ideal"; meaning $K/I=\mathbb{F}$.

Definition. Say that K is quadratic if its associated graded $\operatorname{gr}K=\bigoplus_{p=0}^{\infty}I^p/I^{p+1}$ is a quadratic algebra. Alternatively, let $A=Q(K)=\langle V=I/I^2\rangle/\langle\ker(\bar\mu_2:V\otimes V\to I^2/I^3)\rangle$ be the "quadratic approximation" to K (Q is a lovely functor). Then K is quadratic iff the obvious $\mu:A\to\operatorname{gr}K$ is an isomorphism. If G is a group, we say it is quadratic if its group ring is, with its augmentation ideal.

Why Care? • In abstract generality, $\operatorname{gr}K$ is a simplified version of K and if it is quadratic it is as simple as it may be without being silly. • In some concrete (somewhat generalized) knot theoretic cases, A is a space of "universal Lie algebraic formulas" and the "primary approach" for proving (strong) quadraticity, constructing an appropriate homomorphism $Z:K\to\hat A$, becomes wonderful mathematics:

K	u-Knots and Braids	v-Knots	w-Knots
A	Metrized Lie algebras [BN1]	Lie bialgebras [Hav]	Finite dimensional Lie algebras [BN3]
Z	Associators [Dri, BND]	Etingof-Kazhdan quantization [EK, BN2]	Kashiwara-Vergne-Alekseev-Torossian [KV, AT]

PvB_n is the group

$$\langle\sigma_{ij}:1\le i\ne j\le n\rangle\Big/\begin{matrix}\sigma_{ij}\sigma_{ik}\sigma_{jk}=\sigma_{jk}\sigma_{ik}\sigma_{ij}\\ \sigma_{ij}\sigma_{kl}=\sigma_{kl}\sigma_{ij}\end{matrix}$$

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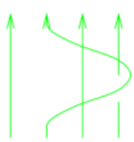
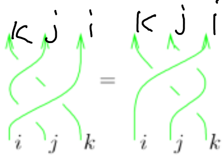
L. Kauffman

of "pure virtual braids" ("braids when you look", [ref] "blunder braids"):

$\sigma_{24}=$  R3: 

The Main Theorem [Lee]. PvB_n is quadratic.

The Overall Strategy. Consider the "singularity tower" of (K,I) (here ":" means $\otimes_K$ and μ is (always) multiplication):

$$\cdots\ I^{:p+1}\xrightarrow{\mu_{p+1}}I^{:p}\xrightarrow{\mu_p}I^{:p-1}\longrightarrow\cdots\longrightarrow K$$

We care as $\operatorname{im}(\mu^p=\mu_1\circ\cdots\circ\mu_p)=I^p$, so $I^p/I^{p+1}=\operatorname{im}\mu^p/\operatorname{im}\mu^{p+1}$. Hence we ask:

- What's $I^{:p}/\mu(I^{:p+1})$?
- How injective is this tower?

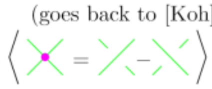
Lemma. $I^{:p}/\mu(I^{:p+1})\simeq(I/I^2)^{\otimes p}=V^{\otimes p}$.

Flow Chart.

$$\begin{tikzcd}
 & \boxed{\text{Any }(K,I)} \arrow[r, "\text{Prop 1}"] & \boxed{\text{2-local}} \arrow[r, "\text{Prop 2}"] \arrow[dl, dashed] & \boxed{\text{Quadratic}} \\
\boxed{K=\mathit{PvB}_n} \arrow[r, "\text{Thm}", "\text{by Peter}"'] & \boxed{\text{Hutchings' Criterion}} \arrow[r] & \boxed{\text{2-injective}} \arrow[u] &
\end{tikzcd}$$

Example. (goes back to [Koh])

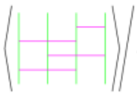

T. Kohno

$K=$  $I=$ 

$$(K/I^{p+1})^\star=(\text{invariants of type }p)=:\mathcal{V}_p$$

$$(I^p/I^{p+1})^\star=\mathcal{V}_p/\mathcal{V}_{p-1}\quad C=\langle t^{ij}\,|\,t^{ij}=t^{ji}\rangle=$$ 

$$\ker\bar\mu_2=\langle[t^{ij},t^{kl}]=0=[t^{ij},t^{ik}+t^{jk}]\rangle=\langle\text{4T relations}\rangle$$

$A=\begin{pmatrix}\text{horizontal chord dia-}\\ \text{grams mod 4T}\end{pmatrix}=$  $/4T$

Z: universal finite type invariant, the Kontsevich integral.

Proposition 1. The sequence

$$R_p:=\bigoplus_{j=1}^{p-1}\left(I^{:j-1}:R_2:I^{:p-j-1}\right)\xrightarrow{\partial}I^{:p}\xrightarrow{\mu_p}I^{:p-1}$$

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is exact, where $R_2:=\ker\mu:I^{:2}\to I$; so (K,I) is "2-local".

Reread.

The Free Case. If J is an augmentation ideal in $K=F=\langle x_i\rangle$, denote $F\to F/J=\mathbb{F}$ by $x\mapsto[x]$ and define $\psi:F\to F$ by $x_i\mapsto x_i+[x_i]$. Then $J_0:=\psi(J)$ is $\{w\in F:\deg w>0\}$. For J_0 it is easy to check that $R_2=R_p=0$, and hence the same is true for every J.

The General Case. If $K=F/M$ and $I\subset K$, then $I=J/M$ where $J\subset F$. Then $I^{:p}=J^{:p}/\sum J^{:j-1}:M:J^{:p-j}$ and we have

$$\begin{tikzcd}
J^{:p} \arrow[rr, "\mu^F", "\text{1-1}"'] \arrow[d, "\text{onto}"', "\pi_p"] & & J^{:p-1} \arrow[d, "\pi_{p-1}"', "\text{onto}"] \\
I^{:p}=J^{:p}/\sum J^{:}:M:J^{:} \arrow[rr, "\mu"] & & I^{:p-1}=J^{:p-1}/\sum J^{:}:M:J^{:}.
\end{tikzcd}$$

So $\ker(\mu)=\pi_p\left(\mu_F^{-1}(\ker\pi_{p-1})\right)=\pi_p\left(\sum\mu_F^{-1}(J^{:}:M:J^{:})\right)=\pi_p\left(\sum J^{:}:\mu_F^{-1}(M):J^{:}\right)=\sum I^{:}:R_2:I^{:}$.

2-Injectivity. A (one-sided infinite) sequence

$$\cdots\longrightarrow K_{p+1}\xrightarrow{\delta_{p+1}}K_p\xrightarrow{\delta_p}\cdots\longrightarrow K_0=K$$

is "injective" if for all $p>0$, $\ker\delta_p=0$. It is "2-injective" if its "1-reduction"

$$\cdots\longrightarrow\frac{K_{p+1}}{\ker\delta_{p+1}}\xrightarrow{\bar\delta_{p+1}}\frac{K_p}{\ker\delta_p}\xrightarrow{\bar\delta_p}\frac{K_{p-1}}{\ker\delta_{p-1}}\longrightarrow\cdots$$

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is injective; i.e. if for all p, $\ker(\delta_p\circ\delta_{p+1})=\ker\delta_{p+1}$. A pair (K,I) is "2-injective" if its singularity tower is 2-injective.

Proposition 2. If (K,I) is 2-local and 2-injective, it is quadratic.

Reread

Proof. Staring at the 1-reduced sequence $\frac{I^{:p+1}}{\ker\mu_{p+1}}\xrightarrow{\mu_{p+1}}\frac{I^{:p}}{\ker\mu_p}\xrightarrow{\mu_p}\cdots\longrightarrow K$, get $\frac{I^p}{I^{p+1}}\simeq\frac{I^{:p}/\ker\mu_p}{\mu(I^{:p+1}/\ker\mu_{p+1})}\simeq\frac{I^{:p}}{\mu(I^{:p+1})+\ker\mu_p}$. But trivially $\frac{I^{:p}}{\mu(I^{:p+1})}\simeq(I/I^2)^{\otimes p}$, so the above is $(I/I^2)^{\otimes p}/\sum\left(I^{:j-1}:R_2:I^{:p-j-1}\right)$. But that's the degree p piece of $Q(K)$.

The Pure Virtual Braid Group is Quadratic, II

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The X Lemma (inspired by [Hut]).

$$\begin{tikzcd} A_0 \arrow[dr, "\alpha_0"] & & C_0 \\ & B \arrow[ur, "\beta_0"] \arrow[dr, "\beta_1"] & \\ A_1 \arrow[ur, "\alpha_1"] & & C_1 \end{tikzcd}$$

If the above diagram is Conway ($\asymp$) exact, then its two diagonals have the same "2-injectivity defect". That is, if $A_0 \to B \to C_0$ and $A_1 \to B \to C_1$ are exact, then $\ker(\beta_1 \circ \alpha_0)/\ker\alpha_0 \simeq \ker(\beta_0 \circ \alpha_1)/\ker\alpha_1$.

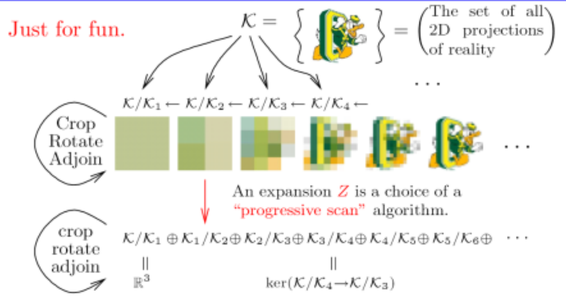
Proof. $\frac{\ker(\beta_1\circ\alpha_0)}{\ker\alpha_0} \xrightarrow[\alpha_0]{\sim} \ker\beta_1 \cap \operatorname{im}\alpha_0$
$= \ker\beta_0 \cap \operatorname{im}\alpha_1 \xleftarrow[\alpha_1]{\sim} \frac{\ker(\beta_0\circ\alpha_1)}{\ker\alpha_1}$.

The Hutchings Criterion [Hut]. The singularity tower of (K, I) is 2-injective iff on the right, $\ker(\pi \circ \partial) = \ker(\partial)$. That is, iff every "diagrammatic syzygy" lifts to a "topological syzygy".

$$\begin{tikzcd} R_p \arrow[dr, "\partial"] & & I^{:p-1} \\ & I^{:p} \arrow[ur, "\mu_p"] \arrow[dr, "\pi"] & \\ I^{:p+1} \arrow[ur, "\mu_{p+1}"] & & V^{\otimes p} \end{tikzcd}$$

add emph
(and understand better...)

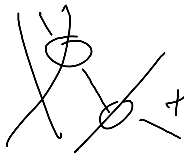
Just for fun.



[Polyak]

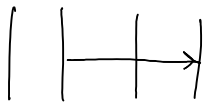
Add a box for $A_n = \mathbb{Q}(PvB_n)$.

I:



$\sigma_{ij} =$ ✕ $=$ ✕ the "semi-virtual"

$V = I/I^2 : \langle$ v-braids w/ one ✕ $\rangle / $ ✕ $=$ ✕



$= \langle a_{ij} \rangle_{i \neq j}$

Derive loc, GT.
& draw

A box for $I^{:p}$, like Peter's.

A box for R_2: