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Subdivide page 1: "The core material"
page 2: "Examples & interpretations" ?

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The Pure Virtual Braid Group is Quadratic[1]

Dror Bar–Natan and Peter Lee in Oregon, August 2011
http://www.math.toronto.edu/~drorbn/Talks/Oregon-1108/
Foots & refs on PDF version, page 3.

Let K be ~~an~~ algebra over a field $\mathbb{F}$ with char $\mathbb{F}=0$, and let $I \subset K$ be an "augmentation ideal"; meaning $K/I=\mathbb{F}$.
Definition. Say that K is quadratic if its associated graded $\operatorname{gr} K=\bigoplus_{p=0}^{\infty} I^p/I^{p+1}$ is a quadratic algebra. Alternatively, let $A=Q(K)=\langle V=I/I^2\rangle/\langle \ker(\bar\mu_2 : V\otimes V\to I^2/I^3)\rangle$ be the "quadratic approximation" to K (Q is a lovely functor). Then K is quadratic iff the obvious $\mu : A \to \operatorname{gr} K$ is an isomorphism. If G is a group, we say it is quadratic if its group ring is, with its augmentation ideal.

Why Care? • In abstract generality, $\operatorname{gr} K$ is a simplified version of K and if it is quadratic it is as simple as it may be without being silly. • In some concrete (somewhat generalized) knot theoretic cases, A is a space of "universal Lie algebraic formulas" and the "primary approach" for proving (strong) quadraticity, constructing an appropriate homomorphism $Z : K \to \hat{A}$, becomes wonderful mathematics:

	u-Knots and Braids	v-Knots	w-Knots
K	u-Knots and Braids	v-Knots	w-Knots
A	Metrized Lie algebras [BN1]	Lie bialgebras [Hav]	Finite dimensional Lie algebras [BN3]
Z	Associators [Dri, BND]	Etingof-Kazhdan quantization [EK, BN2]	Kashiwara-Vergne-Alekseev-Torrosian [KV, AT]

$P\!vB_n$ is the group

L. Kauffman

$$\langle \sigma_{ij} : 1\le i\ne j\le n\rangle \Big/ \begin{matrix}\sigma_{ij}\sigma_{ik}\sigma_{jk}=\sigma_{jk}\sigma_{ik}\sigma_{ij}\\ \sigma_{ij}\sigma_{kl}=\sigma_{kl}\sigma_{ij}\end{matrix}$$

of "pure virtual braids" ("braids when you look", "blunder braids"):

Do Now!

The Main Theorem [Lee]. $P\!vB_n$ is quadratic.

The Overall Strategy. Consider the "singularity tower" of (K,I) (here ":" means $\otimes_K$ and μ is (always) multiplication):

$$\cdots\; I^{:p+1} \xrightarrow{\mu_{p+1}} I^{:p} \xrightarrow{\mu_p} I^{:p-1} \longrightarrow \cdots \longrightarrow K$$

We care as $\operatorname{im}(\mu^p=\mu_1\circ\cdots\circ\mu_p)=I^p$, so $I^p/I^{p+1}=\operatorname{im}\mu^p/\operatorname{im}\mu^{p+1}$. Hence we ask:

- What's $I^{:p}/\mu(I^{:p+1})$?
- How injective is this tower?

Lemma. $I^{:p}/\mu(I^{:p+1})\simeq (I/I^2)^{\otimes p}=V^{\otimes p}$.

Flow Chart.

$$\boxed{\text{Any }(K,I)} \xrightarrow[\cancel{12}]{\cancel{\text{Thm}}\ \text{Prop 1}} \boxed{\text{2-local}} \xrightarrow{\cancel{\text{Thm 1}}\ \text{Prop 2}} \boxed{\text{Quadratic}}$$

$$\boxed{K=P\!vB_n} \xrightarrow[\text{by Peter}]{\text{Thm 4}} \boxed{\text{Hutchings' Criterion}} \xrightarrow[\cancel{3}]{\cancel{\text{Thm}}} \boxed{\text{2-injective}}$$

Just for fun.

$\mathcal{K}=\{\ \}=$ (The set of all 2D projections of reality)

$\mathcal{K}/\mathcal{K}_1\leftarrow\mathcal{K}/\mathcal{K}_2\leftarrow\mathcal{K}/\mathcal{K}_3\leftarrow\mathcal{K}/\mathcal{K}_4\leftarrow \cdots$

Crop Rotate Adjoin

An expansion Z is a choice of a "progressive scan" algorithm.

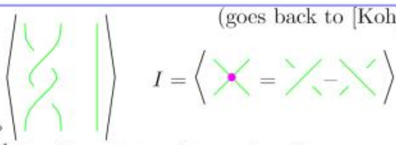
crop rotate adjoin

$$\mathcal{K}/\mathcal{K}_1\oplus\mathcal{K}_1/\mathcal{K}_2\oplus\mathcal{K}_2/\mathcal{K}_3\oplus\mathcal{K}_3/\mathcal{K}_4\oplus\mathcal{K}_4/\mathcal{K}_5\oplus\mathcal{K}_5/\mathcal{K}_6\oplus\cdots$$

$\mathcal{K}/\mathcal{K}_1 = \mathbb{R}^3$, $\mathcal{K}_3/\mathcal{K}_4=\ker(\mathcal{K}/\mathcal{K}_4\to\mathcal{K}/\mathcal{K}_3)$

Example. (goes back to [Koh])

T. Kohno



$K=\langle\ \rangle \qquad I=\langle\ =\ -\ \rangle$

$$(K/I^{p+1})^\star=(\text{invariants of type }p)=:\mathcal{V}_p$$

$$(I^p/I^{p+1})^\star=\mathcal{V}_p/\mathcal{V}_{p-1}\qquad C=\langle t^{ij} \mid t^{ij}=t^{ji}\rangle=\langle\ \rangle$$

$$\ker\bar\mu_2=\langle [t^{ij},t^{kl}]=0=[t^{ij},t^{ik}+t^{jk}]\rangle=\langle 4T\text{ relations}\rangle$$

$$A=\begin{pmatrix}\text{horizontal chord dia-}\\ \text{grams mod 4T}\end{pmatrix}=\langle\ \rangle\Big/4T$$

Z: universal finite type invariant, the Kontsevich integral.

2-Injectivity. A (one-sided infinite) sequence

$$\cdots \longrightarrow K_{p+1} \xrightarrow{\delta_{p+1}} K_p \xrightarrow{\delta_p} \cdots \longrightarrow K_0=K$$

is "injective" if for all $p>0$, $\ker\delta_p=0$. It is "2-injective" if its "1-reduction"

$$\cdots \longrightarrow \frac{K_{p+1}}{\ker\delta_{p+1}} \xrightarrow{\bar\delta_{p+1}} \frac{K_p}{\ker\delta_p} \xrightarrow{\bar\delta_p} \frac{K_{p-1}}{\ker\delta_{p-1}} \longrightarrow \cdots$$

is injective; i.e. if for all p, $\ker(\delta_p\circ\delta_{p+1})=\ker\delta_{p+1}$. A pair (K,I) is "2-injective" if its singularity tower is 2-injective.

~~**2-Locality.** The pair (K,I) is "2-local" if~~ the sequence

Prop 1

$$R_p:=\bigoplus_{j=1}^{p-1}\left(I^{:j-1}:R_2:I^{:p-j-1}\right) \xrightarrow{\partial} I^{:p} \xrightarrow{\mu_p} I^{:p-1}$$

is exact, where $R_2:=\ker\mu : I^{:2}\to I$.; so (K,I) is "2-local".

Proof of Prop 1

Prop 2

~~**Theorem 1.**~~ If (K,I) is 2-local and 2-injective, it is quadratic.

Proof. Staring at the 1-reduced sequence $\frac{I^{:p+1}}{\ker\mu_{p+1}} \xrightarrow{\mu_{p+1}} \frac{I^{:p}}{\ker\mu_p} \xrightarrow{\mu_p} \cdots \longrightarrow K$, get $\frac{I^p}{I^{p+1}}\simeq \frac{I^{:p}/\ker\mu_p}{\mu(I^{:p+1}/\ker\mu_{p+1})}\simeq\frac{I^{:p}}{\mu(I^{:p+1})+\ker\mu_p}$. But trivially $\frac{I^{:p}}{\mu(I^{:p+1})}\simeq (I/I^2)^{\otimes p}$, so the above is $(I/I^2)^{\otimes p}/\sum\left(I^{:j-1}:R_2:I^{:p-j-1}\right)$. But that's the degree p piece of $Q(K)$.

proved

Prop 1, the free case. If J is an augmentation ideal in $K=F=\langle x_i\rangle$, denote $F\to F/J=\mathbb{F}$ by $x\mapsto[x]$, and define $\psi: F\to F$ by $x_i\mapsto x_i+[x_i]$.

$x \mapsto \lfloor x \rfloor$, and define $\Psi : F \to F$ by $x_i \mapsto x_i + \lfloor x_i \rfloor$.
Then $J_0 = \Psi(J) = \{w \in F : \deg w > 0\}$. For J_0 it easy to check that $R_2 = R_p = 0$, and hence the same is true for every J.

Prop 1, the general case. If $K = F/M$ and $I \subset K$, then $I = J/M$ where $J \subset F$. Then $I^{:p} = J^{:p} / \sum J^{:j-1} : M : J^{:p-j}$ and we have

$$\begin{tikzcd}
J^{:p} \arrow[r, "\mu^F", "1-1"'] \arrow[d, "\pi_p", "\text{onto}"'] & J^{:p-1} \arrow[d, "\pi_{p-1}"', "\text{onto}"] \\
I^{:p} = J^{:p} / \sum J^{:\bullet} : M : J^{:\bullet} \arrow[r, "\mu^K"] & I^{:p-1} = J^{:p-1} / \sum J^{:\bullet} : M : J^{:\bullet}
\end{tikzcd}$$

So
$$\begin{aligned}
\ker(\mu^K) &= \pi_p\left(\mu_F^{-1}\left(\ker(\pi_{p-1})\right)\right) = \\
&= \pi_p\left(\sum \mu_F^{-1}\left(J^{:j-1} : M : J^{:p-j-1}\right)\right) \\
&= \pi_p\left(\sum J^{:j-1} : \mu_F^{-1}(M) : J^{:p-j-1}\right) \\
&= \sum I^{:j-1} : R_2 : I^{:p-j-1}.
\end{aligned}$$

re-confirm after typesetting.

Footnotes

1. Following a homonymous paper and thesis by Peter Lee [Lee]. All serious work here is his and was extremely patiently conveyed to DBN; page design by ~~DBN~~ the latter.

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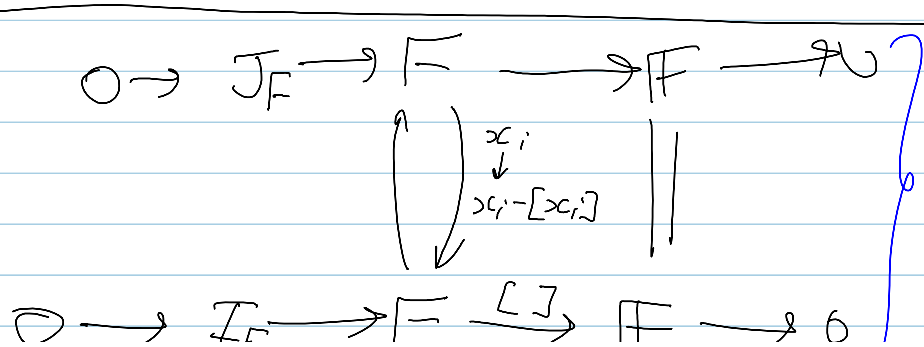

"In the free case, all augmentation ideals are equivalent"

$$\begin{tikzcd}
0 \arrow[r] & I_F \arrow[r] & F \arrow[r, "{[\,]}"] \arrow[d] & \mathbb{F} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & I \arrow[r] & K \arrow[r] & \mathbb{F} \arrow[r] & 0
\end{tikzcd}$$

ideals are equivalent.

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J_F \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & \mathbb{F} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & I \arrow[r] & K \arrow[r] & \mathbb{F} \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
& \sum J^{j-1} \otimes M \otimes J^{p-1-j} \arrow[d] & \\
J^{:p} \arrow[r] \arrow[d] & J^{:p-1} \arrow[r] \arrow[d] & 0 \\
I^{:p} \arrow[r] & I^{:p-1} &
\end{tikzcd}$$

Claim If

$$\begin{tikzcd}
A \arrow[r, hook, "\iota"] \arrow[d, "\pi_A"] & B \arrow[d, "\pi_B"] \\
C \arrow[r, "\gamma"] & D
\end{tikzcd}$$

Then $\ker \gamma =$

$\pi_A(\iota^{-1} \ker(\pi_B)) =$

$= \ . \ - \ - \ -$

$$\begin{tikzcd}
& A \arrow[dl] \arrow[dr] & \\
A/B \arrow[rr, "\mu"] & & A/C
\end{tikzcd}$$

$\ker \mu = C/B$

Recycling:

Proof of Prop 1. The free case If $K = F = \langle x_i \rangle$ is free, replace its augmentation ideal I_F by $J_F := \langle w \in F : \deg w > 0 \rangle$, using

$$(F, I_F) \underset{x_{i-1} \leftarrow\!\shortmid x_i}{\overset{x_i \mapsto x_{i+1}}{\rightleftarrows}} (F, J_F).$$

For J_F trivially $R_2 = 0$ and so is R_p, so The same holds for J_F.

Can this be skipped if all maximal ideals in F are "the same"?