Oregon Handout as of July 23, 2011

July-23-11
4:52 PM

Weakly?

Keep as is, for now.

The Pure Virtual Braid Group is Quadratic[1]

Dror Bar-Natan and Peter Lee in Oregon, August 2011

http://www.math.toronto.edu/~drorbn/Talks/Oregon-1108/
Foots & refs on PDF version, page 2.

Presented to the great algebra masters of the Oregon School, in pursuit of their wisdom and advice, in acceptance that they know all and have seen all, and in dread that we will inflict boredom upon them.

Let K be an algebra over a field $\mathbb{F}$ with $\operatorname{char}\mathbb{F}=0$, and let $I\subset K$ be an "augmentation ideal"; meaning $K/I=\mathbb{F}$.

Definition. Say that K is **quadratic** if its associated graded $\operatorname{gr}K=\bigoplus_{m=0}^{\infty}$ is a quadratic algebra. Alternatively, let $A=Q(K)=\langle V=I/I^2\rangle/\ker(\mu_{11}:V\otimes V\to I^2/I^3)$ be the "quadratic approximation" to K (Q is a lovely functor). Then K is quadratic iff the obvious $\mu:A\to\operatorname{gr}K$ is an isomorphism.

Why Care? • In abstract generality, $\operatorname{gr}K$ is a simplified version of K and if it is quadratic it is as simple as it may be without being silly. • In some concrete (somewhat generalized) knot theoretic cases, A is a space of "universal Lie algebraic formulas" and the "primary approach" for proving quadraticity, constructing an appropriate homomorphism $Z:K\to\hat{A}$, becomes wonderful mathematics:

strong

beatify & add refs

K	u-knots and braids	v-knots	w-knots
A	metrized Lie algebras	Lie bialgebras	f.d. Lie algebras
Z	associators	Etingof-Kazhdan quantization	Kashiwara-Vergne-Alekseev-Torrosian

PvB_n

Flow chart.

$$\begin{tikzcd}
\boxed{\text{Any } (K,I)} \arrow[r, "\text{Thm 2}"] & \boxed{\text{2-local}} \arrow[r, "\text{Thm 1}"] \arrow[d, dashed] & \boxed{\text{Weakly Quadratic}} \\
\boxed{K=PvB_n} \arrow[r, "\text{Thm 4 by Peter}"] & \boxed{\text{Syz Comp}} \arrow[r, "\text{Thm 3}"] & \boxed{\text{2-injective}} \arrow[u, bend left]
\end{tikzcd}$$

Define 2-injective
(first for general SQA's then for (K,I)).

Define 2-local

Prove Thm 1

Prove Thm 2.

Abstract Generalities. (K,I): an algebra and an "augmentation ideal" in it. $\hat{K}:=\varprojlim K/I^m$ the "I-adic completion". $\operatorname{gr}_I K:=\widehat{\bigoplus}I^m/I^{m+1}$ has a product μ, especially, $\mu_{11}:(C=I/I^2)^{\otimes 2}\to I^2/I^3$. The "quadratic approximation" $\mathcal{A}_I(K):=\widehat{FC}/\langle\ker\mu_{11}\rangle$ of K surjects using μ on $\operatorname{gr}K$.

The Prized Object. A "homomorphic $\mathcal{A}$-expansion": a homomorphic filterred $Z:K\to\mathcal{A}$ for which $\operatorname{gr}Z:\operatorname{gr}K\to\mathcal{A}$ inverts μ. (given homomorphicity, $\Leftrightarrow$ Z induces the identity on $I/I^2=C$)

Dror's Dream. All interesting graded objects and equations, especially those around quantum groups, arise this way.

Example.

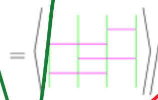
T. Kohno

$K=$  $I=$ 

$$(K/I^{m+1})^\star=(\text{invariants of type } m)=:\mathcal{V}_m$$

$$(I^m/I^{m+1})^\star=\mathcal{V}_m/\mathcal{V}_{m-1}\qquad C=\langle t^{ij}\,|\,t^{ij}=t^{ji}\rangle=$$ 

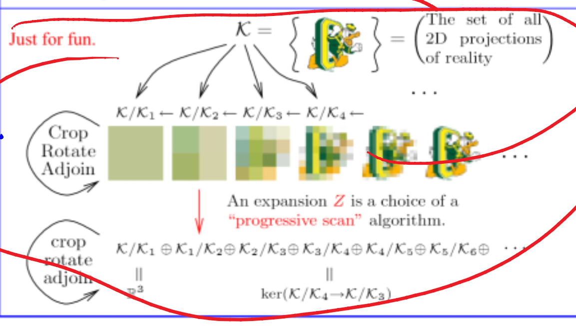
$$\ker\mu_{11}=\langle[t^{ij},t^{kl}]=0=[t^{ij},t^{ik}+t^{jk}]\rangle=\langle\text{4T relations}\rangle$$

$$\mathcal{A}\ \cancel{=\mathcal{A}_n}=\left(\begin{array}{l}\text{horizontal chord dia-}\\\text{grams mod 4T}\end{array}\right)=$$  4T

Z: universal finite type invariant, the Kontsevich integral.

Why Prized? Sizes K and shows it "as big" as $\mathcal{A}$; reduces "topological" questions to quadratic algebra questions; gives life and meaning to questions in graded algebra; universalizes those more than "universal enveloping algebras" and allows for richer quotients.

Just for fun.



Def. 2-injective:

$$I^{:p+1} \to I^{:p} \to I^{:p-1} \to \cdots \to I \to k$$

is 2-injective, where

$$K_{p+1} \xrightarrow{d_{p+1}} K_p \xrightarrow{d_p} K_{p-1} \xrightarrow{d_{p-1}} \cdots \to k$$

is 2-injective means that $\ker d_p \circ d_{p+1} = \ker d_{p+1}$;

$\iff$ it's reduction

$$\frac{K_{p+1}}{\ker d_{p+1}} \to \frac{K_p}{\ker d_p} \to \frac{K_{p-1}}{\ker d_{p-1}} \to \cdots$$

is injective.

Def. 2-local:

$$\ker\left(I^{:p} \xrightarrow{\mu} I^{:p-1}\right) = \sum_{j=1}^{p-1} I^{:j-1} : (\ker I^{:2} \to I) : I^{:p-j-1}$$

Thm! 2-injective + 2 local $\Rightarrow$ quadratic

Pf.

$$(I/I^2)^{\otimes p} \xrightarrow{\sim} I^{:p} / \mu(I^{:p+1})$$

$$\frac{I^p}{I^{p+1}} \simeq \frac{I^{:p}/\ker \mu_p}{\mu(I^{:p+1}/\ker \mu_{p+1})} = \frac{I^{:p}}{\mu(I^{:p+1}) + \ker \mu_p}$$

$$= (I/I^2)^{\otimes p} / \ker \mu_p = \cdots$$

Syzygy complete at: [Quadraticity: 2-Injectivity]

Footnotes

1. Following a homonymous paper and thesis by Peter Lee (see ???). All serious work here is his; page design by DBN.

References

[BEER] L. Bartholdi, B. Enriquez, P. Etingof, and E. Rains, *Groups and Lie algebras corresponding to the YangBaxter equations,* Jornal of Algebra **305-2** (2006) 742-764, arXiv:math.RA/0509661.