Oregon Handout as of August 4, 2011

August-04-11
10:40 AM

The Pure Virtual Braid Group is Quadratic[1]

Abstract Generalities

Dror Bar–Natan and Peter Lee in Oregon, August 2011
http://www.math.toronto.edu/~drorbn/Talks/Oregon-1108/
foots & refs on PDF version, page 3

Let K be a unital algebra over a field $\mathbb{F}$ with char $\mathbb{F}=0$, and let $I\subset K$ be an "augmentation ideal"; so $K/I \xrightarrow[\epsilon]{\sim} \mathbb{F}$.

Definition. Say that K is quadratic if its associated graded $\operatorname{gr} K=\bigoplus_{p=0}^{\infty} I^p/I^{p+1}$ is a quadratic algebra. Alternatively, let $A=q(K)=\langle V=I/I^2\rangle/\langle R_2=\ker(\bar\mu_2: V\otimes V\to I^2/I^3)\rangle$ be the "quadratic approximation" to K (q is a lovely functor). Then K is quadratic iff the obvious $\mu: A\to \operatorname{gr} K$ is an isomorphism. If G is a group, we say it is quadratic if its group ring is, with its augmentation ideal.

The Overall Strategy. Consider the "singularity tower" of (K,I) (here ":" means $\otimes_K$ and μ is (always) multiplication):

$$\cdots\; I^{:p+1} \xrightarrow{\mu_{p+1}} I^{:p} \xrightarrow{\mu_p} I^{:p-1} \longrightarrow \cdots \longrightarrow K$$

We care as $\operatorname{im}(\mu^p=\mu_1\circ\cdots\circ\mu_p)=I^p$, so $I^p/I^{p+1}=\operatorname{im}\mu^p/\operatorname{im}\mu^{p+1}$. Hence we ask:

- How injective is this tower?
- What's $I^{:p}/\mu(I^{:p+1})$?

Lemma. $I^{:p}/\mu(I^{:p+1})\simeq (I/I^2)^{\otimes p}=V^{\otimes p}$.

Flow Chart.

$$\begin{array}{l} \boxed{\text{Any } (K,I)} \xrightarrow{\text{Prop 1}} \boxed{\text{2-local}} \xrightarrow{\text{Prop 2}} \boxed{\text{Quadratic}} \\ \boxed{K=PvB_n} \xrightarrow[\text{by Peter}]{\text{Thm S}} \boxed{\text{Hutchings' Criterion}} \longrightarrow \boxed{\text{2-injective}} \end{array}$$

Proposition 1. The sequence

$$\mathfrak{R}_p := \bigoplus_{j=1}^{p-1}\left(I^{:j-1}:\mathfrak{R}_2:I^{:p-j-1}\right) \xrightarrow{\partial} I^{:p} \xrightarrow{\mu_p} I^{:p-1}$$

is exact, where $\mathfrak{R}_2 := \ker\mu: I^{:2}\to I$; so (K,I) is "2-local".

The Free Case. If J is an augmentation ideal in $K=F=\langle x_i\rangle$, define $\psi: F\to F$ by $x_i\mapsto x_i+\epsilon(x_i)$. Then $J_0:=\psi(J)$ is $\{w\in F:\deg w>0\}$. For J_0 it is easy to check that $\mathfrak{R}_2=\mathfrak{R}_p=0$, and hence the same is true for every J.

The General Case. If $K=F/\langle M\rangle$ (where M is a vector space of "moves") and $I\subset K$, then $I=J/\langle M\rangle$ where $J\subset F$. Then $I^{:p}=J^{:p}/\sum J^{:j-1}:\langle M\rangle:J^{:p-j}$ and we have

$$\begin{tikzcd} J^{:p} \arrow[rr, "\mu_F", "\text{1-1}"'] \arrow[d, "\pi_p", "\text{onto}"'] & & J^{:p-1} \arrow[d, "\text{onto}", "\pi_{p-1}"'] \\ I^{:p}=J^{:p}/\sum J^{:}:\langle M\rangle:J^{:} \arrow[rr, "\mu"] & & I^{:p-1}=J^{:p-1}/\sum J^{:}:\langle M\rangle:J^{:} \end{tikzcd}$$

So $\ker(\mu)=\pi_p\left(\mu_F^{-1}(\ker\pi_{p-1})\right)=\pi_p\left(\sum\mu_F^{-1}(J^{:}:\langle M\rangle:J^{:})\right)=\sum\pi_p\left(J^{:}:\mu_F^{-1}\langle M\rangle:J^{:}\right)=\sum I^{:}:\mathfrak{R}_2:I^{:}=:\sum_{j=1}^{p-1}\mathfrak{R}_{p,j}$.

2-Injectivity. A (one-sided infinite) sequence

$$\cdots\longrightarrow K_{p+1}\xrightarrow{\delta_{p+1}} K_p \xrightarrow{\delta_p}\cdots\longrightarrow K_0=K$$

is "injective" if for all $p>0$, $\ker\delta_p=0$. It is "2-injective" if its "1-reduction"

$$\cdots\longrightarrow \frac{K_{p+1}}{\ker\delta_{p+1}} \xrightarrow{\bar\delta_{p+1}} \frac{K_p}{\ker\delta_p} \xrightarrow{\bar\delta_p} \frac{K_{p-1}}{\ker\delta_{p-1}} \longrightarrow\cdots$$

is injective; i.e. if for all p, $\ker(\delta_p\circ\delta_{p+1})=\ker\delta_{p+1}$. A pair (K,I) is "2-injective" if its singularity tower is 2-injective.

Why Care? • In abstract generality, $\operatorname{gr} K$ is a simplified version of K and if it is quadratic it is as simple as it may be without being silly. • In some concrete (somewhat generalized) knot theoretic cases, A is a space of "universal Lie algebraic formulas" and the "primary approach" for proving (strong) quadraticity, constructing an appropriate homomorphism $Z: K\to\hat{A}$, becomes wonderful mathematics:

	u-Knots and Braids	v-Knots	w-Knots
K			
A	Metrized Lie algebras [BN1]	Lie bialgebras [Hav]	Finite dimensional Lie algebras [BN3]
Z	Associators [Dri, BND]	Etingof-Kazhdan quantization [EK, BN2]	Kashiwara-Vergne-Alekseev-Torossian [KV, AT]

Proposition 2. If (K,I) is 2-local and 2-injective, it is quadratic.

Proof. Staring at the 1-reduced sequence $\frac{I^{:p+1}}{\ker\mu_{p+1}}\xrightarrow{\mu_{p+1}}\frac{I^{:p}}{\ker\mu_p}\xrightarrow{\mu_p}\cdots\longrightarrow K$, get $\frac{I^p}{I^{p+1}}\simeq\frac{I^{:p}/\ker\mu_p}{\mu(I^{:p+1}/\ker\mu_{p+1})}\simeq\frac{I^{:p}}{\mu(I^{:p+1})+\ker\mu_p}$. But $\frac{I^{:p}}{\mu(I^{:p+1})}\simeq(I/I^2)^{\otimes p}$, so the above is $(I/I^2)^{\otimes p}/\sum\left(I^{:j-1}:\mathfrak{R}_2:I^{:p-j-1}\right)$. But that's the degree p piece of $q(K)$.

The X Lemma (inspired by [Hut]).

$$\begin{tikzcd} A_0 \arrow[dr, "\alpha_0"] & & C_0 \\ & B \arrow[ur, "\beta_0"] \arrow[dr, "\beta_1"] & \\ A_1 \arrow[ur, "\alpha_1"] & & C_1 \end{tikzcd}$$

If the above diagram is Conway ($\asymp$) exact, then its two diagonals have the same "2-injectivity defect". That is, if $A_0\to B\to C_0$ and $A_1\to B\to C_1$ are exact, then $\ker(\beta_1\circ\alpha_0)/\ker\alpha_0\simeq\ker(\beta_0\circ\alpha_1)/\ker\alpha_1$.

Proof. $\frac{\ker(\beta_1\circ\alpha_0)}{\ker\alpha_0}\xrightarrow[\alpha_0]{\sim}\ker\beta_1\cap\operatorname{im}\alpha_0$
$=\ker\beta_0\cap\operatorname{im}\alpha_1\xleftarrow[\alpha_1]{\sim}\frac{\ker(\beta_0\circ\alpha_1)}{\ker\alpha_1}$.

The Hutchings Criterion [Hut]. The singularity tower of (K,I) is 2-injective iff on the right, $\ker(\pi\circ\partial)=\ker(\partial)$. That is, iff every "diagrammatic syzygy" is also a "topological syzygy".

$$\begin{tikzcd} \mathfrak{R}_p \arrow[dr, "\partial"] & & I^{:p-1} \\ & I^{:p} \arrow[ur, "\mu_p"] \arrow[dr, "\pi"] & \\ I^{:p+1} \arrow[ur, "\mu_{p+1}"] & & V^{\otimes p} \end{tikzcd}$$

Conclusion. **We need to know that (K,I) is "syzygy complete"** — that every diagrammatic syzygy is also a topological syzygy, that $\ker(\pi\circ\partial)=\ker(\partial)$.

James Gillespie's Sightline #2 (1984) is a syzygy, and (arguably) Toronto's largest sculpture. Find it next to University of Toronto's Hart House.

The Pure Virtual Braid Group is Quadratic, II

Examples and Interpretations

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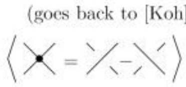
Example. (goes back to [Koh])

T. Kohno

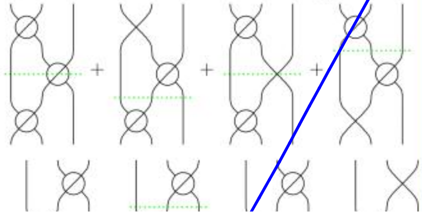
$K=$  $I=$ 

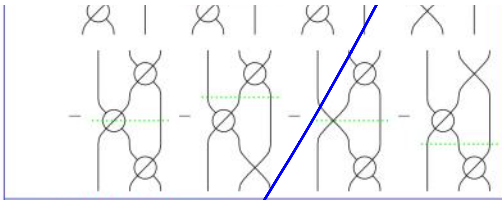
$(K/I^{p+1})^\star=(\text{invariants of type } p)=:\mathcal{V}_p$

$(I^p/I^{p+1})^\star=\mathcal{V}_p/\mathcal{V}_{p-1}\quad V=\langle t^{ij}\,|\,t^{ij}=t^{ji}\rangle=\langle\,|\ \vdash\!\dashv\,\rangle$

$\ker\bar\mu_2=\langle[t^{ij},t^{kl}]=0=[t^{ij},t^{ik}+t^{jk}]\rangle=\langle\text{4T relations}\rangle$

$\mathfrak{R}_2(PvB_n)$ is generated as a vector space by C^{ij}_{kl} and

$Y_{ijk}:=$ 



$(I^p/I^{p+1})^\star = \mathcal{V}_p/\mathcal{V}_{p-1}$ $\quad V = \langle t^{ij} | t^{ij} = t^{ji} \rangle = \langle | \ \vdash\!\dashv \rangle$

$\ker \bar\mu_2 = \langle [t^{ij}, t^{kl}] = 0 = [t^{ij}, t^{ik} + t^{jk}] \rangle = \langle 4\text{T relations} \rangle$

$A = q(K) = \begin{pmatrix}\text{horizontal chord dia-}\\ \text{grams mod 4T}\end{pmatrix} = \langle \cdots \rangle / 4\text{T}$

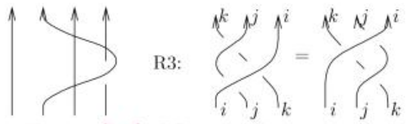
Z: universal finite type invariant, the Kontsevich integral.

PvB_n is the group

$$\langle \sigma_{ij} : 1 \le i \ne j \le n \rangle \Big/ \begin{matrix}\sigma_{ij}\sigma_{ik}\sigma_{jk} = \sigma_{jk}\sigma_{ik}\sigma_{ij}\\ \sigma_{ij}\sigma_{kl} = \sigma_{kl}\sigma_{ij}\end{matrix}$$

L. Kauffman [Kau, KL]

of "pure virtual braids" ("braids when you look", "blunder braids"):

$\sigma_{24} =$ 

The Main Theorem [Lee]. PvB_n is quadratic.

$A_n = q(PvB_n)$.

[GPV] 

Goussarov-Polyak-Viro

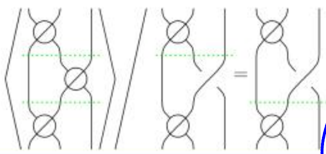
$I =$  with $\tilde\sigma_{ij} = \sigma_{ij} - 1$, the "semi-virtual crossing".

$V = I/I^2 = \langle \text{v-braids with one semi-virtual crossing} \rangle / (\ldots) = \langle a_{ij} \rangle_{1 \le i \ne j \le n}$ $\qquad a_{24} =$

$A_n = TV/\langle [a_{ij}, a_{ik}] + [a_{ij}, a_{jk}] + [a_{ik}, a_{jk}],\ c^{ij}_{kl} = [a_{ij}, a_{kl}] \rangle$,

$y_{ijk} = \ldots + \ldots + \ldots - \ldots - \ldots - \ldots$

$I^{:p}$.



Figuring out $\mathfrak{R}_2$ and R_2.

$$\ker \mu = \pi_2\left(\mu_F^{-1}(M)\right)$$

is in principle computable, and then R_2 follows as $V^{\otimes 2} = (I/I^2)^{\otimes 2} = I^{:2}/\mu_3(I^{:3})$.

$$\begin{array}{ccc} J^{:2} & \xrightarrow[\text{1-1}]{\mu_F} & J \supset M \\ \downarrow \pi_2 & & \downarrow \pi_1 \\ I^{:2} & \xrightarrow{\mu} & I = J/\langle M \rangle \end{array}$$

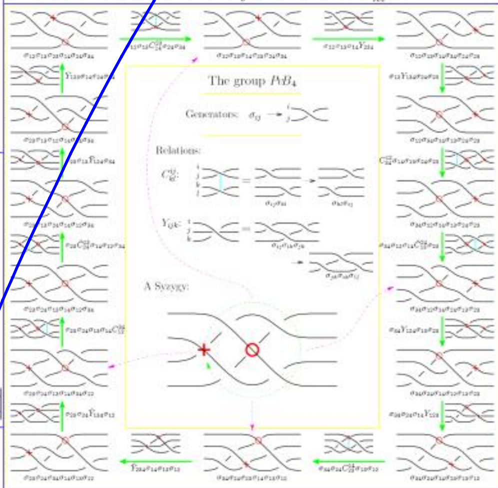
Syzygy Completeness, for PvB_n, means:

$$\mathfrak{R}_p = \bigoplus_{j=1}^{p-1} \mathfrak{R}_{p,j} \xrightarrow{\partial} I^{:p} \xrightarrow{\pi} V^{\otimes p}$$

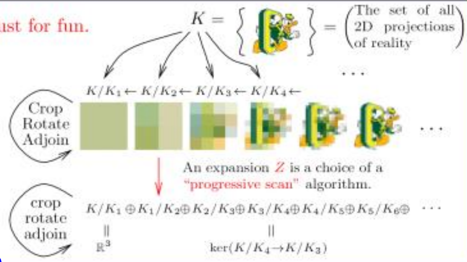
$\{\tilde\sigma_{12} : \underline{Y_{345}} : \sigma_{67}\tilde\sigma_{89} : \ldots\} \longrightarrow$

$\{\tilde\sigma_{12} : Y_{345} : \sigma_{67}\tilde\sigma_{89} : \ldots\} \longrightarrow \{a_{12}y_{345}a_{89}\ldots\}$

Is every relation between the y_{ijk}'s and the c^{ij}_{kl}'s also a relation between the Y_{ijk}'s and the C^{ij}_{kl}'s?



Just for fun.



The Pure Virtual Braid Group is Quadratic, III

Scratch / Recycling

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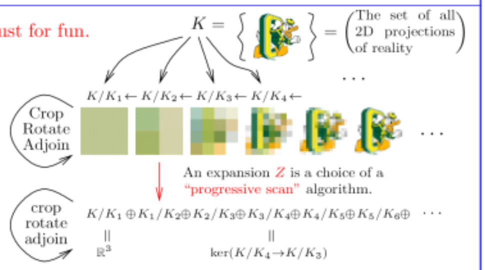
$\mathfrak{R}_2$'s simpler than it seems! It's an "augmentation bimodule" ($I\mathfrak{R}_2 = 0 = \mathfrak{R}_2 I$ thus $xr = \epsilon(x)r = r\epsilon(x) = rx$ for $x \in K$ and $r \in \mathfrak{R}_2$), and hence $\mathfrak{R}_2 = \pi_2(\mu_F^{-1}M)$.

$$\begin{array}{ccc} J^{:2} & \xrightarrow[\text{1-1}]{\mu_F} & J \supset M \\ \downarrow \pi_2 & & \downarrow \pi_1 \\ I^{:2} & \xrightarrow{\mu} & I = J/\langle M \rangle \end{array}$$

$\mathfrak{R}_p$'s simpler than it seems! In $\mathfrak{R}_{p,j} = I^{:j-1} : \mathfrak{R}_2 : I^{p-j-1}$ the I factors may be replaced by $V = I/I^2$. Hence

$$\mathfrak{R}_p \simeq \bigoplus_{j=1}^{p-1} V^{\otimes j-1} \otimes \pi_2(\mu_F^{-1}M) \otimes V^{\otimes p-j-1}.$$

Just for fun.



K =
= The set of all 2D projections of reality
K/K1 ← K/K2 ← K/K3 ← K/K4 ←
Crop Rotate Adjoin
An expansion Z is a choice of a "progressive scan" algorithm.
crop rotate adjoin
K/K1 ⊕ K1/K2 ⊕ K2/K3 ⊕ K3/K4 ⊕ K4/K5 ⊕ K5/K6 ⊕ ⋯
ℝ3
ker(K/K4 → K/K3)

Footnotes

1. Following a homonymous paper and thesis by Peter Lee [Lee]. All serious work here is his and was extremely patiently explained by him to DBN. Page design by the latter.

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