Oregon Handout as of August 3, 2011

August-03-11
3:17 PM

$\sqrt{}\ K/I \xrightarrow[\sim]{\in} \mathbb{F}$

The Pure Virtual Braid Group is Quadratic[1]

Abstract Generalities

Dror Bar–Natan and Peter Lee in Oregon, August 2011

http://www.math.toronto.edu/~drorbn/Talks/Oregon-1108/
foots & refs on PDF version, page 3

Let K be a unital algebra over a field $\mathbb{F}$ with char $\mathbb{F}=0$, and let $I\subset K$ be an "augmentation ideal"; meaning $K/I=\mathbb{F}$.

Definition. Say that K is quadratic if its associated graded $\operatorname{gr} K=\bigoplus_{p=0}^{\infty} I^p/I^{p+1}$ is a quadratic algebra. Alternatively, let $A=q(K)=\langle V=I/I^2\rangle/\langle R_2=\ker\bar\mu_2 : V\otimes V\to I^2/I^3\rangle$ be the "quadratic approximation" to K (q is a lovely functor). Then K is quadratic iff the obvious $\mu: A\to \operatorname{gr} K$ is an isomorphism. If G is a group, we say it is quadratic if its group ring is, with its augmentation ideal.

blue

The Overall Strategy. Consider the "singularity tower" of (K,I) (here ":" means $\otimes_K$ and μ is (always) multiplication):

$$\cdots\ I^{:p+1}\xrightarrow{\mu_{p+1}} I^{:p}\xrightarrow{\mu_p} I^{:p-1}\longrightarrow\cdots\longrightarrow K$$

We care as $\operatorname{im}(\mu^p=\mu_1\circ\cdots\circ\mu_p)=I^p$, so $I^p/I^{p+1}=\operatorname{im}\mu^p/\operatorname{im}\mu^{p+1}$. Hence we ask:

- How injective is this tower?
- What's $I^{:p}/\mu(I^{:p+1})$?

Lemma. $I^{:p}/\mu(I^{:p+1})\simeq (I/I^2)^{\otimes p}=V^{\otimes p}$.

Flow Chart.

$$\begin{array}{ccccccc}
\boxed{\text{Any } (K,I)} & \xrightarrow[1]{\text{Prop}} & \boxed{\text{2-local}} & \xrightarrow{\text{Prop 2}} & \boxed{\text{Quadratic}} \\
\boxed{K=PvB_n} & \xrightarrow[\text{by Peter}]{\text{Thm S}} & \boxed{\text{Hutchings' Criterion}} & \longrightarrow & \boxed{\text{2-injective}}
\end{array}$$

Proposition 1. The sequence

$$\mathfrak{R}_p:=\bigoplus_{j=1}^{p-1}\left(I^{:j-1}:\mathfrak{R}_2:I^{:p-j-1}\right)\xrightarrow{\partial} I^{:p}\xrightarrow{\mu_p} I^{:p-1}$$

is exact, where $\mathfrak{R}_2:=\ker\mu: I^{:2}\to I$; so (K,I) is "2-local".

The Free Case. If J is an augmentation ideal in $K=F=\langle x_i\rangle$, ~~denote $F\to F/J=\mathbb{F}$ by $x\mapsto[x]$ and~~ define $\psi:F\to F$ by $x_i\mapsto x_i+\cancel{[x_i]}$ $\epsilon(x_i)$. Then $J_0:=\psi(J)$ is $\{w\in F:\deg w>0\}$. For J_0 it is easy to check that $\mathfrak{R}_2=\mathfrak{R}_p=0$, and hence the same is true for every J. $\langle M\rangle$ $\langle M\rangle$

The General Case. If $K=F/M$ and $I\subset K$, then $I=J/M$ where $J\subset F$. Then $I^{:p}=J^{:p}/\sum J^{:j-1}\langle M\rangle J^{:p-j}$ and we have

$$\begin{array}{ccc}
J^{:p} & \xrightarrow[\text{1-1}]{\mu_F} & J^{:p-1}\\
{\scriptstyle\text{onto}}\downarrow{\scriptstyle\pi_p} & & {\scriptstyle\pi_{p-1}}\downarrow{\scriptstyle\text{onto}}\\
I^{:p}=J^{:p}/\sum J^{:}\langle M\rangle J^{:} & \xrightarrow{\mu} & I^{:p-1}=J^{:p-1}/\sum J^{:}\langle M\rangle J^{:}
\end{array}$$

So $\ker(\mu)=\pi_p\left(\mu_F^{-1}(\ker\pi_{p-1})\right)=\pi_p\left(\sum\mu_F^{-1}\left(J^{:}\langle M\rangle J^{:}\right)\right)=\sum\pi_p\left(J^{:}:\mu_F^{-1}\langle M\rangle J^{:}\right)=\sum I^{:}:\mathfrak{R}_2:I^{:}$.

2-Injectivity. A (one-sided infinite) sequence

$$\cdots\longrightarrow K_{p+1}\xrightarrow{\delta_{p+1}} K_p\xrightarrow{\delta_p}\cdots\longrightarrow K_0=K$$

is "injective" if for all $p>0$, $\ker\delta_p=0$. It is "2-injective" if its "1-reduction"

$$\cdots\longrightarrow\frac{K_{p+1}}{\ker\delta_{p+1}}\xrightarrow{\bar\delta_{p+1}}\frac{K_p}{\ker\delta_p}\xrightarrow{\bar\delta_p}\frac{K_{p-1}}{\ker\delta_{p-1}}\longrightarrow\cdots$$

is injective; i.e. if for all p, $\ker(\delta_p\circ\delta_{p+1})=\ker\delta_{p+1}$. A pair (K,I) is "2-injective" if its singularity tower is 2-injective.

Why Care? • In abstract generality, $\operatorname{gr} K$ is a simplified version of K and if it is quadratic it is as simple as it may be without being silly. • In some concrete (somewhat generalized) knot theoretic cases, A is a space of "universal Lie algebraic formulas" and the "primary approach" for proving (strong) quadraticity, constructing an appropriate homomorphism $Z:K\to\hat A$, becomes wonderful mathematics:

K	u-Knots and Braids	v-Knots	w-Knots
A	Metrized Lie algebras [BN1]	Lie bialgebras [Hav]	Finite dimensional Lie algebras [BN3]
Z	Associators [Dri, BND]	Etingof-Kazhdan quantization [EK, BN2]	Kashiwara-Vergne-Alekseev-Torossian [KV, AT]

Proposition 2. If (K,I) is 2-local and 2-injective, it is quadratic.

Proof. Staring at the 1-reduced sequence $\frac{I^{:p+1}}{\ker\mu_{p+1}}\xrightarrow{\mu_{p+1}}\frac{I^{:p}}{\ker\mu_p}\xrightarrow{\mu_p}\cdots\longrightarrow K$, get $\frac{I^p}{I^{p+1}}\simeq\frac{I^{:p}/\ker\mu_p}{\mu(I^{:p+1}/\ker\mu_{p+1})}\simeq\frac{I^{:p}}{\mu(I^{:p+1})+\ker\mu_p}$. But $\frac{I^{:p}}{\mu(I^{:p+1})}\simeq(I/I^2)^{\otimes p}$, so the above is $(I/I^2)^{\otimes p}/\sum\left(I^{:j-1}:\mathfrak{R}_2:I^{:p-j-1}\right)$. But that's the degree p piece of $q(K)$.

The X Lemma (inspired by [Hut]).

$$\begin{array}{ccccc}
A_0 & & & & C_0\\
 & \searrow^{\alpha_0} & & \nearrow^{\beta_0} & \\
 & & B & & \\
 & \nearrow_{\alpha_1} & & \searrow_{\beta_1} & \\
A_1 & & & & C_1
\end{array}$$

Michael Hutchings

will compute homology for food

If the above diagram is Conway ($\asymp$) exact, then its two diagonals have the same "2-injectivity defect". That is, if $A_0\to B\to C_0$ and $A_1\to B\to C_1$ are exact, then $\ker(\beta_1\circ\alpha_0)/\ker\alpha_0\simeq\ker(\beta_0\circ\alpha_1)/\ker\alpha_1$.

Proof. $\frac{\ker(\beta_1\circ\alpha_0)}{\ker\alpha_0}\xrightarrow[\alpha_0]{\sim}\ker\beta_1\cap\operatorname{im}\alpha_0$
$=\ker\beta_0\cap\operatorname{im}\alpha_1\xleftarrow[\alpha_1]{\sim}\frac{\ker(\beta_0\circ\alpha_1)}{\ker\alpha_1}$.

The Hutchings Criterion [Hut]. The singularity tower of (K,I) is 2-injective iff on the right, $\ker(\pi\circ\partial)=\ker(\partial)$. That is, iff every "diagrammatic syzygy" is also a "topological syzygy".

$$\begin{array}{ccccc}
\mathfrak{R}_p & & & & I^{:p-1}\\
 & \searrow^{\partial} & & \nearrow^{\mu_p} & \\
 & & I^{:p} & & \\
 & \nearrow_{\mu_{p+1}} & & \searrow_{\pi} & \\
I^{:p+1} & & & & V^{\otimes p}
\end{array}$$

Conclusion. We need to know that (K,I) is "syzygy complete" — that every diagrammatic syzygy is also a topological syzygy, that $\ker(\pi\circ\partial)=\ker(\partial)$.

James Gillespie's Sightline #2 (1984) is a syzygy, and (arguably) Toronto's largest sculpture. Find it next to University of Toronto's Hart House.

as a ~~bimodule~~ vector space

The Pure Virtual Braid Group is Quadratic, II

Examples and Interpretations

Dror Bar–Natan and Peter Lee in Oregon, August 2011

http://www.math.toronto.edu/~drorbn/Talks/Oregon-1108/

Example. (goes back to [Koh]) $\mathfrak{R}_2(PvB_n)$ is generated by C^{ij}_{kl} and as a ~~K-bimodule~~ vector space

The Pure Virtual Braid Group is Quadratic, II

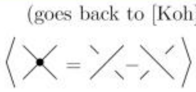
Examples and Interpretations

Dror Bar-Natan and Peter Lee in Oregon, August 2011

http://www.math.toronto.edu/~drorbn/Talks/Oregon-1108/

Example. (goes back to [Koh])

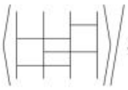
T. Kohno

$K =$  $\quad I =$ 

$(K/I^{p+1})^\star =$ (invariants of type p) $=: \mathcal{V}_p$

$(I^p/I^{p+1})^\star = \mathcal{V}_p/\mathcal{V}_{p-1} \quad V = \langle t^{ij} | t^{ij} = t^{ji} \rangle = \langle | \,|\!\!-\!\!|\, \rangle$

$\ker \bar\mu_2 = \langle [t^{ij}, t^{kl}] = 0 = [t^{ij}, t^{ik} + t^{jk}] \rangle = \langle$4T relations$\rangle$

$A = q(K) =$ (horizontal chord diagrams mod 4T) $= \langle$  $\rangle / 4T$

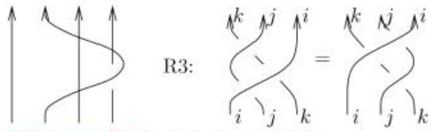
Z: universal finite type invariant, the Kontsevich integral.

PvB_n is the group

$$\langle \sigma_{ij} : 1 \le i \ne j \le n \rangle \Big/ \begin{matrix} \sigma_{ij}\sigma_{ik}\sigma_{jk} = \sigma_{jk}\sigma_{ik}\sigma_{ij} \\ \sigma_{ij}\sigma_{kl} = \sigma_{kl}\sigma_{ij} \end{matrix}$$

L. Kauffman [Kau, KL]

of "pure virtual braids" ("braids when you look", "blunder braids"):

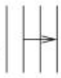
$\sigma_{24} =$ R3: 

The Main Theorem [Lee]. PvB_n is quadratic.

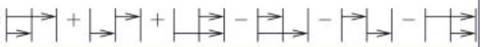
$A_n = q(PvB_n)$. [GPV]

Goussarov-Polyak-Viro

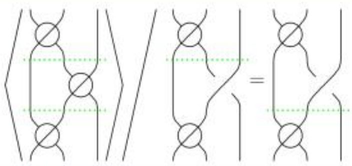
$I =$  with $\bar\sigma_{ij} = \sigma_{ij} - 1$, the "semi-virtual crossing".

$V = I/I^2 = \langle$ v-braids with one semi-virtual crossing $\rangle / (\text{virtual crossing relation})$ $= \langle a_{ij} \rangle_{1 \le i \ne j \le n}$ $\quad a_{24} =$ 

$A_n = TV/\langle [a_{ij}, a_{ik}] + [a_{ij}, a_{jk}] + [a_{ik}, a_{jk}],\ c^{ij}_{kl} = [a_{ij}, a_{kl}] \rangle,$

$y_{ijk} =$ 

$I^{:P}$



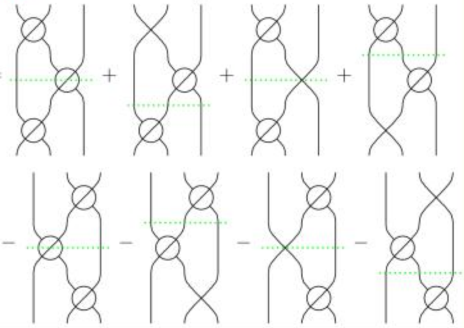
Figuring out $\mathfrak{R}_2$ and R_2.

$$\ker \mu = \pi_2\left(\mu_F^{-1}(M)\right)$$

$$\begin{tikzcd} J^{:2} \arrow[r, "\mu_F", "1\text{-}1"'] \arrow[d, "\pi_2"'] & J \supset M \arrow[d, "\pi_1"] \\ I^{:2} \arrow[r, "\mu"] & I = J/M \end{tikzcd}$$

is in principle computable, and then R_2 follows as $V^{\otimes 2} = (I/I^2)^{\otimes 2} = I^{:2}/\mu_3(I^{:3})$.

$\mathfrak{R}_2(PvB_n)$ is generated by C^{ij}_{kl} and

$Y_{ijk} :=$ 

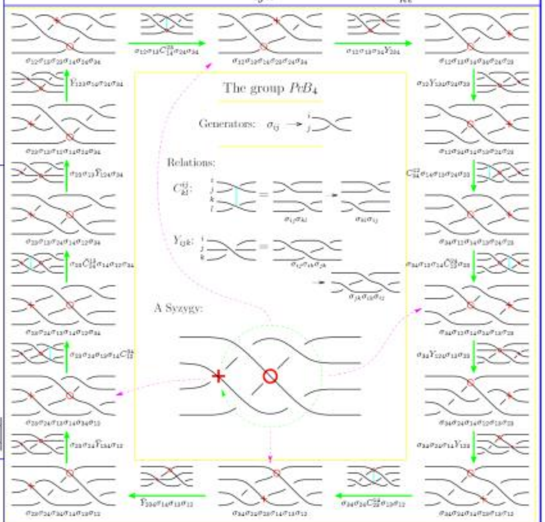
Syzygy Completeness, for PvB_n, means:

$$\mathfrak{R}_p = \bigoplus_{j=1}^{p-1} \mathfrak{R}_{p,j} \xrightarrow{\partial} I^{:p} \xrightarrow{\pi} V^{\otimes p}$$

$\{\tilde\sigma_{12} : \underline{Y_{345}} : \sigma_{67}\tilde\sigma_{89} : \ldots\} \longrightarrow$

$\{\tilde\sigma_{12} : Y_{345} : \sigma_{67}\tilde\sigma_{89} : \ldots\} \longrightarrow \{a_{12}y_{345}a_{89}\ldots\}$

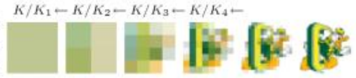
Is every relation between the y_{ijk}'s and the c^{ij}_{kl}'s also a relation between the Y_{ijk}'s and the C^{ij}_{kl}'s?



Just for fun. $K =$  $=$ (The set of all 2D projections of reality)

$K/K_1 \leftarrow K/K_2 \leftarrow K/K_3 \leftarrow K/K_4 \leftarrow \ldots$

Crop Rotate Adjoin



An expansion Z is a choice of a "progressive scan" algorithm.

crop rotate adjoin

$K/K_1 \oplus K_1/K_2 \oplus K_2/K_3 \oplus K_3/K_4 \oplus K_4/K_5 \oplus K_5/K_6 \oplus \cdots$

$\| \quad \mathbb{R}^3 \qquad \| \quad \ker(K/K_4 \to K/K_3)$

Footnotes

1. Following a homonymous paper and thesis by Peter Lee [Lee]. All serious work here is his and was extremely patiently explained by him to DBN. Page design by the latter.

References

[AT] A. Alekseev and C. Torossian, *The Kashiwara-Vergne conjecture and Drinfeld's associators*, arXiv:0802.4300.

[BN1] D. Bar-Natan, *On the Vassiliev knot invariants*, Topology **34** (1995) 423–472.

[BN2] D. Bar-Natan, *Facts and Dreams About v-Knots and Etingof-Kazhdan*, talk presented at the Swiss Knots 2011 conference. Video and more at `http://www.math.toronto.edu/~drorbn/Talks/SwissKnots-1105/`.

[BN3] D. Bar-Natan, *Finite Type Invariants of W-Knotted Objects: From Alexander to Kashiwara and Vergne*, paper and related files at `http://www.math.toronto.edu/~drorbn/papers/WKO/`.

[BND] D. Bar-Natan, and Z. Dancso, *Homomorphic Expansions for Knotted Trivalent Graphs*, arXiv:1103.1896.

[BEER] L. Bartholdi, B. Enriquez, P. Etingof, and E. Rains, *Groups and Lie algebras corresponding to the Yang-Baxter equations*, Journal of Algebra **305-2** (2006) 742-764, arXiv:math.RA/0509661.

[Dri] V. G. Drinfel'd, *Quasi-Hopf Algebras*, Leningrad Math. J. **1** (1990) 1419–1457 and *On Quasitriangular Quasi-Hopf Algebras and a Group Closely Connected with* $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$, Leningrad Math. J. **2** (1991) 829–860.

[EK] P. Etingof and D. Kazhdan, *Quantization of Lie Bialgebras, I*, Selecta Mathematica, New Series **2** (1996) 1–41, arXiv:q-alg/9506005, and *Quantization of Lie Bialgebras, II*, Selecta Mathematica, New Series **4** (1998) 213–231, arXiv:q-alg/9701038.

[GPV] M. Goussarov, M. Polyak, and O. Viro, *Finite type invariants of classical and virtual knots*, Topology **39** (2000) 1045–1068, arXiv:math.GT/9810073.

[Hav] A. Haviv, *Towards a diagrammatic analogue of the Reshetikhin-Turaev link invariants*, Hebrew University PhD thesis, September 2002, arXiv:math.QA/0211031.

[Hut] M. Hutchings, *Integration of singular braid invariants and graph cohomology*, Transactions of the AMS **350** (1998) 1791–1809.

[KV] M. Kashiwara and M. Vergne, *The Campbell-Hausdorff Formula and Invariant Hyperfunctions*, Invent. Math. **47** (1978) 249–272.

[Kau] L. H. Kauffman, *Virtual Knot Theory*, European J. Comb. **20** (1999) 663–690, arXiv:math.GT/9811028.

[KL] L. H. Kauffman and S. Lambropoulou, *Virtual Braids*, Fundamenta Mathematicae **184** (2005) 159–186, arXiv:math.GT/0407349.

[Koh] T. Kohno, *Monodromy representations of braid groups and Yang-Baxter equations*, Ann. Inst. Fourier **37** (1987) 139–160.

[Lee] P. Lee, *The Pure Virtual Braid Group is Quadratic*, in preparation.

linked at . . . .