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The Pure Virtual Braid Group is Quadratic[1]

Dror Bar–Natan and Peter Lee in Oregon, August 2011
http://www.math.toronto.edu/~drorbn/Talks/Oregon-1108/
foots & refs on PDF version, page 3

Abstract Generalities

Let K be a unital algebra over a field $\mathbb{F}$ with char $\mathbb{F}=0$, and let $I\subset K$ be an "augmentation ideal"; meaning $K/I=\mathbb{F}$.

Definition. Say that K is quadratic if its associated graded $\operatorname{gr}K=\bigoplus_{p=0}^{\infty}I^p/I^{p+1}$ is a quadratic algebra. Alternatively, let $A=q(K)=\langle V=I/I^2\rangle/\langle R_2=\ker(\bar\mu_2:V\otimes V\to I^2/I^3)\rangle$ be the "quadratic approximation" to K (q is a lovely functor). Then K is quadratic iff the obvious $\mu:A\to\operatorname{gr}K$ is an isomorphism. If G is a group, we say it is quadratic if its group ring is, with its augmentation ideal.

The Overall Strategy. Consider the "singularity tower" of (K,I) (here ":" means $\otimes_K$ and μ is (always) multiplication):

$$\cdots\ I^{:p+1}\xrightarrow{\mu_{p+1}}I^{:p}\xrightarrow{\mu_p}I^{:p-1}\longrightarrow\cdots\longrightarrow K$$

We care as $\operatorname{im}(\mu^p=\mu_1\circ\cdots\circ\mu_p)=I^p$, so $I^p/I^{p+1}=\operatorname{im}\mu^p/\operatorname{im}\mu^{p+1}$. Hence we ask:

- How injective is this tower?
- What's $I^{:p}/\mu(I^{:p+1})$?

Lemma. $I^{:p}/\mu(I^{:p+1})\simeq(I/I^2)^{\otimes p}=V^{\otimes p}$.

Flow Chart.

$$\boxed{\text{Any }(K,I)}\xrightarrow{\text{Prop 1}}\boxed{\text{2-local}}\xrightarrow{\text{Prop 2}}\boxed{\text{Quadratic}}$$

$$\boxed{K=PvB_n}\xrightarrow[\text{by Peter}]{\text{Thm S}}\boxed{\text{Hutchings' Criterion}}\longrightarrow\boxed{\text{2-injective}}$$

Proposition 1. The sequence

$$\mathcal{R}_p:=\bigoplus_{j=1}^{p-1}\left(I^{:j-1}:\mathcal{R}_2:I^{:p-j-1}\right)\xrightarrow{\partial}I^{:p}\xrightarrow{\mu_p}I^{:p-1}$$

is exact, where $\mathcal{R}_2:=\ker\mu:I^{:2}\to I$; so (K,I) is "2-local".

The Free Case. If J is an augmentation ideal in $K=F=\langle x_i\rangle$, denote $F\to F/J=\mathbb{F}$ by $x\mapsto[x]$ and define $\psi:F\to F$ by $x_i\mapsto x_i+[x_i]$. Then $J_0:=\psi(J)$ is $\{w\in F:\deg w>0\}$. For J_0 it is easy to check that $\mathcal{R}_2=\mathcal{R}_p=0$, and hence the same is true for every J.

The General Case. If $K=F/M$ and $I\subset K$, then $I=J/M$ where $J\subset F$. Then $I^{:p}=J^{:p}/\sum J^{:j-1}:M:J^{:p-j}$ and we have

$$\begin{tikzcd}
J^{:p} \arrow[rr, "\mu_F", "1\text{-}1"'] \arrow[d, "\pi_p", "\text{onto}"'] & & J^{:p-1} \arrow[d, "\pi_{p-1}"', "\text{onto}"] \\
I^{:p}=J^{:p}/\sum J^{:}:M:J^{:} \arrow[rr, "\mu"] & & I^{:p-1}=J^{:p-1}/\sum J^{:}:M:J^{:}
\end{tikzcd}$$

So $\ker(\mu)=\pi_p\left(\mu_F^{-1}(\ker\pi_{p-1})\right)=\pi_p\left(\sum\mu_F^{-1}(J^{:}:M:J^{:})\right)=\sum\pi_p\left(J^{:}:\mu_F^{-1}(M):J^{:}\right)=\sum I^{:}:\mathcal{R}_2:I^{:}$.

2-Injectivity. A (one-sided infinite) sequence

$$\cdots\longrightarrow K_{p+1}\xrightarrow{\delta_{p+1}}K_p\xrightarrow{\delta_p}\cdots\longrightarrow K_0=K$$

is "injective" if for all $p>0$, $\ker\delta_p=0$. It is "2-injective" if its "1-reduction"

$$\cdots\longrightarrow\frac{K_{p+1}}{\ker\delta_{p+1}}\xrightarrow{\bar\delta_{p+1}}\frac{K_p}{\ker\delta_p}\xrightarrow{\bar\delta_p}\frac{K_{p-1}}{\ker\delta_{p-1}}\longrightarrow\cdots$$

is injective; i.e. if for all p, $\ker(\delta_p\circ\delta_{p+1})=\ker\delta_{p+1}$. A pair (K,I) is "2-injective" if its singularity tower is 2-injective.

Why Care?

- In abstract generality, $\operatorname{gr}K$ is a simplified version of K and if it is quadratic it is as simple as it may be without being silly.
- In some concrete (somewhat generalized) knot theoretic cases, A is a space of "universal Lie algebraic formulas" and the "primary approach" for proving (strong) quadraticity, constructing an appropriate homomorphism $Z:K\to\hat{A}$, becomes wonderful mathematics:

	u-Knots and Braids	v-Knots	w-Knots
K	u-Knots and Braids	v-Knots	w-Knots
A	Metrized Lie algebras [BN1]	Lie bialgebras [Hav]	Finite dimensional Lie algebras [BN3]
Z	Associators [Dri, BND]	Etingof-Kazhdan quantization [EK, BN2]	Kashiwara-Vergne-Alekseev-Torossian [KV, AT]

Proposition 2. If (K,I) is 2-local and 2-injective, it is quadratic.

Proof. Staring at the 1-reduced sequence $\frac{I^{:p+1}}{\ker\mu_{p+1}}\xrightarrow{\mu_{p+1}}\frac{I^{:p}}{\ker\mu_p}\xrightarrow{\mu_p}\cdots\longrightarrow K$, get $\frac{I^p}{I^{p+1}}\simeq\frac{I^{:p}/\ker\mu_p}{\mu(I^{:p+1}/\ker\mu_{p+1})}\simeq\frac{I^{:p}}{\mu(I^{:p+1})+\ker\mu_p}$. But $\frac{I^{:p}}{\mu(I^{:p+1})}\simeq(I/I^2)^{\otimes p}$, so the above is $(I/I^2)^{\otimes p}/\sum\left(I^{:j-1}:\mathcal{R}_2:I^{:p-j-1}\right)$. But that's the degree p piece of $q(K)$.

The X Lemma (inspired by [Hut]).

$$\begin{tikzcd}
A_0 \arrow[dr, "\alpha_0"] & & C_0 \\
& B \arrow[ur, "\beta_0"] \arrow[dr, "\beta_1"] & \\
A_1 \arrow[ur, "\alpha_1"] & & C_1
\end{tikzcd}$$


Michael Hutchings
will compute homology for food

If the above diagram is Conway ($\asymp$) exact, then its two diagonals have the same "2-injectivity defect". That is, if $A_0\to B\to C_0$ and $A_1\to B\to C_1$ are exact, then $\ker(\beta_1\circ\alpha_0)/\ker\alpha_0\simeq\ker(\beta_0\circ\alpha_1)/\ker\alpha_1$.

Proof. $\frac{\ker(\beta_1\circ\alpha_0)}{\ker\alpha_0}\xrightarrow[\alpha_0]{\sim}\ker\beta_1\cap\operatorname{im}\alpha_0$
$=\ker\beta_0\cap\operatorname{im}\alpha_1\xleftarrow[\alpha_1]{\sim}\frac{\ker(\beta_0\circ\alpha_1)}{\ker\alpha_1}$

The Hutchings Criterion [Hut]. The singularity tower of (K,I) is 2-injective iff on the right, $\ker(\pi\circ\partial)=\ker(\partial)$. That is, iff every "diagrammatic syzygy" lifts to a "topological syzygy".

$$\begin{tikzcd}
\mathcal{R}_p \arrow[dr, "\partial"] & & I^{:p-1} \\
& I^{:p} \arrow[ur, "\mu_p"] \arrow[dr, "\pi"] & \\
I^{:p+1} \arrow[ur, "\mu_{p+1}"] & & V^{\otimes p}
\end{tikzcd}$$

Conclusion. **We need to know that (K,I) is "syzygy complete"** — that every diagrammatic syzygy lifts to a topological syzygy, that $\ker(\pi\circ\partial)=\ker(\partial)$. ~~Namely, that every relation between the y_{ijk}'s and c^{ij}_{kl}'s lifts to a relation between the Y_{ijk}'s and C^{ij}_{kl}'s.~~

is also

next page

James Gillespie's Sightline #2 (1984) is a syzygy, and (arguably) Toronto's largest sculpture. Find it next to University of Toronto's Hart House.

The Pure Virtual Braid Group is Quadratic, II

Examples and Interpretations

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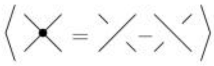
Example. (goes back to [Koh])

T. Kohno

$K =$  $\quad I =$ 

$$(K/I^{p+1})^\star = (\text{invariants of type } p) =: \mathcal{V}_p$$

$$(I^p/I^{p+1})^\star = \mathcal{V}_p/\mathcal{V}_{p-1} \quad V = \langle t^{ij} | t^{ij} = t^{ji}\rangle = \langle | \vdash\!\dashv \rangle$$

$$\ker \bar\mu_2 = \langle [t^{ij}, t^{kl}] = 0 = [t^{ij}, t^{ik} + t^{jk}]\rangle = \langle \text{4T relations}\rangle$$

$A = q(K) = \left(\begin{matrix}\text{horizontal chord dia-}\\ \text{grams mod 4T}\end{matrix}\right) =$  4T

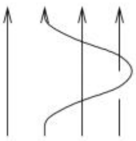
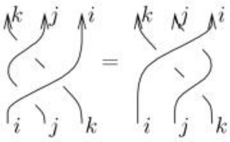
Z: universal finite type invariant, the Kontsevich integral.

PvB_n is the group

$$\langle \sigma_{ij} : 1 \le i \ne j \le n \rangle \Big/ \begin{matrix}\sigma_{ij}\sigma_{ik}\sigma_{jk} = \sigma_{jk}\sigma_{ik}\sigma_{ij}\\ \sigma_{ij}\sigma_{kl} = \sigma_{kl}\sigma_{ij}\end{matrix}$$

L. Kauffman [Kau, KL]

of "pure virtual braids" ("braids when you look", "blunder braids"):

$\sigma_{24} =$  R3: 

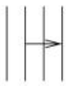
The Main Theorem [Lee]. PvB_n is quadratic.

$A_n = q(PvB_n)$. [GPV]

Goussarov-Polyak-Viro

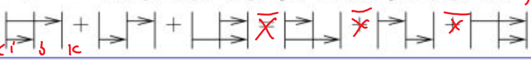
$I =$  with $\tilde{\sigma}_{ij} = \sigma_{ij} - 1 =$ ⤫ − ✕, the "semi-virtual crossing".

$V = I/I^2 = \left\langle \begin{matrix}\text{v-braids}\\ \text{with one} \end{matrix} \right\rangle \Big/ (⤫ = ✕)$

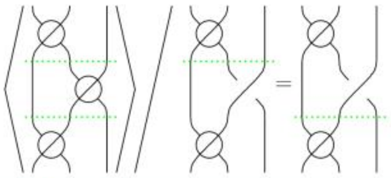
$= \langle a_{ij}\rangle_{1\le i\ne j\le n}$ $\qquad a_{24} =$ 

$y_{ijk} =$ $\qquad C^{ij}_{kl} =$

~~So~~ $A_n = TV/\langle [a_{ij}, a_{ik}] + [a_{ij}, a_{jk}] + [a_{ik}, a_{jk}],\ [a_{ij}, a_{kl}]\rangle$ 0,

y_{ijk} i j k 

$I^{:3}$.



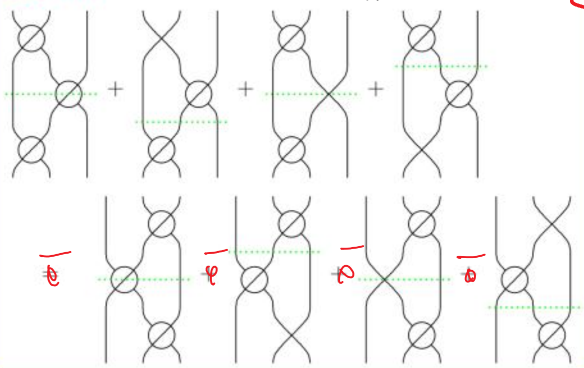
Figuring out $\mathcal{R}_2$ and R_2.

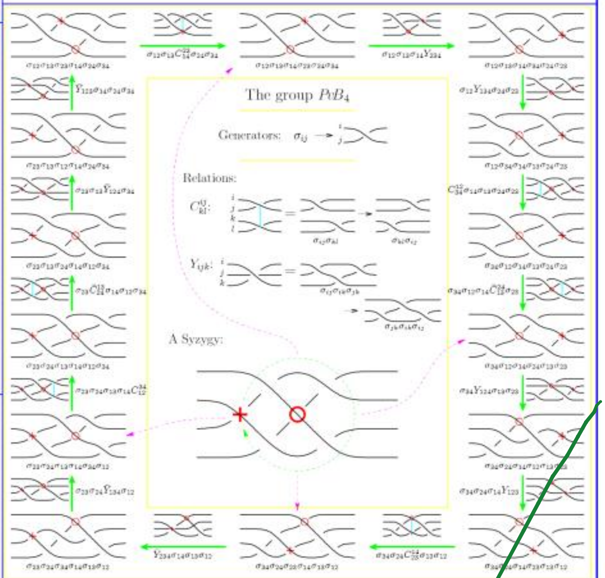
$$\ker \mu = \pi_2\left(\mu_F^{-1}(M)\right)$$

$$\begin{tikzcd} J^{:2} \arrow[r, "\mu_F", "1\text{-}1"'] \arrow[d, "\pi_2"'] & J \supset M \arrow[d, "\pi_1"] \\ I^{:2} \arrow[r, "\mu"] & I = J/M \end{tikzcd}$$

is in principle computable, and then R_2 follows as $V^{\otimes 2} = (I/I^2)^{\otimes 2} = I^{:2}/\mu_3(I^{:3})$.

$\mathcal{R}_2(PvB_n)$ 0 is generated by $Y_{ijk} =$





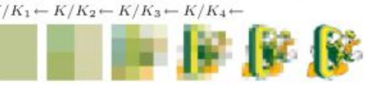
$$R_p \longrightarrow I^{:p} \longrightarrow V^{\otimes p}$$

$$\{\tilde\sigma_{12} : Y_{345} : \sigma_{67}\tilde\sigma_{89} : \ldots\} \longrightarrow \{a_{12}\, y_{345}\, a_{89} \ldots\}$$

Is every relation between the y_{ijk}'s & C^{ij}_{kl}'s also a relation between the Y_{ijk}'s & C^{ij}_{kl}'s?

Just for fun. $K = \left\{ \right.$  $\left.\right\} = \left(\begin{matrix}\text{The set of all}\\ \text{2D projections}\\ \text{of reality}\end{matrix}\right)$ …

$K/K_1 \leftarrow K/K_2 \leftarrow K/K_3 \leftarrow K/K_4 \leftarrow$

Crop Rotate Adjoin  …

An expansion Z is a choice of a "progressive scan" algorithm.

crop rotate adjoin $\quad K/K_1 \oplus K_1/K_2 \oplus K_2/K_3 \oplus K_3/K_4 \oplus K_4/K_5 \oplus K_5/K_6 \oplus \cdots$

$\| \quad \mathbb{R}^3 \qquad \| \quad \ker(K/K_4 \to K/K_3)$