Oregon Handout Post Mortem

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The Pure Virtual Braid Group is Quadratic[1]

Abstract Generalities

Dror Bar–Natan and Peter Lee in Oregon, August 2011

http://www.math.toronto.edu/~drorbn/Talks/Oregon-1108/
foots & refs on PDF version, page 3

Broken 2

Let K be a unital algebra over a field $\mathbb{F}$ with $\operatorname{char}\mathbb{F}=0$, and let $I\subset K$ be an "augmentation ideal"; so $K/I \xrightarrow[\epsilon]{\sim} \mathbb{F}$.

Definition. Say that K is quadratic if its associated graded $\operatorname{gr}K=\bigoplus_{p=0}^{\infty} I^p/I^{p+1}$ is a quadratic algebra. Alternatively, let $A=q(K)=\langle V=I/I^2\rangle/\langle R_2=\ker(\bar\mu_2: V\otimes V\to I^2/I^3)\rangle$ be the "quadratic approximation" to K (q is a lovely functor). Then K is quadratic iff the obvious $\mu: A\to\operatorname{gr}K$ is an isomorphism. If G is a group, we say it is quadratic if its group ring is, with its augmentation ideal.

The Overall Strategy. Consider the "singularity tower" of (K,I) (here ":" means $\otimes_K$ and μ is (always) multiplication):

$$\cdots\; I^{:p+1}\xrightarrow{\mu_{p+1}} I^{:p}\xrightarrow{\mu_p} I^{:p-1}\longrightarrow\cdots\longrightarrow K$$

We care as $\operatorname{im}(\mu^p=\mu_1\circ\cdots\circ\mu_p)=I^p$, so $I^p/I^{p+1}=\operatorname{im}\mu^p/\operatorname{im}\mu^{p+1}$. Hence we ask:

- What's $I^{:p}/\mu(I^{:p+1})$? • How injective is this tower?

A

Lemma. $I^{:p}/\mu(I^{:p+1})\simeq(I/I^2)^{\otimes p}=V^{\otimes p}$; set $\pi: I^{:p}\to V^{\otimes p}$.

Flow Chart.

$$\begin{array}{ccccccc}
\boxed{\text{Any }(K,I)} & \xrightarrow{\text{Prop 1}} & \boxed{\text{2-local}} & \xrightarrow{\text{Prop 2}} & \boxed{\text{Quadratic}} & & \\
\boxed{K=PvB_n} & \xrightarrow[\text{by Peter}]{\text{Thm S}} & \boxed{\text{Hutchings' Criterion}} & \longrightarrow & \boxed{\text{2-injective}} & &
\end{array}$$

Proposition 1. The sequence

$$\mathfrak{R}_p:=\bigoplus_{j=1}^{p-1}\left(I^{:j-1}:\mathfrak{R}_2:I^{:p-j-1}\right)\xrightarrow{\partial} I^{:p}\xrightarrow{\mu_p} I^{:p-1}$$

is exact, where $\mathfrak{R}_2:=\ker\mu: I^{:2}\to I$; so (K,I) is "2-local".

The Free Case. If J is an augmentation ideal in $K=F=\langle x_i\rangle$, define $\psi:F\to F$ by $x_i\mapsto x_i+\epsilon(x_i)$. Then $J_0:=\psi(J)$ is $\{w\in F:\deg w>0\}$. For J_0 it is easy to check that $\mathfrak{R}_2=\mathfrak{R}_p=0$, and hence the same is true for every J.

The General Case. If $K=F/\langle M\rangle$ (where M is a vector space of "moves") and $I\subset K$, then $I=J/\langle M\rangle$ where $J\subset F$. Then $I^{:p}=J^{:p}/\sum J^{:j-1}:\langle M\rangle:J^{:p-j}$ and we have

$$\begin{array}{ccc}
J^{:p} & \xrightarrow[\text{1-1}]{\mu_F} & J^{:p-1}\\
{\scriptstyle\text{onto}}\downarrow{\scriptstyle\pi_p} & & {\scriptstyle\pi_{p-1}}\downarrow{\scriptstyle\text{onto}}\\
I^{:p}=J^{:p}/\sum J^{:}:\langle M\rangle:J^{:} & \xrightarrow{\mu} & I^{:p-1}=J^{:p-1}/\sum J^{:}:\langle M\rangle:J^{:}
\end{array}$$

C

So[2] $\ker(\mu)=\pi_p\left(\mu_F^{-1}(\ker\pi_{p-1})\right)=\pi_p\left(\sum\mu_F^{-1}\left(J^{:}:\langle M\rangle:J^{:}\right)\right)=\sum\pi_p\left(J^{:}:\mu_F^{-1}\langle M\rangle:J^{:}\right)=\sum I^{:}:\mathfrak{R}_2:I^{:}=:\sum_{j=1}^{p-1}\mathfrak{R}_{p,j}$.

trivial bimodule

B

$\mathfrak{R}_2$ is simpler than may seem! It's an ~~"augmentation bimodule"~~ ($I\mathfrak{R}_2=0=\mathfrak{R}_2I$ thus $xr=\epsilon(x)r=r\epsilon(x)=rx$ for $x\in K$ and $r\in\mathfrak{R}_2$), and hence $\mathfrak{R}_2=\pi_2(\mu_F^{-1}M)$.

$$\begin{array}{ccc}
J^{:2} & \xrightarrow[\text{1-1}]{\mu_F} & J\supset M\\
\downarrow{\scriptstyle\pi_2} & & \downarrow{\scriptstyle\pi_1}\\
I^{:2} & \xrightarrow{\mu} & I=J/\langle M\rangle
\end{array}$$

$\mathfrak{R}_p$ is simpler than may seem! In $\mathfrak{R}_{p,j}=I^{:j-1}:\mathfrak{R}_2:I^{:p-j-1}$ the I factors may be replaced by $V=I/I^2$. Hence

$$\mathfrak{R}_p\simeq\bigoplus_{j=1}^{p-1}V^{\oplus j-1}\otimes\pi_2(\mu_F^{-1}M)\otimes V^{\otimes p-j-1}.$$

Claim. $\pi(\mathfrak{R}_{p,j})=R_{p,j}$; namely,

$$\pi\left(I^{:j-1}:\mathfrak{R}_2:I^{:p-j-1}\right)=V^{\otimes j-1}\otimes R_2\otimes V^{\otimes p-j-1}.$$

Why Care? • In abstract generality, $\operatorname{gr}K$ is a simplified version of K and if it is quadratic it is as simple as it may be without being silly. • In some concrete (somewhat generalized) knot theoretic cases, A is a space of "universal Lie algebraic formulas" and the "primary approach" for proving (strong) quadraticity, constructing an appropriate homomorphism $Z:K\to\hat A$, becomes wonderful mathematics:

K	u-Knots and Braids	v-Knots	w-Knots
A	Metrized Lie algebras [BN1]	Lie bialgebras [Hav]	Finite dimensional Lie algebras [BN3]
Z	Associators [Dri, BND]	Etingof-Kazhdan quantization [EK, BN2]	Kashiwara-Vergne-Alekseev-Torossian [KV, AT]

2-Injectivity. A (one-sided infinite) sequence

$$\cdots\longrightarrow K_{p+1}\xrightarrow{\delta_{p+1}}K_p\xrightarrow{\delta_p}\cdots\longrightarrow K_0=K$$

is "injective" if for all $p>0$, $\ker\delta_p=0$. It is "2-injective" if its "1-reduction"

$$\cdots\longrightarrow\frac{K_{p+1}}{\ker\delta_{p+1}}\xrightarrow{\bar\delta_{p+1}}\frac{K_p}{\ker\delta_p}\xrightarrow{\bar\delta_p}\frac{K_{p-1}}{\ker\delta_{p-1}}\longrightarrow\cdots$$

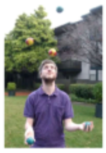
is injective; i.e. if for all p, $\ker(\delta_p\circ\delta_{p+1})=\ker\delta_{p+1}$. A pair (K,I) is "2-injective" if its singularity tower is 2-injective.

Proposition 2. If (K,I) is 2-local and 2-injective, it is quadratic.

Proof. Staring at the 1-reduced sequence $\frac{I^{:p+1}}{\ker\mu_{p+1}}\xrightarrow{\mu_{p+1}}\frac{I^{:p}}{\ker\mu_p}\xrightarrow{\mu_p}\cdots\longrightarrow K$, get $\frac{I^p}{I^{p+1}}\simeq\frac{I^{:p}/\ker\mu_p}{\mu(I^{:p+1}/\ker\mu_{p+1})}\simeq\frac{I^{:p}}{\mu(I^{:p+1})+\ker\mu_p}$. But $\frac{I^{:p}}{\mu(I^{:p+1})}\simeq(I/I^2)^{\otimes p}$, so the above is $(I/I^2)^{\otimes p}/\sum\left(I^{:j-1}:\mathfrak{R}_2:I^{:p-j-1}\right)$. But that's the degree p piece of $q(K)$.

The X Lemma (inspired by [Hut]).

$$\begin{array}{ccccc}
A_0 & & & & C_0\\
& \searrow^{\alpha_0} & & \nearrow^{\beta_0} & \\
& & B & & \\
& \nearrow^{\alpha_1} & & \searrow^{\beta_1} & \\
A_1 & & & & C_1
\end{array}$$



If the above diagram is Conway ($\asymp$) exact, then its two diagonals have the same "2-injectivity defect". That is, if $A_0\to B\to C_0$ and $A_1\to B\to C_1$ are exact, then $\ker(\beta_1\circ\alpha_0)/\ker\alpha_0\simeq\ker(\beta_0\circ\alpha_1)/\ker\alpha_1$.

Proof. $\frac{\ker(\beta_1\circ\alpha_0)}{\ker\alpha_0}\xrightarrow[\alpha_0]{\sim}\ker\beta_1\cap\operatorname{im}\alpha_0$

$$=\ker\beta_0\cap\operatorname{im}\alpha_1\xleftarrow[\alpha_1]{\sim}\frac{\ker(\beta_0\circ\alpha_1)}{\ker\alpha_1}.$$

The Hutchings Criterion [Hut]. The singularity tower of (K,I) is 2-injective iff on the right, $\ker(\pi\circ\partial)=\ker(\partial)$. That is, iff every "diagrammatic syzygy" is also a "topological syzygy".

$$\begin{array}{ccccc}
\mathfrak{R}_p & & & & I^{:p-1}\\
& \searrow^{\partial} & & \nearrow^{\mu_p} & \\
& & I^{:p} & & \\
& \nearrow^{\mu_{p+1}} & & \searrow^{\pi} & \\
I^{:p+1} & & & & V^{\otimes p}
\end{array}$$

Conclusion. We need to know that (K,I) is "syzygy complete" — that every diagrammatic syzygy is also a topological syzygy, that $\ker(\pi\circ\partial)=\ker(\partial)$.

A. There should have been a second lemma here:

Lemma 2 (. Under the right projection,

$$\pi(\ker \mu_2) = R_2$$

2. In the proper circumstances,

$$\pi(\ker \mu_p) = \sum I^{\otimes j-1} \otimes R_2 \otimes I^{\otimes p-j-1}$$

B. Why "M" is enough there, rather than $\langle M \rangle$, requires a better explanation.

C. If we rephrase this commutative diagrams as

$$\begin{array}{ccc} J^{:p} & \xrightarrow{\mu_F} & J \cdot J^{:p-1} \\ \downarrow{\scriptstyle \pi_p} & & \downarrow{\scriptstyle \pi_{p-1}} \\ I^{:p} & \xrightarrow{\mu} & I \cdot I^{:p-1} \end{array}$$

Then μ_F is an isomorphism and the original Polishchuk issue disappears. The issue now is whether

$$\ker(J \cdot J^{:p-1} \to I \cdot I^{:p-1}) \overset{?}{=} \ker(J^{:p-1} \to I^{:p-1}),$$

and that's the same as

$$\sum J^{:i} : \langle M \rangle : J^{:} \underset{?}{\subseteq} J \cdot J^{:p-1}$$

and that's a much lighter issue; the only problem may come from $a^+ + a^- + a^+ a^- = 0$, and it probably doesn't really come.

Aside: If $A, B \subset \mathrm{im}(f)$, then $f^{-1}(A) + f^{-1}(B) = f^{-1}(A+B)$.

proof: $\subset$ if $z \in f^{-1}(A)+f^{-1}(B)$ then $z = f(x)+f(y)$ w/ $x \in A$ & $y \in B$ so $z = f(x+y)$ w/ $x+y \in A+B$ $\square$

$\supset$ if $z \in f^{-1}(A+B)$ then $f(z) = x+y$ w/ $x \in A$ & $y \in B$ so as $A \subset \text{im}(f)$ & $B \subset \text{im}(f)$, $\exists\, x_1, y_1$ s.t. $x = f(x_1)$, $y = f(y_1)$. Hence

$$f(z - x_1 - y_1) = f(z) - f(x_1) - f(y_1)$$
$$= x + y - x - y = 0$$

so $z - x_1 - y_1 \in \ker f$ so $\exists\, w \in \ker f$ s.t. $z = (x_1 + w) + y_1$. But then

$f(x_1 + w) = f(x_1) = x \in A \Rightarrow x_1 + w \in f^{-1}(A)$

& $f(y_1) = y \in B \qquad \Rightarrow y_1 \in f^{-1}(B)$

So $z \in f^{-1}(A) + f^{-1}(B)$.

The Pure Virtual Braid Group is Quadratic, II

Examples and Interpretations

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Example. (goes back to [Koh])

T. Kohno

$K = \langle \ \rangle \qquad I = \langle \ = \ - \ \rangle$

$$(K/I^{p+1})^\star = (\text{invariants of type } p) =: \mathcal{V}_p$$

$$(I^p/I^{p+1})^\star = \mathcal{V}_p/\mathcal{V}_{p-1} \quad V = \langle t^{ij} | t^{ij} = t^{ji} \rangle = \langle \ \rangle$$

$$\ker \bar\mu_2 = \langle [t^{ij}, t^{kl}] = 0 = [t^{ij}, t^{ik} + t^{jk}] \rangle = \langle \text{4T relations} \rangle$$

$$A = q(K) = \begin{pmatrix}\text{horizontal chord dia-}\\ \text{grams mod 4T}\end{pmatrix} = \langle \ \rangle / \text{4T}$$

Z: universal finite type invariant, the Kontsevich integral.

PvB_n is the group

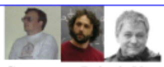
$$\langle \sigma_{ij} : 1 \le i \ne j \le n \rangle \Big/ \begin{array}{c} \sigma_{ij}\sigma_{ik}\sigma_{jk} = \sigma_{jk}\sigma_{ik}\sigma_{ij} \\ \sigma_{ij}\sigma_{kl} = \sigma_{kl}\sigma_{ij} \end{array}$$

L. Kauffman
[Kau, KL]

of "pure virtual braids" ("braids when you look", "blunder braids"):

The Main Theorem [Lee]. PvB_n is quadratic.

$A_n = q(PvB_n)$. [GPV]



Goussarov-Polyak-Viro

$I =$ with $\tilde\sigma_{ij} = \sigma_{ij} - 1$, the "semi-virtual crossing".

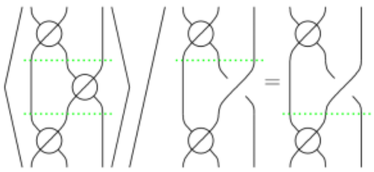
$$V = I/I^2 = \left\langle \begin{array}{c}\text{v-braids}\\ \text{with one semi-virtual crossing}\end{array} \right\rangle \Big/ (\ = \ ) = \langle a_{ij} \rangle_{1 \le i \ne j \le n}$$

$a_{24} =$

$$A_n = TV/\langle [a_{ij}, a_{ik}] + [a_{ij}, a_{jk}] + [a_{ik}, a_{jk}],\ c^{ij}_{kl} = [a_{ij}, a_{kl}] \rangle,$$

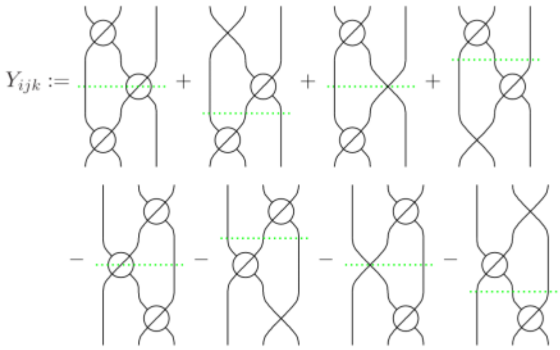
$y_{ijk} = \ + \ + \ - \ - \ - $

$I^{:p}$.



James Gillespie's Sightline #2 (1984) is a syzygy, and (arguably) Toronto's largest sculpture. Find it next to University of Toronto's Hart House.

$\mathfrak{R}_2(PvB_n)$ is generated as a vector space by C^{ij}_{kl} and



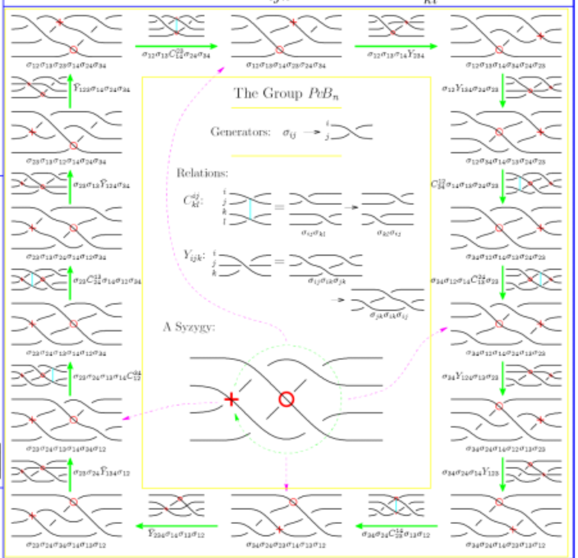
Syzygy Completeness, for PvB_n, means:

$$\mathfrak{R}_p = \bigoplus_{j=1}^{p-1} \mathfrak{R}_{p,j} \xrightarrow{\partial} I^{:p} \xrightarrow{\pi} V^{\otimes p}$$

$\{\tilde\sigma_{12} : \underline{Y_{345}} : \tilde\sigma_{67} : \ldots\} \longrightarrow$

$$\{\tilde\sigma_{12} : Y_{345} : \tilde\sigma_{67} : \ldots\} \longrightarrow \{a_{12} y_{345} a_{67} \ldots\}$$

Is every relation between the y_{ijk}'s and the c^{ij}_{kl}'s also a relation between the Y_{ijk}'s and the C^{ij}_{kl}'s?



Theorem S. Let D be the free associative algebra generated by symbols a_{ij}, y_{ijk} and c^{ij}_{kl}, where $1 \le i, j, k, l \le n$ are distinct integers. Let D_0 be the part of D with only a_{ij} symbols and let D_1 be the span of the monomials in D having only a_{ij} symbols, with exactly one exception that may be either a y_{ijk} or a c^{ij}_{kl}. Let $\partial : D_1 \to D_0$ be the map defined by

$$y_{ijk} \mapsto [a_{ij}, a_{ik}] + [a_{ij}, a_{jk}] + [a_{ik}, a_{jk}],$$
$$c^{ij}_{kl} \mapsto [a_{ij}, a_{kl}].$$

Then $\ker \partial$ is generated by a family of elements readable from the picture above and by a few similar but lesser families.

Footnotes

1. Following a homonymous paper and thesis by Peter Lee [Lee]. All serious work here is his and was extremely patiently explained by him to DBN. Page design by the latter.
2. The ~~proof~~ work presented here is broken. Specifically, at the very end of the proof of the "general case" of Proposition 1 the sum that makes up $\ker \pi_{p-1}$ is interchaged with μ_F^{-1}. This is invalid; in general it is not true that $T^{-1}(U+V) = T^{-1}(U) + T^{-1}(V)$, when T is a linear transformation and U and V are subspaces of its target space. We thank Alexander Polishchuk for noting this gap. ... more added here

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[BN2] D. Bar-Natan, *Facts and Dreams About v-Knots and Etingof-Kazhdan,* talk presented at the Swiss Knots 2011 conference. Video and more at `http://www.math.toronto.edu/~drorbn/Talks/SwissKnots-1105/`.

[BN3] D. Bar-Natan, *Finite Type Invariants of W-Knotted Objects: From Alexander to Kashiwara and Vergne,* paper and related files at `http://www.math.toronto.edu/~drorbn/papers/WKO/`.

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