Lee in Geneva: The Pure Virtual Braid Group is Quadratic

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5:10 AM

Thm K be an algebra w/ aug ideal I;

$$\hat{K} = \varprojlim K/I^m, \quad gr_I K = \widehat{\bigoplus} I^m/I^{m+1}$$

$$R := \ker\left(\mu : I/I^2 \otimes I/I^2 \longrightarrow I^2/I^3\right)$$

The "quadratic approximation"

$$qK = T(I/I^2)/\langle R\rangle$$

$gr_I K$ & qK are both generated in deg 1, & have the same quadratic relations. When are they the same?

Always $\exists$ surjection $qK \twoheadrightarrow gr_I K$.

An "expansion" is a filtered map $Z : \hat{K} \to qK$ s.t. $gr Z$ inverts that surjection.

If $K = \mathbb{Q}PB_n$, $qK = A_n$, Z exists by the Kontsevich integral. $\Rightarrow qK \cong gr_I K$ as algebras.

$$PvB_n := \frac{\langle R_{ij}\rangle_{1\le i\ne j\le n}}{R\text{III},\ LS} \qquad R\text{III}: R_{ij}R_{ik}R_{jk} = R_{jk}R_{ik}R_{ij}$$
$$LS: [R_{ij}, R_{kl}] = 1.$$

$I = \langle R_{ij} - 1\rangle = \langle$  $\rangle = \langle \overline{R_{ij}}\rangle$

$$R = \langle \overline{R_{ij}} : Loc: \ ,\ 6T: [\overline{R_{ij}}, \overline{R_{ik}} + \overline{R_{jk}}] + [\overline{R_{ik}}, \overline{R_{jk}}] = 0\rangle$$

So

$$qK = T\langle r_{ij}\rangle/Loc, 6T$$

[BEER]: $\nexists$ homomorphic Z. "PvB_n isn't 1-formal".

<u>Thm</u> (Lee) $gr_I PvB_n \cong qPvB_n$ as graded vector spaces.

$T^{\otimes_K m}$, $T^{\otimes_K (m-1)}$

Let $\mu_m : I^{\otimes_K m} \to I^{\otimes_K (m-1)}$ "multiply any two adjacent components"

By analogy w/ R, let

$$R^k := \ker\left(\mu_2 : I \otimes_K I \to I^2\right)$$

and

$$R^k_m := \bigoplus_j I^{\otimes j} \otimes R^k \otimes I^{\otimes (m-j-1)}$$

$\exists$ map $\partial_K : R^k_m \longrightarrow I^{\otimes m}$

by construction, $\partial_K(R^k_m) \subset \ker \mu_m$.

<u>Lemma</u> Equality holds: $\partial_K(R^k_m) = \ker \mu_m$.

Consider $I^{\otimes_K (m+1)} \xrightarrow{\mu} I^{\otimes_K m} \longrightarrow \operatorname{cok} \mu \cong \dfrac{I^{\otimes m}}{I \cdot I^{\otimes m}}$

$$\underset{\text{claim}}{\cong} \left(I/I^2\right)^{\otimes m}$$

Consider

$$\begin{tikzcd}
\ker \partial^{Ind} \arrow[dr] & & \ker \partial_K \arrow[ll, "F^{syz}"'] \arrow[dl] \\
& R^k_m \arrow[dl, "\partial_K"'] \arrow[dr, "\partial^{Ind}"] & \\
I^{\otimes m} \arrow[rr, "F^0"] & & \left(I/I^2\right)^{\otimes m}
\end{tikzcd}$$

<u>Thm</u> K is quadratic iff F^{syz} is surjective.

[in PB_n this is Hutchings]

Modulo mild assumptions,

$$R^k_n \underset{F'}{\cong} R_m := \langle R \rangle_{\deg m}$$

and $\partial^{Ind} = d_\mu \circ F'$ w/ $d_\mu : R_m \to \left(I/I^2\right)^{\otimes m}$ the obvious.

Then A. $\ker d_u \cong \ker \partial^{Ind}$ is "relations among relations in qK"

B. Can interpret $I^{\otimes m}$ as "n-singular PVB"

so $\ker \partial_K =$ "topological syzygies"

So we need to show the surjectivity of

$$\text{"topological syzygies"} \xrightarrow{F^{syz}} \text{"combinatorial syzygies"}$$

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