# A Narrative for Quadraticity

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$$\left(I/I^2\right)^{\otimes p} \Big/ \left\langle \ker \frac{I}{I^2}\otimes\frac{I}{I^2} \to \frac{I^2}{I^3} \right\rangle \longrightarrow I^p/I^{p+1}$$

$\ker(A\otimes A \xrightarrow{\mu} A) = \{ab\otimes c - a\otimes bc\}$

$$I^{:p} = I^{:p-1}:I \to I^{:p-2}:I^2 \to \quad \dots \to I:I:I^{p-2} \to I:I^{p-1} \to I^p$$

There are three reasons why $I^{\otimes p}$ is not $I^p/I^{p+1}$:

1. Hot potatos: $ab\otimes c \longleftrightarrow a\otimes bc$ "exchanges" } exist even in the free case.
2. Tor (name?): $\ker(I:I\to A)$ may be non-trivial; that is, $\ker(I\otimes I\to A)$ may be more than $\{ab\otimes c - a\otimes bc : a,c\in I\}$.
3. And then the obvious, the presence of $I^{p+1}$ on the right side.

Example for 2: $(x-1)\otimes(y-1)-(y-1)\otimes(x-1)$ in $\mathbb{Q}[x,y]$.

as opposed to $x\otimes y - y\otimes x = (x\otimes y - xy\otimes 1) + (yx\otimes 1 - y\otimes x)$

(even simpler: in same ring, take $I=\{p : p(0,0)=0\}$, and then $x\otimes y - y\otimes x$ is in $\ker\mu$, but is not a combination of exchanges.)

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Perhaps the trick would be to understand the totality of all "barrier removal" maps:

$$I^{p_1}:I^{p_2}:\dots:I^{p_i}:I^{p_{i+1}}:\dots:I^{p_n}$$
$$\longrightarrow I^{p_1}:I^{p_2}:\dots:I^{p_i+p_{i+1}}:\dots:I^{p_n}$$