2-Injectivity

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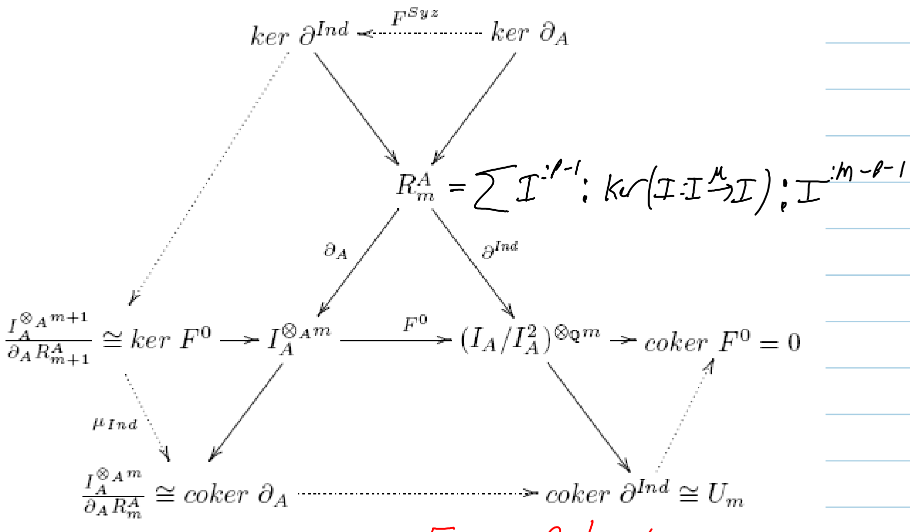
9:52 AM

$$\begin{tikzcd}
& & \ker\alpha \arrow[d] & \\
& & A \arrow[dl, "\alpha"'] \arrow[dr, "\gamma"] & \\
\ker\beta \arrow[r] & B \arrow[rr, "\beta"'] \arrow[d] & & C \\
& \operatorname{Cok}\alpha & &
\end{tikzcd}$$



From Peter's paper

$$\begin{tikzcd}
& I^{:p} \arrow[dl, "\mu"'] \\
I^{:p-1} \arrow[r, "\mu"'] & I^{:p-2}
\end{tikzcd}$$

$$\frac{\ker \mu^2}{\ker\mu} \cong \ker\mu \cap \operatorname{im}\mu$$

$$\cong \operatorname{im}\partial_A \cap \operatorname{im}\mu$$

$$\begin{tikzcd}
& & R \arrow[d, "\gamma"] & S \arrow[d, "\delta"] & \\
0 \arrow[r] & \ker\mu \arrow[r] & A \arrow[r, "\mu"] \arrow[d] & B \arrow[r, "\mu"] \arrow[d] & C \\
& & A/\operatorname{im}\gamma \arrow[r] \arrow[d] & B/\operatorname{im}\delta \arrow[d] & \\
& & 0 & 0 &
\end{tikzcd}$$

I need the diagram for the story, given $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ we try to force injectivity by replacing $A \to \frac{A}{\ker\alpha}$ and $B \to \frac{B}{\ker\beta}$. Have we succeeded?

Except as $\ker\alpha$ & $\ker\beta$ are not constructive, we use $\operatorname{im}\partial_A$ and $\operatorname{im}\partial_B$, instead.

$$\begin{tikzcd}
A_1 \arrow[d, "\partial_A"'] & B_1 \arrow[d, "\partial_B"'] & \\
A_0 \arrow[r, "\alpha"] \arrow[d] & B_0 \arrow[r, "\beta"] \arrow[d] \arrow[dr, "\pi"] & C_0 \\
A_0/\operatorname{im}\partial_A = \ker\alpha \arrow[r, "\bar\alpha"] & B_0/\operatorname{im}\partial_B = \ker\beta & B_0/\operatorname{im}\alpha
\end{tikzcd}$$

not used

Quick lesson: 2-injectivity is easier in the presence of surjectivity.

$$\frac{\ker \beta\circ\alpha}{\ker\alpha} \xrightarrow[\alpha]{\sim} \operatorname{im}\alpha \cap \ker\beta = \operatorname{im}\alpha \cap \operatorname{im}\partial_B$$

$$= \ker\pi \cap \operatorname{im}\partial_B \xleftarrow[\partial_B]{\sim} \frac{\ker \pi\circ\partial_B}{\ker\partial_B}$$

("$\ker\pi\circ\partial_B = \ker\partial_B$" := "Syzygy Complete"?)

Comment. Chord diagrams, that is $B_0/\operatorname{im}\alpha$, arise even without any mention of finite type invariants, that is, even without "moding out by I^{n+1}".

Question. Perhaps the right notion of "quadraticity" should be about "2-injectivity" of the sequence

$$\cdots \to I^{\circ p+1} \to I^{\circ p} \to I^{\circ p-1} \to \cdots,$$

with no mention of $(I/I^2)^{\otimes p}$ etc.

$$\ker\left(I^{\otimes p-j} \otimes I^{j} \to I^{\otimes(p-j-1)} \otimes I^{j+1}\right).$$

$$\begin{tikzcd}
I_F^{\otimes(p-j)} \otimes I_F^{j} \arrow[r, "\text{not}", "\text{injective}"'] \arrow[d] & I_F^{\otimes(p-j-1)} \otimes I_F^{j+1} \arrow[d] \\
I_A^{\otimes(p-j)} \otimes I_A^{j+1} \arrow[r] & I_A^{\otimes(p-j-1)} \otimes I_A^{j+1}
\end{tikzcd}$$

So the "injectivization" of this must be performed first.

$\ker\left(I : I^p \to I^{p+1}\right).$

$$\begin{tikzcd} I_F : I_F^p \arrow[r, "\sim"] \arrow[d] & I_F^{p+1} \arrow[d] \\ I_A : I_A^p \arrow[r] & I_A^{p+1} \end{tikzcd}$$

To do this I need to know $\ker(I_F^p \to I_A^p)$, which factors

$$I_F^p \cong I_F^{:p} \longrightarrow I_A^{:p} \longrightarrow I_A^p$$

That's what we started with.

Also, $\ker(I_F^p \to I_A^p) = M \cap I_F^p$

I should attempt everything from the perspective of the GPV/BHLR intuition!