COMMENTS ON THE PAPER "GOLDMAN–TURAEV FORMALITY FROM THE KONTSEVICH INTEGRAL"

1. Comments and questions

Eq. (2.1): This is a rather sophisticated way to use the "Snake Lemma" for the following: Given a homomorphism $\lambda : B \to E$ and two submodules $A \leq B$, $D \leq E$, if $\lambda(A) = 0$ and $\lambda(B) \subset D$, then λ induces a map $\eta : B/A \to D, \{b\} \mapsto \lambda(b)$.

Eq. (2.1): It is written two lines before that "λ induces a unique map η": since the unicity is used several times in the sequel, it would be better to specify in which sense η is induced by λ (i.e., have the commutative quadrilateral with vertices B, C, D, E). Thus, the reader will understand what you mean, for instance in the proof of Cor. 2.4, when you write "makes the diagram commute."

Example 2.1: As it is defined here, A/M_1 is not the abelianization of A, but it is zero. Indexes have to be shifted in some way.

Lemma 3.1: I think that the order of composition has been reversed: what we wish is a filtered map Z such that $\mathrm{Gr}Z \circ \psi$ is the identity of $\mathcal{C}$; since ψ is surjective, this identity implies that ψ is an isomorphism, and so is $\mathrm{Gr}Z$. Furthermore, in practice, this is what happens: we have a presumed "universal" invariant Z, we propose a "surgery" map ψ and we compute $\mathrm{Gr}Z \circ \psi$. (Same remark in Lemma 4.8.)

P. 11: What you call "enhanced cobracket" here is actually a variation of the self-intersection operation μ that Turaev introduced long before his cobracket (*Math. Sb.* 1978): see §3.2 of [Mas18] and the operation $\delta_{\vec{\mu}}$ that is defined there. I think that those two references should be added.

§3.3: Some references could be added here too: first, the Lie bialgebra structure that is described here on |FA| is known as the "necklace Lie bialgebra", whose cobracket has been introduced by Schedler (*IMRN* 2005); second, that the associated graded of the Goldman–Turaev Lie bialgebra of a punctured disk is the necklace Lie bialgebra was also observed in [Mas18, §5.4 & §7].

Pages 32–33: It took me time to understand your description of the maps λ_a and λ_d. I would have said things as follows: (1) For $\gamma \in \pi$, define $\lambda_a(\gamma) \in \tilde{\mathcal{T}}(\cap)$ as the ascending lift of γ (which is a well-defined unframed bottom knot), and then set its framing number to be +1; the map λ_d is defined similarly; (2) Explain how to draw a "floor diagram" of $\lambda_a(\gamma)$ (resp. $\lambda_d(\gamma)$) using the regular homotopy class $\tilde{\gamma}$ in D_p and inserting the appropriate "kinks" to get the framing number equal to +1 (here, Whitney's formula is needed to relate the rotation number to the algebraic self-intersection of a regular curve in the disk); (3) compare the "floor diagrams" of $\lambda_a(\gamma)$ and $\lambda_d(\gamma)$ that are obtained in this way.

Page 37: The lack of formality of the self-intersection map μ (or its variant $\vec{\mu}$) is also proved in [Mas18, Th. 7.1]: the correcting term (which is denoted there by q) is expressed in terms of the Gamma function of the associator underlying the construction of the Kontsevich integral.

Page 38, footnote: It could be specified, somewhere in §3 or §4, which version of the Kontsevich integral is used in the paper. In particular, it could be said why you prefer to work with the "analytic" construction rather than the "combinatorial" construction.

Page 44: I don't understand the "+" sign in $\mathrm{cl}(|1| \otimes B + |B| \otimes 1)$ at the very end of the proof.

2. Typos and other minor comments

P.4, l.6: "Another approach" $\rightsquigarrow$ "Another approach to"

P.4, l.-19: "biaglebra" $\rightsquigarrow$ "bialgebra"

P.4, l.-17: "Gwenel" $\rightsquigarrow$ "Gwenaël"

P.5, l.-3: "$\pi(B^n)$" $\rightsquigarrow$ "$\pi(B^i)$"

P.5, l.6: Repetition in the sentence "Therefore the Kontsevich integral therefore (...)".

P.7, l.-12: "Kontsevich lntegral" $\rightsquigarrow$ "Kontsevich integral"

P.7, l.-8: "presentated" $\rightsquigarrow$ "presented"

P.9, l.-13: "which is contracts" $\rightsquigarrow$ "which contracts"

P.10, l.8: "Let D_p denote" $\rightsquigarrow$ "Let D_p denote the"

P.10, l.-10: It should be specified that the path ν is from $\star$ to $\bullet$ in the negative direction of the oriented boundary of D_p. Moreover, the second base point has three different notations in your paper: $\star$, $*$ and this kind of "virus" symbol in your pictures.

P.12, l.13: "There I-adic filtration" $\rightsquigarrow$ "The I-adic filtration"

P.13, l.8: The text is here refering to Figure 7, but the notations "z and w" are not shown in that figure.

P.13, l.-5: "As_p" $\rightsquigarrow$ "FA"

P.14, l.10: "$|As_p|$" $\rightsquigarrow$ "|FA|"

P.15, l.6: "schematic diagram" $\rightsquigarrow$ "a schematic diagram"

P.18, l.-9: "that it ψ" $\rightsquigarrow$ "that ψ"

P.19, l.-17: "by the same names" $\rightsquigarrow$ "denoted by the same names"

P.19, l.-3: "as in see Figure" $\rightsquigarrow$ "as in Figure"

P.19, l.-1: There is an unexpected "18" here.

P.20, l.8: "subscripts" $\rightsquigarrow$ "superscripts"

P.21, l.-13: Two occurrences of ψ should be changed to ψ_∇.

P.22, l.8: This third identity here should be decorated by a "∇", since it is only true in the Conway quotient.

P.22, l.-9: "bialebra" $\rightsquigarrow$ "bialgebra"

P.22, l.-7: "$\tilde{\mathcal{T}}/\tilde{\mathcal{T}}^n$" $\rightsquigarrow$ "$\mathbb{C}\tilde{\mathcal{T}}/\tilde{\mathcal{T}}^n$"

P.24, l.-8: "Goldman Bracket" $\rightsquigarrow$ "Goldman bracket"

P.24, l.-1: It might be specified that the domain of β is endowed with the t-filtration.

P.25, l.-19: "pole-stand" $\rightsquigarrow$ "pole-strand"

P.25, l.-6: The subspace $\tilde{\mathcal{A}}^{\geq i}(O)$ could be denoted $\tilde{\mathcal{A}}^i(O)$ for coherence with your other notations for filtrations.

P.26, l.l.1: "$x_3^2x^2x_1^2x_2x_3$" $\rightsquigarrow$ "$x_3^2x_2x_1^2x_2x_3$"

P.26, l.-12: It seems that a circle skeleton component is missing in the target of the map λ_1.

P.27, l.5: "be a crossing" $\rightsquigarrow$ "is a crossing"

P.27, l.6: "speparate components" $\rightsquigarrow$ "separate components"

P.27, l.-9: "identifies" $\rightsquigarrow$ "and, so, identifies"

P.27, l.-8: "lies in in" $\rightsquigarrow$ "lies in"

P.28, l.9: "Golman bracket" $\rightsquigarrow$ "Goldman bracket"

P.31, l.-12: I would say that there is no Conway quotient on the source of this map β.

P.31, l.-2: The skeleton does not seem to be correct in this description of the kernel.

P.32, l.5: The Conway quotient does not seem to be relevant in considering $\tilde{\mathcal{T}}^{/1}$. (Here and twice below in the proof of Prop. 5.11.)

P.34, l.7: $k := 1$ and $\ell := 1$ in this occurrence of the map β.

P.34, l.-14: "$\mu = \beta \circ \hat{\eta} \circ \beta^{-1}$" $\rightsquigarrow$ "$\mu = \beta \circ \hat{\eta} \circ \beta^{-1}$ up to the framing term"

P.35, l.2: The orientations of the closed components have not been indicated.

P.35, l.-2: "follows from" $\rightsquigarrow$ "follow from"

P.36, l.1: "Recall from 5.4" $\rightsquigarrow$ "Recall from Remark 5.4"

P.39, l.1: It might be better not to call β the straight vertical strand, since the notation β is already used intensively for something else.

P.39, l.-11: "$Z(\Phi_1)$" $\rightsquigarrow$ "$Z(\Phi)$"

P.39, l.-4: "$\tilde{\mathcal{A}}/2$" $\rightsquigarrow$ "$\tilde{\mathcal{A}}^{/2}$"

P.41, l.-13: "satatement" $\rightsquigarrow$ "statement"

P.41, l.-6: "$\mathrm{gr}_a\lambda$" $\rightsquigarrow$ "$\mathrm{gr}\lambda_a$"

P.42, l.-1: "λ_1^{alg}" $\rightsquigarrow$ "λ_a^{alg}"