GOLDMAN-TURAEV FORMALITY FROM THE KONTSEVITCH INTEGRAL

DROR BAR-NATAN, ZSUZSANNA DANCSO, TAMARA HOGAN, JESSICA LIU,
AND NANCY SCHERICH

ABSTRACT. We present a three dimensional realisation of the Goldman-Turaev
Lie biaglebra, and construct Goldman-Turaev homomorphic expansions from
the Kontsevich integral.

CONTENTS

1

To do list for Zsuzsi

(1) (BIG COMMENT) Section 3.1, reconsider the depth for which we discuss the Kontsevich integral. Who is our audience?
(2) Section 3.3, make 2 dummy figures–described in the side notes
(3) Section 3.3, read over the added informal descriptions of the operations to tighten up.
(4) Section 3.3, there is an old note from Jessica about signs. Do we need to keep that comment, or can we delete it?
(5) Find the reference for Proposition 3.6– Quillen66? Or new reference for Magnus expansion.
(6) I added a footnote for the Magnus expansion. Do we need it? Should we say more there?
(7) add a reference for Proposition 3.8.
(8) Section 4, make it clear where the proof for Theorem 4.9 ends.
(9) Section 4, make dummy figure for chord diagram stacking
(10) I reordered the intro section according to Dror's comments. Have you read it over? It probably needs proof reading again.

1. Introduction

In 1986, Goldman defined a Lie bracket [Gol86] on the space of homotopy classes of free loops on a compact oriented surface. Shortly after in 1991, Turaev defined a cobracket [Tur91] on the same space[1]. This bracket and cobracket make the space of free loops into a Lie bialgebra – known as the Goldman-Turaev (GoTu) Lie bialgebra – which forms the basis for the field of string topology [?] and has been an object of study from many perspectives.

In this paper we, describe a 3-dimensional lift of the Goldman-Turaev Lie bialgebra into a space of tangles in a handlebody. We recover the bracket and cobracket maps as projections of intuitive operations on tangles. We show the Kontsevich integral is homomorphic with respect to these tangle operations. Our main result is informally summarised as follows:

Main Result. *Let $\tilde{\mathcal{T}}$ denote the space of formal linear combinations of tangles in a punctured disc cross an interval $M_p = D_p \times I$. Projecting to the bottom $D_p \times 0$, one obtains curves on a punctured disc, and the Goldman–Turaev operations on these curves are induced[2] by the* stacking *and* flipping *operations on the tangles. The Kontsevich integral is a homomorphic expansion for tangles in M_p, and descends to a Goldman–Turaev homomorphic expansion on D_p.*

This result is parallel to Massuyeau's [Mas18], however, our approach to the cobracket is significantly different and simpler, hence, more likely to lead to give

---

[1]Turaev's version required factoring out by the constant loop; there is a lift to the full space of homotopy classes of loops, given a framing on the surface [AKKN20].

[2]In a specific sense defined in Section 2

insight into the motivational application descried below. Another related result is [?], which constructs Goldman–Turaev expansions from the Khnizhnik-Zamolodchikov connection, a geometric incarnation of the Kontsevich integral.

In more detail, we describe a space $\widetilde{\mathcal{T}}$ of formal linear combintations of framed tangles in the handlebody $\mathcal{D}_p \times I$ and operations on this space, which induce the Goldman-Turaev operations in the bottom projection to $D_p \times \{0\}$. The Goldman bracket arises from the commutator associated to the stacking product in a Conway skein quotient of $\widetilde{\mathcal{T}}$, defined in Section 4.7, and the Turaev cobracked from taking the difference between a tangle and its vertical flip, again in a Conway quotient. We study the associated graded spaces and operations, and show that the Kontsevich integral is a homomorphic expansion for these tangles, in other words, intertwines the operations with their associated graded counterparts. We show that therefore, the Kontsevich integral descends to a homomorphic expansion for the Goldman-Turaev Lie bialgebra. For the flipping operation and the Turaev cobracket, the precise statements are subtle, and care needs to be taken with the technical details.

1.1. **Motivation.** The Kashiwara–Vergne equations originally arose from the study of convolutions on Lie groups [?]. The equations were reformulated algebraically in terms of automorphisms of free Lie algebras [?], it this form they are a refinement of the Baker-Campbell-Hausdorff formula for products of exponentials of non-commuting variables.

Kashiwara–Vergne theory has multiple topological interpretations in which Kashiwara–Vergne solutions correspond to certain invariants – called *homomorphic expansions* – of topological objects. The existence of a homomorphic expansion is also called *formality* in the literature, this language is inspired by rational homotopy theory and group theory [?].

One of these topological interpretations is due to the first two authors [BND17], who showed that homomorphic expansions of welded foams – a class of 4-dimensional tangles – are in one to one correspondence with solutions to the KV equations. Recently, a series of papers by Alekseev, Kawazumi, Kuno and Naef [AKKN20,AKKN18b,AKKN18a] drew an analogous connection between KV solutions and homomorphic expansions for the Goldman-Turaev Lie bialgebra for the disc with two punctures (up to non-negligible differences in the technical details). This correspondence was used to generalise the Kashiwara–Vergne equations via considering different surfaces, including those of higher genus.

In other words, there is an intricate algebraic connection between four-dimensional welded foams and the GT Lie bi-algebra, which strongly suggests that there is a topological connection as well. In addition to the inherent interest in tangles in handlebodies, one goal for this paper is to work towards this connection between the two-dimensional Goldman–Turaev Lie bialgebra and four-dimesnional welded foams, by constructing a three-dimensional realisation of the Goldman-Turaev Lie bialgebra, with homomorphic expansions which descend to Goldman-Turaev expansions.

The paper is organised as follows: Section 2 gives a general algebraic framework for how the Goldman–Turaev operations are induced by tangle operations. In Section 3 we give a brief overview of the Kontsevich integral and the Goldman Turaev Lie bialgebra. In Section 4, we define tangles in handlebodies, relevant operations and Vassiliev filtrations. We identify the associated graded space of tangles as a space of chord diagrams, and introduce the Conway skein quotient. In Section **??**, we identify the GoTu Lie biaglebra in a low filtration degree, and prove the main theorem.

2. Conceptual summary "Executive Summary"

`sec:conceptsum`

We induce the genus zero Goldman-Turaev operations from tangle operations, in the spirit of "connecting homomorphisms": this Section is a summary of the basic approach. We use the words *associated graded structures*, *homomorphic expansions*, and *Goldman-Turaev operations* without definition, only mentioning their basic properties which make this conceptual outline coherent; the definitions follow in Section 3.

In the diagram (2.1), top and bottom rows are exact and the right and left vertical maps are zero, and therefore, by minor diagram chasing, the middle vertical map λ induces a unique map $\eta : C \to D$, a degenerate case of a connecting homomorphism. In our applications λ is a difference of two maps λ_1 and λ_2, whose values differ in E but coincide in a quotient F.

`eq:Snake` (2.1)

$$
\begin{array}{ccccccccc}
 & & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle \lambda=\lambda_1-\lambda_2} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & D & \longrightarrow & E & \longrightarrow & F & &
\end{array}
$$

with the map η indicated by a dashed arrow from C to D.

In Section 5 we present two constructions which produce the Goldman bracket and the Turaev cobracket, respectively, as induced homomorphisms η, from corresponding tangle operations λ_1 and λ_2. The following example is a schematic version of what will become the argument for the Goldman bracket:

$$a[b,c] = [ab,c] - [a,c]b$$

Example 2.1. Let A be an associative algebra, and let $\{L_i\}$ denote the lower central series of A. That is, $L_1 := A$, and $L_{i+1} := [L_i, A]$. Then the L_i are Lie ideals, and let $M_i = AL_i$ denote the two-sided ideal generated by L_i. The quotient A/M_1 is the abelianisation of A, denoted by A^{ab}. Then we have the following diagram:

$$= L_i A$$

eq:SnakeExample   (2.2)

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & \frac{A}{M_2} \otimes \frac{A}{M_2} & \longrightarrow & A^{ab} \otimes A^{ab} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle [\cdot,\cdot]} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & \frac{M_1}{M_2} & \longrightarrow & \frac{A}{M_2} & \longrightarrow & A^{ab} & \longrightarrow & 0
\end{array}
$$

with map η shown above.

Here λ is the algebra commutator, which is indeed the difference between two maps: the multiplication (λ_1) and the multiplication in the opposite order (λ_2). The kernel K of the projection to $A^{ab} \otimes A^{ab}$ is generated by the subalgebras $\left\{ \frac{M_1}{M_2} \otimes \frac{A}{M_2}, \frac{A}{M_2} \otimes \frac{M_1}{M_2} \right\}$ in $\frac{A}{M_2} \otimes \frac{A}{M_2}$. The map η is a well defined commutator map $A^{ab} \otimes A^{ab} \to \frac{M_1}{M_2}$, given by $\eta(x \otimes y) = [x, y] \bmod M_2$.   □

In summary:

Proposition 2.2. *If the rows of the following diagram are exact, then there exists a unique map $\eta : C \to D$ that makes the diagram commute.*

eq:inducedconnhom   (2.3)

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & D & \longrightarrow & E & \longrightarrow & F & \longrightarrow & 0
\end{array}
$$

with map η shown above.

The goal of this paper is to construct homomorphic expansions (aka formality isomorphisms) for the Goldman-Turaev Lie bialgebra from the Kontsevich integral. In outline, this follows from the naturality property of the construction above, under the associated graded functor, as follows.

Given a short exact sequence

$$0 \longrightarrow A \xhookrightarrow{\iota} B \xtwoheadrightarrow{\pi} C \longrightarrow 0,$$

and a descending filtration on B

$$B = B^0 \supseteq B^1 \supseteq B^2 \supseteq \cdots \supseteq B^n \supseteq \cdots,$$

there is an induced filtration on A given by

$$A = A^0 \supseteq A^1 \supseteq A^2 \supseteq \cdots \supseteq A^n \supseteq \cdots,$$

(handwritten margin note: *unnecessary repeat.*)

where $A^i = \iota^{-1}(\iota A \cap B^i)$. Similarly, there is an induced filtration on C given by

$$C = C^0 \supseteq C^1 \supseteq C^2 \supseteq \cdots \supseteq C^n \supseteq \ldots$$

where $C^i = \pi(B^n)$.

Lemma 2.3. *If the rows of the diagram (2.3) are exact and filtered so that the filtrations on the left and right are induced from the filtration in the middle, then the induced map η is also filtered.*

Proof. Basic diagram chasing: given $c \in C^n$, since $C^n = \pi(B^n)$, there is a $b \in B^n$ such that $\pi(b) = c$. Since λ is filtered, $\lambda(b) \in E^n$, and $\lambda(b) \in \iota(D)$ by exactness. Since $D^n = \iota^{-1}(\iota(D) \cap E^n)$, we have that $\lambda(b) = \iota(d)$ for a $d \in D^n$. By uniqueness of the induced map, $d = \eta(c)$. □

The associated graded functor is a functor from the category of filtered algebras (or vector spaces) to the category of graded algebras (or vector spaces). For a filtered algebra

$$A = A^0 \supseteq A^1 \supseteq A^2 \supseteq \cdots \supseteq A^n \supseteq \ldots,$$

the (degree completed) associated graded algebra is defined to be

$$\operatorname{gr} A = \Pi_{n=0}^{\infty} A^n/A^{n+1}.$$

The associated graded map of a filtered map is defined in the natural way (as in the proof of Lemma 2.4 below). In general, gr is not an exact functor, but it does preserve exactness for the special class of filtered short exact sequences where the filtrations on A and C are induced from the filtration on B:

`lem:ExaxtGr` **Lemma 2.4.** *If in the filtered short exact sequence*

$$0 \longrightarrow A \overset{\iota}{\longhookrightarrow} B \overset{\pi}{\twoheadrightarrow} C \longrightarrow 0$$

the filtrations on A and C are induced from the filtration on B, then the associated graded sequence is also exact:

$$0 \longrightarrow \operatorname{gr} A \overset{\operatorname{gr} \iota}{\longhookrightarrow} \operatorname{gr} B \overset{\operatorname{gr} \pi}{\twoheadrightarrow} \operatorname{gr} C \longrightarrow 0.$$

Proof. Since gr is a functor, we know that $\operatorname{gr} \pi \circ \operatorname{gr} \iota = 0$, hence $\operatorname{im} \operatorname{gr} \iota \subseteq \ker \operatorname{gr} \pi$. It remains to show that $\ker \operatorname{gr} \pi \subseteq \operatorname{im} \operatorname{gr} \iota$.

Let $[b] \in B^n/B^{n+1}$, and assume that $\operatorname{gr} \pi([b]) = 0$. Since $\operatorname{gr} \pi([b]) = [\pi(b)] \in C^n/C^{n+1}$, we have $\operatorname{gr} \pi([b]) = 0$ if and only if $\pi(b) \in C^{n+1}$. As the filtration on C is induced from B, we know that $C^{n+1} = \pi(B^{n+1})$. Thus, $\pi(b) \in \pi(B^{n+1})$. Or in other words, there exists $x \in B^{n+1}$ such that $\pi(b) = \pi(x)$. This implies that $\pi(b - x) = 0$ and hence that $b - x \in \iota(A)$ by exactness.

Therefore, $b = x + \iota(a)$ for some $x \in B^{n+1}$ and $a \in A$. It follows that $[b] = [\iota(a)] = \operatorname{gr} \iota([a])$ in B^n/B^{n+1} and hence $\ker \operatorname{gr} \pi \subseteq \operatorname{im} \operatorname{gr} \iota$ as required. □

Corollary 2.5. *If the rows of the diagram in Equation 2.3 are exact, and the filtrations on the left and right are induced from the filtration in the middle, then*

the rows of the associated graded diagram are also exact, and the unique connecting homomorphism is $\mathrm{gr}\,\eta$.

(2.4)

$$
\begin{array}{ccccccccc}
& & & & \overset{\mathrm{gr}\,\eta}{\dashrightarrow} & & & & \\
0 & \longrightarrow & \mathrm{gr}\,A & \longrightarrow & \mathrm{gr}\,B & \longrightarrow & \mathrm{gr}\,C & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \mathrm{gr}\,\lambda} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & \mathrm{gr}\,D & \longrightarrow & \mathrm{gr}\,E & \longrightarrow & \mathrm{gr}\,F & \longrightarrow & 0
\end{array}
$$

Proof. The exactness of the rows is Lemma 2.4. The induced map is $\mathrm{gr}\,\eta$ as $\mathrm{gr}\,\eta$ makes the diagram commute, and the induced map is unique. $\qquad\square$

An expansion for an algebraic structure X is a filtered homomorphism $Z :$ $X \to \mathrm{gr}\,X$ (with special properties). Thus, if expansions exist for each of the spaces A through F, we obtain a multi-cube:
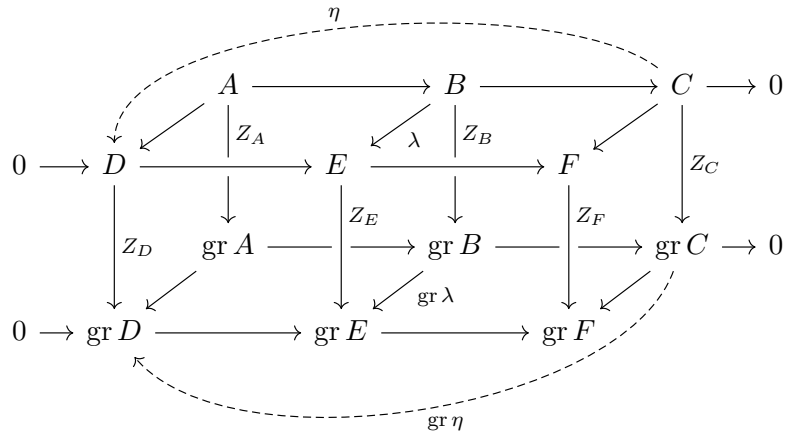
eq:Cube   (2.5)

lem:Naturality **Lemma 2.6.** *If, in the multi-cube* (2.5) *all vertical faces commute, then so does the square:*

eq:HomExp   (2.6)

$$
\begin{array}{ccc}
D & \overset{\eta}{\dashleftarrow} & C \\
\downarrow{\scriptstyle Z_D} & & \downarrow{\scriptstyle Z_C} \\
\mathrm{gr}\,D & \underset{\mathrm{gr}\,\eta}{\dashleftarrow} & \mathrm{gr}\,C
\end{array}
$$

Proof. Follows from the uniqueness of the induced maps. $\qquad\square$

The commutativity of the square (2.6), where η represents the Goldman bracket and the Turaev cobracket, respectively, is – by definition – the homomorphicity property of the expansion. This homomorphicity is our main result. The non-trivial vertical face of the multi-cube is the one containing λ, and the commutativity of this for each Goldman-Turaev operation will follow from homomorphicity
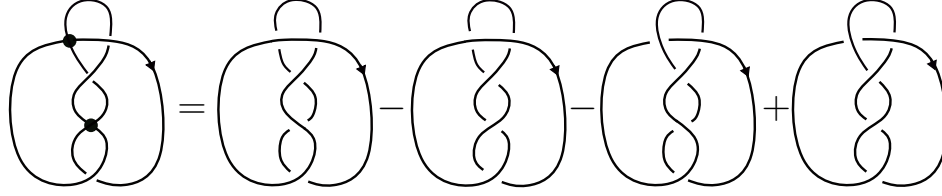
FIGURE 1. A knot with two double points written as a signed sum four knots.

fig:pumpkins

properties of the Kontsevich integral. Namely, the Kontsevich integral (standing in for Z_B and Z_E) intertwines the appropriate tangle operations λ_0 and λ_1 with their associated graded counterparts. This is the idea behind the approach of this paper.

3. PRELIMINARIES: HOMOMORPHIC EXPANSIONS AND THE GOLDMAN-TURAEV LIE BIALGEBRA

secs:PreKIms

Should we say formality instead of/in addition to homomorphic expansion?

3.1. **The Kontsevich Integral.** The Kontsevich Integral is the knot theoretic prototype of a *homomorphic expansion*. Homomorphic expansions (a.k.a. formality isomorphisms, universal finite type invariants) provide a connection between knot theory and quantum algebra/Lie theory. Many detailed expositions on the Kontsevich Integral exist in the literature, we recommend [CDM12, Section 8], or [Kon93, BN95, Dan10]. We briefly review the basics here from an algebraic perspective, which is outlined – in a slightly different, finitely presented case – in [BND17, Section 2].

Let $\mathcal{K}$ denote the set of oriented knots in $\mathbb{R}^3$, and allow formal linear combinations of knots with coefficients in $\mathbb{C}$. There is a filtration on this infinite dimensional vector space called the Vassiliev filtration, which is defined in terms of resolutions of *double points*. Namely, a *double point* is defined to be the difference of an over and under crossing:

$$\times = \times - \times.$$

A knot with k double points is a signed sum of 2^k knots. See Figure 1 for an example. The Vassiliev filtration is the decreasing filtration

$$\mathcal{K} = \mathcal{K}_0 \supseteq \mathcal{K}_1 \supseteq \mathcal{K}_2 \supseteq ...$$

where $\mathcal{K}_i$ is linearly generated by knots with at least i double points.

The degree completed associated graded space of $\mathcal{K}$ with respect to the Vassiliev filtration is defined as

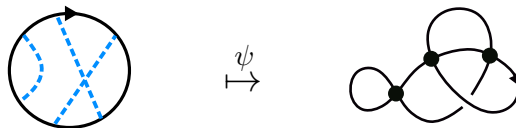$$\mathcal{A} := \prod_{n \geq 0} \mathcal{K}_n/\mathcal{K}_{n+1}.$$

FIGURE 2. Example of ψ mapping a chord diagram to a singular knot where the right-hand side is viewed as an element of $\mathcal{K}_3/\mathcal{K}_4$.

fig:psionchord

Since $\mathcal{A}$ is a graded vector space, it lends itself naturally to recursive calculations and inductive arguments. An *expansion* Z is a filtered linear map of knots taking values in $\mathcal{A}$, which retains as much information as possible. Rigorously, this means that the associated graded map of Z is the identity map of $\mathcal{A}$:

$$Z : \mathcal{K} \to \mathcal{A} \quad \text{such that} \quad \operatorname{gr} Z = id_{\mathcal{A}}.$$

An expansion is *homomorphic* with respect to some operations (such as connected sum) if it also intertwines these operations with their associated graded counterparts. This allows for a study of these operations via the associated graded space as well.

A crucial step towards making effective use of this machinery is to get a handle on the space $\mathcal{A}$ in concrete terms: namely, $\mathcal{A}$ has a combinatorial description as a space of *chord diagrams*. A chord diagram of degree k on an oriented circle is a perfect matching[3] on a set of $2k$ points arranged around the circle, up to orientation preserving diffeomorphism. The circle which supports the chord diagram is called the *skeleton*. In other words, a chord diagram on a circle is a combinatorial object consisting of $2k$ cyclically ordered points, partitioned into pairs. In diagrams, each pair is indicated by a *chord*, as in the left of Figure 2.

There is a natural map ψ from chord diagrams with i chords to $\mathcal{K}_i/\mathcal{K}_{i+1}$, as shown in Figure 2. Namely, by contracting each chord into a double point, we obtain an i-singular knot. This is well-defined only up to crossing changes – as crossings other than double points may appear – however, the difference between the over/under choices for any additional crossing is in $\mathcal{K}_{i+1}$.

It is not difficult to establish that ψ is surjective, and that there are two relations in its kernel: the 4-Term (4T) and Framing Independence (FI) relations, shown in Figure 3. In fact, these two relations generate the kernel, and ψ descends to an isomorphism on the quotient; this, however, is significantly harder to prove.

The key technique is to construct an expansion as in the following Lemma, [BND17, Proposition 2.7]:

lem:assocgradyoga

Lemma 3.1. *[BND17] Let $\mathcal{K}$ be a filtered space of formal linear combinations of knotted objects[4], and $\mathcal{A}$ the associated graded space of $\mathcal{K}$. Let $\mathcal{C}$ be a "candidate*

---

[3] A perfect matching on a set is a partitioning of the set by 2-element subsets.

[4] "Knotted objects" may mean knots, links, tangles, knotted graphs, etc, depending on context.

FIGURE 3. The 4T and FI relations, understood as local relations: the strand(s) are part(s) of the skeleton circle, and the skeleton may support additional chords outside the picture shown, which are the same throughout all terms of the relation.

fig:4TFI

model" for $\mathcal{A}$: a graded linear space equipped with a surjective graded map $\psi : \mathcal{C} \to \mathcal{A}$. If there exists a filtered map $Z : \mathcal{K} \to \mathcal{C}$, such that $\psi \circ \mathrm{gr}\, Z = id_{\mathcal{A}}$, then ψ is an isomorphism and Z is an expansion for $\mathcal{K}$.

In other words, once one finds a candidate model $\mathcal{C}$ for $\mathcal{A}$, finding an *expansion valued in* $\mathcal{C}$ also implies that ψ is an isomorphism. In classical Vassiliev theory, $\mathcal{K}$ is the space of oriented knots, $\mathcal{C}$ is the space of chord diagrams, and a $\mathcal{C}$-valued expansion is the Kontsevich integral [Kon93].

For a detailed introduction to the Kontsevich integral we recommend [CDM12, Section 8]. The definition an explicit integral formula associated to a Morse representation of a knot or link in $\mathbb{C} \times \mathbb{R}$, as in Figure 4.

Definition 3.2. Let K be an oriented link in $\mathbb{R}^3 \cong \mathbb{R}_t \times \mathbb{C}$, where t parametrises the vertical real dimension, and the embedding K is Morse with respect to t: that is, critical points are cups or caps. The unnormalised Kontsevich integral $Z'(K)$ of K is defined as:

eq:Kint

$$(3.1) \qquad Z'(K) := \sum_{m=0}^{\infty} \int_{\substack{t_{min}<t_m<\ldots<t_1<t_{max} \\ t_i\,non-critical}} \sum_{P=\{(z_i,z_i')\}_i} \frac{(-1)^{P_\downarrow}}{(2\pi i)^m} \mathfrak{D}_P \bigwedge_{i=1}^{m} \frac{dz_i - dz_i'}{z_i - z_i'}.$$

Here the values t_{min} and t_{max} denote the minimum and maximum heights of K, and each summand is an integral over an m-simplex determined by $t_m < \cdots < t_1$. The summation is over choices P of "pairings" of two points on the knot of height t_i, each of which, when projected to the plane at $t = 0$, yields a complex pair (z_i, z_i'). We denote by $\mathfrak{D}_P$ the chord diagram given by interpreting the m pairings
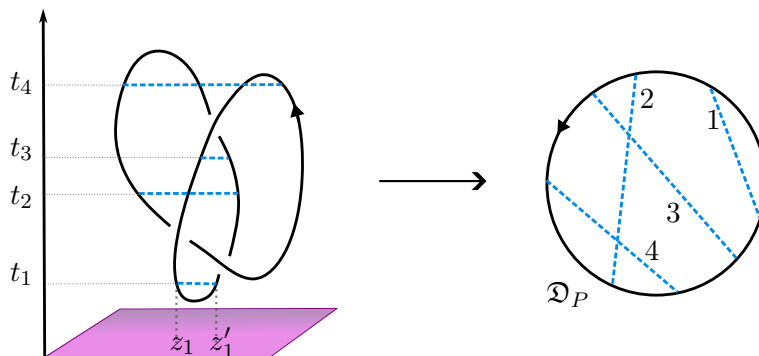
FIGURE 4. The Kontsevich Integral is computed from a Morse embedding of the knot

fig:Kint

(z_i, z'_i) as chords, as shown in Figure 4. Finally, $P_\downarrow$ is the number of points in P where (t_i, z_i) or (t_i, z'_i) is on a t-descending arc in K.

Kontsevich's famous result [Kon93] is that $Z(K) := \frac{Z'(K)}{Z'(\smile)^{c/2}}$ is an invariant of unframed links, where c denotes the number of critical points – minima and maxima – in the Morse embedding of K. The Kontsevich integral takes values in the space of chord diagrams (for links, with chords on multiple circles) modulo the 4T and FI relations. The Kontsevich integral Z satisfies $\psi \circ \mathrm{gr}\, Z = id_\mathcal{C}$. Therefore, ψ is an isomorphism, and Z is an expansion for unframed links. In light of this, we do not distinguish between $\mathcal{C}$ and $\mathcal{A}$, and use $\mathcal{A}$ to mean In addition, Z has a number of good properties, for example, it is homomorphic with respect to knot connected sum.

subsec:FramedKon

3.2. **The framed Kontsevich Integral.** Kontsevich's original construction gives an invariant of unframed links. However, in this paper we work primarily with framed links and tangles, thus we briefly review the framed version; see also [CDM12, Sections 3.5 and 9.1] and [LM96].

First we need a framed version of the Vassiliev filtration. Let $\tilde{\mathcal{K}}$ denote the set of *framed* links in $\mathbb{R}^3$: that is, links along with a non-zero section of the normal bundle. A knot diagram is interpreted as a framed knot using the blackboard framing. The Reidemeister move R1 move changes the blackboard framing, and by ommitting it, one obtains a Reidemeister theory for framed links. In analogy with a double point, a *framing change* is defined to be the difference

$$\uparrow\!\!\!\!\!\rightarrow := \overset{\curvearrowleft}{\rho} - \uparrow.$$

The framed Vassiliev filtration is the descending filtration

$$\tilde{\mathcal{K}} = \tilde{\mathcal{K}}_0 \supseteq \tilde{\mathcal{K}}_1 \supseteq \tilde{\mathcal{K}}_2 \supseteq \ldots$$

where $\tilde{\mathcal{K}}_i$ is linearly generated by knots with at least i double points *or framing changes*. The degree completed associated graded space of $\tilde{\mathcal{K}}$ with respect to the

framed Vassiliev filtration is

$$\tilde{\mathcal{A}} := \prod_{n \geq 0} \tilde{\mathcal{K}}_n / \tilde{\mathcal{K}}_{n+1}.$$

A natural first guess for a combinatorial description of $\tilde{\mathcal{A}}$ is in terms of chord diagrams with "framing change markings" on the skeleton, graded by the number of chords and markings. There is a natural surjective graded map $\tilde{\psi}$ from marked chord diagrams onto $\tilde{\mathcal{A}}$, which is defined like ψ for chords, and which replaces each marking with a framing change. The kernel of $\tilde{\psi}$ includes the $4T$ relation as before.

In place of the FI relation ($=0$), a weaker relation arises from the equality $\tilde{\rho} - \tilde{\rho} = \tilde{\rho}$ in $\tilde{\mathcal{K}}$. In fact, $\tilde{\rho} = \tilde{\rho} - \tilde{\rho} = (\tilde{\rho} - \uparrow) + (\uparrow - \tilde{\rho})$, and $\uparrow - \tilde{\rho} = \tilde{\rho} - \uparrow$ modulo $\tilde{\mathcal{K}}_2$. In other words, the following relation is in the kernel of $\tilde{\psi}$:

$$ = 2 .$$

Therefore, it is not necessary to have dedicated notation for the framing change markings, since $\frac{1}{2}$. The candidate model for the associated graded space is simply chord diagrams modulo the $4T$ relation, and no FI relation. We denote this space by $\tilde{\mathcal{C}}$.

To show that $\tilde{\psi} : \tilde{\mathcal{C}} \to \tilde{\mathcal{A}}$ is an isomorphism, the strategy is the same as before: construct a $\tilde{\mathcal{C}}$-valued expansion and use Lemma 3.1. This $\tilde{\mathcal{C}}$-valued expansion is the framed version $\tilde{Z}$ of the Kontsevich integral. The definition is similar to (3.1), the main issue is that in the absence of the FI relation, the integral diverges at cups and caps. This is resolved with a renormalisation using the framing, for details see [CDM12, Section 9.1], or [LM96, Gor99].

3.3. **The Goldman-Turaev Lie bialgebra.** In order to define the Goldman-Turaev Lie bialgebra, we need to recall some basic definitions and notation.

Let D_p denote p-punctured disc, with $p+1$ circle boundary components $\partial_0, \partial_1, ..., \partial_p$, embedded in the complex plane so that ∂_0 is the outer boundary, as in Figure 5. In particular, the plane-embedding specifies a framing on D_p, and thus loops in D_p are equipped with a notion of rotation number. Let $\pi = \pi_1(D_p, *)$ denote the fundamental group of D_p with basepoint $* \in \partial_0$. We denote by $\mathbb{C}\pi$ the group algebra of π, and by $\overline{\mathbb{C}\pi} = \mathbb{C}\pi / \mathbb{C}\mathbb{1}$ the linear quotient by the constant loop.

Let $\bullet$ and $*$ be two "nearby" basepoints on ∂_0 and ξ be the inward pointing normal vector to ∂_0 at $\bullet$ and $*$ and let $\tilde{\pi} = \tilde{\pi}_{\bullet *}$ denote the group of immersed curves $\gamma : ([0,1], 0, 1) \to (D_p, \bullet, *)$ under regular homotopy, so that $\dot{\gamma}(0) = \xi$, and $\dot{\gamma}(1) = -\xi$, as shown in Figure 5. Note that rotation number is invariant under regular homotopy. Recall that $\tilde{\pi}$ is a group, as shown in Figure 6:

(1) Let ν denote the path from $\star$ to $\bullet$ along ∂_0. The group product $\gamma_1 \cdot \gamma_2$ is the smooth concatenation of γ_1 with ν followed by γ_2.
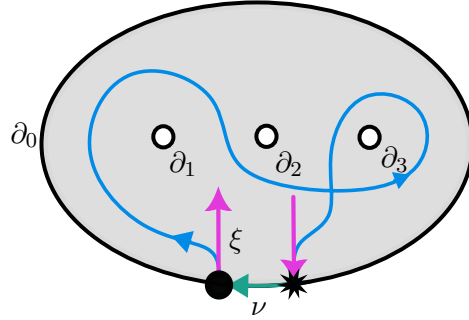
FIGURE 5. D_3 with an immersed loop from $\bullet$ to $*$ with initial tangent vector ξ and terminal tangent vector $-\xi$. The path along the boundary from $*$ to $\bullet$ is ν.
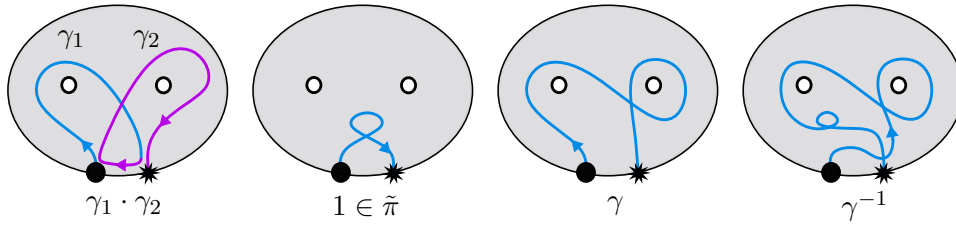
fig:DP

FIGURE 6. The group structure on $\tilde{\pi}$.

fig:DPGroup

(2) The identity is the path that is contractible and of rotation number zero when composed with ν.

(3) The inverse of γ is the concatenation $\bar{\nu}\,\bar{\gamma}\,\nu^*$ where the overline denotes the reverse path, and ν^* includes a negative twist (to ensure that the rotation number of $\gamma \cdot \gamma^{-1}$ is 0). The beginning and end of the path is adjusted in an epsilon neighbourhood of the base points to have inward and outward pointing tangent vectors, as in Figure 6.

Denote by $\mathbb{C}\tilde{\pi}$ the group algebra of $\tilde{\pi}$. There is a forgetful map $\tilde{\pi} \to \pi$ which maps γ to the (non-regular) homotopy class of $\gamma\,\nu$. This linearly extends to a forgetful map $\mathbb{C}\tilde{\pi} \to \mathbb{C}\pi$.

For an algebra A we denote by $|A|$ the *linear* quotient $A/[A, A]$ (not the abelianization), where $[A, A]$ denotes the subspace spanned by commutators $[x, y] = xy - yx$ for $x, y \in A$. We denote the quotient (trace) map by $|\cdot| : A \to |A|$. In our context, elements of $|\mathbb{C}\pi|$ have an explicit description as the $\mathbb{C}$-vector space generated by homotopy classes of free loops in D_p. Similarly, $|\mathbb{C}\tilde{\pi}|$ is comprised of regular homotopy classes of free loops, where $|\gamma|$ denotes the class of $\gamma\nu$ as a free loop.

The Goldman–Turaev Lie bialgebra comes in two flavours: *original* and *enhanced*. The original construction of the Goldman bracket is a Lie bracket on $|\mathbb{C}\pi|$. However, the original Turaev cobracket is only well-defined on $|\overline{\mathbb{C}\pi}| = |\mathbb{C}\pi|/\mathbb{C}\mathbb{1}$, the linear quotient by the homotopy class of the constant loop. The space $|\overline{\mathbb{C}\pi}|$ is a Lie bialgebra with this cobracket and the Goldman bracket, which descends from $|\mathbb{C}\pi|$. There is an enhancement of the cobracket which promotes it to $|\mathbb{C}\pi|$, thereby making $|\mathbb{C}\pi|$ a Lie bialgebra under the Goldman bracket and the enhanced cobracket. In [AKKN18b] this enhancement was constructed in order to establish the relationship to Kashiwara–Vergne theory. To define the enhanced cobracket, a curve in $|\mathbb{C}\pi|$ is lifted to an immersed curve with rotation number zero. Below we define the Goldman bracket and this enhanced version of the Turaev cobracket.

The Goldman Bracket, in summary, takes in two free loops and at each intersection between the two loops smooths the intersection to join the loops into one longer connected loop, then sums over each intersection. For a free loop α in D_p and a point q on α, we denote by α_q be the loop α based at q.

sort of clunky

def:bracket **Definition 3.3** (The Goldman bracket). Let $\alpha, \beta \in |\mathbb{C}\pi|$ be free loops with homotopy representatives chosen so that there are only finitely many transverse double intersections between α and β. The Goldman bracket $[\cdot, \cdot]_G : |\mathbb{C}\pi| \otimes |\mathbb{C}\pi| \to |\mathbb{C}\pi|$ is given by

$$[\alpha, \beta]_G := -\sum_{q \in \alpha \cap \beta} \varepsilon_q |\alpha_q \beta_q|,$$

where $\varepsilon_q = \varepsilon(\dot{\alpha}_q, \dot{\beta}_q) \in \{\pm 1\}$ is the local intersection number of α and β at q, $\alpha_q \beta_q$ is the concatenation of α_q and β_q, and the extension to $|\mathbb{C}\pi|$ is linear. Then $[\cdot, \cdot]_G$ is a Lie bracket on $|\mathbb{C}\pi|$.

The sign here (with the minus sign in front) matches with AKKN genus 0, but is the opposite of AKKN higher genus and Goldman's original definition. Our current multiplication and bracket matches the sign here, so if we change the sign then we should change the stacking order of our multiplication.

The original definition of the Turaev cobracket is similar, but uses self intersections of a curve in place of the intersections between two curves, but one has to take care for well definedness. We first construct the (enhanced) cobracket via a self-intersection map for *based* curves, as in [AKKN18b, Section 5.2]. This definition is suitable for direct comparison with the three-dimensional operations of Section 5. Figure 7 shows an example computation for the self intersection map μ. For a based curve γ in $\mathbb{C}\pi$, the idea behind μ is to "cut off" portions of γ at self intersection points to get two curves, one that is based, one that is free, and then sum over all self intersections.

def:mu **Definition 3.4** (The self-intersection map). For $\gamma \in \mathbb{C}\pi$, let $\tilde{\gamma} : [0,1] \to D_p$ denote an immersed representative of γ in $\mathbb{C}\tilde{\pi}$ with only transverse double points, and with rotation number $\text{rot}(\tilde{\gamma}) = 0$. Let $\tilde{\gamma} \cap \tilde{\gamma}$ denote the set of double points. The self intersection map μ is defined as follows:

$$\mu : \mathbb{C}\pi \to |\mathbb{C}\pi| \otimes \mathbb{C}\pi$$

$$\mu(\gamma) = -\sum_{p \in \tilde{\gamma} \cap \tilde{\gamma}} \varepsilon_p |\tilde{\gamma}_{t_1^p t_2^p}| \otimes \tilde{\gamma}_{0 t_1^p} \tilde{\gamma}_{t_2^p 1},$$

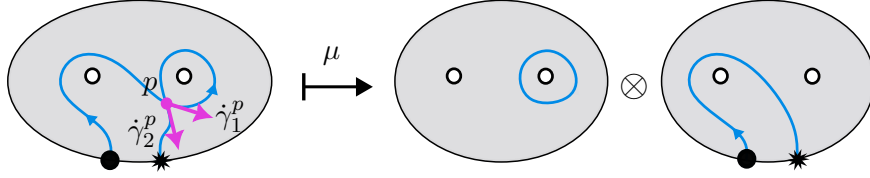FIGURE 7. Example of the self intersection map μ where $\epsilon_p = -1$.

where t_1^p and t_2^p are the first and second time parameter in $[0, 1]$ where $\tilde{\gamma}$ goes through p; where $\tilde{\gamma}_{rs}$ denotes the path traced by $\tilde{\gamma}$ from $t = r$ to $t = s$; and $\varepsilon_p = \varepsilon\left(\dot{\tilde{\gamma}}(t_1^p), \dot{\tilde{\gamma}}(t_2^p)\right) \in \{\pm 1\}$ is the local self-intersection number, and the formula extends to $\mathbb{C}\pi$ liearly.

See Figure 7 for an example calculation of the self-intersection map.

Definition 3.5 (The Turaev co-bracket). The Turaev cobracket δ is the unique linear map which makes the following diagram commute, where $\mathrm{Alt}(x \otimes y) = x \otimes y - y \otimes x = x \wedge y$:

$$
\begin{array}{ccccc}
\mathbb{C}\pi & \xrightarrow{\mu} & |\mathbb{C}\pi| \otimes \mathbb{C}\pi & \xrightarrow{1 \otimes |\cdot|} & |\mathbb{C}\pi| \otimes |\mathbb{C}\pi| \\
\downarrow{\scriptstyle |\cdot|} & & & & \downarrow{\scriptstyle \mathrm{Alt}} \\
|\mathbb{C}\pi| & \dashrightarrow{\hspace{1cm}\delta\hspace{1cm}} & & & |\mathbb{C}\pi| \wedge |\mathbb{C}\pi|
\end{array}
$$

3.4. **Associated graded Goldman-Turaev Lie bialgebra.** There I-adic filtration on $\mathbb{C}\pi$ is the filtration by powers of the augmentation ideal $\mathcal{I} = \langle \{\alpha - 1\}_{\alpha \in \pi} \rangle$:

$$\mathbb{C}\pi = \mathcal{I}^0 \supseteq \mathcal{I} \supseteq \mathcal{I}^2 \supseteq \ldots$$

This descends to a filtration on $|\mathbb{C}\pi|$:

$$|\mathbb{C}\pi| = |\mathcal{I}^0| \supseteq |\mathcal{I}| \supseteq |\mathcal{I}^2| \supseteq \ldots$$

The completed associated graded algebra for $|\mathbb{C}\pi|$ with respect to this filtration

> We should add in a remark about how the bracket and cobracket are degree shifting maps.

$$\mathrm{gr}\,|\mathbb{C}\pi| = \prod_{n=0}^{\infty} |\mathcal{I}^n| / |\mathcal{I}^{n+1}|$$

has an explicit description in terms of *cyclic words*, as follows. Let $\mathrm{FA} = \mathrm{FA}\langle x_1, \cdots, x_p \rangle$ denote the free associative algebra on r generators. Then elements of $|\mathrm{FA}|$ are cyclic words in letters $x_1, \cdots, x_p$, that is, words modulo cyclic permutations of the letters. The following result is due to [].

> complete with citation: Quillen66?
>
> I couldn't find the reference. Can you give more info?

Proposition 3.6. *Given the set of standard generators $\{\gamma_i\}_{i=1}^p$ for π, the map $\varphi(\gamma_i^{\pm 1}) = e^{\pm x_i}$ defines an isomorphism[5] of algebras $\mathrm{gr}\,|\mathbb{C}\pi| \to \mathrm{FA}$, which descends*

---

[5] φ is the Magnus expansion

FIGURE 8. The graded Goldman bracket.

`fig:grbracket`

to a linear isomorphism

$$\mathrm{gr}\,|\mathbb{C}\pi| \cong |\,\mathrm{FA}\,|.$$

Therefore, $|\,\mathrm{FA}\,|$ carries the structure of a Lie bialgebra under $\mathrm{gr}[\cdot,\cdot]_G$ and $\mathrm{gr}\,\delta$.

Representing cyclic words diagrammatically as letters along a circle, the graded Goldman bracket sums over matching pairs of letters in z and w, joins the circles at the matching letter, and takes the difference of the two ways of including only one copy of the letter in the new cyclic word, as in Figure 8.

`def:grbracket`

Definition 3.7. [The graded Goldman bracket] Let $z = z_1 \cdots z_l$ and $w = w_1 \cdots w_m$ be two cyclic words in $|\,\mathrm{FA}\,|$. The *graded Goldman bracket*

$$[-,-]_{\mathrm{gr}\,G} : |\,\mathrm{FA}\,| \otimes |\,\mathrm{FA}\,| \to |\,\mathrm{FA}\,|$$

of z and w is given by:

$$[z,w]_{\mathrm{gr}\,G} = \sum_{j,k} \delta_{z_j,w_k}(w_1 \ldots w_{k-1} z_{j+1} \ldots z_l z_1 \ldots z_j w_{k+1} \ldots w_m -$$

$$w_1 \ldots w_{k-1} z_j \ldots z_l z_1 \ldots z_{j-1} w_{k+1} \ldots w_m)$$

where δ_{z_j,w_k} is the Kronecker delta.

Proposition 3.8. *The graded Goldman bracket $[-,-]_{\mathrm{gr}\,G}$ is the associated graded map of the Goldman bracket under the filtration induced by the I-adic filtration.*

add reference or proof

Using the same diagrammatic representation of cyclic words as letters along a circle, the graded Turaev co-bracket of a cyclic word is computed by a summation of *pairing cuts*, an example is shown in Figure 9. For a pairing cut, identify two matching letters in the cyclic word, call the letters x. Draw a chord between the two x's and split the circle into to two separate circles along this chord. Remove the two x's. Take the difference of the two words where an x is placed on the left chord and an x is placed on the right chord. (The cyclic word has an orientation which determines the 'left' and 'right'.)

Definition 3.9 (The graded Turaev cobracket). Let $w = w_1 \ldots w_m \in |As_p|$. The graded Turaev cobracket for w is given by

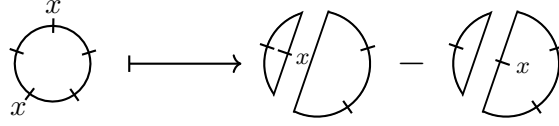$$\delta_{\mathrm{gr}} : |\,\mathrm{FA}\,| \to |\,\mathrm{FA}\,| \wedge |\,\mathrm{FA}\,|$$

FIGURE 9. An example pairing cut of a cyclic word as a term in the graded Turaev cobracket.

$$\delta_{\mathrm{gr}}(w) = \sum_{j<k} \delta_{w_j,w_k}(|w_j \dots w_{k-1}| \wedge |w_{k+1} \dots w_n w_1 \dots w_{j-1}| +$$

$$|w_k \dots w_n w_1 \dots w_{j-1}| \wedge |w_{j+1} \dots w_{k-1}|),$$

where δ_{w_j,w_k} is (unfortunately) the Kronecker delta.

Proposition 3.10. *The graded Turaev cobracket δ_{gr} is the associated graded map of the Turaev cobracket under the filtration induced by the I-adic filtration.*

An diagrammatic example of a term in the definition of the graded cobracket is given in Figure 9.

4. EXPANSIONS FOR TANGLES IN HANDLEBODIES

4.1. **The space $\mathbb{C}\tilde{\mathcal{T}}$.** In this paper we consider the space $\mathbb{C}\tilde{\mathcal{T}}$ of framed, oriented tangles in a genus p handlebody, and show that homomorphic expansions on this space descend to homomorphic expansions on the Goldman-Turaev Lie biagebra as defined in [AKKN20]. This section describes the space $\mathbb{C}\tilde{\mathcal{T}}$.

Let M_p denote the manifold $D_p \times I$ where D_p is a disc in the complex plane with p points removed. While M_p is not a compact manifold, knot theory in M_p is equivalent to knot theory in a genus p handlebody. For the purpose of the Kontsevich integral, we identify D_p with a square in the complex plane with p points removed, so M_p can be drawn as a cube with p vertical lines removed; we call these lines the *poles*, as shown in Figure 11. We refer to $D_p \times \{0\}$ as the "floor" or "bottom", and $D_p \times \{1\}$ as the "ceiling" or "top". The "back wall" is the north ($i \in \mathbb{C}$) edge of D_p times $[0,1]$.

Definition 4.1. An oriented tangle T in M_p is an embedding of an oriented compact 1-manifold

$$(S, \partial S) \hookrightarrow (M_p, D_p \times \{0\} \cup D_p \times \{1\}).$$

The interior of S lies in the interior of M_p, and the boundary points of S are mapped to the top or bottom. Oriented tangles in M_p are considered up to ambient isotopy fixing the boundary. We denote the set of isotopy classes by $\mathcal{T}$.

Definition 4.2. A *framing* for an oriented tangle T in M_p is a continuous choice of unit normal vector at each point of T, which is fixed pointing in the north direction ($i \in \mathbb{C}$) at the boundary points. *Framed oriented tangles* in M_p are
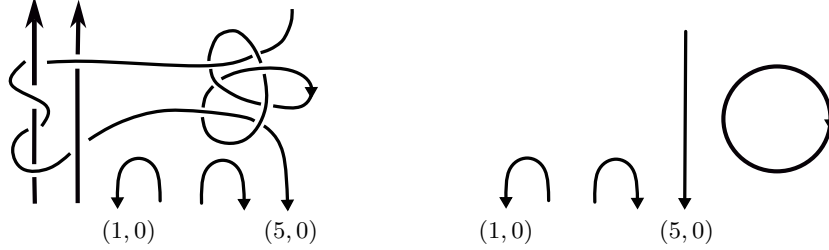
FIGURE 10. On the left is a tangle in M_2, and on the right is schematic diagram of the skeleton of the tangle. The skeleton of the tangle is the combinatorial data given by the following set of order pairs and the integer 1:
$$\{[((2,0),0),((1,0),0))], [((3,0),0),(4,0),0))], [((5,0),1),(5,0),0))], 1\}$$
.

fig:skeleton

considered up to ambient isotopy fixing the boundary. We denote the set of isotopy classes of framed oriented tangles by $\tilde{\mathcal{T}}$.

Henceforth, any tangle is assumed to be framed and oriented unless otherwise stated. The skeleton of a tangle is the underlying combinatorial information with the topology forgotten:

def:skeleton

Definition 4.3. The *skeleton* $\sigma(T)$ of a tangle $T = (S \hookrightarrow M_p)$ – see Figure 10 – is the set of tangle endpoints $P_{bot} \subseteq D_p \times \{0\}$ and $P_{top} \subseteq D_p \times \{1\}$, along with

(1) A partition of $P_{bot} \cup P_{top}$ into ordered pairs given by the oriented intervals of S.
(2) A non-negative integer k: the number of circles in S.

Maybe it would be better to define $P_{bot}, P_{top} \subseteq D_p$ and then say $P_{bot} \times \{0\}$ and $P_\top \times \{1\}$ are the tangle endpoints. Then it would make descriptions of tangle operations easier, as well as the info in figure 9.

The skeleton of a framed tangle is the same as the skeleton of the underlying unframed tangle. The set of framed tangles in M_p with skeleton S is denoted $\tilde{\mathcal{T}}(S)$. For example, $\tilde{\mathcal{T}}(\bigcirc)$ is the set of framed knots in M_p.

The linear extension of $\tilde{\mathcal{T}}(S)$, denoted $\mathbb{C}\tilde{\mathcal{T}}(S)$, is the vector space of $\mathbb{C}$-linear combinations of tangles in $\tilde{\mathcal{T}}(S)$. We denote by $\mathbb{C}\tilde{\mathcal{T}}$ the disjoint union $\sqcup_S \tilde{\mathcal{T}}(S)$ over all skeleta S, identified at 0. Tangles with different skeleta cannot be linearly combined.

We can look at tangles in M_p using tangle diagrams in two different ways, by projecting either to the back wall of M_p or to the floor.

If we project to the back wall, an ℓ-component tangle in M_p can be diagrammatically represented as a tangle diagram with p straight vertical "poles", and ℓ tangle "strands" of circle and interval components. The strands pass over (in front of) and under (behind) the poles and other strands, as shown on the right in Figure 11. The poles are equipped with an orientation coming from the parametrization in $M_p \cong D_p \times I$, and in figures we draw them oriented upwards, unless otherwise stated. By Reidemeister's theorem, $\tilde{\mathcal{T}}$ is equivalent to such diagrams modulo the Reidemeister moves R2 and R3. (No R1, as the tangles are framed.)

on the the left hand side of this figure something is missing
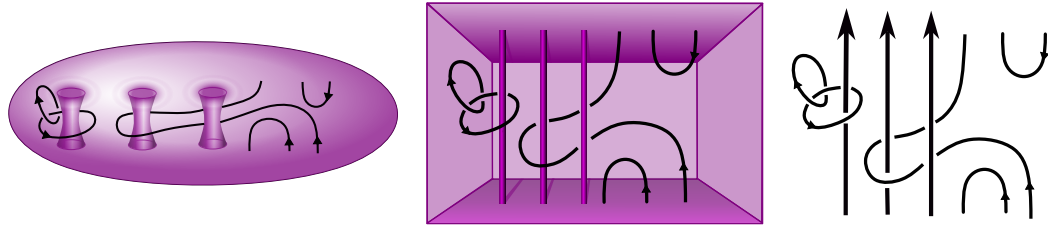
no its not–it is a refresh rate problem.

FIGURE 11. An example of a tangle in M_3, drawn first in a handlebody, secondly in a cube with poles, and lastly as a tangle diagram projected to the back wall of the cube.
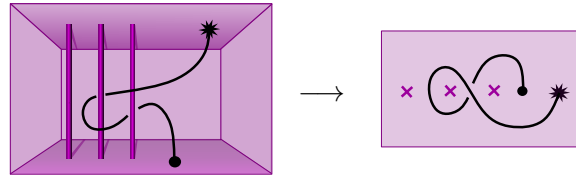
fig:polestudio

FIGURE 12. An example of a tangle in M_3 projected to the bottom floor of the cube. Strands of a tangle diagram can pass over bottom endpoints (dot) or under top endpoints (star).

fig:BottomDiagram

By projecting instead to the floor $D_p \times \{0\}$ of the cube, a tangle in M_p is represented by a tangle diagram in D_p. The R2 and R3 moves continue to apply. The endpoints of the tangle are fixed: bottom endpoints are denoted by dots, top endpoints are denoted by stars. Strands of the tangle diagram can pass over bottom endpoints, or under top endpoints, as shown in Figure 12. However, the strands cannot pass over the punctures in D_p.

sec:opsonT

4.2. **Operations on $\tilde{\mathcal{T}}$.** There are several useful operations defined on $\tilde{\mathcal{T}}$. These operations extend naturally to $\mathbb{C}\tilde{\mathcal{T}}$, and are used in Section 5 to relate quotients of $\mathbb{C}\tilde{\mathcal{T}}$ to the Goldman-Turaev Lie bialgebra.

  • *Stacking:* Given tangles $T_1, T_2 \in M_p$, if the top endpoints of $\sigma(T_1)$ match the bottom endpoints of $\sigma(T_2)$ in D_p, and the orientations on the strands of T_1 and T_2 agree at the matching endpoints, then we can stack T_2 on top of T_1 and shrink the height to get a new tangle $T_1 T_2 \in M_p$.
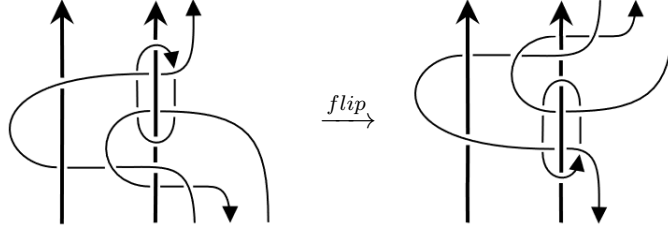
FIGURE 13. A tangle in M_2 and its flip

- *Strand addition:* The *strand addition* operation adds a non-interacting additional strand to a tangle T at a point $q \in D_p$ to get a new tangle $T \sqcup_q \uparrow$. More precisely, pick a contractible $U \subseteq D_p$ such that T is contained entirely in $U \times [0,1]$ and a point $q \in D_p$ outside of U. The tangle $T \sqcup_q \uparrow$ is T together with an upward-oriented vertical strand $q \times I$ at q.
- *Strand orientation switch:* This operation reverses the orientation of a given strand of the tangle.
- *Flip:* Given a tangle T in M_p, the flip of a tangle T in M_p, denoted $\overline{T}$, is the mirror image of T with respect to the ceiling, as shown in Figure 13. When T is flipped, each top boundary point $(q,1)$ becomes a bottom boundary point $(q,0)$, and vice versa. The orientations and framing of the strands of T are reflected along with the strands. However, the orientations of the poles remain ascending. Equivalently, we can define the flip operation as reversing the parametrisation of I in $M_p \cong D_p \times I$. This, in effect, flips the orientation of the poles but changes nothing else.

In section 5.1, we relate commutator of tangles with respect to stacking, given by $[T_1, T_2] = T_1 T_2 - T_2 T_1$, to the Goldman bracket, and in section 5.2 we relate the flip operation to the Turaev cobracket.

4.3. **The t-filtration on $\tilde{\mathcal{T}}$ and the associated graded $\tilde{\mathcal{A}}$.** There are different filtrations on the space $\mathbb{C}\tilde{\mathcal{T}}$ that one might consider in setting up a Vassiliev theory. In line with classical notation of Vassiliev invariants, we denote by a double point the difference between an over-crossing and an under-crossing:

$$\times = \nearrow\!\!\!\!\nwarrow - \nwarrow\!\!\!\!\nearrow$$

In the context of tangles in M_p, double points come in two varieties: *pole-strand*, if the crossing occurs between a pole and a tangle strand, and *strand-strand*, if the crossing occurs between two tangle strands.

The main filtration we consider on $\mathbb{C}\tilde{\mathcal{T}}$ is the filtration by the total number of double points of either type, as well as strand framing changes (as described in Section 3.2). We call this the *total filtration*, or simply t-filtration, and write it as

$$\mathbb{C}\tilde{\mathcal{T}} = \tilde{\mathcal{T}}_0 \supseteq \tilde{\mathcal{T}}_1 \supseteq \tilde{\mathcal{T}}_2 \supseteq \tilde{\mathcal{T}}_3 \supseteq \cdots$$
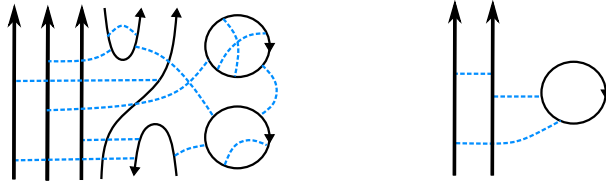
(A) Two chord diagrams: an admissible one (left) that doesn't contain any pole-pole chords, and non-admissible one (right) that does contain a pole-pole chord.

ssibleNonAdmissible

(B) The 4T relation, which is admissible if at most one of the three skeleton components is a pole.

fig:Admissible 4T

fig:admissible

FIGURE 14. Examples of admissible and non-admissible chord diagrams, and the 4T relation

where $\tilde{\mathcal{T}}_t$ is the set of linear combinations of framed tangle diagrams with at least t total double points and strand framing changes.

Definition 4.4. The associated graded space of $\mathbb{C}\tilde{\mathcal{T}}$ with respect to the total filtration is
$$\tilde{\mathcal{A}} := \operatorname{gr} \mathbb{C}\tilde{\mathcal{T}} = \prod_{t \geq 0} \tilde{\mathcal{T}}_t/\tilde{\mathcal{T}}_{t+1}.$$
The degree t component of $\tilde{\mathcal{A}}$ is $\tilde{\mathcal{A}}_t := \tilde{\mathcal{T}}_t/\tilde{\mathcal{T}}_{t+1}$.

rem:2frame=double

Remark 4.5. Modulo $\tilde{\mathcal{T}}_2$, $\updownarrow = \overset{\uparrow}{\rho} - \uparrow = \uparrow - \underset{\downarrow}{b}$. As a result, in $\tilde{\mathcal{A}}$, a framing change can always be represented as $\frac{1}{2}$ a double point as
$$\overset{\uparrow}{b} = \overset{\uparrow}{\rho} - \underset{\downarrow}{b} = (\overset{\uparrow}{\rho} - \uparrow) + (\uparrow - \underset{\downarrow}{b}) = 2\updownarrow.$$

As in classical Vassiliev theory (cf. section 3.2), the associated graded space $\tilde{\mathcal{A}}$ has a combinatorial description in terms of *chord diagrams*.

Definition 4.6. A *chord diagram* on a tangle skeleton is an even number of marked points on the poles and skeleton strands, up to orientation preserving diffeomorphism, along with a perfect matching on the marked points – that is, a partition of marked points into unordered pairs. In diagrams, the pairs are connected by a *chord*, indicated by a dotted line, as in Figure 14(A).

def:admissible

Definition 4.7. A chord diagram is *admissible* if all chords connect strands to strands, or strands to poles. That is, there are no pole-pole chords in an admissible diagram, see Figure 14(A) for an example.
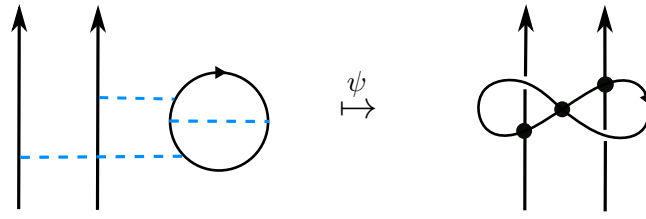
FIGURE 15. Example of ψ with the right hand side viewed as an element of $\tilde{\mathcal{T}}_3/\tilde{\mathcal{T}}_4$. Different choices of over or under crossings with the poles only differ by elements of $\tilde{\mathcal{T}}_4$.

fig:psi

def:cdspace

Definition 4.8. The space $\mathcal{D}(S)$ of *admissible chord diagrams on a diagram S* is the space of $\mathbb{C}$-linear combinations of admissible chord diagrams on the skeleton S factored out by *admissible $4T$* relations, shown in Figure **??**. Admissible $4T$ relations are $4T$ relations in the classical sense, subject to the condition that all four terms are admissible[6]. That is,

$$\mathcal{D}(S) = \frac{\left\{\text{linear combinations of admissible chord diagrams on } S\right\}}{\left\{\text{admissible } 4T \text{ relations}\right\}}$$

The space $\mathcal{D}(S)$ is a graded vector space, where the degree is given by the number of chords. Denote the degree t component of $\mathcal{D}(S)$ by $D_t(S)$. Let $\mathcal{D}$ be the disjoint union $\sqcup_S \mathcal{D}(S)$, identified at 0. We denote the degree t component of $\mathcal{D}$ by $\mathcal{D}_t = \sqcup_S \mathcal{D}_t(S)$.

There is a familiar isomorphism from classical finite type theory

$$\psi : \mathcal{D} \to \tilde{\mathcal{A}}.$$

In degree t, $\psi_t : \mathcal{D}_t \to \tilde{\mathcal{T}}_t/\tilde{\mathcal{T}}_{t+1}$, is defined as before by contracting chords to double points, as shown in Figure 15. This may create other crossings, but modulo $\tilde{\mathcal{T}}_{t+1}$ it does not matter which skeleton component is over or under at these crossings.

thm:tassocgraded

Theorem 4.9. *The map $\psi : \mathcal{D} \to \tilde{\mathcal{A}}$ is an isomorphism.*

We prove that ψ is an isomorphism by showing that it is well-defined and surjective, then using lemma 3.1 to show that it is an isomorphism.

Lemma 4.10. *The map ψ is well-defined and surjective.*

Proof. To show ψ is well-defined, it suffices to show that admissible $4T$ relations in $\mathcal{D}_t$ are in the kernel of ψ. This is shown in Figure 16. For surjectivity, a framing change in $\tilde{\mathcal{A}}$ can always be written as one half a double point, as described in Remark 4.5. So all framing changes are in the image of ψ, and ψ is surjective. □

---

[6]Equivalently, a 4T relation is admissible if at most one of the three skeleton components involved is a pole.

$$\psi\left(-\left|\begin{array}{c}\end{array}\right| + \left|\begin{array}{c}\end{array}\right| + \left|\begin{array}{c}\end{array}\right| - \left|\begin{array}{c}\end{array}\right|\right) = -\;\;\;\;\;\;\; + \;\;\;\;\;\;\; + \;\;\;\;\;\;\; - \;\;\;\;\;\;\;$$

$$= \;\;\;\;\;\;\; - \;\;\;\;\;\;\; = 0$$

FIGURE 16. The proof that $\psi : \mathcal{D} \to \tilde{\mathcal{A}}$ is well defined. The figure is understood locally: If the figure is a map in the degree t component, then the chord diagrams have $t - 2$ other chords that are not shown but in the same position throughout all four terms, and similarly, the tangles have $t - 2$ other double points that are not shown, but in the same positions throughout all the terms.

fig:psicomputation

thm:Zwelldefined

Lemma 4.11. *The Kontsevich integral Z is a well-defined filtered map from $\mathbb{C}\tilde{\mathcal{T}}$ to $\mathcal{D}$ such that $\psi \circ \operatorname{gr} Z = \operatorname{id}_{\tilde{\mathcal{A}}}$.*

Proof. The image of Z on an element in $\mathbb{C}\tilde{\mathcal{T}}$ will be a chord diagram on a skeleton with p poles and some number of circles. Since the poles in M_p are parallel, any pair of points (z_i, z_i') on the poles will be constant, the form $dz_i - dz_i' = 0$, and the contribution to the integral will be zero. Therefore chord diagrams in the image of Z don't contain pole-pole chords, so they are always admissible. So Z indeed always lands in $\mathcal{D}$.

It remains to show that $\psi \circ \operatorname{gr} Z = id_{\tilde{\mathcal{A}}}$.

$$\begin{array}{ccc} \mathbb{C}\tilde{\mathcal{T}} \xrightarrow{\;Z\;} \mathcal{D} & & \tilde{\mathcal{A}} \xrightarrow{\;\operatorname{gr} Z\;} \mathcal{D} \\ \downarrow{\scriptstyle\psi} & \overset{\operatorname{gr}}{\Longmapsto} & \quad{\scriptstyle\psi\circ\operatorname{gr} Z = id_{\tilde{\mathcal{A}}}} \;\; \downarrow{\scriptstyle\psi} \\ \tilde{\mathcal{A}} & & \tilde{\mathcal{A}} \end{array}$$

Recall that for a filtered map $f : A \to B$, the associated graded $\operatorname{gr} f : \operatorname{gr} A \to \operatorname{gr} B$ is defined on graded components by $[a] \in A_t/A_{t+1} \mapsto [f(a)] \in B_t/B_{t+1}$. We consider $\operatorname{gr} Z : \tilde{\mathcal{A}} \to \mathcal{D}$. Let $[T] \in \tilde{\mathcal{T}}_t/\tilde{\mathcal{T}}_{t+1}$ so that is T is a tangle in M_p with at least t double points. Note that it's always possible to pick such a representative, since a framing change can be written as $\frac{1}{2}$ times a double point in $\tilde{\mathcal{T}}_t/\tilde{\mathcal{T}}_{t+1}$. Then $Z(T)$ is a sum of chord diagrams with $e^{\frac{C}{2}} - e^{-\frac{C}{2}}$ at each chord C corresponding to each double point in T. All terms with degree less than t are zero, so the value of $\operatorname{gr} Z(T)$ depends only on the degree t term of $Z(T)$. The degree t term is a single chord diagram with a single chord for each double point, so applying ψ to

$$\mathbf{0} = \overrightarrow{\underset{i}{\diamondsuit}} - \hat{\underset{1}{\diamondsuit}} = -\overrightarrow{\underset{i}{\diamondsuit}} + \overrightarrow{\diamondsuit} + \hat{\diamondsuit} - \hat{\diamondsuit}$$

FIGURE 17. DUMMY IMAGE!!! placeholder for picture of chord diagram stacking and flip

chorddiagoperations

this turns all the chords back to double points, which up to crossing changes in $\tilde{\mathcal{T}}_{t+1}$, is just $[T]$. Therefore $\psi \operatorname{gr} Z = \operatorname{id}_{\tilde{\mathcal{A}}}$. Since $\psi \operatorname{gr} Z = id_{\tilde{\mathcal{A}}}$. $\qquad \square$

The next corollary is immediate from lemma 3.1.

Corollary 4.12. *The map $\psi : \mathcal{D} \to \tilde{\mathcal{A}}$ is an isomorphism and Z is an expansion for $\tilde{\mathcal{T}}$.*

Now it is established that $\tilde{\mathcal{A}}$ can be identified with the space of admissible chord diagrams $\mathcal{D}$. For a skeleton S, define $\tilde{\mathcal{A}}(S)$ to be the space of admissible chord diagrams on the skeleton S, so that $\tilde{\mathcal{A}}(S)$ is the associated graded of $\mathbb{C}\tilde{\mathcal{T}}(S)$. For example, $\tilde{\mathcal{A}}(\bigcirc)$ is the associated graded of $\mathbb{C}\tilde{\mathcal{T}}(\bigcirc)$, the space of knots in M_p.

4.4. **Operations on $\tilde{\mathcal{A}}$.** The operations *stacking* and *flip* on $\mathcal{T}$ induce operations by the same names on $\tilde{\mathcal{A}}$. In view of Theorem 4.9, we give descriptions of these operations using chord diagrams.

The operation *stacking* is given by stacking D_1 on top of D_2 by concatenating the the top ends of the poles in D_2 to the bottom ends of the poles in D_2 to get $D_1 D_2$, see Figure 17. It is clear from the definition of ψ that this is the correct chord diagram description of stacking, and as in $\mathcal{T}$, is only defined when the endpoints of D_1 and D_2 match appropriately.

The operation *flip* reflects a chord diagram with respect to a "mirror on the ceiling", reverses the orientations of the poles so that they are the same as they were originally, and adds a factor of $(-1)^m$, where m is the total number of marked points on the poles. The factor of $(-1)^m$ comes from the fact that reversing the orientation of one strand at a double point is the same as multiplying by a factor of -1. See Figure 17.

Proposition 4.13. *The Kontsevich integral Z is homomorphic with respect to stacking, strand additions and flips.*

Proof. It is clear for stacking and strand addition. When the orientation of the poles are reversed, every chord diagram D_P in the output of the Kontsevich integral will be multiplied get $(-1)^m$, where m is the total number of chord endings on poles, because m points in P will change whether they are on a descending arc or not, so $P_\downarrow$ will change by $m \bmod 2$.

$\qquad \square$

sec:s-sfiltration

4.5. **The s-filtration on $\tilde{\mathcal{T}}$ and $\tilde{\mathcal{A}}$.** As described in Section 4.3, the space $\mathbb{C}\tilde{\mathcal{T}}$ (and therefore $\tilde{\mathcal{A}}$) has a total filtration given by strand framing changes and double

points of either type, strand-pole and strand-strand. In this section we look at a second filtration on $\mathbb{C}\tilde{\mathcal{T}}$ and $\tilde{\mathcal{A}}$, where we still look at strand framing changes, but only consider the number of strand-strand double points. This filtration will be called the *strand filtration*, or simply *s*-filtration. The *s*-filtration is given by

$$\mathbb{C}\tilde{\mathcal{T}} = \tilde{\mathcal{T}}^0 \supseteq \tilde{\mathcal{T}}^1 \supseteq \tilde{\mathcal{T}}^2 \supseteq \tilde{\mathcal{T}}^3 \supseteq \cdots$$

where $\tilde{\mathcal{T}}^s \subseteq \mathbb{C}\tilde{\mathcal{T}}$ are linear combinations of link diagrams with at least s strand framing changes and strand double points.

Remark 4.14. We do *not* consider the full associated graded of $\mathbb{C}\tilde{\mathcal{T}}$ with respect to the *s*-filtration, but instead use it to identify the Goldman-Turaev spaces in low degrees in Section 5. The associated graded of $\mathbb{C}\tilde{\mathcal{T}}$ with respect to the *s*-filtration has been studied by Habiro and Massuyeau in [HM21], where they consider "bottom tangles". Note the language – if we project to the "bottom" instead of the "back wall", then all double points are of type strand-strand, so the *s*-filtration is just the usual Vassiliev filtration in the bottom projection.

The *s*-filtration also induces a filtration on $\tilde{\mathcal{A}}$ as follows. Combining the notations for the *t*- and *s*-filtrations, let $\tilde{\mathcal{T}}_t^s$ denote the set of linear combinations of tangle diagrams in $\mathbb{C}\tilde{\mathcal{T}}$ that have at least t double points, at least s of which are strand-strand.

Definition 4.15. The *s*-filtered component of $\tilde{\mathcal{A}}$ denoted $\tilde{\mathcal{A}}^{\geq s} := \prod \tilde{\mathcal{T}}_t^s / \tilde{\mathcal{T}}_{t+1}^s$ is the set of linear combinations of chord diagrams with at least s strand-strand chords, or rather at least s chords between the non-pole skeleton components.

Note that the number of s chords is not a grading on $\tilde{\mathcal{A}}$ because the 4T relation is not homogeneous with respect to strand-strand chords.

Proposition 4.16. *The Kontsevich integral is a filtered map with respect to the s-filtration.*

Proof. This follows immediately from Theorem 4.11: Z is an expansion with respect to the total filtration, and strand-strand double points correspond to strand-strand chords via the identification of the associated graded space as a space of chord diagrams. $\square$

4.6. **Notation conventions.** Throughout this paper we consider the t and s filtrations on $\mathbb{C}\tilde{\mathcal{T}}$ and $\tilde{\mathcal{A}}$, as well as on their various quotients and subspaces. We summarize the notation in the section below:

- $\mathbb{C}\tilde{\mathcal{T}}$ is the space of $\mathbb{C}$-linear combinations of framed tangles in M_p
- $\mathbb{C}\tilde{\mathcal{T}}(\bigcirc)$ is the space of C-linear combinations of framed knots in M_p
- $\tilde{\mathcal{T}}_t$ is the t'th filtered component of $\mathbb{C}\tilde{\mathcal{T}}$ with respect to the t-filtration, which contains all linear combinations of framed tangles in M_p with at least t double points(both strand-strand and strand-pole types) and framing changes.

[margin notes:] maybe this is a (trivial) proposition
I changed it to a remark. is that better?

[margin labels:] ionQuotientNotation
prop:ZrespectsS
sec:notation

- $\tilde{\mathcal{T}}^s$ is the s'th filtered component of $\mathbb{C}\tilde{\mathcal{T}}$ with respect to the s-filtration, which contains all linear combinations of framed tangles in M_p with at least s strand-strand double points and framing changes.
- $\tilde{\mathcal{T}}_t^s := \tilde{\mathcal{T}}_t \cap \tilde{\mathcal{T}}^s$, which is the set of elements of $\mathbb{C}\tilde{\mathcal{T}}$ with at least s framing changes and strand-strand double points, and at least t framing changes and double points of any type.
- $\tilde{\mathcal{T}}^{/s} := \mathbb{C}\tilde{\mathcal{T}}/\tilde{\mathcal{T}}^s$, is the quotient of $\mathbb{C}\tilde{\mathcal{T}}$ where diagrams with more than s strand-strand double points or framing changes are in the kernel.
- $\tilde{\mathcal{T}}^{1/2} := \tilde{\mathcal{T}}^1/\tilde{\mathcal{T}}^2$, is the quotient of $\mathbb{C}\tilde{\mathcal{T}}$ where diagrams with 0 or greater than 1 strand-strand double point or framing change are in the kernel.
- $\tilde{\mathcal{A}}$ is the associated graded space of $\mathbb{C}\tilde{\mathcal{T}}$ under the t-filtration, and is the space of admissible chord diagrams modulo admissible $4T$ relations.
- $\tilde{\mathcal{A}}_t := \tilde{\mathcal{T}}_t/\tilde{\mathcal{T}}_{t+1}$ is the degree t component of $\tilde{\mathcal{A}}$ which consists of all admissible chord diagrams in $\tilde{\mathcal{A}}$ with exactly t chords of any type.
- $\tilde{\mathcal{A}}^{\geq s} := \prod_t \tilde{\mathcal{T}}_t^s/\tilde{\mathcal{T}}_{t+1}^s$ is the s'th filtered component of $\tilde{\mathcal{A}}$
- $\tilde{\mathcal{A}}^{/s} := \tilde{\mathcal{A}}/\tilde{\mathcal{A}}^{\geq s}$

Theses notations are extended to subspaces and quotients of $\mathbb{C}\tilde{\mathcal{T}}$ and $\tilde{\mathcal{A}}$ in the natural way.

4.7. **The Conway quotient.** In this section we introduce the Conway quotient of $\mathbb{C}\tilde{\mathcal{T}}$: essentially, a Conway skein module of tangles in M_p without fixing the value of the unknot. The Conway relation respects the t and s filtrations and the Kontsevich integral descends to the Conway quotient.

Definition 4.17. The Conway quotient of $\mathbb{C}\tilde{\mathcal{T}}$ is defined as

$$\mathbb{C}\tilde{\mathcal{T}}_\nabla := \mathbb{C}\tilde{\mathcal{T}}[\![a]\!] \Big/ \text{✗} - \text{✗} = (e^{\frac{a}{2}} - e^{-\frac{a}{2}}) \text{)(},$$

where a is a formal variable with t and s degree 1. The skein relation is applied only to strand-strand crossings, not strand-pole crossings. We will use the variable b as a shorthand for $b = e^{\frac{a}{2}} - e^{-\frac{a}{2}}$.

The t and s filtrations on $\mathbb{C}\tilde{\mathcal{T}}$ induce filtrations on $\mathbb{C}\tilde{\mathcal{T}}_\nabla$. Following the notation conventions in Section 4.6, let $\tilde{\mathcal{T}}_{\nabla,t}$ denote the t'th filtered component of $\mathbb{C}\tilde{\mathcal{T}}_\nabla$ and $\tilde{\mathcal{A}}_\nabla := \operatorname{gr}_t \mathbb{C}\tilde{\mathcal{T}}_\nabla = \prod \tilde{\mathcal{T}}_{\nabla,t}/\tilde{\mathcal{T}}_{\nabla,t+1}$ denote the associated graded algebra of $\mathbb{C}\tilde{\mathcal{T}}_\nabla$ with respect the total filtration. We now show that $\tilde{\mathcal{A}}_\nabla$ has a diagrammatic description similar to $\tilde{\mathcal{A}}$, where $\tilde{\mathcal{A}} \cong \mathcal{D}$ as in Theorem 4.9.

Definition 4.18. Let

$$\mathcal{D}_\nabla := \mathcal{D}[\![a]\!] \Big/ \text{)⋯(} = a\,\overset{\uparrow}{\underset{\downarrow}{\longmapsto}}, \quad \text{)⋯(} = a\,\text{✗}$$

where a is a formal variable of degree 1 as above, and the relations locally apply only when all skeleton components involved are strands, not poles.

Note that the quotient relations in $\mathcal{D}_\nabla$ preserve the t-grading on $\mathcal{D}$ and the grading descends to $\mathcal{D}_\nabla$. The next theorem shows that $\tilde{\mathcal{A}}_\nabla \cong \mathcal{D}_\nabla$. This theorem essentially follows from the results of [LM95], and we present a brief direct proof.

thm:Z_conway

Theorem 4.19. *The Kontsevich integral descends to an expansion $Z_\nabla : \mathbb{C}\tilde{\mathcal{T}}_\nabla \to \mathcal{D}_\nabla$ and $\tilde{\mathcal{A}}_\nabla \cong \mathcal{D}_\nabla$.*

Proof. This proof follows the general schema introduced in Section 3.1, in particular Lemma 3.1 and the map ψ, which assigns singular tangles to chord diagrams.

First we show that ψ descends to a graded surjection $\psi : \mathcal{D}_\nabla \to \tilde{\mathcal{A}}_\nabla$. To show that ψ is well-defined, we need to show that the Conway relation in $\mathcal{D}_\nabla$ is in the kernel. Locally,

$$\psi\left(\,\text{⌣⌣} - a\,\text{⊓⊔}\,\right) = \text{⌣} - a\,\text{⌣}\,;$$

and denote the (global) total degree on both sides by t. In other words, the (global) right hand side is interpreted as an element of $\tilde{\mathcal{T}}_{\nabla,t}/\tilde{\mathcal{T}}_{\nabla,t+1}$. Using the Conway skein relation in $\tilde{\mathcal{A}}_\nabla$, the right had side can be simplified

$$\text{⌣} - a\,\text{⌣} = (e^{\frac{a}{2}} - e^{-\frac{a}{2}})\,\text{⌣} - a\,\text{⌣} = (e^{\frac{a}{2}} - e^{-\frac{a}{2}} - a)\,\text{⌣} + a(\text{⌣} - \text{⌣})$$

Observe that $a(\text{⌣} - \text{⌣})$ and the lowest degree term of $e^{\frac{a}{2}} - e^{-\frac{a}{2}} - a$ are both of degree 2, hence $(\text{⌣} - a\,\text{⌣}) \in \tilde{\mathcal{T}}_{\nabla,t+1}$, and therefore is zero in $\tilde{\mathcal{T}}_{\nabla,t}/\tilde{\mathcal{T}}_{\nabla,t+1}$.

We now verify that the Kontsevich integral Z descends to the quotient $\mathbb{C}\tilde{\mathcal{T}}_\nabla$ by checking the relations in $\mathbb{C}\tilde{\mathcal{T}}_\nabla$ directly. Recall that $Z(\text{⤬}) = (e^{\frac{C}{2}})\text{⤬}$ and $Z(\text{⤬}) = (e^{-\frac{C}{2}})\text{⤬}$, where C denotes a chord, the exponential is interpreted formally as a power series, and C^k denotes stacking k chords. Using the Conway relation, we compute:

$$C^k = \left.\begin{matrix}\text{⫴}\end{matrix}\right\}k \;\overset{\nabla}{=}\; a^k\left.\begin{matrix}\text{⧓}\end{matrix}\right\}k \; = a^k(\text{⤬})^k = \begin{cases} a^k\uparrow\uparrow, & \text{if } k \text{ is even} \\[2ex] a^k\text{⤬}, & \text{if } k \text{ is odd} \end{cases}$$

Now applying Z to the left hand side of the Conway relation, we obtain

$$Z(\text{✕}) - Z(\text{✕}) = (e^{\frac{C}{2}} - e^{\frac{-C}{2}})\text{✕}$$

$$= \sum_{k=0}^{\infty} \left(\frac{C^k}{2^k k!} - \frac{(-1)^k C^k}{2^k k!} \right) \text{✕}$$

$$= \sum_{k=0}^{\infty} \frac{C^{2k+1}}{2^{2k}(2k+1)!} \text{✕}$$

$$= \sum_{k=0}^{\infty} \frac{a^{2k+1}\text{✕}}{2^{2k}(2k+1)!} \text{✕}$$

$$= \sum_{k=0}^{\infty} \frac{a^{2k+1}}{2^{2k}(2k+1)!} \text{↑ ↑}$$

$$= (e^{\frac{a}{2}} - e^{-\frac{a}{2}})\text{↑ ↑}$$

$$= Z\left((e^{\frac{a}{2}} - e^{-\frac{a}{2}})\text{)(} \right).$$

Thus, Z descends to the Conway quotient $\mathbb{C}\tilde{\mathcal{T}}_\nabla$.

Therefore, by Lemma 3.1, Z is a homomorphic expansion for $\mathbb{C}\tilde{\mathcal{T}}_\nabla$ and $\psi : \mathcal{D}_\nabla \to \tilde{\mathcal{A}}_\nabla$ is an isomorphism. $\qquad\square$

While our main focus is the t-filtration on $\mathbb{C}\tilde{\mathcal{T}}_\nabla$ and its associated graded space $\tilde{\mathcal{A}}_\nabla$, the low degree components of the associated graded of $\mathbb{C}\tilde{\mathcal{T}}_\nabla$ with respect to the s-filtration arise when identifying the Goldman-Turaeav Lie bialgebra, as will be detailed in the coming Section 5. One space that arises is $\tilde{\mathcal{T}}_\nabla^{/1}$, the quotient of $\tilde{\mathcal{T}}_\nabla$ by the s-degree 1 component (recall the notation conventions from Section 4.6). On this quotient, the Conway relation has no effect and $\tilde{\mathcal{T}}_\nabla^{/1}$ is actually isomorphic to $\tilde{\mathcal{T}}^{/1}$.

Proposition 4.20. $\tilde{\mathcal{T}}_\nabla^{/1} \cong \tilde{\mathcal{T}}^{/1}$

Proof. The quotients $\tilde{\mathcal{T}}^{/1}$ and $\tilde{\mathcal{T}}_\nabla^{/1}$ are both spanned by the classes of tangles T. Such a tangle T in $\tilde{\mathcal{T}}^{/1}$ are only defined up to crossing changes. crossing changes yield a double point and thus are killed in the quotient. So, tangles in $\tilde{\mathcal{T}}^{/1}$ are only defined up to crossing changes. Similarly, in $\tilde{\mathcal{T}}_\nabla^{/1}$ the Conway quotient only applies to crossing changes, (the Conway relation occurs in degree 1), so further quotienting by the Conway relation has no effect on $\tilde{\mathcal{T}}^{/1}$, and so $\tilde{\mathcal{T}}_\nabla^{/1} \cong \tilde{\mathcal{T}}^{/1}$. $\quad\square$

In particular, there are well-defined "multiplication by b" and "division by b" maps, denoted m_b and q_b, where $q_b : \tilde{\mathcal{T}}_\nabla^{/2} \to \tilde{\mathcal{T}}_\nabla^{/1}$ restricts to an isomorphism on $\tilde{\mathcal{T}}_\nabla^1/\tilde{\mathcal{T}}_\nabla^2$. We now show that this map exists by defining it explicitly as the inverse of multiplication by b.

prop:divbybexists **Proposition 4.21.** *The multiplication by b map $m_b : \tilde{\mathcal{T}}^{/1} \to \tilde{\mathcal{T}}_\nabla^{/2}$ is injective, and its image is $\tilde{\mathcal{T}}_\nabla^{1/2}$.*

Proof. We first prove that the image of m_b is $\tilde{\mathcal{T}}_\nabla^{1/2}$. The quotient $\tilde{\mathcal{T}}^{/1}$ is spanned by the classes of tangles T. For a tangle T, the image $m_b(T) = bT$ is in $\tilde{\mathcal{T}}^1$, and represents an element in $\tilde{\mathcal{T}}^{1/2}$. Thus, the image of m_b is contained in $\tilde{\mathcal{T}}^{1/2}$.

Conversely, any element y of $\tilde{\mathcal{T}}^{1/2}$ is (non-uniquely) represented as a sum of the form $\sum_{i=1}^k T_i + b \sum_{j=1}^l T_j$, where T_i are tangles with one double point each, and T_j are arbitrary tangles. Then, by the Conway relation, each $T_i = b \cdot T_i^C$, where T_i^C denotes the tangle where the double point in T_i has been smoothed. Thus, $y = b\left(\sum_{i=1}^k T_i^C + \sum_{j=1}^l T_j\right)$, and therefore y is in the image of m_b.

To prove injectivity of m_b, it is enough to provide an inverse, division by b map, on $\tilde{\mathcal{T}}_\nabla^{1/2}$, but in fact there is a one sided inverse defined on all of $\tilde{\mathcal{T}}_\nabla^{/2}$ which is defined as follows. For a tangle T and a crossing x of T, let $\epsilon(x) \in \{\pm 1\}$ be the sign of x, and $T|_{x \to \asymp}$ be the tangle T with x replaced by a smoothing. There is a well defined "division by b" map $q_b : \tilde{\mathcal{T}}_\nabla^{/2} \to \tilde{\mathcal{T}}^{/1}$ given by the linear extension of the following:

$$bT \overset{q_b}{\mapsto} T$$

$$T \overset{q_b}{\mapsto} \frac{1}{2} \sum_{x \text{ crossing of } T} \epsilon(x) T|_{x \to \asymp}$$

For well-definedness, it is straightforward to check that q_b preserves the Reidemeister moves. We also need to check that $\tilde{\mathcal{T}}_\nabla^2$ and the Conway relation are in the kernel. For $b^k T \in \tilde{\mathcal{T}}_\nabla^2$, if $k = 1$, then $T \in \tilde{\mathcal{T}}^1$, so $q_b(bT) = 0$. If instead $k = 0$, then T has at least two double points. Replacing a crossing by a smoothing only changes the crossing that is replaced, so other crossings (and therefore double points) remain unchanged. Therefore $q_b(T)$ can be written as a sum where each term has at least one double point, so $q_b(T) = 0$ as well.

To show that the Conway relation also vanishes, note that the terms in $q_b(\times) = q_b(\nearrow - \nwarrow)$ come from either smoothing a crossing that is a part of the double point, or smoothing a crossing that is not. In the latter, the double points outside the local relation remain unchanged, so those terms are in $\tilde{\mathcal{T}}_\nabla^1$. The only remaining terms are those where the crossings forming the double point are smoothed, so we get

$$q_b\left(\times - \times\right) = \frac{1}{2})(- (-1)\frac{1}{2})(=)(= q_b(b))($$

showing q_b is well-defined.

Restricting $q_b : \tilde{\mathcal{T}}_\nabla^{1/2} \to \tilde{\mathcal{T}}_\nabla^{/1}$ is clearly surjective. To show it's injective, note that the restriction is simply given by $bT \mapsto T$, and if $T \in \tilde{\mathcal{T}}_\nabla^1$, then $bT \in \tilde{\mathcal{T}}_\nabla^2$. $\square$

cor:divbyb

Corollary 4.22. *The map $m_b : \tilde{\mathcal{T}}^{/1} \to \tilde{\mathcal{T}}_\nabla^{1/2}$ is an isomorphism with inverse $q_b : \tilde{\mathcal{T}}_\nabla^{1/2} \to \tilde{\mathcal{T}}^{/1}$.*

m_b and q_b are filtered degree shifting maps.

finish this remark

rem:grdivbyb

Remark 4.23. The associated graded of q_b is an isomorphism $\operatorname{gr} q_B : \tilde{\mathcal{A}}^{1/2} \to \tilde{\mathcal{A}}^{/1}$ given by drawing the chord diagram as with one s-s chord, smoothing that chord using ∇ and getting a factor of b with no remaining s-s chords, and then diving off the b.

Clean up

for N: make this formal.

In general, the Conway relation changes the skeleton of a diagram. So when we write $\tilde{\mathcal{T}}_\nabla(\bigcirc)$ this is the set of diagrams for which there is a representative expressed in terms of knots and no factors of b– the grading comes from the actual skeleton, not from factors of b. Similarly, $\tilde{\mathcal{A}}_\nabla^{/r}(S)$ represents the associated graded space of $\tilde{\mathcal{T}}_\nabla^{/r}(S)$, and a nontrivial chord diagram in $\tilde{\mathcal{A}}_\nabla^{/r}(S)$ has a representative with a chord diagram on skeleton S with at most r strand-strand chords.

prop:/1conway

Proposition 4.24. $\tilde{\mathcal{T}}^{/1}(\bigcirc) \cong \tilde{\mathcal{T}}_\nabla^{/1}(\bigcirc)$.

cor:gr/1conway

Corollary 4.25. $\tilde{\mathcal{A}}^{/1}(\bigcirc) \cong \tilde{\mathcal{A}}_\nabla^{/1}(\bigcirc)$.

However, Proposition 4.24 and Corollary 4.25 are not true for higher degree quotients because of the skeleton changing issue induced by the Conway relation.

5. IDENTIFYING THE GOLDMAN-TURAEV LIE BIALGEBRA

::IdentifyingGTinCON

In this section we establish our main results: we identify the Goldman-Turaev Lie bialgebra in the low s-filtered degree quotients of $\mathbb{C}\tilde{\mathcal{T}}$, and show that the Kontsevich integral induces a homomorphic expansion on this space with respect to the s-filtration. Appealing to the principle summarized in Section 2 we present diagrams like (2.1), where the induced map η is the Goldman bracket and the self intersection map μ, repsectively. We deduce the homomorphicity of the expansion from the naturality of the construction as in (2.5).

identifybracketinCON

5.1. **The Goldman Bracket.** Recall from Section 3.3 that D_p denotes the p-punctured disc, π is its fundamental group, and $|\mathbb{C}\pi|$ is the linear quotient $|\mathbb{C}\pi| := \mathbb{C}\pi/[\mathbb{C}\pi, \mathbb{C}\pi]$, which is linearly generated by homotopy classes of free loops in D_p. The Goldman bracket (Definition 3.3) is a lie bracket $[\cdot, \cdot]_G : |\mathbb{C}\pi| \otimes |\mathbb{C}\pi| \to |\mathbb{C}\pi|$. Recall from Section 4.7 the space $\mathbb{C}\tilde{\mathcal{T}}(\bigcirc)$ is the vector space of $\mathbb{C}$-linear combinations of framed knots in $M_p = D_p \times I$.

prop:BotProj

Proposition 5.1. *The bottom projection $M_p \to D_p \times \{0\}$ induces a surjective map $\mathbb{C}\tilde{\mathcal{T}}(\bigcirc) \to |\mathbb{C}\tilde{\pi}|$. Post-composing this with the projection $|\mathbb{C}\tilde{\pi}| \to |\mathbb{C}\pi|$ results in a surjective filtered map*

$$\beta : \mathbb{C}\tilde{\mathcal{T}}(\bigcirc) \to |\mathbb{C}\pi|.$$

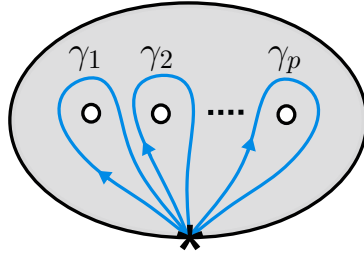FIGURE 18. The standard generating curves of π.

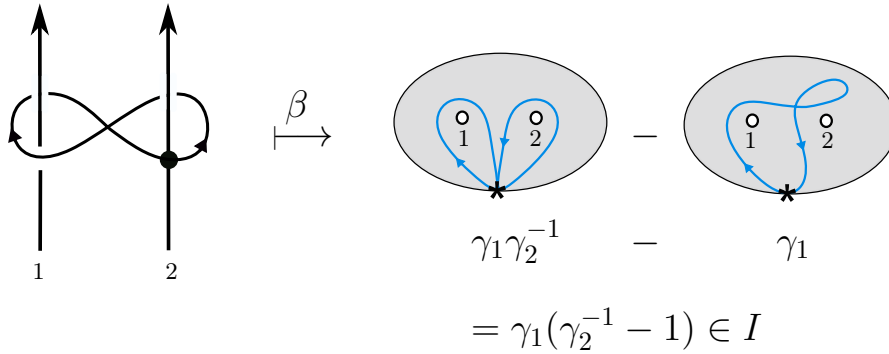$$\gamma_1\gamma_2^{-1} \quad - \quad \gamma_1$$

$$= \gamma_1(\gamma_2^{-1} - 1) \in I$$

FIGURE 19. Example calculation demonstrating that β is a filtered map.

Proof. By Reidemeister's Theorem, framed knots in $\mathbb{C}\tilde{\mathcal{T}}(\bigcirc)$ are faithfully represented by knot diagrams in $D_p \times \{0\}$ – regular projections to the bottom with over/under information – modulo the Reidemeister moves (R2, R3). The bottom projection sends the Reidemeister moves for knots to the corresponding moves generating regular homotopies of immersed free loops, hence β is well-defined. The projection is clearly surjective as any loop can be lifted to a knot by introducing arbitrary under/over information at the crossings.

The statement that β is filtered means that step k of the the Vassiliev t-filtration in $\mathbb{C}\tilde{\mathcal{T}}(\bigcirc)$ projects to step k of the filtration on $|\mathbb{C}\pi|$ induced by the I-adic filtration of π. Note that strand-strand double points and framing changes map to 0 under β, thus, we only have something to prove for knots with k strand-pole double points.

Let $\gamma_1, ..., \gamma_p$ denote the standard generators of π as in Figure 18. A knot $K \in \mathbb{C}\tilde{\mathcal{T}}(\bigcirc)$ maps to a free loop in $|\mathbb{C}\pi|$, whose conjugacy class in π is represented as a word in the generators γ_i. A pole-stand double point on pole j maps to a difference between two curves passing on either side of the j'th puncture (see Figure 19 for an example). Therefore, the words in π representing these curves

differ in a single instance of $\gamma_j^{\pm 1}$. Thus, a knot with k pole-strand double points maps to a product with k factors of the form $\pm(\gamma_j^{\pm 1} - 1)$. This is by definition an element in $\mathcal{I}^k$. □

`prop:kerbeta` **Proposition 5.2.** *The kernel of β is $\tilde{\mathcal{T}}^1(\bigcirc)$, and β descends to a filtered (with respect to the t-filtration) linear isomorphism $\beta : \tilde{\mathcal{T}}^{/1}(\bigcirc) \to |\mathbb{C}\pi|$.*

Proof. Two framed knots in $\mathbb{C}\tilde{\mathcal{T}}(\bigcirc)$ project to the same loop in $|\mathbb{C}\pi|$ if and only if they differ by framing changes and (strand-strand) crossing changes, which generate precisely the step 1 of the s-filtration, that is, $\tilde{\mathcal{T}}^1(\bigcirc)$. □

Recall from Proposition 4.24 that $\tilde{\mathcal{T}}^{/1}(\bigcirc) = \tilde{\mathcal{T}}_\nabla^{/1}(\bigcirc)$. Hence, we get the following:

`cor:loopsasknots` **Corollary 5.3.** *The map β descends to an isomorphism $\beta : \tilde{\mathcal{T}}_\nabla^{/1}(\bigcirc) \to |\mathbb{C}\pi|$.*

Recall that $\tilde{\mathcal{A}}$ is the associated graded space of $\mathbb{C}\tilde{\mathcal{T}}$ with respect to the t-filtration, and $\mathbb{C}\tilde{\mathcal{T}}$ is also filtered by the s-filtration. Explicitly, $\tilde{\mathcal{A}}(\bigcirc)$ is the space of admissible chord diagrams on a circle skeleton as in Definition 4.8, $\tilde{\mathcal{A}}^{\geq i}(\bigcirc)$ is the s-degree i filtered component of $\tilde{\mathcal{A}}(\bigcirc)$, and $\tilde{\mathcal{A}}^{/i}(\bigcirc) = \tilde{\mathcal{A}}(\bigcirc)/\tilde{\mathcal{A}}^{\geq i}(\bigcirc)$. Recall from Section 3.3 that the associated graded vector space of $|\mathbb{C}\pi|$ is $|\mathrm{FA}|$, where $\mathrm{FA} = \mathrm{FA}\langle x_1, \cdots, x_p \rangle$ denotes the free associative algebra over $\mathbb{C}$, and the linear quotient $|\mathrm{FA}| = \mathrm{FA}/[\mathrm{FA}, \mathrm{FA}]$ is the $\mathbb{C}$-vector space generated by cyclic words in the letters $x_1, ..., x_p$.

Proposition 5.4. *The associated graded map $\mathrm{gr}\,\beta : \tilde{\mathcal{A}}(\bigcirc) \to |\mathrm{FA}|$ has kernel $\tilde{\mathcal{A}}^{\geq 1}(\bigcirc)$. Hence, $\mathrm{gr}\,\beta$ descends to an isomorphism $\mathrm{gr}\,\beta : \tilde{\mathcal{A}}^{/1}(\bigcirc) \to |\mathrm{FA}|$.*

Proof. The statement follows from applying the associated graded functor to the filtered short exact sequence

$$0 \longrightarrow \tilde{\mathcal{T}}^1(\bigcirc) \longrightarrow \tilde{\mathcal{T}}(\bigcirc) \overset{\beta}{\longrightarrow} |\mathbb{C}\pi| \longrightarrow 0.$$

□

Remark 5.5. In $\tilde{\mathcal{A}}^{/1}(\bigcirc)$ chord diagrams with any strand-strand chords are zero. Thus, non-zero elements of this space are represented as chord diagrams on poles and a single circle strand, with strand-pole chords only, as in Figure 20. Such a chord diagram corresponds naturally to a cyclic word by labelling the poles with $x_1, ..., x_p$ and reading the word along the circle skeleton, as shown. Indeed, this is the map $\mathrm{gr}\,\beta$.

We are now ready to derive the Goldman bracket from the stacking commutator on $\mathbb{C}\tilde{\mathcal{T}}$:

`thm:bracketsnake` **Theorem 5.6.** *Let $\lambda_1 : \tilde{\mathcal{T}}_\nabla^{/2}(\bigcirc) \otimes \tilde{\mathcal{T}}_\nabla^{/2}(\bigcirc) \to \tilde{\mathcal{T}}_\nabla^{/2}(\bigcirc)$ denote the stacking product. Let λ_2 denote the opposite product, that is, $\lambda_2(K_1, K_2) = K_2 K_1$. Then $\lambda = \lambda_1 - \lambda_2$ induces the Goldman bracket on $|\mathbb{C}\pi|$: the commutative diagram in*
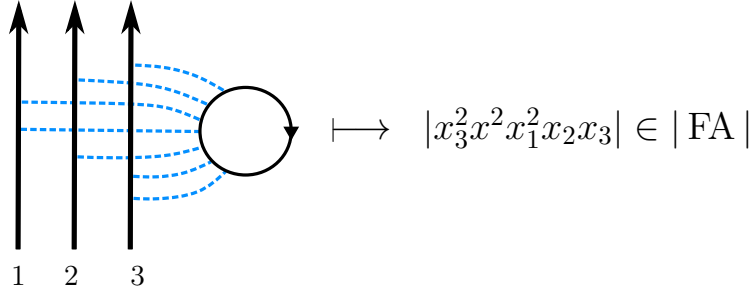
FIGURE 20. Chord diagrams with no strand-strand chords can be read as cyclic words.

fig:CycWord
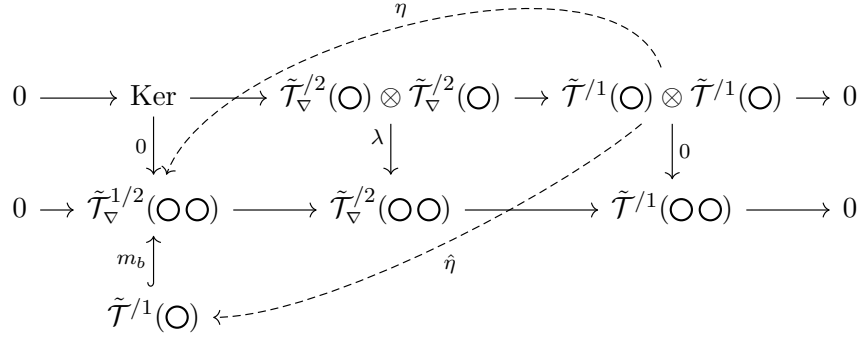
FIGURE 21. The nontrivial horizontal maps are the respective quotient and inclusion maps. The space Ker is the kernel of the projection map on the top right.

fig:Snakeforbracket

Figure 21 the induced homomorphism η agrees with the Goldman Bracket under the identification $\beta : \tilde{\mathcal{T}}^{/1}(\bigcirc) \to |\mathbb{C}\pi|$ as

$$[-, -]_G = \beta \circ q_b \circ \eta \circ (\beta^{-1} \otimes \beta^{-1}).$$

Proof. For $K_1 \otimes K_2$ in $\tilde{\mathcal{T}}_\triangledown^{/2}(\bigcirc) \otimes \tilde{\mathcal{T}}_\triangledown^{/2}(\bigcirc)$, $\lambda(K_1 \otimes K_2) = K_1 K_2 - K_2 K_1$. Project $K_1 K_2$ and $K_2 K_1$ to the bottom to obtain link diagrams. Let a *mixed crossing* of such a diagram be a crossing where one strand belongs to K_1 and the other strand belongs to K_2. Notice that in $K_2 K_1$ all mixed crossings are flipped compared to $K_1 K_2$, while other crossings – those belonging to K_1 or K_2 only – are the same.

Using the double point notation, write positive mixed crossings in $K_1 K_2$ as $\asymp = \diagup\!\!\!\!\diagdown + \diagdown\!\!\!\!\diagup$ and negative mixed crossings as $\asymp = \diagdown\!\!\!\!\diagup - \diagup\!\!\!\!\diagdown$, where each double point has one strand belongs to K_1 and the other belongs to K_2. Rewriting all the mixed crossings of $K_1 K_2$ in this way yields a sum of tangles indexed by subsets

of the mixed crossings. Denote the set of mixed crossings by M, and for a subset $X \subseteq M$, denote by L_X the singular link obtained by changing the crossings in X to double points, and flipping the other mixed crossings (those in $M \setminus X$). Also, let ϵ_X be the product of the signs of all crossings in X. Then

$$(5.1) \qquad K_1 K_2 = \sum_{X \subseteq M} \epsilon_X L_X.$$

Notice that $L_\emptyset = K_2 K_1$, and if $|X| = i$ then $L_X \in \tilde{\mathcal{T}}_\nabla^i(\bigcirc\bigcirc)$. Therefore, $\lambda(K_1 K_2)$ is in $\tilde{\mathcal{T}}_\nabla^1(\bigcirc\bigcirc)$, and therefore the right hand square commutes. Furthermore, we have

$$(5.2) \qquad \lambda(K_1, K_2) = \sum_{X \subseteq M, |X|=1} L_X \in \tilde{\mathcal{T}}_\nabla^{1/2}(\bigcirc\bigcirc).$$

Now for the left square, the kernel K of the projection map from $\tilde{\mathcal{T}}_\nabla^{/2}(\bigcirc) \otimes \tilde{\mathcal{T}}_\nabla^{/2}(\bigcirc) \to \tilde{\mathcal{T}}_\nabla^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}_\nabla^{/1}(\bigcirc)$ is generated by $\tilde{\mathcal{T}}_\nabla^{1/2}(\bigcirc) \otimes \tilde{\mathcal{T}}_\nabla^{/2}(\bigcirc)$ in $\tilde{\mathcal{T}}_\nabla^{/2}(\bigcirc) \otimes \tilde{\mathcal{T}}_\nabla^{/2}(\bigcirc)$. and $\tilde{\mathcal{T}}_\nabla^{/2}(\bigcirc) \otimes \tilde{\mathcal{T}}_\nabla^{1/2}(\bigcirc)$. Suppose that $K_1 \otimes K_2$ is in $\tilde{\mathcal{T}}_\nabla^{1/2}(\bigcirc) \otimes \tilde{\mathcal{T}}_\nabla^{/2}(\bigcirc)$, in other words, there is a double point in K_1. Then, by the same computation as in Equation 5.1, $\lambda(K_1 \otimes K_2)$ is in $\tilde{\mathcal{T}}_\nabla^2(\bigcirc\bigcirc)$, as every term contains the pre-existing double point in K_1, and at least one additional mixed double point. Therefore, the left hand square commutes.

As in Section 2, then λ induces a unique well defined homomorphism $\eta : \tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc) \to \tilde{\mathcal{T}}_\nabla^{1/2}(\bigcirc\bigcirc)$. We identify η as the Goldman bracket. We have that the isomorphism β give $\tilde{\mathcal{T}}^{/1}(\bigcirc) \cong |\mathbb{C}\pi|$ (Proposition 5.2), identifying the domain of η with the domain of the Goldman bracket. We now argue that η has image in $\tilde{\mathcal{T}}^{/1}(\bigcirc) \cong |\mathbb{C}\pi|$.

By Equation (5.2), $\lambda(K_1, K_2)$ is a sum of terms, each with a single mixed double point. Applying the Conway relation to smooth each of these mixed double points changes the skeleton from two circles to one circle, and introduces a factor of b. In other words, $\lambda(K_1, K_2) \in b\tilde{\mathcal{T}}_\nabla^{/2}(\bigcirc) \subseteq \tilde{\mathcal{T}}_\nabla^{1/2}(\bigcirc\bigcirc)$. By Corollary 4.22, restricted to a circle skeleton, we know that $b\tilde{\mathcal{T}}_\nabla^{/2}(\bigcirc) \cong \tilde{\mathcal{T}}^{/1}(\bigcirc)$ via the map q_b. (The map $\hat{\eta}$ in the diagram is $q_b \circ \eta$). In turn, $\tilde{\mathcal{T}}^{/1}(\bigcirc) \cong |\mathbb{C}\pi|$ again via the map β.

In summary, the map η is induced from λ in the following way. For curves $\gamma_1 \otimes \gamma_2 \in \tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc)$, let $K_1 \otimes K_2$ be an arbitrary vertical lift of $\gamma_1 \otimes \gamma_2$ to knots in $\tilde{\mathcal{T}}_\nabla^{/2}(\bigcirc) \otimes \tilde{\mathcal{T}}_\nabla^{/2}(\bigcirc)$. Then

$$\eta(\gamma_1 \otimes \gamma_2) = \frac{\lambda(K_1 \otimes K_2)}{b} \in \tilde{\mathcal{T}}^{/1}(\bigcirc),$$

where we use the notation $\frac{1}{b}$ to mean composition with q_b. We need to show that this agrees with the Goldman bracket (Definition 3.3). This is clear from the definition: the Goldman bracket is a sum of smoothings of the mixed crossings of γ_1 and γ_2, exactly as above, and the signs in the sum match the signs of the crossings. See Figure 22 for an example calculation. $\qquad \square$
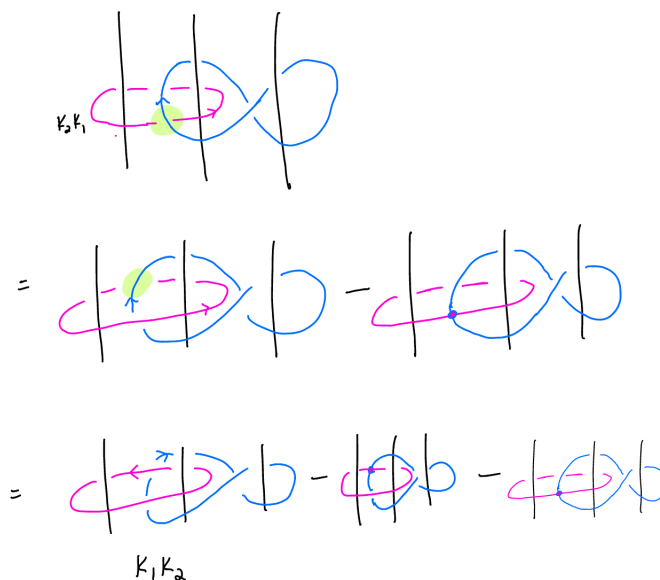
FIGURE 22. Example commutator bracket computation.

The graded Goldman bracket is a map $[-,-]_{\mathrm{gr}\,G} : |\,\mathrm{FA}\,| \otimes |\,\mathrm{FA}\,| \to |\,\mathrm{FA}\,|$, as in Definition 3.7. By taking the associated graded of the diagram in Figure 21 we arrive at the commutative diagram in Figure 23 and recover the associated graded Goldman bracket:

Corollary 5.7. *The diagram in Figure 23 commutes, the rows are exact, $\mathrm{gr}\,\eta$ is the induced connecting homomorphism. Therefore, $\mathrm{gr}\,\hat{\eta}$ is the associated graded Goldman bracket via the identification $\mathcal{A}^{/1}(\bigcirc) \cong |\,\mathrm{FA}\,|$.*

Proof. This follows from   □

Theorem 5.8. *The Kontsevich integral descends to a homomorphic expansion for the Goldman Bracket. That is, the outside square of the following diagram commutes:*

$$
\begin{array}{ccccccc}
 & & & \xrightarrow{\quad [\cdot,\cdot]_G \quad} & & & \\
|\mathbb{C}\pi| & \xleftarrow[\beta]{\cong} & \tilde{\mathcal{T}}^{/1}(\bigcirc) & \xleftarrow{\hat\eta} & \tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc) & \xleftarrow[\beta^{-1}\otimes\beta^{-1}]{\cong} & |\mathbb{C}\pi| \otimes |\mathbb{C}\pi| \\
\downarrow{Z^{/1}} & & \downarrow{Z^{/1}} & & \downarrow{Z^{/1}\otimes Z^{/1}} & & \downarrow{Z^{/1}\otimes Z^{/1}} \\
|\mathrm{FA}| & \xleftarrow[\mathrm{gr}\,\beta]{\cong} & \mathcal{A}^{/1}(\bigcirc) & \xleftarrow{\mathrm{gr}\,\hat\eta} & \mathcal{A}^{/1}(\bigcirc) \otimes \mathcal{A}^{/1}(\bigcirc) & \xleftarrow[\mathrm{gr}\,\beta^{-1}\otimes\mathrm{gr}\,\beta^{-1}]{\cong} & |\mathrm{FA}| \otimes |\mathrm{FA}| \\
 & & & \xrightarrow[\mathrm{gr}[\cdot,\cdot]_G]{} & & &
\end{array}
$$

FIGURE 23. The associated graded commutative diagram of Figure 21.

`Snakefor_gr_bracket`

FIGURE 24. Commutative cube showing the formality of the Goldman bracket from the Kontsevich integral.

`fig:Cube_for_bracket`

Proof. The top and bottom squares commute by Theorem 5.8 and Corollary 5.7. All that needs to be shown is the commutativity of the middle square. This middle square occurs as the diagonal square of the multi-cube in Figure 24.

Using the construction in Section 2, we only need to show that the faces of the multi-cube in Figure 24 commute; this implies desired commutativity of the diagonal square. We have already established that the top and bottom faces commute from Theorem 5.8 and Corollary 5.7. The front and back vertical faces commute because Z is a filtered map with respect to the s-filtration (Proposition 4.16). The left and right vertical sides commute trivially because of the zero maps.

The Kontsevich integral is homomorphic with respect to the stacking product (Proposition 4.13). Since λ is the difference between the stacking product and

its opposite product, Z is homomorphic with respect to λ. In other words, the middle vertical face of Figure 24) commutes:

$$
\begin{array}{ccc}
& & \tilde{\mathcal{T}}_\nabla^{/2}(\bigcirc) \otimes \tilde{\mathcal{T}}_\nabla^{/2}(\bigcirc) \\
& \overset{\lambda}{\swarrow} & \downarrow {\scriptstyle Z^{/2} \otimes Z^{/2}} \\
\tilde{\mathcal{T}}_\nabla^{/2}(\bigcirc\bigcirc) & & \\
{\scriptstyle Z^{/2}} \downarrow & \mathcal{A}_\nabla^{/2}(\bigcirc) \otimes \mathcal{A}_\nabla^{/2}(\bigcirc) & \\
\mathcal{A}_\nabla^{/2}(\bigcirc\bigcirc) & \overset{\mathrm{gr}\,\lambda}{\nwarrow} &
\end{array}
$$

In summary, all faces of the multi-cube in diagram in Figure 24 commutes, and therefore so does the induced diagonal square, completing the proof. $\qquad\square$

5.2. **The Tureav co-bracket.** In Section 3.3 we reviewed the definition of the Turaev cobracket on $|\mathbb{C}\pi|$ via the map $\mu : \mathbb{C}\tilde{\pi} \to |\mathbb{C}\pi| \otimes \mathbb{C}\pi$, which required choosing a rotation number $-1/2$ representative for curves in $\mathbb{C}\tilde{\pi}$. Our lift for the cobracket imitates this construction.

We start by interpreting $\mathbb{C}\tilde{\pi}$ in the context of tangles. Let $\frown$ denote an interval skeleton component where both endpoints are on the bottom $D_p \times \{0\}$. We call a tangle with skeleton $\frown$ a *bottom tangle*. We mark the endpoints of the interval by $\bullet$ and $*$, as in Figure 25. Furthermore, we denote by $\tilde{\mathcal{T}}(\bigcirc^k \frown^\ell)$ tangles with k circle skeleton components, and ℓ bottom intervals.

We extend the projection map β (Proposition 5.1) to such tangles to obtain an isomorphism similar to Corollary 5.3:

Proposition 5.9. *There is a well-defined natural bottom projection*

$$\beta : \tilde{\mathcal{T}}_\nabla(\bigcirc^k \frown^\ell) \to |\mathbb{C}\pi|^{\otimes k} \otimes \mathbb{C}\pi^{\otimes \ell},$$

which descends to an isomorphism $\beta : \tilde{\mathcal{T}}^{/1}(\bigcirc^k \frown^\ell) \overset{\cong}{\to} |\mathbb{C}\pi|^{\otimes k} \otimes \mathbb{C}\pi^{\otimes \ell}$.

Proof. Identical to the proof of Proposition 5.1. $\qquad\square$

Proposition 5.10. *The division by b map, q_b, descends to an isomorphism*

$$q_b : \tilde{\mathcal{T}}_\nabla^{1/2}(\frown) \overset{\cong}{\to} \tilde{\mathcal{T}}_\nabla^{/1}(\bigcirc\frown).$$

Proof. The map q_b uses the Conway relation to smooth double points to get a two-component tangle, where one component has interval skeleton and the other component has circle skeleton. $\qquad\square$

Next, we will recover μ as the connecting homomorphism induced from the difference between two ways to lift a bottom tangle.
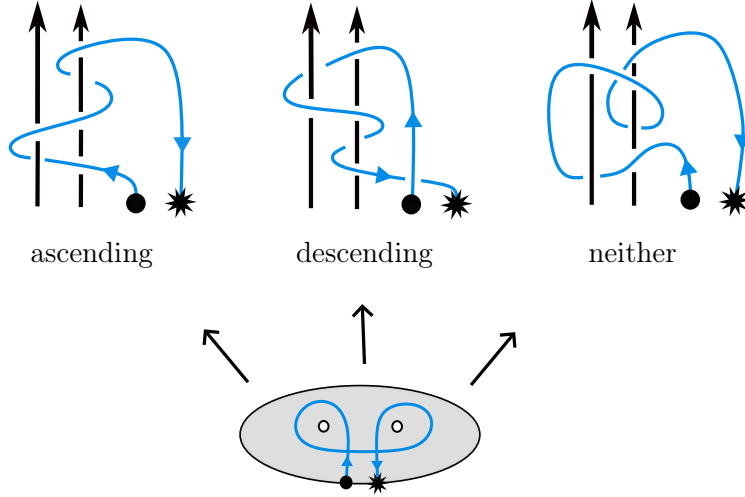
FIGURE 25. An example curve in $\mathbb{C}\pi$ lifted to bottom tangles. The left lift is an ascending tangle, the middle lift is a descending tangle, and the last lift is neither ascending nor descending. All three tangles are equivalent in $\tilde{\mathcal{T}}^{/1}$, but distinct in $\tilde{\mathcal{T}}$.

fig:ascending

def:asc+desc

Definition 5.11. Let $\bullet$ and $*$ be two points on the boundary of D_p that are close together. An embedding

$$T : (I, \{0, 1\}) \hookrightarrow (M_p, \{\bullet, *\})$$

representing a bottom tangle is called *ascending* if it "first goes up, and then goes *straight* down". More precisely, if (z, s) is a global coordinate system for $M_p = D_p \times I$, then T is an ascending tangle if there exists $c \in (0, 1)$ such that when $t \in (0, c)$, the $\frac{d}{ds}$ component of $\dot{T}$ is positive, when $t \in (c + \epsilon, 1)$, $\dot{T}$ is a negative constant multiple of $\frac{d}{ds}$, and when $t \in (c, c + \epsilon)$, T smoothly transitions through a maximum (no sharp corner).

Likewise, such an embedding representing a bottom tangle T is *descending* if it "first goes straight up, and then goes down". So there is $c \in (0, 1)$ such that when $t \in (0, c)$, $\dot{T}$ is a positive constant multiple of $\frac{d}{ds}$ and when $t \in (c + \epsilon, 1)$ the $\frac{d}{ds}$ component of $\dot{T}$ is negative, and when $t \in (c, c + \epsilon)$, T smoothly transitions through a maximum.

Definition 5.12. An *ascending tangle* is a bottom tangle in M_p whose ambient isotopy class has an ascending embedding. Similarly, a *descending tangle* is a bottom tangle in M_p whose ambient isotopy class has an descending embedding. See Figure 25 for an example.

Given a curve K in $\mathbb{C}\pi$, through the isomorphism β, K can be lifted to a bottom tangle in $\tilde{\mathcal{T}}^{/1}(\cap)$. Because we are in the quotient by degree 1 terms, crossings

can be changed at will to make the lifted tangle be ascending or descending. However, to lift K to a framed tangle takes some care. For any framed curve K in $\mathbb{C}\pi$, we can choose a homotopy class representative with rotation number 0 that is a sailing curve. A *sailing curve* is a curve whose tangent vector never points in a fixed specified direction. For this context, viewing $D_p \times 0$ as a subset of $\mathbb{C}$ a we fix the north direction $\vec{n}$ to be in the direction of i, and sailing curves never point north. For a curve to avoid pointing north when turning from west to east, (instead of tacking like a sailboat with your nose to the wind) a kink can be added to loop the curve back around through the south direction and then continue heading east (do a jib turn like a sailboat with your back to the wind). See Figure 26 for an example sailing curve. When taking a lift of a sailing curve K, there is an ascending lift of the curve where the north vector is never tangent to the curve. We will denote this lift as $\lambda_a(K)$. We can choose a framing at each point p on $\lambda_a(K)$ by taking the tangent vector $\dot{T}$ at p and the projection of $\vec{n}$ on to the plane normal to $\dot{T}$ (since $\dot{T}$ is never parallel to $\vec{n}$). Thus $\lambda_a(K)$ is a framed ascending bottom tangle. Similarly we can lift K to a framed descending bottom tangle denoted $\lambda_d(K)$. Finally, we define $\bar{\lambda} : \tilde{\mathcal{T}}^{/1}(\cap) \to \tilde{\mathcal{T}}^{/2}(\cap)$ by

$$\bar{\lambda}(K) = \lambda_a(K) - \lambda_b(K)$$

to be the difference between the framed ascending bottom tangle and the framed descending bottom tangle. In $\tilde{\mathcal{T}}^{/2}(\cap)$, crossing changes matter so $\bar{\lambda}$ is not the zero map.

Theorem 5.13. *The diagram in Figure 27 commutes and the unique induced map η when composed with isomorphisms q_b and β is the self intersection map μ, that is $\mu = \beta q_b \eta \beta^{-1}$.*

Proof. Let $\gamma \in |\mathbb{C}\pi|$ and let $T = \lambda_a(\gamma)$ be an ascending lift. We can rewrite each strand-strand crossings (s-crossing for short) of T as a sum or difference of a double point and its counterpart in T^{f_b}. As in the proof of **??**, rewriting each s-crossing of T in this way yields a sum indexed by the subsets of its s-crossings. For every subset X of s-crossings, there is a term in the sum where the s-crossings in X are replaced by double points, and those not in X are replaced with their counterparts in T^{f_b}. In particular, the term corresponding to $X = \emptyset$ is exactly T^{f_b}, so $T - T^{f_b}$ lives in $\tilde{\mathcal{T}}^1(\cap)$.

By passing to the quotient $\tilde{\mathcal{T}}_{\triangledown}^1/\tilde{\mathcal{T}}_{\triangledown}^2(\cap)$, only the terms that have a single double point remain, so $T - T^{f_b}$ becomes a sum over the s-crossings of T, where in each term the s-crossing is replaced by a double point. The map q_b uses the Conway relation to smooth these double points to get a two-component tangle, where one component has interval skeleton and the other component has circle skeleton. Thus we land in $\tilde{\mathcal{T}}_{\triangledown}^{/1}(\bigcirc\cap)$, which is isomorphic to $|\mathbb{C}\pi| \otimes \mathbb{C}\pi$ via β. $\qquad\square$

For a bottom tangle, there is a closure map from $cl : \tilde{\mathcal{T}}(\cap) \to \tilde{\mathcal{T}}(\bigcirc)$ by connecting the endpoints of the bottom tangle, $\bullet$ and $*$, by a canonical path in the boundary of the disk. Recall from Section 3.3 that the cobracket δ is constructed
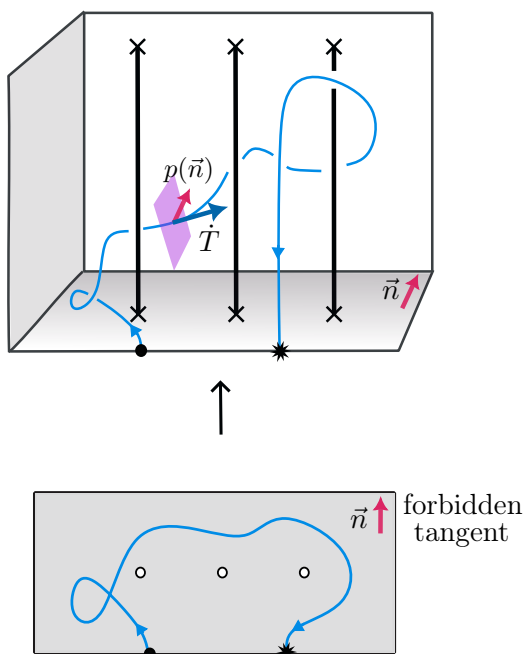
FIGURE 26. A rotation number 0 sailing curve in $\mathbb{C}\pi$ lifts to a framed bottom tangle in M_p.

raming_from_sailing

FIGURE 27. The nontrivial horizontal maps are the respective quotient maps.

fig:Snakeformu

from μ by post composing with the closure map and then antisymetrizing. In the context of tangle diagrams, this construction is shown in Figure 28. The closure map $cl : \tilde{\mathcal{T}}^{/1}(\bigcirc\frown) \to \tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc)$ orders the components by placing the closed bottom tangle in the second slot. The intermediate induced map after

$$\begin{array}{ccccc}
\tilde{\mathcal{T}}_{\triangledown}^{1/2}(\frown) & \longrightarrow & \tilde{\mathcal{T}}_{\triangledown}^{/2}(\frown) & \longrightarrow & \tilde{\mathcal{T}}^{/1}(\frown) \\
{\scriptstyle 0}\downarrow & & {\scriptstyle \lambda}\downarrow & & {\scriptstyle 0}\downarrow \\
\tilde{\mathcal{T}}_{\triangledown}^{1/2}(\frown) & \longrightarrow & \tilde{\mathcal{T}}_{\triangledown}^{/2}(\frown) & \longrightarrow & \tilde{\mathcal{T}}^{/1}(\frown) \\
{\scriptstyle \cong}\uparrow\downarrow {\scriptstyle q_b} & & & & \\
\tilde{\mathcal{T}}^{/1}(\bigcirc\frown) & \xleftarrow{\hat{\eta}} & & & \\
\downarrow {\scriptstyle cl} & & & & \\
\tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc) & \xleftarrow{\hat{\delta}} & & & \\
\downarrow {\scriptstyle Alt} & & & & \\
\tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc) & \xleftarrow{\delta} & & &
\end{array}$$

FIGURE 28. Constructing δ from $\hat{\eta}$.

$$\begin{array}{ccccccc}
& & \tilde{\mathcal{A}}_{\triangledown}^{1/2}(\frown) & \longrightarrow & \tilde{\mathcal{A}}_{\triangledown}^{/2}(\frown) & \longrightarrow & \tilde{\mathcal{A}}^{/1}(\frown) & \longrightarrow & 0 \\
& & {\scriptstyle 0}\downarrow & & {\scriptstyle \mathrm{gr}\,\lambda}\downarrow & & {\scriptstyle 0}\downarrow \\
0 & \longrightarrow & \tilde{\mathcal{A}}_{\triangledown}^{1/2}(\frown) & \longrightarrow & \tilde{\mathcal{A}}_{\triangledown}^{/2}(\frown) & \longrightarrow & \tilde{\mathcal{A}}^{/1}(\frown) \\
& & {\scriptstyle \cong}\uparrow\downarrow {\scriptstyle q_b} & & & & \\
& & \tilde{\mathcal{A}}^{/1}(\bigcirc\frown) & \xleftarrow{\mathrm{gr}\,\mu} & & & \\
& & \downarrow {\scriptstyle \mathrm{gr}\,cl} & & & & \\
& & \tilde{\mathcal{A}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{A}}^{/1}(\bigcirc) & \xleftarrow{\mathrm{gr}\,\hat{\delta}} & & &
\end{array}$$

FIGURE 29. Associated graded diagram constructing the graded ordered Turaev cobracket.

closing, but before antisymmtrizing, is denoted in the figure by $\hat{\delta}$ and is called the *ordered* Turaev cobracket. We will show the Kontsevich integral is homomorphic with respect to $\hat{\delta}$. The homomorphicity of δ with respect to Z follows from immediately the homomorphicity of $\hat{\delta}$ with respect to Z because $\mathrm{gr}(Alt) = Alt$.

Taking the associated graded of the diagram in Figure 21 we arrive at the diagram in Figure 29

$$
\begin{array}{ccccc}
\tilde{\mathcal{T}}_{\triangledown}^{1/2}(\cap) & \longrightarrow & \tilde{\mathcal{T}}_{\triangledown}^{/2}(\cap) & \longrightarrow & \tilde{\mathcal{T}}^{/1}(\cap) \\
{\scriptstyle 0}\downarrow & & {\scriptstyle \lambda}\downarrow & & \downarrow{\scriptstyle 0} \\
\tilde{\mathcal{T}}_{\triangledown}^{1/2}(\cap) & \longrightarrow & \tilde{\mathcal{T}}_{\triangledown}^{/2}(\cap) & \longrightarrow & \tilde{\mathcal{T}}^{/1}(\cap) \\
{\scriptstyle \cong}\updownarrow{\scriptstyle q_b} & & & & \\
\tilde{\mathcal{T}}^{/1}(\bigcirc\cap) & \xleftarrow{\hspace{0.6cm}} & {\scriptstyle \mu} & & \downarrow{\scriptstyle 0} \\
\downarrow{\scriptstyle cl} & & \downarrow{\scriptstyle cl} & & \\
\tilde{\mathcal{T}}^{/1}(\bigcirc)\otimes\tilde{\mathcal{T}}^{/1}(\bigcirc) & \xrightarrow{\exists\rho} & \tilde{\mathcal{T}}_{\triangledown}^{/2}(\bigcirc) & \longrightarrow & \tilde{\mathcal{T}}^{/1}(\bigcirc) \longrightarrow 0
\end{array}
$$

FIGURE 30. Commutative diagram for Lemma 5.15.

Theorem 5.14. *The diagram in Figure 29 commutes and the induced map* $\operatorname{gr}\hat{\delta}$ *is the associated graded ordered Turaev cobracket.*

Proof. The maps in the diagram of Figure 28 are filtered maps, and therefore Figure 29 is obtained by applying the associated graded functor to it. As a result, the diagram of Figure 29 commutes, $\operatorname{gr}\mu$ is the induced map from the snake lemma for this diagram, and so $\operatorname{gr}\hat{\delta}$ coincides with the graded ordered Turaev cobracket. $\square$

Lemma 5.15. *There exists a map* $\rho : \tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc) \to \tilde{\mathcal{T}}_{\triangledown}^{/2}(\bigcirc)$ *that makes the diagram in Figure 30 commute.*

Proof. There is an isomorphism from $\tilde{\mathcal{T}}^{/1}(\bigcirc)\otimes\tilde{\mathcal{T}}^{/1}(\bigcirc)$ to $\tilde{\mathcal{T}}^{/1}(\bigcirc\bigcirc)$ by combining the two tangles into a single tangle and forgetting the order of the components. Since we are modding out by s degree 1, there is no notion of over or under, these are just curves in the disc.

    The map $\rho : \tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc) \to \tilde{\mathcal{T}}^{/2}(\bigcirc)$ is defined to be the following composition of maps.

$$
\tilde{\mathcal{T}}^{/1}(\bigcirc)\otimes\tilde{\mathcal{T}}^{/1}(\bigcirc) \xrightarrow{\text{forget}} \tilde{\mathcal{T}}^{/1}(\bigcirc\bigcirc) \xleftarrow{\;q_b\;} \tilde{\mathcal{T}}_{\triangledown}^{1/2}(\bigcirc) \xhookrightarrow{\hspace{0.6cm}} \tilde{\mathcal{T}}_{\triangledown}^{/2}(\bigcirc)
$$

with the overarching arrow labeled ρ.

Since the image of ρ in $\tilde{\mathcal{T}}_{\triangledown}^{/2}$ is all of $\tilde{\mathcal{T}}^{1/2}$ we get the following short exact sequence.

$$
\tilde{\mathcal{T}}^{/1}(\bigcirc)\otimes\tilde{\mathcal{T}}^{/1}(\bigcirc) \xrightarrow{\;\rho\;} \tilde{\mathcal{T}}_{\triangledown}^{/2}(\bigcirc) \longrightarrow \tilde{\mathcal{T}}^{/1}(\bigcirc) \longrightarrow 0
$$

pcubesimplification

akefor_gr_cobracket

pcubesimplification

In this figure, do we need μ in it still?

$$\begin{array}{ccccccc}
\tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc) & \xrightarrow{forget} & \tilde{\mathcal{T}}^{/1}(\bigcirc\bigcirc) & \xleftarrow{q_b} & \tilde{\mathcal{T}}^{1/2}_{\nabla}(\bigcirc) & \longrightarrow & \tilde{\mathcal{T}}^{/2}_{\nabla}(\bigcirc) \\
\downarrow{\scriptstyle Z^{/1} \otimes Z^{/1}} & & \downarrow{\scriptstyle Z^{/1}} & & \downarrow{\scriptstyle Z^{/1}} & & \downarrow{\scriptstyle Z^{/2}} \\
\tilde{\mathcal{A}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{A}}^{/1}(\bigcirc) & \xrightarrow{forget} & \tilde{\mathcal{A}}^{/1}(\bigcirc\bigcirc) & \xleftarrow{\operatorname{gr} q_b} & \tilde{\mathcal{A}}^{1/2}_{\nabla}(\bigcirc) & \longrightarrow & \tilde{\mathcal{A}}^{/2}_{\nabla}(\bigcirc)
\end{array}$$

with ρ arcing over the top and $\operatorname{gr}\rho$ arcing under the bottom.

FIGURE 31. Commutative diagram for Lemma 5.16

The commutativity of the diagram in Figure 30 relies finally on the commutativity of the bottom left square. We single this square out below and verify the commutativity.

$$\begin{array}{ccc}
\tilde{\mathcal{T}}^{1/2}_{\nabla}(\cap) & \xrightarrow{\hspace{6cm}} & \tilde{\mathcal{T}}^{/2}_{\nabla}(\cap) \\
{\scriptstyle \cong}\Big\updownarrow{\scriptstyle q_b} & & \Big\downarrow{\scriptstyle cl} \\
\tilde{\mathcal{T}}^{/1}(\bigcirc\cap) & & \\
\Big\downarrow{\scriptstyle cl} & \xdashrightarrow{\rho} & \\
\tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc) & \longrightarrow \tilde{\mathcal{T}}^{/1}(\bigcirc\bigcirc) \xleftarrow[\cong]{q_b} \tilde{\mathcal{T}}^{1/2}_{\nabla}(\bigcirc) \longrightarrow & \tilde{\mathcal{T}}^{/2}_{\nabla}(\bigcirc)
\end{array}$$

Let $T \in \tilde{\mathcal{T}}^{1/2}_{\nabla}(\cap)$, then T is a bottom tangle with exactly one double point. Following along the top and right of the diagram in Figure 30, when T is closed, we get a closed loop with one double point inside $\tilde{\mathcal{T}}^{/2}_{\nabla}(\bigcirc)$. Following along the right and bottom, $q_b(T)$ uses the Conway relation to snip off a loop of T to get a tangle in $\tilde{\mathcal{T}}^{/1}(\bigcirc\cap)$ with one closed loop and a bottom tangle, with no double points. Closing the bottom tangle and forgetting the order of the closed loops gives a tangle in $\tilde{\mathcal{T}}^{/1}(\bigcirc\bigcirc)$ with two closed loops and no double points. Reversing the Conway relation along q_b glues together the two closed loops to get a single closed loop with one double point then included into $\tilde{\mathcal{T}}^{/2}(\bigcirc)$. This arrives at the same closed loop with one double point as if we had closed T in the first place. □

Lemma 5.16. *The diagram in Figure 31 commutes.*

Proof. The right square commutes because Z is a filtered map and respects filtered inclusions.

For the middle square, we use the map q_b from right to left and show commutativity on a double point.

$$Z^{/1}(q_b(\asymp)) = Z^{/1}(\mathord{)}\,\mathord{(}\,) = \mathord{)}\,\mathord{(}$$
$$Z^{/1}(\asymp) = e^{C/2} - e^{-C/2}$$
$$= \frac{C}{2} - (-\frac{C}{2}) + \text{higher degree terms} \in \tilde{\mathcal{A}}_{\triangledown}^{/2}(\bigcirc)$$
$$= C = \underset{.....}{\asymp} = a\asymp = a\mathord{)}\,\mathord{(}$$
$$\mathrm{gr}\, q_b(Z^{/1}(\asymp)) = \mathrm{gr}(a\mathord{)}\,\mathord{(}) = \mathord{)}\,\mathord{(}$$

For the left square, Z compatible with forgetful is because we land in $/1$, where there are no s-s chords. $\qquad\square$

Theorem 5.17. *The Kontsevich integral descends to a homomorphic expansion for the ordered Turaev cobracket. That is, the following square commutes:*

$$
\begin{array}{ccc}
\tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc) & \xleftarrow{\quad\hat{\delta}\quad} & \tilde{\mathcal{T}}^{/1}(\cap) \\
\downarrow{\scriptstyle Z^{/1} \otimes Z^{/1}} & & \downarrow{\scriptstyle Z^{/1}} \\
\mathcal{A}^{/1}(\bigcirc) \otimes \mathcal{A}^{/1}(\bigcirc) & \xleftarrow[\mathrm{gr}\,\hat{\delta}]{} & \mathcal{A}^{/1}(\cap)
\end{array}
$$

Proof. The diagram in Figure 32 is attained by taking the Kontsevich integral of the commutative diagram in Figure 30 (with the middle layers omitted). We have already established that the top and bottom faces commute by Lemma 5.15 and Theorem 5.14. The left and right vertical sides trivially commute because of the zero maps. The front-left vertical square commutes by Lemma 5.16. The front-right and back faces commute because Z respects the s-filtration and is homomorphic with respect to the inclusion and quotient maps of the filtered components.

The middle vertical face of Figure 32 is the following square.

$$
\begin{array}{ccc}
 & & \tilde{\mathcal{T}}_{\triangledown}^{/2}(\cap) \\
 & \overset{cl\circ\lambda}{\swarrow} & \downarrow{\scriptstyle Z^{/2}} \\
\tilde{\mathcal{T}}_{\triangledown}^{/2}(\bigcirc) & & \\
\downarrow{\scriptstyle Z^{/2}} & & \mathcal{A}_{\triangledown}^{/2}(\cap) \\
 & \overset{\mathrm{gr}(cl\circ\lambda)}{\swarrow} & \\
\mathcal{A}_{\triangledown}^{/2}(\bigcirc) & &
\end{array}
$$

The Kontsevich integral is homomorphic with respect to the flip operation, as shown in Proposition 4.13. The map $cl \circ \lambda$ applied to a bottom tangle outputs the difference between the closed ascending lift and the closed descending lift. The closed descending lift is the flip of the closed ascending lift. So $cl \circ \lambda =$
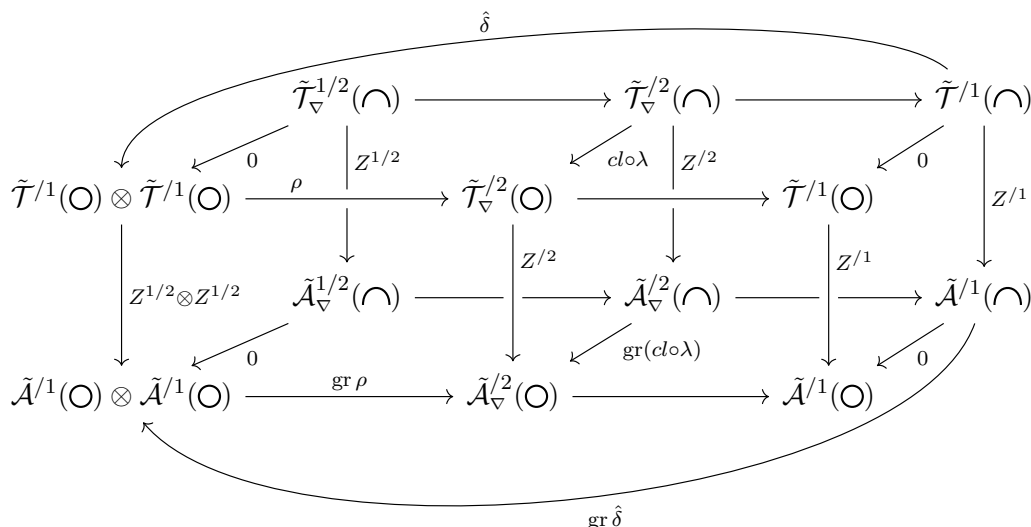
FIGURE 32. Commutative cube showing the formality of the ordered Turaev cobracket from the Kontsevich integral.

$(id - flip) \circ cl$ acting on ascending representatives. Z is homomorphic with respect to $(id - flip) \circ cl$.

The commutativity of all vertical faces of the cube diagram in Figure 32 implies that the induces diagonal square also commutes, which gives the desired formality of the theorem statement. $\square$

where does conjugation come into play?? Something about flipping first then dragging the ends down and then closing.

remark–if we were doing this with μ is it wouldn't work because flip of a bottom tangle is not a bottom tangle. It is much cleaner to just pass to the closures.

REFERENCES

[AKKN18a] Anton Alekseev, Nariya Kawazumi, Yusuke Kuno, and Florian Naef. The goldman-turaev lie bialgebra and the kashiwara-vergne problem in higher genera, 2018.

[AKKN18b] Anton Alekseev, Nariya Kawazumi, Yusuke Kuno, and Florian Naef. The goldman–turaev lie bialgebra in genus zero and the kashiwara–vergne problem. *Advances in Mathematics*, 326:1–53, 2018.

[AKKN20] Anton Alekseev, Nariya Kawazumi, Yusuke Kuno, and Florian Naef. Goldman–turaev formality implies kashiwara–vergne. *Quantum Topology*, 11(4):657–689, 2020.

[BN95] D. Bar-Natan. On the vassiliev knot invariants. *Topology*, 34:423–472, 1995.

[BND17] Dror Bar-Natan and Zsuzsanna Dancso. Finite type invariants of w-knotted objects II: tangles, foams and the Kashiwara-Vergne problem. *Math. Ann.*, 367(3-4):1517–1586, 2017.

[CDM12] S. Chmutov, S. Duzhin, and J. Mostovoy. *Introduction to Vassiliev Knot Invariants*. Cambridge University Press, 2012.

Da   [Dan10]   Zsuzsanna Dancso. On the kontsevich integral for knotted trivalent graphs. *Alg. Geom. Topol.*, 10(3):1317–1365, 2010.

Gol   [Gol86]   William Mark Goldman. Invariant functions on lie groups and hamiltonian flows of surface group representations. *Invent. Math.*, 85:263–302, 1986.

Gor   [Gor99]   Viktor Goryunov. Vassiliev invariants of knots in $\mathbb{R}^3$ and in a solid torus. *Differential and symplecttic topology of knots and curves, Amer. Math. Soc. Transl.*, 190(2):37–59, 1999.

HM   [HM21]   Kazuo Habiro and Gwénaël Massuyeau. The Kontsevich integral for bottom tangles in handlebodies. *Quantum Topol.*, 12(4):593–703, 2021.

Kon   [Kon93]   Maxim Kontsevich. Vassiliev's knot invariants. *Adv. in Soviet Math.*, 16(2):137–150, 1993.

LM95   [LM95]   Tu Quoc Thang Le and Jun Murakami. Kontsevich's integral for the homfly polynomial and relations between values of multiple zeta functions. *Topology and its applications*, 62:193–206, 1995.

LM96   [LM96]   Tu Quoc Thang Le and Jun Murakami. The universal Vassiliev-Kontsevich invariant for framed oriented links. *Compositio Mathematica*, 102(1):41–64, 1996.

Mas   [Mas18]   Géwna el Massuyeau. Formal descriptions of turaev's loop operations. *Quantum Topol.*, 9:39–117, 2018.

Tur   [Tur91]   Vladimir Turaev. Skein quantization of poisson algebras of loops on surfaces. *Ann. Sci. École Norm. Sup.*, 24:635–704, 1991.

Department of Mathematics, University of Toronto, Toronto, Ontario, Canada
Email address: drorbn@math.toronto.edu
URL: http://www.math.toronto.edu/~drorbn

School of Mathematics and Statistics, The University of Sydney, Sydney, NSW, Australia
Email address: zsuzsanna.dancso@sydney.edu.au

School of Mathematics and Statistics, The University of Melbourne, Melbourne, Victoria, Australia
Email address: hogant@student.unimelb.edu.au
URL: https://www.tamaramaehogan.com/

Department of Mathematics, University of Toronto, Toronto, Ontario, Canada
Email address: chengjin.liu@mail.utoronto.ca

Department of Mathematics and Statistics, Elon University, Elon, North Carolina
Email address: nscherich@elon.edu
URL: http://www.nancyscherich.com