GOLDMAN-TURAEV FORMALITY FROM THE KONTSEVITCH INTEGRAL

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Abstract. We present a three dimensional realisation of the Goldman-Turaev Lie biaglebra, and construct Goldman-Turaev homomorphic expansions from the Kontsevich integral.

Contents

Key words and phrases. knots, links in a handlebody, expansions, finite type invariants, Lie algebras .

To do list for Zsuzsi

(1) (BIG COMMENT) Section 3.1, reconsider the depth for which we discuss the Kontsevich integral. Who is our audience?
(2) Section 3.3, read over the added informal descriptions of the operations to tighten up.
(3) Section 3.3, there is an old note from Jessica about signs. Do we need to keep that comment, or can we delete it?
(4) Find the reference for Proposition 3.6– Quillen66? Or new reference for Magnus expansion.
(5) I added a footnote for the Magnus expansion. Do we need it? Should we say more there?
(6) add a reference for Proposition 3.8.
(7) Section 4, make it clear where the proof for Theorem 4.9 ends.
(8) Section 4, make dummy figure for chord diagram stacking
(9) I reordered the intro section according to Dror's comments. Have you read it over? It probably needs proof reading again.

1. Introduction

In 1986, Goldman defined a Lie bracket [Gol86] on the space of homotopy classes of free loops on a compact oriented surface. Shortly after in 1991, Turaev defined a cobracket [Tur91] on the same space[1]. This bracket and cobracket make the space of free loops into a Lie bialgebra – known as the Goldman-Turaev (GoTu) Lie bialgebra – which forms the basis for the field of string topology [?] and has been an object of study from many perspectives.

add referemnces: chas-sullivan, kashiwara-vergne, AN, AT, Formality paper

In this paper we, describe a 3-dimensional lift of the Goldman-Turaev Lie bialgebra into a space of tangles in a handlebody. We recover the bracket and cobracket maps as projections of intuitive operations on tangles. We show the Kontsevich integral is homomorphic with respect to these tangle operations. Our main result is informally summarised as follows:

Main Result. *Let $\tilde{\mathcal{T}}$ denote the space of formal linear combinations of tangles in a punctured disc cross an interval $M_p = D_p \times I$. Projecting to the bottom $D_p \times 0$, one obtains curves on a punctured disc, and the Goldman–Turaev operations on these curves are induced[2] by the* stacking *and* flipping *operations on the tangles. The Kontsevich integral is a homomorphic expansion for tangles in M_p, and descends to a Goldman–Turaev homomorphic expansion on D_p.*

This result is parallel to Massuyeau's [Mas18], however, our approach to the cobracket is significantly different and simpler, hence, more likely to lead to give

[1] Turaev's version required factoring out by the constant loop; there is a lift to the full space of homotopy classes of loops, given a framing on the surface [AKKN20].

[2] In a specific sense defined in Section 2

insight into the motivational application descried below. Another related result is [?], which constructs Goldman–Turaev expansions from the Khnizhnik-Zamolodchikov connection, a geometric incarnation of the Kontsevich integral.

In more detail, we describe a space $\widetilde{\mathcal{T}}$ of formal linear combintations of framed tangles in the handlebody $\mathcal{D}_p \times I$ and operations on this space, which induce the Goldman-Turaev operations in the bottom projection to $D_p \times \{0\}$. The Goldman bracket arises from the commutator associated to the stacking product in a Conway skein quotient of $\widetilde{\mathcal{T}}$, defined in Section 4.7, and the Turaev cobracked from taking the difference between a tangle and its vertical flip, again in a Conway quotient. We study the associated graded spaces and operations, and show that the Kontsevich integral is a homomorphic expansion for these tangles, in other words, intertwines the operations with their associated graded counterparts. We show that therefore, the Kontsevich integral descends to a homomorphic expansion for the Goldman-Turaev Lie bialgebra. For the flipping operation and the Turaev cobracket, the precise statements are subtle, and care needs to be taken with the technical details.

1.1. **Motivation.** The Kashiwara–Vergne equations originally arose from the study of convolutions on Lie groups [?]. The equations were reformulated algebraically in terms of automorphisms of free Lie algebras [?], it this form they are a refinement of the Baker-Campbell-Hausdorff formula for products of exponentials of non-commuting variables.

Kashiwara–Vergne theory has multiple topological interpretations in which Kashiwara–Vergne solutions correspond to certain invariants – called *homomorphic expansions* – of topological objects. The existence of a homomorphic expansion is also called *formality* in the literature, this language is inspired by rational homotopy theory and group theory [?].

One of these topological interpretations is due to the first two authors [BND17], who showed that homomorphic expansions of welded foams – a class of 4-dimensional tangles – are in one to one correspondence with solutions to the KV equations. Recently, a series of papers by Alekseev, Kawazumi, Kuno and Naef [AKKN20,AKKN18b,AKKN18a] drew an analogous connection between KV solutions and homomorphic expansions for the Goldman-Turaev Lie bialgebra for the disc with two punctures (up to non-negligible differences in the technical details). This correspondence was used to generalise the Kashiwara–Vergne equations via considering different surfaces, including those of higher genus.

In other words, there is an intricate algebraic connection between four-dimensional welded foams and the GT Lie bi-algebra, which strongly suggests that there is a topological connection as well. In addition to the inherent interest in tangles in handlebodies, one goal for this paper is to work towards this connection between the two-dimensional Goldman–Turaev Lie bialgebra and four-dimesnional welded foams, by constructing a three-dimensional realisation of the Goldman-Turaev Lie bialgebra, with homomorphic expansions which descend to Goldman-Turaev expansions.

There are other papers by Turaev and Massuyeau-Turaev that are not mentioned here.

There are also some references that Yusuke mentioned that we should include

Turaev's paper- we can probably pull some of our lemmas from his paper, reference for relationship with HOMFLY, but he does not mention the free associative algebra at all. Our paper is not a subset of his. Skein algebra quantizes — symmetric lie algebra generated by the goldman lie algebra–you can get a poison algebra, These skien modules quantize that poisson algebra

The paper is organised as follows: Section 2 gives a general algebraic framework for how the Goldman–Turaev operations are induced by tangle operations. In Section 3 we give a brief overview of the Kontsevich integral and the Goldman Turaev Lie bialgebra. In Section 4, we define tangles in handlebodies, relevant operations and Vassiliev filtrations. We identify the associated graded space of tangles as a space of chord diagrams, and introduce the Conway skein quotient. In Section **??**, we identify the GoTu Lie biaglebra in a low filtration degree, and prove the main theorem.

Acknowledgements. We are grateful to Anton Alekseev, Gwenel Massuyeau, and Yusuke Kuno for fruitful conversations. DBN was supported by NSERC RGPIN 262178 and RGPIN-2018-04350, and by The Chu Family Foundation (NYC). ZD was partially supported by the ARC DECRA DE170101128. NS was supported by the NSF under Grant No. DMS-1929284 while in residence at the Institute for Computational and Experimental Research in Mathematics in Providence, RI, during the Braids Program. We thank the Sydney Mathematical Research Institute and the University of Sydney for their hospitality, and funding for multiple research visits.

sec:conceptsum

2. Conceptual summary

We induce the genus zero Goldman-Turaev operations from tangle operations, in the spirit of "connecting homomorphisms": this Section is a summary of the basic approach. We provide some proofs which are not immediate and use the words *homomorphic expansions*, and *Goldman-Turaev operations* without definition, only mentioning their basic properties which make this conceptual outline coherent; the definitions follow in Section 3.

In the diagram (2.1), the top and bottom rows are exact and the right and left vertical maps are zero, and therefore, by minor diagram chasing, the middle vertical map λ induces a unique map $\eta : C \to D$, a degenerate case of a connecting homomorphism. In our applications λ is a difference of two maps λ_1 and λ_2, whose values differ in E but coincide in a quotient F.

eq:inducedconnhom

$$\begin{array}{ccccccc} & & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle \lambda=\lambda_1-\lambda_2} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & D & \longrightarrow & E & \longrightarrow & F & & \end{array} \qquad \eta : C \dashrightarrow D \tag{2.1}$$

In Section 5 we present two constructions which produce the Goldman bracket and the Turaev cobracket, respectively, as induced homomorphisms η, from corresponding tangle operations λ_1 and λ_2. The following example is a schematic version of what will become the argument for the Goldman bracket:

Example 2.1. Let A be an associative algebra, and let $\{L_i\}$ denote the lower central series of A. That is, $L_1 := A$, and $L_{i+1} := [L_i, A]$. Then the L_i are Lie ideals, and let $M_i = AL_i = L_iA$ denote the two-sided ideal generated by L_i. The quotient A/M_1 is the abelianisation of A, denoted by A^{ab}. Then we have the following diagram:

eq:SnakeExample

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & \frac{A}{M_2}\otimes\frac{A}{M_2} & \longrightarrow & A^{ab}\otimes A^{ab} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle [\cdot,\cdot]} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & \frac{M_1}{M_2} & \longrightarrow & \frac{A}{M_2} & \longrightarrow & A^{ab} & \longrightarrow & 0 \end{array} \qquad \eta: A^{ab}\otimes A^{ab} \dashrightarrow \frac{M_1}{M_2} \tag{2.2}$$

Here λ is the algebra commutator, which is indeed the difference between two maps: the multiplication (λ_1) and the multiplication in the opposite order (λ_2). The kernel K of the projection to $A^{ab} \otimes A^{ab}$ is generated by the subalgebras $\left\{\frac{M_1}{M_2} \otimes \frac{A}{M_2}, \frac{A}{M_2} \otimes \frac{M_1}{M_2}\right\}$ in $\frac{A}{M_2} \otimes \frac{A}{M_2}$. The map η is a well defined commutator map $A^{ab} \otimes A^{ab} \to \frac{M_1}{M_2}$, given by $\eta(x \otimes y) = [x, y] \bmod M_2$. □

The goal of this paper is to construct homomorphic expansions (aka formality isomorphisms) for the Goldman-Turaev Lie bialgebra from the Kontsevich integral. In outline, this follows from the naturality property of the construction above, under the associated graded functor, as follows.

Given a short exact sequence

$$0 \longrightarrow A \overset{\iota}{\hookrightarrow} B \overset{\pi}{\twoheadrightarrow} C \longrightarrow 0,$$

and a descending filtration on B

$$B = B^0 \supseteq B^1 \supseteq B^2 \supseteq \cdots \supseteq B^n \supseteq \dots,$$

there is an induced filtration on A given by

$$A = A^0 \supseteq A^1 \supseteq A^2 \supseteq \cdots \supseteq A^n \supseteq \dots,$$

where $A^i = \iota^{-1}(\iota A \cap B^i)$. Similarly, there is an induced filtration on C given by

$$C = C^0 \supseteq C^1 \supseteq C^2 \supseteq \cdots \supseteq C^n \supseteq \dots$$

where $C^i = \pi(B^n)$.

Lemma 2.2. *If the rows of the diagram* (2.1) *are exact and filtered so that the filtrations on the left and right are induced from the filtration in the middle, then the induced map η is also filtered.*

Proof. Basic diagram chasing: given $c \in C^n$, since $C^n = \pi(B^n)$, there is a $b \in B^n$ such that $\pi(b) = c$. Since λ is filtered, $\lambda(b) \in E^n$, and $\lambda(b) \in \iota(D)$ by exactness. Since $D^n = \iota^{-1}(\iota(D) \cap E^n)$, we have that $\lambda(b) = \iota(d)$ for a $d \in D^n$. By uniqueness of the induced map, $d = \eta(c)$. □

The associated graded functor is a functor from the category of filtered algebras (or vector spaces) to the category of graded algebras (or vector spaces). For a filtered algebra

$$A = A^0 \supseteq A^1 \supseteq A^2 \supseteq \cdots \supseteq A^n \supseteq \dots,$$

the (degree completed) associated graded algebra is defined to be

$$\operatorname{gr} A = \Pi_{n=0}^{\infty} A^n/A^{n+1}.$$

The associated graded map of a filtered map is defined in the natural way (as in the proof of Lemma 2.3 below). In general, gr is not an exact functor, but it does preserve exactness for the special class of filtered short exact sequences where the filtrations on A and C are induced from the filtration on B:

lem:ExaxtGr

Lemma 2.3. *If in the filtered short exact sequence*

$$0 \longrightarrow A \overset{\iota}{\hookrightarrow} B \overset{\pi}{\twoheadrightarrow} C \longrightarrow 0$$

the filtrations on A and C are induced from the filtration on B, then the associated graded sequence is also exact:

$$0 \longrightarrow \operatorname{gr} A \overset{\operatorname{gr}\iota}{\hookrightarrow} \operatorname{gr} B \overset{\operatorname{gr}\pi}{\twoheadrightarrow} \operatorname{gr} C \longrightarrow 0.$$

Proof. Since gr is a functor, we know that $\operatorname{gr}\pi \circ \operatorname{gr}\iota = 0$, hence $\operatorname{im}\operatorname{gr}\iota \subseteq \ker \operatorname{gr}\pi$. It remains to show that $\ker\operatorname{gr}\pi \subseteq \operatorname{im}\operatorname{gr}\iota$.

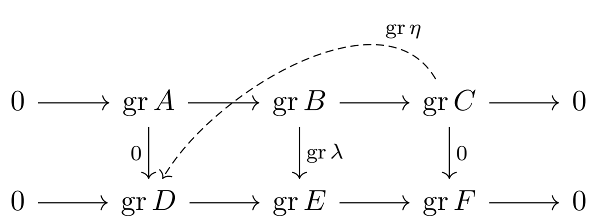
Let $[b] \in B^n/B^{n+1}$, and assume that $\operatorname{gr}\pi([b]) = 0$. Since $\operatorname{gr}\pi([b]) = [\pi(b)] \in C^n/C^{n+1}$, we have $\operatorname{gr}\pi([b]) = 0$ if and only if $\pi(b) \in C^{n+1}$. As the filtration on C is induced from B, we know that $C^{n+1} = \pi(B^{n+1})$. Thus, $\pi(b) \in \pi(B^{n+1})$. Or in other words, there exists $x \in B^{n+1}$ such that $\pi(b) = \pi(x)$. This implies that $\pi(b - x) = 0$ and hence that $b - x \in \iota(A)$ by exactness.

Therefore, $b = x + \iota(a)$ for some $x \in B^{n+1}$ and $a \in A$. It follows that $[b] = [\iota(a)] = \operatorname{gr}\iota([a])$ in B^n/B^{n+1} and hence $\ker\operatorname{gr}\pi \subseteq \operatorname{im}\operatorname{gr}\iota$ as required. □

r_induced_is_unique

Corollary 2.4. *If the rows of the diagram in Equation 2.1 are exact, and the filtrations on the left and right are induced from the filtration in the middle, then the rows of the associated graded diagram are also exact, and the unique connecting homomorphism is $\operatorname{gr}\eta$.*

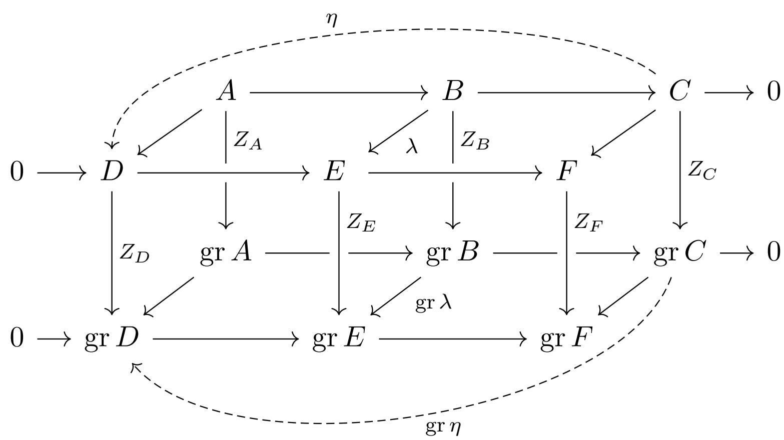
(2.3)



Proof. The exactness of the rows is Lemma 2.3. The induced map is $\operatorname{gr}\eta$ as $\operatorname{gr}\eta$ makes the diagram commute, and the induced map is unique. □

An expansion for an algebraic structure X is a filtered homomorphism $Z : X \to \operatorname{gr} X$ (with special properties as explained in Section 3.1). Thus, if expansions exist for each of the spaces A through F, we obtain a multi-cube:

eq:Cube (2.4)



lem:Naturality **Lemma 2.5.** *If, in the multi-cube* (2.4) *all vertical faces commute, then so does the square:*

eq:HomExp (2.5)
$$\begin{array}{ccc} D & \overset{\eta}{\dashleftarrow} & C \\ \downarrow Z_D & & \downarrow Z_C \\ \operatorname{gr} D & \overset{\operatorname{gr}\eta}{\dashleftarrow} & \operatorname{gr} C \end{array}$$

Proof. Follows from the uniqueness of the induced maps. □

In Section 5.1, we will show how the Goldman bracket and Turaev cobracket each arise as induced maps η, where $\lambda = \lambda_1 - \lambda_2$ is a difference of tangle operations. Therefore the Kontsevich integral therefore induces an expansion for the Goldman–Turaev operations, and the commutativity of the square (2.5) for each operation is – by definition – the homomorphicity property of the expansion. This homomorphicity is our main result. The non-trivial vertical face of the multi-cube is the one containing λ, and the commutativity of this for each Goldman-Turaev operation will follow from homomorphicity properties of the Kontsevich integral. Namely, the Kontsevich integral (standing in for Z_B and Z_E) intertwines the appropriate tangle operations λ_0 and λ_1 with their associated graded counterparts. This is the idea behind the approach of this paper.

3. Preliminaries: Homomorphic expansions and the Goldman-Turaev Lie bialgebra

subsec:FramedKon

3.1. **Homomorphic expansions and the framed Kontsevich integral.** The Kontsevich Integral is the knot theoretic prototype of a *homomorphic expansion.*

Should we say formality instead of/in addition to homomorphic expansion? Add the reference to the formality/ Lie algebra paper.

Homomorphic expansions (a.k.a. formality isomorphisms, well-behaved universal finite type invariants) provide a connection between knot theory and quantum algebra/Lie theory. We begin with a short review of homomorphic expansions from an algebraic perspective, which is outlined – in a slightly different, finitely presentated case – in [BND17, Section 2]. Kontsevich's original construction gives an invariant of unframed links; for a detailed introduction, we recommend [CDM12, Section 8], or [Kon93, BN95, Dan10]. In this paper we work primarily with framed links and tangles, thus we briefly review the framed versions of the Vassiliev filtration and Kontsevich integral; for more detail see [CDM12, Sections 3.5 and 9.1] and [LM96].

3.1.1. *Homomorphic expansions.* Let $\mathcal{K}$ denote a given set of knots, links or tangles in $\mathbb{R}^3$ (e.g., oriented knots), and allow formal linear combinations with coefficients in $\mathbb{C}$. For links and tangles, allow only linear combinations of embeddings of the same skeleton[3]. The *Vassiliev filtration* – defined in terms of resolutions of double points $\doublepoint = \overcrossing - \undercrossing$ – is a decreasing filtration on this linear extension:

$$\mathbb{C}\mathcal{K} = \mathcal{K}_0 \supseteq \mathcal{K}_1 \supseteq \mathcal{K}_2 \supseteq \dots$$

The degree completed associated graded space of $\mathbb{C}\mathcal{K}$ with respect to the Vassiliev filtration is

$$\mathcal{A} := \prod_{n\geq 0} \mathcal{K}_n/\mathcal{K}_{n+1}.$$

An *expansion* is a filtered linear map $\mathbb{Z} : \mathbb{C}\mathcal{K} \to \mathcal{A}$, such that the associated graded map of Z is the identity $\operatorname{gr} Z = id_{\mathcal{A}}$.

Usually, $\mathcal{K}$ is equipped with additional operations: examples are knot connected sum, tangle composition, strand orientation reversal, etc. Homomorphic expansions are compatible with these operations, and thus allow for a study of $\mathcal{K}$ via the more tractable associated graded spaces.

Specifically, an expansion is *homomorphic* with respect to an operation m, if it intertwines m with its associated graded operation on $\mathcal{A}$. That is, $Z \circ m = \operatorname{gr} m \circ Z$. A crucial step towards making effective use of this machinery is to get a handle on the space $\mathcal{A}$ in concrete terms: for example, in classical knot theory, $\mathcal{A}$ has a combinatorial description as a space of *chord diagrams* [CDM12, Chapter 4].

There is a natural map ψ from chord diagrams with i chords to $\mathcal{K}_i/\mathcal{K}_{i+1}$, defined by "contracting chords" as in Figure 1. It is not difficult to establish that ψ is surjective. In the case of classical (oriented, unframed) knots, there are two relations in the kernel of ψ: the 4-Term (4T) and Framing Independence (FI) relations, shown in Figure 2. In fact, these two relations generate the kernel,

[3]The *skeleton* of a knotted object is the underlying combinatorial object. For example: the skeleton of a link is the number of components; the skeleton of a braid is the underlying permutation; the skeleton of a tangle is the number of strands, connectivity, and number of circle components. In these contexts $\mathbb{C}\mathcal{K}$ is a disjoint union of vector spaces, rather than a single vector space.

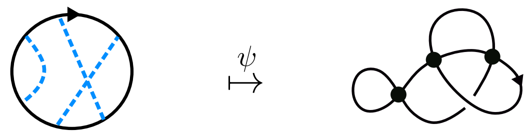


FIGURE 1. Example of ψ mapping a chord diagram to a knot with double points by contracting the chords. The right-hand side represents a well-defined element in $\mathcal{K}_3/\mathcal{K}_4$.

fig:psionchord

and ψ descends to an isomorphism on the quotient; this, however, is significantly harder to prove.

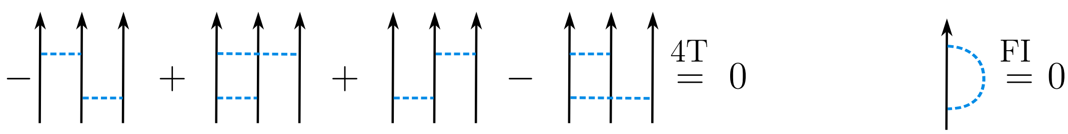


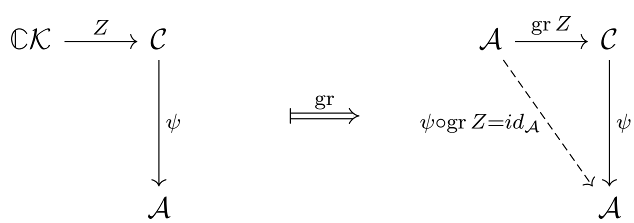
FIGURE 2. The 4T and FI relations, understood as local relations: the strand(s) are part(s) of the skeleton circle, and the skeleton may support additional chords outside the picture shown.

fig:4TFI

The key technique is to construct an expansion as in the following Lemma, [BND17, Proposition 2.7]:

lem:assocgradyoga

Lemma 3.1. *[BND17] Let $\mathbb{C}\mathcal{K}$ be a filtered vector space (or union of vector spaces), and $\mathcal{A}$ the associated graded space of $\mathbb{C}\mathcal{K}$. Let $\mathcal{C}$ be a "candidate model" for $\mathcal{A}$: a graded linear space equipped with a surjective homogeneous map $\psi : \mathcal{C} \to \mathcal{A}$. If there exists a filtered map $Z : \mathbb{C}\mathcal{K} \to \mathcal{C}$, such that $\psi \circ \operatorname{gr} Z = id_{\mathcal{A}}$, then ψ is an isomorphism and $\psi \circ Z$ is an expansion for $\mathcal{K}$.*

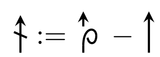


In other words, once one finds a candidate model $\mathcal{C}$ for $\mathcal{A}$, finding an *expansion valued in* $\mathcal{C}$ also implies that ψ is an isomorphism. In classical Vassiliev theory, $\mathcal{K}$ is the space of oriented knots, $\mathcal{C}$ is the space of chord diagrams, and a $\mathcal{C}$-valued expansion is the Kontsevich integral [Kon93].

subsubsec:Framing

3.1.2. *Framed theory.* In this paper we work with *framed* links and tangles, so we give a brief introduction to the framed version of the general theory summarised in the previous section. For simplicity, we consider links for now.

Let $\tilde{\mathcal{K}}$ denote the set of *framed* links in $\mathbb{R}^3$: that is, links along with a non-zero section of the normal bundle. A link diagram is interpreted as a framed link using the blackboard framing. The Reidemeister move R1 move changes the blackboard framing, and by ommitting it, one obtains a Reidemeister theory for framed links. In analogy with a double point, a *framing change* is defined to be the difference



The framed Vassiliev filtration is the descending filtration

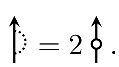
$$\tilde{\mathcal{K}} = \tilde{\mathcal{K}}_0 \supseteq \tilde{\mathcal{K}}_1 \supseteq \tilde{\mathcal{K}}_2 \supseteq ...$$

where $\tilde{\mathcal{K}}_i$ is linearly generated by knots with at least i double points *or framing changes*. The degree completed associated graded space of $\tilde{\mathcal{K}}$ with respect to the framed Vassiliev filtration is

$$\tilde{\mathcal{A}} := \prod_{n\geq 0} \tilde{\mathcal{K}}_n/\tilde{\mathcal{K}}_{n+1}.$$

A natural first guess for a combinatorial description of $\tilde{\mathcal{A}}$ is in terms of chord diagrams with "framing change markings" on the skeleton, graded by the number of chords and markings. There is a natural surjective graded map $\tilde{\psi}$ from marked chord diagrams onto $\tilde{\mathcal{A}}$, which is contracts chords as in the classical case, and which replaces each marking with a framing change. The kernel of $\tilde{\psi}$ includes the $4T$ relation as before.

In place of the FI relation (=0), a weaker relation arises from the equality in $\tilde{\mathcal{K}}$. In fact, , and modulo $\tilde{\mathcal{K}}_2$. In other words, the following relation is in the kernel of $\tilde{\psi}$:



Therefore, it is not necessary to have dedicated notation for the framing change markings, since . The candidate model for the associated graded space is simply chord diagrams modulo the $4T$ relation, and no FI relation. We denote this space by $\tilde{\mathcal{C}}$.

To show that $\tilde{\psi} : \tilde{\mathcal{C}} \to \tilde{\mathcal{A}}$ is an isomorphism, it is enough to construct a $\tilde{\mathcal{C}}$-valued expansion and use Lemma 3.1. This $\tilde{\mathcal{C}}$-valued expansion is the framed version $\tilde{Z}$ of the Kontsevich integral. For details of this construction see [CDM12, Section 9.1], or [LM96, Gor99].

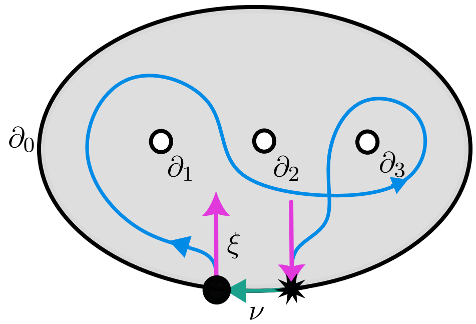


FIGURE 3. D_3 with an immersed loop from $\bullet$ to $*$ with initial tangent vector ξ and terminal tangent vector $-\xi$. The path along the boundary from $*$ to $\bullet$ is ν.

fig:DP

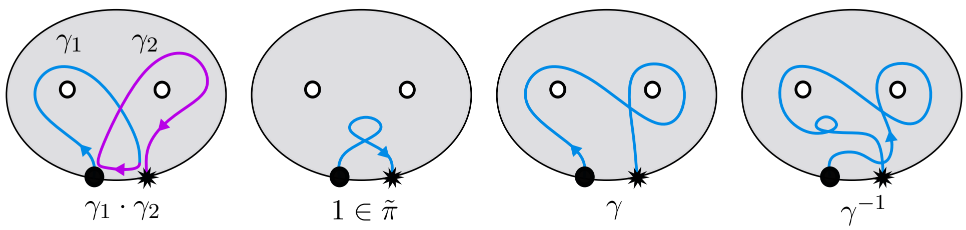


FIGURE 4. The group structure on $\tilde{\pi}$.

fig:DPGroup

subsec:IntroGT

3.2. **The Goldman-Turaev Lie bialgebra.** In order to define the Goldman-Turaev Lie bialgebra, we need to recall some basic definitions and notation.

Let D_p denote p-punctured disc, with $p+1$ circle boundary components $\partial_0, \partial_1, ..., \partial_p$, embedded in the complex plane so that ∂_0 is the outer boundary, as in Figure 3. In particular, the plane-embedding specifies a framing (trivialisation of the tangent bundle) on D_p, and thus immersed loops in D_p are equipped with a notion of *rotation number*.

Let $\pi = \pi_1(D_p, *)$ denote the fundamental group of D_p with basepoint $* \in \partial_0$. We denote by $\mathbb{C}\pi$ the group algebra of π.

We also need to consider based paths. Let $\bullet$ and $*$ be two "nearby" basepoints on ∂_0 and ξ be the direction of the inward pointing normal vector to ∂_0 at $\bullet$ and $*$. Let $\tilde{\pi} = \tilde{\pi}_{\bullet *}$ denote the set of regular homotopy classes of immersed curves $\gamma : ([0,1], 0, 1) \to (D_p, \bullet, *)$, so that $\dot{\gamma}(0) = \xi$, and $\dot{\gamma}(1) = -\xi$, as shown in Figure 3. Note that the rotation number is invariant under regular homotopy. Recall that $\tilde{\pi}$ is in fact a group, illustrated in Figure 4 and defined as follows:

(1) Let ν denote the path from $\star$ to $\bullet$ along ∂_0. The group product $\gamma_1 \cdot \gamma_2$ is the smooth concatenation of γ_1 with ν followed by γ_2.

(2) The group identity is the class of paths which, when composed with ν, become contractible loops of rotation number zero.
(3) The inverse of γ is the concatenation $\overline{\nu}\,\overline{\gamma}\,\nu^*$ where the overline denotes the reverse path, and ν^* includes a negative twist (to ensure that the rotation number of $\gamma\cdot\gamma^{-1}$ is 0). The beginning and end of the path is adjusted in an epsilon neighbourhood of the base points to have inward and outward pointing tangent vectors, as in Figure 4.

Denote by $\mathbb{C}\tilde{\pi}$ the group algebra of $\tilde{\pi}$. There is a forgetful map $\tilde{\pi}\to\pi$ which maps γ to the (non-regular) homotopy class of $\gamma\,\nu$. This linearly extends to a forgetful map $\mathbb{C}\tilde{\pi}\to\mathbb{C}\pi$.

For an algebra A we denote by $|A|$ the *linear*[4] quotient $A/[A,A]$, where $[A,A]$ denotes the subspace spanned by commutators $[x,y]=xy-yx$ for $x,y\in A$. We denote the quotient (trace) map by $|\cdot|:A\to|A|$. In our context, $|\mathbb{C}\pi|$ has an explicit description as the $\mathbb{C}$-vector space generated by homotopy classes of free loops in D_p. In a similar but more subtle fashion, $|\mathbb{C}\tilde{\pi}|$ is spanned by *regular* homotopy classes of immersed free loops, where $|\gamma|$ denotes the class of $\gamma\nu$ as a free immersed loop.

The Goldman–Turaev Lie bialgebra comes in two flavours: *original* and *enhanced*. The original construction of the Goldman bracket is a Lie bracket on $|\mathbb{C}\pi|$. However, the original Turaev cobracket is only well-defined on $|\overline{\mathbb{C}\pi}|=|\mathbb{C}\pi|/\mathbb{C}\mathbb{1}$, the linear quotient by the homotopy class of the constant loop. The space $|\overline{\mathbb{C}\pi}|$ is a Lie bialgebra with this cobracket and the Goldman bracket, which descends from $|\mathbb{C}\pi|$. There is an enhancement [AKKN18b] of the cobracket, which promotes it to $|\mathbb{C}\pi|$, thereby making $|\mathbb{C}\pi|$ a Lie bialgebra under the Goldman bracket and the enhanced cobracket. In [AKKN18b] this enhancement is necessary in order to establish the relationship between the Goldman-Turaev Lie bialgebra and Kashiwara–Vergne theory. To define the enhanced cobracket, a curve in $|\mathbb{C}\pi|$ is lifted to an immersed curve with a fixed rotation number. Below we review the definitions of the Goldman bracket and the enhanced version of the Turaev cobracket.

The Goldman Bracket sums over smoothing intersections between two free loops. For a free loop α in $|\mathbb{C}\pi|$ and a point q on α, denote by α_q the loop α based at q.

def:bracket **Definition 3.2** (The Goldman bracket)**.** Let $\alpha,\beta\in|\mathbb{C}\pi|$ be free loops with homotopy representatives chosen so that there are only finitely many transverse double intersections between α and β. The Goldman bracket $[\cdot,\cdot]_G:|\mathbb{C}\pi|\otimes|\mathbb{C}\pi|\to|\mathbb{C}\pi|$ is given by

$$[\alpha,\beta]_G\coloneqq-\sum_{q\in\alpha\cap\beta}\varepsilon_q|\alpha_q\beta_q|,$$

[4]Not to be confused with the abelianisation of A. In particular, $|A|$ does not inherit an algebra structure from A.

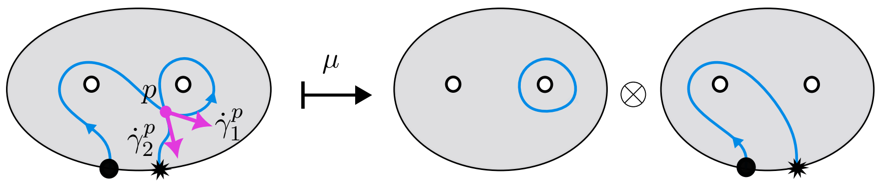


fig:defmu

FIGURE 5. Example of the self intersection map μ where $\epsilon_p = -1$.

where $\varepsilon_q = \varepsilon(\dot{\alpha}_q, \dot{\beta}_q) \in \{\pm 1\}$ is the local intersection number of α and β at q, $\alpha_q\beta_q$ is the concatenation of α_q and β_q, and the extension to $|\mathbb{C}\pi|$ is linear. Then one easily checks that $[\cdot,\cdot]_G$ is a Lie bracket on $|\mathbb{C}\pi|$.

The original definition of the Turaev cobracket is similar, but uses self intersections of a curve in place of the intersections between two curves. Unfortunately, it is not well-defined with respect to the Reidemeister 1 relation for free homotopy curves, hence the need for the enhancement. We construct the (enhanced) cobracket via a self-intersection map for *based* curves, as in [AKKN18b, Section 5.2]; this definition lends itself well to direct comparison with the three-dimensional operations of Section 5. For a based curve γ in $\mathbb{C}\pi$, the idea is to "snip off" portions of γ at self intersection points to get two curves, one of which is based and the other free. Figure 5 shows an example.

The sign here (with the minus sign in front) matches with AKKN genus 0, but is the opposite of AKKN higher genus and Goldman's original definition. Our current multiplication and bracket matches the sign here, so if we change the sign then we should change the stacking order of our multiplication.

def:mu

Definition 3.3 (The self-intersection map)**.** For $\gamma \in \mathbb{C}\pi$, let $\tilde{\gamma} \in \mathbb{C}\tilde{\pi}$ denote a path such that $\tilde{\gamma}\nu$ is homotopic to γ; and such that $\tilde{\gamma}$ has only transverse double points, and $\mathrm{rot}(\tilde{\gamma}) = 1/2$ (hence, $\mathrm{rot}(\tilde{\gamma}\nu) = 0$). Let $\tilde{\gamma} \cap \tilde{\gamma}$ denote the set of double points. The self intersection map μ is defined as folows:

$$\mu : \mathbb{C}\pi \to |\mathbb{C}\pi| \otimes \mathbb{C}\pi$$

$$\mu(\gamma) = -\sum_{p \in \tilde{\gamma} \cap \tilde{\gamma}} \varepsilon_p |\tilde{\gamma}_{t_1^p t_2^p}| \otimes \tilde{\gamma}_{0 t_1^p} \tilde{\gamma}_{t_2^p 1},$$

where t_1^p and t_2^p are the first and second time parameter in $[0,1]$ where $\tilde{\gamma}$ goes through p; where $\tilde{\gamma}_{rs}$ denotes the path traced by $\tilde{\gamma}$ from $t = r$ to $t = s$; the sign $\varepsilon_p = \varepsilon\left(\dot{\tilde{\gamma}}(t_1^p), \dot{\tilde{\gamma}}(t_2^p)\right) \in \{\pm 1\}$ is the local self-intersection number; and the formula extends to $\mathbb{C}\pi$ linearly.

The Turaev cobracket is obtained from μ by closing off the path component and making the tensor product alternating: this descends to a map on $|\mathbb{C}\pi|$, as follows.

Definition 3.4 (The Turaev co-bracket)**.** The Turaev cobracket δ is the unique linear map which makes the following diagram commute, where $\mathrm{Alt}(x \otimes y) =$

$x \otimes y - y \otimes x = x \wedge y$:

$$\begin{array}{ccccc}
\mathbb{C}\pi & \xrightarrow{\mu} & |\mathbb{C}\pi| \otimes \mathbb{C}\pi & \xrightarrow{1 \otimes |\cdot|} & |\mathbb{C}\pi| \otimes |\mathbb{C}\pi| \\
\downarrow{\scriptstyle |\cdot|} & & & & \downarrow{\scriptstyle \mathrm{Alt}} \\
|\mathbb{C}\pi| & & \overset{\delta}{\dashrightarrow} & & |\mathbb{C}\pi| \wedge |\mathbb{C}\pi|
\end{array}$$

3.3. **Associated graded Goldman-Turaev Lie bialgebra.** There I-adic filtration on $\mathbb{C}\pi$ is the filtration by powers of the augmentation ideal $\mathcal{I} = \langle \{\alpha - 1\}_{\alpha \in \pi} \rangle$:

$$\mathbb{C}\pi = \mathcal{I}^0 \supseteq \mathcal{I} \supseteq \mathcal{I}^2 \supseteq ...$$

By the 1930's work of Magnus [Mag35], the associated graded algebra of $\mathbb{C}\pi$ with respect to this filtration is the degree completed free algebra $\mathrm{FA} = \mathrm{FA}\langle x_1, \cdots, x_p \rangle$:

Proposition 3.5. *Given the set of standard generators $\{\gamma_i\}_{i=1}^p$ for π, there is an isomorphism of algebras* $\mathrm{gr}\, \mathbb{C}\pi \to \mathrm{FA}$ *and the exponential expansion $\varphi(\gamma_i^{\pm 1}) = e^{\pm x_i}$ is a homomorphic expansion.*

The I-adic filtration of $\mathbb{C}\pi$ descends to a filtration on $|\mathbb{C}\pi|$:

$$|\mathbb{C}\pi| = |\mathcal{I}^0| \supseteq |\mathcal{I}| \supseteq |\mathcal{I}^2| \supseteq ...$$

The completed associated graded vector space for $|\mathbb{C}\pi|$ with respect to this filtration is, by definition

$$\mathrm{gr}\, |\mathbb{C}\pi| = \prod_{n=0}^{\infty} |\mathcal{I}^n| / |\mathcal{I}^{n+1}|.$$

There is an isomorphism $\mathrm{gr}\, |\mathbb{C}\pi| \cong |\,\mathrm{FA}\,|$, where $|\,\mathrm{FA}\,|$ denotes the linear quotient $|\,\mathrm{FA}\,| = \mathrm{FA} / [\mathrm{FA}, \mathrm{FA}]$, and the exponential expansion descends to a homomorphic expansion for $|\mathbb{C}\pi|$. The vector space $|\,\mathrm{FA}\,|$ is spanned by cyclic words in letters $x_1, \cdots, x_p$, that is, words modulo cyclic permutations of the letters.

Therefore, $|\,\mathrm{FA}\,|$ carries the structure of a Lie bialgebra under $\mathrm{gr}[\cdot,\cdot]_G$ and $\mathrm{gr}\,\delta$ [AKKN18a, Section 3]. Note that the Goldman bracket and the Turaev co-bracket are not strictly filtered maps, as they both shift filtered degree down by one[5]. For example, if $x \in |\mathcal{I}^r|$ and $y \in |\mathcal{I}^s|$, then $[x, y]_G \in |\mathcal{I}^{r+s-1}|$. Correspondingly, the associated graded operations are maps of degree -1.

Figure 6 shows a schematic calculation of the graded Goldman bracket, with cyclic words represented diagrammatically as letters along a circle. The graded Goldman bracket sums over matching pairs of letters in z and w, joins the circles at the matching letter, and takes the difference of the two ways of including only one copy of the letter in the new cyclic word. Stated algebraically, this is summarised as follows:

[5]In [AKKN18a, Sections 3.3, 3.4] the down-shifts are by up to two filtered degrees, as the generating curves around genera and those around boundary components carry different weights. In our genus zero setting this translates to a degree shift of -1.

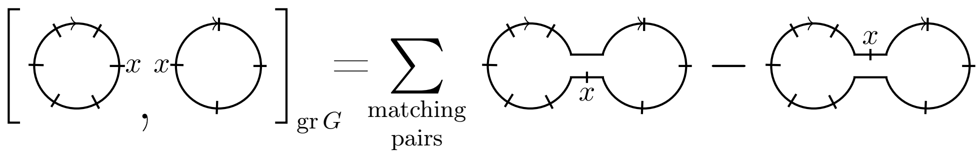


FIGURE 6. A schematic diagrammatic example of the graded Goldman bracket.

fig:grbracket

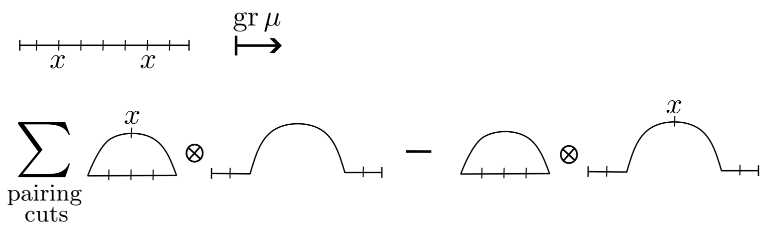


FIGURE 7. A schematic diagrammatic example of the graded Self-intersection map, gr μ.

fig:grmu

Proposition 3.6. *[AKKN18a, Section 3.3] Let $z = |z_1 \cdots z_l|$ and $w = |w_1 \cdots w_m|$ be two cyclic words in $|\,\mathrm{FA}\,|$. The graded Goldman bracket*

$$\mathrm{gr}\,([-,-]_G) = [-,-]_{\mathrm{gr}\,G} : |\,\mathrm{FA}\,| \otimes |\,\mathrm{FA}\,| \to |\,\mathrm{FA}\,|$$

is given by:

$$[z,w]_{\mathrm{gr}\,G} = \sum_{j,k} \delta_{z_j,w_k}(|w_1 \dots w_{k-1} z_{j+1} \dots z_l z_1 \dots z_j w_{k+1} \dots w_m| - \\ |w_1 \dots w_{k-1} z_j \dots z_l z_1 \dots z_{j-1} w_{k+1} \dots w_m|),$$

where δ_{z_j,w_k} is the Kronecker delta.

Figure 7 shows a schematic diagrammatic calculation of the graded self-intersection map μ, as a sum over *pairing cuts.* A pairing cut identifies two matching letters in a word, and splits the word along a chord connecting these matching letters. The graded self-intersection map outputs the tensor product of the resulting cyclic word and the remainder of the associative word. In formulas:

Proposition 3.7. *[AKKN18a, Section 3.4] Let $w = w_1 \dots w_m \in As_p$. The graded self-intersection map*

$$\mathrm{gr}(\mu) = \mu_{\mathrm{gr}} : \mathrm{FA} \to |\,\mathrm{FA}\,| \otimes \mathrm{FA}$$

is given by:

$$\mu_{\mathrm{gr}}(w) = \sum_{j<k} \delta_{w_j,w_k}(|w_j \dots w_{k-1}| \otimes w_1 \dots w_{j-1} w_{k+1} ... w_m - \\ |w_{j+1} \dots w_{k-1}| \otimes w_1 \dots w_j w_{k+1} \dots w_m),$$

where δ_{w_j,w_k} denotes the Kronecker delta.

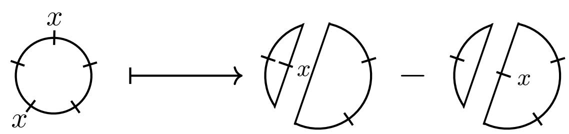


FIGURE 8. An example pairing cut of a cyclic word as a term in the graded Turaev cobracket. Figure is incorrect

fig:paircut

Figure 8 shows a schematic diagrammatic definition of the graded Turaev cobracket, again as a sum over *pairing cuts*. A pairing cut in a cyclic word identifies a pair of coinciding letters, and cuts the cycle into two cycles along the chord connecting the matching letters. To obtain the cobracket, one takes a sum of wedge products of the resulting split cyclic words, adding one copy of the coinciding letter to either side, as shown in Figure 8 and expressed in formulas below:

Proposition 3.8. *[AKKN18a, Section 3.4] Let* $w = w_1 \dots w_m \in |As_p|$. *The graded Turaev cobracket*

$$\mathrm{gr}(\delta) = \delta_{\mathrm{gr}} : |\,\mathrm{FA}\,| \to |\,\mathrm{FA}\,| \wedge |\,\mathrm{FA}\,|$$

is given by

$$\delta_{\mathrm{gr}}(w) = \sum_{j<k} \delta_{w_j,w_k}\big(|w_j \dots w_{k-1}| \wedge |w_{k+1} \dots w_m w_1 \dots w_{j-1}| + |w_k \dots w_m w_1 \dots w_{j-1}| \wedge |w_{j+1} \dots w_{k-1}|\big),$$

where δ_{w_j,w_k} *denotes the Kronecker delta*[6].

sec:TangleSetUp

4. Expansions for tangles in handlebodies

4.1. **Framed oriented tangles.** This section introduces the space $\mathbb{C}\tilde{\mathcal{T}}$ of framed, oriented tangles in a genus p handlebody, with formal linear combinations. Our main result – proven in Section 5 – is that homomorphic expansions on $\mathbb{C}\tilde{\mathcal{T}}$ induce homomorphic expansions on the Goldman-Turaev Lie biagebra.

Let M_p denote the manifold $D_p \times I$ where D_p is a disc in the complex plane with p points removed. While M_p is not a compact manifold, knot theory in M_p is equivalent to knot theory in a genus p handlebody. For the purpose of the Kontsevich integral, we identify D_p with a unit square $[0,1]+[0,i]$ in the complex plane with p points removed, so M_p can be drawn as a cube with p vertical lines removed; we call these lines *poles*, as shown in the middle in Figure 9. We refer to $D_p \times \{0\}$ as the "floor" or "bottom", and $D_p \times \{1\}$ as the "ceiling" or "top". The "back wall" is the face $[i, i+1] \times [0,1]$. We refer to the $i \in \mathbb{C}$ direction as North.

in figure 9 right hand picture, the end points don't quite line up

[6] Apologies for the notation clash.

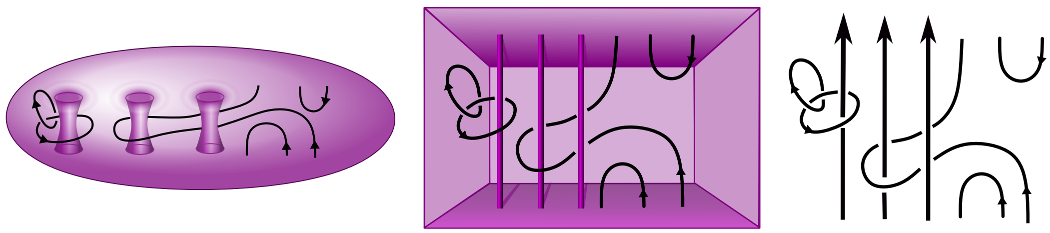


FIGURE 9. An example of a tangle in M_3, drawn first in a handlebody, then in a cube with poles, and lastly as a tangle diagram projected to the back wall of the cube.

fig:polestudio

def:tangle

Definition 4.1. An oriented tangle T in M_p is an embedding of an oriented compact 1-manifold

$$(S, \partial S) \hookrightarrow (M_p, D_p \times \{0\} \cup D_p \times \{1\}).$$

The interior of S lies in the interior of M_p, and the boundary points of S are mapped to the top or bottom. Oriented tangles in M_p are considered up to ambient isotopy fixing the boundary. We denote the set of isotopy classes by $\mathcal{T}$. An example is shown in Figure 9.

Definition 4.2. A *framing* for an oriented tangle T in M_p is a continuous choice of unit normal vector at each point of T, which is fixed pointing North at the boundary points. *Framed oriented tangles* in M_p are also considered up to ambient isotopy fixing the boundary. We denote the set of isotopy classes of framed oriented tangles by $\tilde{\mathcal{T}}$.

Henceforth, any tangle is assumed to be framed and oriented unless otherwise stated. The skeleton of a tangle is the underlying combinatorial information with the topology forgotten:

def:skeleton

Definition 4.3. The *skeleton* $\sigma(T)$ of a tangle $T = (S \hookrightarrow M_p)$ – see Figure 10 – is the set of tangle endpoints $P_{bot} \subseteq D_p \times \{0\}$ and $P_{top} \subseteq D_p \times \{1\}$, along with

(1) A partition of $P_{bot} \cup P_{top}$ into ordered pairs given by the oriented intervals of S.
(2) A non-negative integer k: the number of circles in S.

The skeleton of a framed tangle is the same as the skeleton of the underlying unframed tangle. The set of framed tangles in M_p with skeleton S is denoted $\tilde{\mathcal{T}}(S)$. For example, $\tilde{\mathcal{T}}(\bigcirc)$ is the set of framed knots in M_p.

Maybe it would be better to define $P_{bot}, P_{top} \subseteq D_p$ and then say $P_{bot} \times \{0\}$ and $P_{\top} \times \{1\}$ are the tangle endpoints. Then it would make descriptions of tangle operations easier, as well as the info in figure 9.

The linear extension of $\tilde{\mathcal{T}}(S)$, denoted $\mathbb{C}\tilde{\mathcal{T}}(S)$, is the vector space of $\mathbb{C}$-linear combinations of tangles in $\tilde{\mathcal{T}}(S)$. We denote by $\mathbb{C}\tilde{\mathcal{T}}$ the disjoint union $\sqcup_S \tilde{\mathcal{T}}(S)$ over all skeleta S. Tangles with different skeleta cannot be linearly combined.

One may represent tangles in M_p using tangle diagrams in (at least) two different ways: by projecting to the back wall of M_p or to the floor.

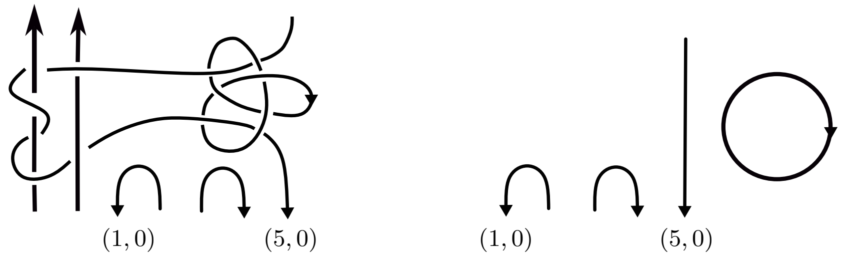


FIGURE 10. On the left is a tangle in M_2, and on the right is schematic diagram of the skeleton of the tangle. The skeleton of the tangle is the combinatorial data given by the following set of order pairs and the integer 1: $\{[((2,0),0),((1,0),0))],[((3,0),0),(4,0),0))],[((5,0),1),(5,0),0))],1\}$
.

fig:skeleton

Projecting to the back wall, an ℓ-component tangle in M_p can be diagrammatically represented as a tangle diagram with p straight vertical *poles*, and ℓ *tangle strands* of circle and interval components. The strands pass over (in front of) and under (behind) the poles and other strands, as shown on the right in Figure 9. The poles are oriented upwards. By Reidemeister's theorem, $\tilde{\mathcal{T}}$ is in bijection with such diagrams modulo the Reidemeister moves R2 and R3, and the framed version of R1.

By projecting instead to the floor $D_p \times \{0\}$, a tangle in M_p is represented by a tangle diagram in D_p. The R2 and R3 moves continue to apply. The endpoints of the tangle are fixed: bottom endpoints are denoted by dots, top endpoints are denoted by stars. Strands of the tangle diagram can pass over bottom endpoints, or under top endpoints, as shown in Figure 11. However, the strands cannot pass across the punctures in D_p.

sec:opsonT

4.2. **Operations on $\tilde{\mathcal{T}}$.** There are several useful operations defined on $\tilde{\mathcal{T}}$. These operations extend linearly to $\mathbb{C}\tilde{\mathcal{T}}$, and are used in Section 5 to relate quotients of $\mathbb{C}\tilde{\mathcal{T}}$ to the Goldman-Turaev Lie bialgebra.

- *Stacking product:* Given tangles $T_1, T_2 \in M_p$, if the top endpoints of $\sigma(T_1)$ coincide with the bottom endpoints of $\sigma(T_2)$ in D_p, and the orientations on the strands of T_1 and T_2 agree, then the product T_1T_2 is the tangle obtained by stacking T_2 on top of T_1.
- *Strand addition:* The *strand addition* operation adds a non-interacting additional strand to a tangle T at a point $q \in D_p$ to get a new tangle $T \sqcup_q \uparrow$. More precisely, pick a contractible $U \subseteq D_p$ such that T is contained entirely in $U \times [0,1]$ and a point $q \in D_p$ outside of U. The tangle $T \sqcup_q \uparrow$ is T together with an upward-oriented vertical strand $q \times I$ at q.
- *Strand orientation switch:* This operation reverses the orientation of a given strand of the tangle.

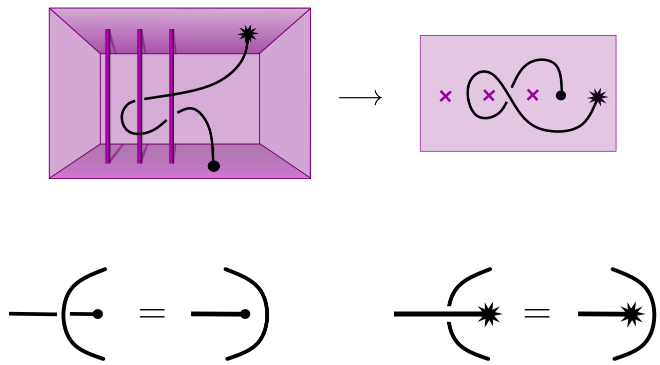


FIGURE 11. An example of a tangle in M_3 projected to the bottom floor of the cube. Strands of a tangle diagram can pass over bottom endpoints (dot) or under top endpoints (star).

fig:BottomDiagram

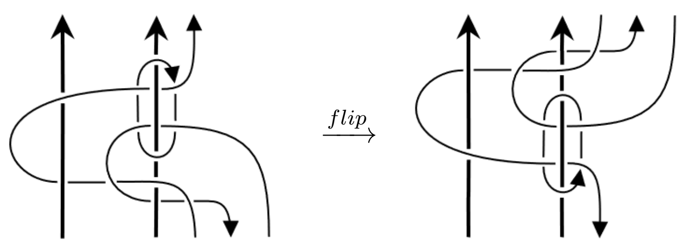


FIGURE 12. A tangle in M_2 and its flip

fig:flip

- *Flip:* Given a tangle T in M_p, the flip of a tangle T in M_p, denoted $\overline{T}$, is the mirror image of T with respect to the ceiling, as shown in Figure 12. When T is flipped, each top boundary point $(q, 1)$ becomes a bottom boundary point $(q, 0)$, and vice versa. The orientations and framing of the strands of T are reflected along with the strands. However, the orientations of the poles remain ascending. Equivalently, the flip operation can be defined as reversing the parametrisation of I in $M_p \cong D_p \times I$. This, in effect, flips the orientation of the poles but changes nothing else.

In Section 5.1, we show that the stacking commutator of tangles, given by $[T_1, T_2] = T_1T_2 - T_2T_1$, induces to the Goldman bracket in the sense of Section!2. In Section 5.2 a similar but more subtle argument relates the flip operation to the Turaev cobracket.

sec:t-filtration

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4.3. **The t-filtration on $\tilde{\mathcal{T}}$ and the associated graded $\tilde{\mathcal{A}}$.** In setting up a theory of Vassiliev invariants for $\tilde{\mathcal{T}}$, there are different filtrations to consider. In line with classical notation of Vassiliev invariants, we denote by a double point

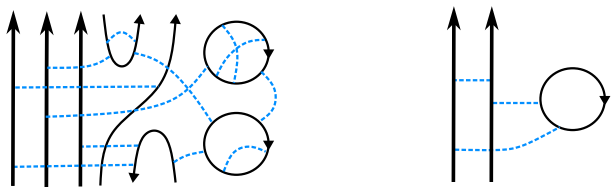


FIGURE 13. Two chord diagrams: an admissible one (left), and a non-admissible one (right) that does contain a pole-pole chord.

ssibleNonAdmissible

the difference between an over-crossing and an under-crossing:

Double points, however, come in two varieties: *pole-strand*, if the crossing occurs between a pole and a tangle strand, and *strand-strand*, if the crossing occurs between two tangle strands. As the poles are fixed, they never cross each other, hence, there are no pole-pole double points.

The main filtration we consider on $\mathbb{C}\tilde{\mathcal{T}}$ is the filtration by the total number of double points of either type, as well as strand framing changes (as in Section 3.1). We call this the *total filtration*, or t-filtration for short, and write it as

$$\mathbb{C}\tilde{\mathcal{T}} = \tilde{\mathcal{T}}_0 \supseteq \tilde{\mathcal{T}}_1 \supseteq \tilde{\mathcal{T}}_2 \supseteq \tilde{\mathcal{T}}_3 \supseteq \cdots$$

where $\tilde{\mathcal{T}}_t$ is the set of linear combinations of framed tangle diagrams with at least t total double points and strand framing changes. ~~In spirit, this filtration comes from the diagrammatic view of projecting to the back wall of the cube.~~

The associated graded space of $\mathbb{C}\tilde{\mathcal{T}}$ with respect to the total filtration is

$$\tilde{\mathcal{A}} := \operatorname{gr}\mathbb{C}\tilde{\mathcal{T}} = \prod_{t\geq 0} \tilde{\mathcal{T}}_t/\tilde{\mathcal{T}}_{t+1}.$$

The degree t component of $\tilde{\mathcal{A}}$ is $\tilde{\mathcal{A}}_t := \tilde{\mathcal{T}}_t/\tilde{\mathcal{T}}_{t+1}$.

As in classical Vassiliev theory (cf. section 3.1), the associated graded space $\tilde{\mathcal{A}}$ has a combinatorial description in terms of *chord diagrams*.

Definition 4.4. A *chord diagram* on a tangle skeleton is an even number of marked points on the poles and skeleton strands, up to orientation preserving diffeomorphism, along with a perfect matching on the marked points – that is, a partition of marked points into unordered pairs. In diagrams, the pairs are connected by a *chord*, indicated by a dashed line, as in Figure 13.

def:admissible

Definition 4.5. A chord diagram is *admissible* if all chords connect strands to strands, or strands to poles, but there are no pole-pole chords. See Figure 13 for examples.

def:cdspace

Definition 4.6. The space $\mathcal{D}(S)$ of *admissible chord diagrams on a diagram* S is the space of $\mathbb{C}$-linear combinations of admissible chord diagrams on the skeleton

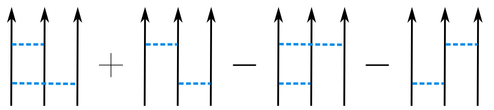


fig:Admissible 4T

FIGURE 14. The 4T relation, which is admissible if at most one of the three skeleton components is a pole.

S, modulo *admissible* $4T$ relations, shown in Figure 14. Admissible $4T$ relations are $4T$ relations where all four terms are admissible[7]. That is,

$$\mathcal{D}(S) = \frac{\mathbb{C}\langle \text{admissible chord diagrams on } S \rangle}{\{\text{admissible } 4T \text{ relations}\}}$$

The space $\mathcal{D}(S)$ is a graded vector space, where the degree is given by the number of chords. Denote the degree t component of $\mathcal{D}(S)$ by $D_t(S)$. Let $\mathcal{D}$ denote the disjoint union $\sqcup_S \mathcal{D}(S)$, and denote the degree t component of $\mathcal{D}$ by $\mathcal{D}_t = \sqcup_S \mathcal{D}_t(S)$.

The well-known map $\psi : \mathcal{D} \to \tilde{\mathcal{A}}$ from classical Vassiliev theory is In degree t, $\psi_t : \mathcal{D}_t \to \tilde{\mathcal{T}}_t/\tilde{\mathcal{T}}_{t+1}$, is defined by "contracting" chords to double points, as shown in Figure 16. This may create other crossings, but modulo $\tilde{\mathcal{T}}_{t+1}$ the over/under information at these crossings does not matter. ?

why is 16 before 15?

Lemma 4.7. *The map ψ is well-defined and surjective.*

Citation needed.

Proof. To show ψ is well-defined, it suffices to show that admissible $4T$ relations in $\mathcal{D}_t$ are in the kernel of ψ. This is the ~~classical~~ "lasso trick" ~~shown~~ in Figure 15. For surjectivity, recall from Section 3.1.2 that a framing change in $\tilde{\mathcal{A}}$ is half of chord. So, both framing changes and double points are in the image of ψ, and thus ψ is surjective. □ standard recalled

According to Lemma 3.1, in order to show that it ψ is an isomorphism, one needs to find an expansion valued in $\mathcal{D}$.

thm:Zwelldefined

Lemma 4.8. *The framed Kontsevich integral $Z : \mathbb{C}\tilde{\mathcal{T}} \to \mathcal{D}$ satisfies the conditions of Lemma 3.1: it is filtered, and $\psi \circ \operatorname{gr} Z = \operatorname{id}_{\tilde{\mathcal{A}}}$.*

ZSUZSI HAS EDITED TO HERE

Proof. The image of Z on an element in $\mathbb{C}\tilde{\mathcal{T}}$ will be a chord diagram on a skeleton with p poles and some number of circles. Since the poles in M_p are parallel, any pair of points (z_i, z_i') on the poles will be constant, the form $dz_i - dz_i' = 0$, and the contribution to the integral will be zero. Therefore chord diagrams in the image of Z don't contain pole-pole chords, so they are always admissible. So Z indeed always lands in $\mathcal{D}$.

It remains to show that $\psi \circ \operatorname{gr} Z = id_{\tilde{\mathcal{A}}}$.

[7]Equivalently, a 4T relation is admissible if at most one of the three skeleton components involved is a pole.

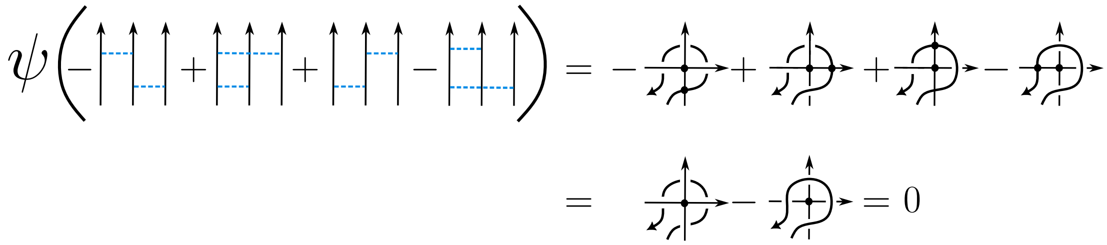


FIGURE 15. Showing that $\psi : \mathcal{D} \to \tilde{\mathcal{A}}$ is well defined. The figure is understood locally: in degree t the chord diagrams have $t-2$ other chords elsewhere, and correspondingly the tangles have $t-2$ other double points elsewhere.

fig:psicomputation

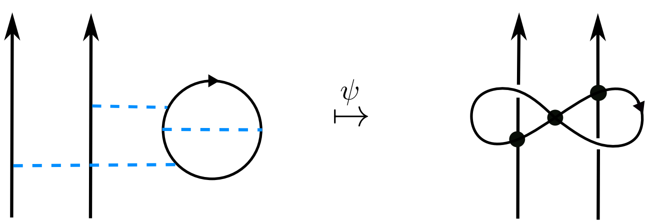
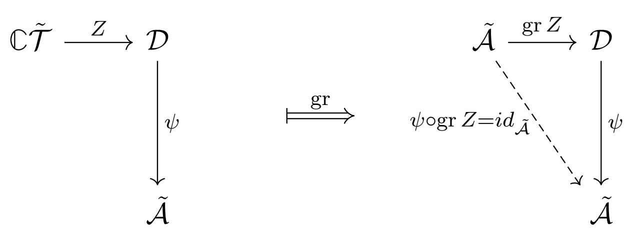


FIGURE 16. Example of ψ with the right hand side viewed as an element of $\tilde{\mathcal{T}}_3/\tilde{\mathcal{T}}_4$. Different choices of over or under crossings with the poles only differ by elements of $\tilde{\mathcal{T}}_4$.

fig:psi



Recall that for a filtered map $f : A \to B$, the associated graded $\mathrm{gr}\, f : \mathrm{gr}\, A \to \mathrm{gr}\, B$ is defined on graded components by $[a] \in A_t/A_{t+1} \mapsto [f(a)] \in B_t/B_{t+1}$. We consider $\mathrm{gr}\, Z : \tilde{\mathcal{A}} \to \mathcal{D}$. Let $[T] \in \tilde{\mathcal{T}}_t/\tilde{\mathcal{T}}_{t+1}$ so that is T is a tangle in M_p with at least t double points. Note that it's always possible to pick such a representative, since a framing change can be written as $\frac{1}{2}$ times a double point in $\tilde{\mathcal{T}}_t/\tilde{\mathcal{T}}_{t+1}$. Then $Z(T)$ is a sum of chord diagrams with $e^{\frac{C}{2}} - e^{-\frac{C}{2}}$ at each chord C corresponding to each double point in T. All terms with degree less than t are zero, so the value of $\mathrm{gr}\, Z(T)$ depends only on the degree t term of $Z(T)$. The degree t term is a single chord diagram with a single chord for each double point, so applying ψ to

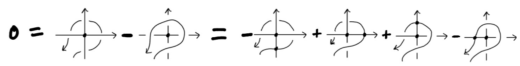


FIGURE 17. DUMMY IMAGE!!! placeholder for picture of chord diagram stacking and flip

chorddiagoperations

this turns all the chords back to double points, which up to crossing changes in $\tilde{\mathcal{T}}_{t+1}$, is just $[T]$. Therefore $\psi\,\mathrm{gr}\,Z = \mathrm{id}_{\tilde{\mathcal{A}}}$. Since $\psi\,\mathrm{gr}\,Z = id_{\tilde{\mathcal{A}}}$. ☐

The next corollary is immediate from lemma 3.1.

Corollary 4.9. *The map $\psi : \mathcal{D} \to \tilde{\mathcal{A}}$ is an isomorphism and Z is an expansion for $\tilde{\mathcal{T}}$.*

Now it is established that $\tilde{\mathcal{A}}$ can be identified with the space of admissible chord diagrams $\mathcal{D}$. For a skeleton S, define $\tilde{\mathcal{A}}(S)$ to be the space of admissible chord diagrams on the skeleton S, so that $\tilde{\mathcal{A}}(S)$ is the associated graded of $\mathbb{C}\tilde{\mathcal{T}}(S)$. For example, $\tilde{\mathcal{A}}(\bigcirc)$ is the associated graded of $\mathbb{C}\tilde{\mathcal{T}}(\bigcirc)$, the space of knots in M_p.

4.4. **Operations on $\tilde{\mathcal{A}}$.** The operations *stacking* and *flip* on $\mathcal{T}$ induce operations by the same names on $\tilde{\mathcal{A}}$. In view of Theorem **??**, we give descriptions of these operations using chord diagrams.

The operation *stacking* is given by stacking D_1 on top of D_2 by concatenating the the top ends of the poles in D_2 to the bottom ends of the poles in D_2 to get D_1D_2, see Figure 17. It is clear from the definition of ψ that this is the correct chord diagram description of stacking, and as in $\mathcal{T}$, is only defined when the endpoints of D_1 and D_2 match appropriately.

The operation *flip* reflects a chord diagram with respect to a "mirror on the ceiling", reverses the orientations of the poles so that they are the same as they were originally, and adds a factor of $(-1)^m$, where m is the total number of marked points on the poles. The factor of $(-1)^m$ comes from the fact that reversing the orientation of one strand at a double point is the same as multiplying by a factor of -1. See Figure 17.

describe the associated graded operations of all the tangle operations

prop:Zhomom

Proposition 4.10. *The Kontsevich integral Z is homomorphic with respect to stacking, strand additions and flips.*

Proof. It is clear for stacking and strand addition. When the orientation of the poles are reversed, every chord diagram D_P in the output of the Kontsevich integral will be multiplied get $(-1)^m$, where m is the total number of chord endings on poles, because m points in P will change whether they are on a descending arc or not, so $P_\downarrow$ will change by m mod 2. ☐

sec:s-sfiltration

4.5. **The s-filtration on $\tilde{\mathcal{T}}$ and $\tilde{\mathcal{A}}$.** As described in Section 4.3, the space $\mathbb{C}\tilde{\mathcal{T}}$ (and therefore $\tilde{\mathcal{A}}$) has a total filtration given by strand framing changes and double points of either type, strand-pole and strand-strand. In this section we look at a second filtration on $\mathbb{C}\tilde{\mathcal{T}}$ and $\tilde{\mathcal{A}}$, where we still look at strand framing changes, but only consider the number of strand-strand double points. This filtration will be called the <u>s</u>*trand filtration*, or simply s-filtration. The s-filtration is given by

$$\mathbb{C}\tilde{\mathcal{T}} = \tilde{\mathcal{T}}^0 \supseteq \tilde{\mathcal{T}}^1 \supseteq \tilde{\mathcal{T}}^2 \supseteq \tilde{\mathcal{T}}^3 \supseteq \cdots$$

where $\tilde{\mathcal{T}}^s \subseteq \mathbb{C}\tilde{\mathcal{T}}$ are linear combinations of link diagrams with at least s strand framing changes and strand double points.

Remark 4.11. We do *not* consider the full associated graded of $\mathbb{C}\tilde{\mathcal{T}}$ with respect to the s-filtration, but instead use it to identify the Goldman-Turaev spaces in low degrees in Section 5. The associated graded of $\mathbb{C}\tilde{\mathcal{T}}$ with respect to the s-filtration has been studied by Habiro and Massuyeau in [HM21], where they consider "bottom tangles". Note the language – if we project to the "bottom" instead of the "back wall", then all double points are of type strand-strand, so the s-filtration is just the usual Vassiliev filtration in the bottom projection.

The s-filtration also induces a filtration on $\tilde{\mathcal{A}}$ as follows. Combining the notations for the t- and s-filtrations, let $\tilde{\mathcal{T}}_t^s$ denote the set of linear combinations of tangle diagrams in $\mathbb{C}\tilde{\mathcal{T}}$ that have at least t double points, at least s of which are strand-strand.

ionQuotientNotation

Definition 4.12. The s-filtered component of $\tilde{\mathcal{A}}$ denoted $\tilde{\mathcal{A}}^{\geq s} := \prod \tilde{\mathcal{T}}_t^s / \tilde{\mathcal{T}}_{t+1}^s$ is the set of linear combinations of chord diagrams with at least s strand-strand chords, or rather at least s chords between the non-pole skeleton components.

Note that the number of s chords is not a grading on $\tilde{\mathcal{A}}$ because the 4T relation is not homogeneous with respect to strand-strand chords.

prop:ZrespectsS

Proposition 4.13. *The Kontsevich integral is a filtered map with respect to the s-filtration.*

Proof. This follows immediately from Theorem 4.8: Z is an expansion with respect to the total filtration, and strand-strand double points correspond to strand-strand chords via the identification of the associated graded space as a space of chord diagrams. ☐

maybe this is a (trivial) proposition

sec:notation

4.6. **Notation conventions.** Throughout this paper we consider the t and s filtrations on $\mathbb{C}\tilde{\mathcal{T}}$ and $\tilde{\mathcal{A}}$, as well as on their various quotients and subspaces. We summarize the notation in the section below:

- $\mathbb{C}\tilde{\mathcal{T}}$ is the space of $\mathbb{C}$-linear combinations of framed tangles in M_p
- $\mathbb{C}\tilde{\mathcal{T}}(\bigcirc)$ is the space of C-linear combinations of framed knots in M_p
- $\tilde{\mathcal{T}}_t$ is the t'th filtered component of $\mathbb{C}\tilde{\mathcal{T}}$ with respect to the t-filtration, which contains all linear combinations of framed tangles in M_p with at least t double points(both strand-strand and strand-pole types) and framing changes.

- $\tilde{\mathcal{T}}^s$ is the s'th filtered component of $\mathbb{C}\tilde{\mathcal{T}}$ with respect to the s-filtration, which contains all linear combinations of framed tangles in M_p with at least s strand-strand double points and framing changes.
- $\tilde{\mathcal{T}}_t^s := \tilde{\mathcal{T}}_t \cap \tilde{\mathcal{T}}^s$, which is the set of elements of $\mathbb{C}\tilde{\mathcal{T}}$ with at least s framing changes and strand-strand double points, and at least t framing changes and double points of any type.
- $\tilde{\mathcal{T}}^{/s} := \mathbb{C}\tilde{\mathcal{T}}/\tilde{\mathcal{T}}^s$, is the quotient of $\mathbb{C}\tilde{\mathcal{T}}$ where diagrams with more than s strand-strand double points or framing changes are in the kernel.
- $\tilde{\mathcal{T}}^{1/2} := \tilde{\mathcal{T}}^1/\tilde{\mathcal{T}}^2$, is the quotient of $\mathbb{C}\tilde{\mathcal{T}}$ where diagrams with 0 or greater than 1 strand-strand double point or framing change are in the kernel.
- $\tilde{\mathcal{A}}$ is the associated graded space of $\mathbb{C}\tilde{\mathcal{T}}$ under the t-filtration, and is the space of admissible chord diagrams modulo admissible $4T$ relations.
- $\tilde{\mathcal{A}}_t := \tilde{\mathcal{T}}_t/\tilde{\mathcal{T}}_{t+1}$ is the degree t component of $\tilde{\mathcal{A}}$ which consists of all admissible chord diagrams in $\tilde{\mathcal{A}}$ with exactly t chords of any type.
- $\tilde{\mathcal{A}}^{\geq s} := \prod_t \tilde{\mathcal{T}}_t^s/\tilde{\mathcal{T}}_{t+1}^s$ is the s'th filtered component of $\tilde{\mathcal{A}}$
- $\tilde{\mathcal{A}}^{/s} := \tilde{\mathcal{A}}/\tilde{\mathcal{A}}^{\geq s}$

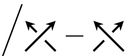
Theses notations are extended to subspaces and quotients of $\mathbb{C}\tilde{\mathcal{T}}$ and $\tilde{\mathcal{A}}$ in the natural way.

sec:Conway

4.7. **The Conway quotient.** In this section we introduce the Conway quotient of $\mathbb{C}\tilde{\mathcal{T}}$: essentially, a Conway skein module of tangles in M_p without fixing the value of the unknot. The Conway relation respects the t and s filtrations and the Kontsevich integral descends to the Conway quotient.

check what happens with framed R1 when we mod out by the first step of the s-filtration...

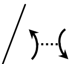
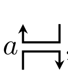
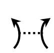
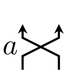
Definition 4.14. The Conway quotient of $\mathbb{C}\tilde{\mathcal{T}}$ is defined as

$$\mathbb{C}\tilde{\mathcal{T}}_\triangledown := \mathbb{C}\tilde{\mathcal{T}}[\![a]\!] \Big/$$  $$= (e^{\frac{a}{2}} - e^{-\frac{a}{2}})$$  ,

where a is a formal variable with t and s degree 1. The skein relation is applied only to strand-strand crossings, not strand-pole crossings. We will use the variable b as a shorthand for $b = e^{\frac{a}{2}} - e^{-\frac{a}{2}}$.

The t and s filtrations on $\mathbb{C}\tilde{\mathcal{T}}$ induce filtrations on $\mathbb{C}\tilde{\mathcal{T}}_\triangledown$. Following the notation conventions in Section 4.6, let $\tilde{\mathcal{T}}_{\triangledown,t}$ denote the t'th filtered component of $\mathbb{C}\tilde{\mathcal{T}}_\triangledown$ and $\tilde{\mathcal{A}}_\triangledown := \operatorname{gr}_t \mathbb{C}\tilde{\mathcal{T}}_\triangledown = \prod \tilde{\mathcal{T}}_{\triangledown,t}/\tilde{\mathcal{T}}_{\triangledown,t+1}$ denote the associated graded algebra of $\mathbb{C}\tilde{\mathcal{T}}_\triangledown$ with respect the total filtration. We now show that $\tilde{\mathcal{A}}_\triangledown$ has a diagrammatic description similar to $\tilde{\mathcal{A}}$, where $\tilde{\mathcal{A}} \cong \mathcal{D}$ as in Theorem **??**.

Definition 4.15. Let

$$\mathcal{D}_\triangledown := \mathcal{D}[\![a]\!] \Big/$$  $= a$  ,  $= a$ 

where a is a formal variable of degree 1 as above, and the relations locally apply only when all skeleton components involved are strands, not poles.

Note that the quotient relations in $\mathcal{D}_\nabla$ preserve the t-grading on $\mathcal{D}$ and the grading descends to $\mathcal{D}_\nabla$. The next theorem shows that $\tilde{\mathcal{A}}_\nabla \cong \mathcal{D}_\nabla$. This theorem essentially follows from the results of [LM95], and we present a brief direct proof.

thm:Z_conway

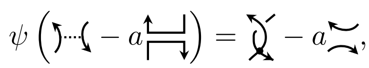
Theorem 4.16. *The Kontsevich integral descends to an expansion* $Z_\nabla : \mathbb{C}\tilde{\mathcal{T}}_\nabla \to \mathcal{D}_\nabla$ *and* $\tilde{\mathcal{A}}_\nabla \cong \mathcal{D}_\nabla$.

This proof uses R1, so I don't know how a framed analogue works exactly, and also not sure that we need it. I commented it out for now.

I believe this theorem is correct with framing changes. Please double check.

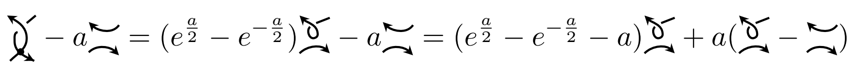
Proof. This proof follows the general schema introduced in Section **??**, in particular Lemma 3.1 and the map ψ, which assigns singular tangles to chord diagrams.

First we show that ψ descends to a graded surjection $\psi : \mathcal{D}_\nabla \to \tilde{\mathcal{A}}_\nabla$. To show that ψ is well-defined, we need to show that the Conway relation in $\mathcal{D}_\nabla$ is in the kernel. Locally,



for Z: We can use (upsmoothing with two arrows up) minus a times double point and RI disapears

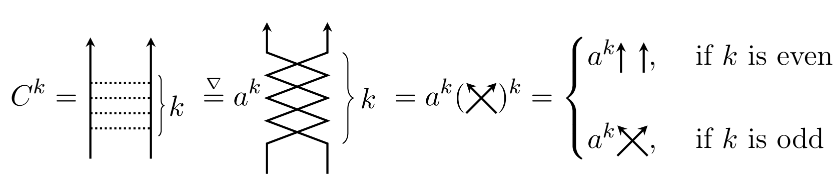
and denote the (global) total degree on both sides by t. In other words, the (global) right hand side is interpreted as an element of $\tilde{\mathcal{T}}_{\nabla,t}/\tilde{\mathcal{T}}_{\nabla,t+1}$. Using the Conway skein relation in $\tilde{\mathcal{A}}_\nabla$, the right had side can be simplified



Observe that $a(\;\;-\;\;)$ and the lowest degree term of $e^{\frac{a}{2}} - e^{-\frac{a}{2}} - a$ are both of degree 2, hence $(\;\;- a\;\;) \in \tilde{\mathcal{T}}_{\nabla,t+1}$, and therefore is zero in $\tilde{\mathcal{T}}_{\nabla,t}/\tilde{\mathcal{T}}_{\nabla,t+1}$.

for Z: add information about how Z acts wrt size of crossing

We now verify that the Kontsevich integral Z descends to the quotient $\mathbb{C}\tilde{\mathcal{T}}_\nabla$ by checking the relations in $\mathbb{C}\tilde{\mathcal{T}}_\nabla$ directly. Recall that $Z(\nearrow\!\!\!\!\nwarrow) = (e^{\frac{C}{2}})\times$ and $Z(\nwarrow\!\!\!\!\nearrow) = (e^{-\frac{C}{2}})\times$, where C denotes a chord, the exponential is interpreted formally as a power series, and C^k denotes stacking k chords. Using the Conway relation, we compute:



Now applying Z to the left hand side of the Conway relation, we obtain

$$
\begin{aligned}
Z(\nwarrow\!\!\!\!\nearrow_{+}) - Z(\nwarrow\!\!\!\!\nearrow_{-}) &= (e^{\frac{C}{2}} - e^{\frac{-C}{2}})\,\nwarrow\!\!\!\!\nearrow \\
&= \sum_{k=0}^{\infty} \left(\frac{C^k}{2^k k!} - \frac{(-1)^k C^k}{2^k k!} \right) \nwarrow\!\!\!\!\nearrow \\
&= \sum_{k=0}^{\infty} \frac{C^{2k+1}}{2^{2k}(2k+1)!} \nwarrow\!\!\!\!\nearrow \\
&= \sum_{k=0}^{\infty} \frac{a^{2k+1} \nwarrow\!\!\!\!\nearrow}{2^{2k}(2k+1)!} \nwarrow\!\!\!\!\nearrow \\
&= \sum_{k=0}^{\infty} \frac{a^{2k+1}}{2^{2k}(2k+1)!} \uparrow\,\uparrow \\
&= (e^{\frac{a}{2}} - e^{-\frac{a}{2}}) \uparrow\,\uparrow \\
&= Z\left((e^{\frac{a}{2}} - e^{-\frac{a}{2}}) \,\rangle\!\uparrow\; \uparrow\!\langle \right).
\end{aligned}
$$

Thus, Z descends to the Conway quotient $\mathbb{C}\tilde{\mathcal{T}}_\triangledown$.

Therefore, by Lemma 3.1, Z is a homomorphic expansion for $\mathbb{C}\tilde{\mathcal{T}}_\triangledown$ and $\psi : \mathcal{D}_\triangledown \to \tilde{\mathcal{A}}_\triangledown$ is an isomorphism. $\square$

While our main focus is the t-filtration on $\mathbb{C}\tilde{\mathcal{T}}_\triangledown$ and its associated graded space $\tilde{\mathcal{A}}_\triangledown$, the low degree components of the associated graded of $\mathbb{C}\tilde{\mathcal{T}}_\triangledown$ with respect to the s-filtration arise when identifying the Goldman-Turaev Lie bialgebra, as will be detailed in the coming Section 5. One space that arises is $\tilde{\mathcal{T}}_\triangledown^{/1}$, the quotient of $\tilde{\mathcal{T}}_\triangledown$ by the s-degree 1 component $\tilde{\mathcal{T}}_\triangledown^{1}$ (recall the notation conventions from Section 4.6). On this quotient, the Conway relation has no effect and $\tilde{\mathcal{T}}_\triangledown^{/1}$ is actually isomorphic to $\tilde{\mathcal{T}}^{/1}$.

prop:nonabneeded

Proposition 4.17. $\tilde{\mathcal{T}}_\triangledown^{/1} \cong \tilde{\mathcal{T}}^{/1}$

Proof. The quotients $\tilde{\mathcal{T}}^{/1}$ and $\tilde{\mathcal{T}}_\triangledown^{/1}$ are both spanned by the classes of tangles, and $\tilde{\mathcal{T}}_\triangledown^{/1}$ is further quotienting $\tilde{\mathcal{T}}^{/1}$ by the Conway relation. Such a tangle T in $\tilde{\mathcal{T}}^{/1}$ is only defined up to strand-strand crossing changes; the difference between two tangles with a crossing change yields a single tangle with a double point in $\tilde{\mathcal{T}}^{1}$. So if two tangles in $\tilde{\mathcal{T}}^{/1}$ differ by an application of the Conway relation, they must also differ by a crossing change, and hence their difference is already in $\tilde{\mathcal{T}}^{1}$. Thus, further quotienting by the Conway relation has no effect on $\tilde{\mathcal{T}}^{/1}$, and so $\tilde{\mathcal{T}}_\triangledown^{/1} \cong \tilde{\mathcal{T}}^{/1}$. $\square$

For Z: Please review this proof for formality and correctness.

The space $\tilde{\mathcal{T}}^{/1}$ is also isomorphic to $\tilde{\mathcal{T}}_\triangledown^{1}/\tilde{\mathcal{T}}_\triangledown^{2}$ through the inverse isomorphisms "multiplication by b" and "division by b" maps, denoted m_b and q_b. We now show this explicitly in the next proposition.

prop:divbybexists

Proposition 4.18. *The multiplication by b map $m_b : \tilde{\mathcal{T}}^{/1} \to \tilde{\mathcal{T}}_\triangledown^{/2}$ is injective, and its image is $\tilde{\mathcal{T}}_\triangledown^{1/2}$.*

Proof. We first prove that the image of m_b is $\tilde{\mathcal{T}}_\triangledown^{1/2}$. The quotient $\tilde{\mathcal{T}}^{/1}$ is spanned by the classes of tangles T. For a tangle T, the image $m_b(T) = bT$ is in $\tilde{\mathcal{T}}^1$, and represents an element in $\tilde{\mathcal{T}}^{1/2}$. Thus, the image of m_b is contained in $\tilde{\mathcal{T}}^{1/2}$.

Conversely, any element y of $\tilde{\mathcal{T}}^{1/2}$ is (non-uniquely) represented as a sum of the form $\sum_{i=1}^k T_i + b\sum_{j=1}^l T_j$, where T_i are tangles with one double point each, and T_j are arbitrary tangles. Then, by the Conway relation, each $T_i = b \cdot T_i^C$, where T_i^C denotes the tangle where the double point in T_i has been smoothed. Thus, $y = b\left(\sum_{i=1}^k T_i^C + \sum_{j=1}^l T_j\right)$, and therefore y is in the image of m_b.

To prove injectivity of m_b, it is enough to provide an inverse, division by b map, on $\tilde{\mathcal{T}}_\triangledown^{1/2}$, but in fact there is a one sided inverse defined on all of $\tilde{\mathcal{T}}_\triangledown^{/2}$ which is defined as follows. For a tangle T and a crossing x of T, let $\epsilon(x) \in \{\pm 1\}$ be the sign of x, and $T|_{x\to)(}$ be the tangle T with x replaced by a smoothing. There is a well defined "division by b" map $q_b : \tilde{\mathcal{T}}_\triangledown^{/2} \to \tilde{\mathcal{T}}^{/1}$ given by the linear extension of the following:

$$bT \overset{q_b}{\mapsto} T$$
$$T \overset{q_b}{\mapsto} \frac{1}{2} \sum_{x \text{ crossing of } T} \epsilon(x) T|_{x\to)(}$$

For well-definedness, it is straightforward to check that q_b preserves the Reidemeister moves. We also need to check that $\tilde{\mathcal{T}}_\triangledown^2$ and the Conway relation are in the kernel. For $b^kT \in \tilde{\mathcal{T}}_\triangledown^2$, if $k = 1$, then $T \in \tilde{\mathcal{T}}^1$, so $q_b(bT) = 0$. If instead $k = 0$, then T has at least two double points. Replacing a crossing by a smoothing only changes the crossing that is replaced, so other crossings (and therefore double points) remain unchanged. Therefore $q_b(T)$ can be written as a sum where each term has at least one double point, so $q_b(T) = 0$ as well.

To show that the Conway relation also vanishes, note that the terms in $q_b(\doublepoint) = q_b(\overcrossing - \undercrossing)$ come from either smoothing a crossing that is a part of the double point, or smoothing a crossing that is not. In the latter, the double points outside the local relation remain unchanged, so those terms are in $\tilde{\mathcal{T}}_\triangledown^1$. The only remaining terms are those where the crossings forming the double point are smoothed, so we get

$$q_b\left(\overcrossing - \undercrossing\right) = \frac{1}{2})(- (-1)\frac{1}{2})(=)(= q_b\left(b)(\right)$$

showing q_b is well-defined.

Restricting $q_b : \tilde{\mathcal{T}}_\triangledown^{1/2} \to \tilde{\mathcal{T}}_\triangledown^{/1}$ is clearly surjective. To show it's injective, note that the restriction is simply given by $bT \mapsto T$, and if $T \in \tilde{\mathcal{T}}_\triangledown^1$, then $bT \in \tilde{\mathcal{T}}_\triangledown^2$. □

cor:divbyb **Corollary 4.19.** *The map* $m_b : \tilde{\mathcal{T}}^{/1} \to \tilde{\mathcal{T}}_\triangledown^{1/2}$ *is an isomorphism with inverse* $q_b : \tilde{\mathcal{T}}_\triangledown^{1/2} \to \tilde{\mathcal{T}}^{/1}$.

Notice that both m_b and q_b are filtered degree shifting maps. These maps are relevant to realize the degree shifting properties of the Goldman bracket and Turaev cobracket in the Conway quotient.

rem:grdivbyb *Remark* 4.20. The associated graded of q_b is an isomorphism $\operatorname{gr} q_b : \tilde{\mathcal{A}}^{1/2} \to \tilde{\mathcal{A}}^{/1}$ given by drawing the chord diagram as with one s-s chord, smoothing that chord using ∇, and getting a factor of b with no remaining s-s chords, and then diving off the b. For Z: Please make this remark formal.

One difficulty that arises in the Conway quotient is that the skeleton of a tangle is not well defined. For example, two disjoint, unlinked circles can be joined together into a figure 8 through an application of the Conway relation. In general, the Conway relation changes the skeleton of a diagram, so we must take some care to define what is meant by notations such as $\tilde{\mathcal{T}}_\triangledown(\bigcirc)$ (recall that without ∇ this notation means take all tangle diagrams with a fixed skeleton $\bigcirc$).

Definition 4.21. For skeleton S, $\tilde{\mathcal{T}}_\triangledown(S)$ is the linear span of the set of diagrams for which there is a representative tangle on S.

For Z: Please review this definition for formality and correctness.

Definition 4.22. For skeleton S, $\tilde{\mathcal{A}}_\triangledown^{/r}(S)$ represents the associated graded space of $\tilde{\mathcal{T}}_\triangledown^{/r}(S)$, and a nontrivial chord diagram in $\tilde{\mathcal{A}}_\triangledown^{/r}(S)$ has a representative with a chord diagram on skeleton S with at most r strand-strand chords.

Are we just saying take the Conway quotient of $\tilde{\mathcal{T}}(S)$, as in $(\tilde{\mathcal{T}}(S))_\triangledown$?

The isomorphism from Proposition 4.17 descends to isomorphisms on $\tilde{\mathcal{T}}^{/1}(\bigcirc)$ and $\tilde{\mathcal{A}}^{/1}(\bigcirc)$. For Z: Please review this definition for formality and correctness.

cor:/1conway **Corollary 4.23.** $\tilde{\mathcal{T}}^{/1}(\bigcirc) \cong \tilde{\mathcal{T}}_\triangledown^{/1}(\bigcirc)$.

cor:gr/1conway **Corollary 4.24.** $\tilde{\mathcal{A}}^{/1}(\bigcirc) \cong \tilde{\mathcal{A}}_\triangledown^{/1}(\bigcirc)$.

However, Corollaries 4.23 and 4.24 are not true for higher degree quotients because of the skeleton changing issue induced by the Conway relation.

5. Identifying the Goldman-Turaev Lie bialgebra

:IdentifyingGTinCON

In this section we establish our main results: we identify the Goldman-Turaev Lie bialgebra in the low s-filtered degree quotients of $\mathbb{C}\tilde{\mathcal{T}}$, and show that the Kontsevich integral induces a homomorphic expansion on this space with respect to the s-filtration. Appealing to the principle summarized in Section 2 we present diagrams like (**??**), where the induced map η is the Goldman bracket and the self intersection map μ, repsectively. We deduce the homomorphicity of the expansion from the naturality of the construction as in (2.4).

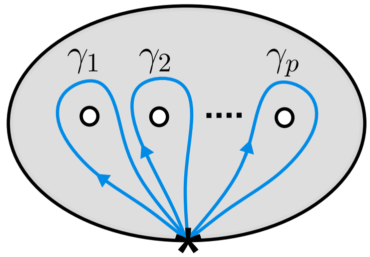


fig:GenCurves

FIGURE 18. The standard generating curves of π.

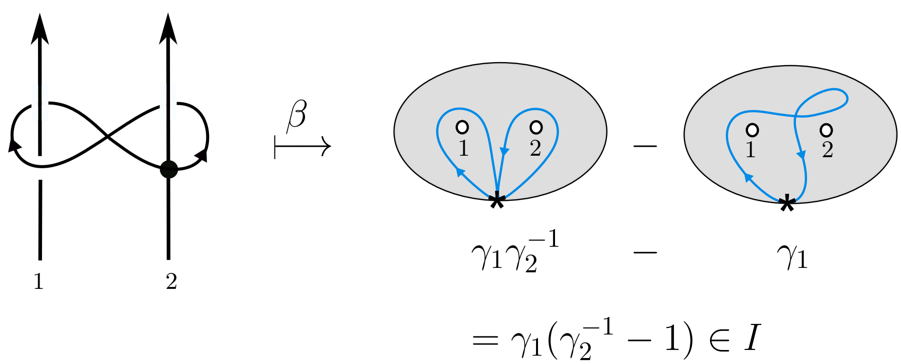


fig:BetaFiltered

FIGURE 19. Example calculation demonstrating that β is a filtered map.

dentifybracketinCON

5.1. **The Goldman Bracket.** Recall from Section 3.2 that D_p denotes the p-punctured disc, π is its fundamental group, and $|\mathbb{C}\pi|$ is the linear quotient $|\mathbb{C}\pi| := \mathbb{C}\pi/[\mathbb{C}\pi, \mathbb{C}\pi]$, which is linearly generated by homotopy classes of free loops in D_p. The Goldman bracket (Definition 3.2) is a lie bracket $[\cdot,\cdot]_G : |\mathbb{C}\pi| \otimes |\mathbb{C}\pi| \to |\mathbb{C}\pi|$. Recall from Section 4.7 the space $\mathbb{C}\tilde{\mathcal{T}}(\bigcirc)$ is the vector space of $\mathbb{C}$-linear combinations of framed knots in $M_p = D_p \times I$.

prop:BotProj

Proposition 5.1. *The bottom projection $M_p \to D_p \times \{0\}$ induces a surjective map $\mathbb{C}\tilde{\mathcal{T}}(\bigcirc) \to |\mathbb{C}\tilde{\pi}|$. Post-composing this with the projection $|\mathbb{C}\tilde{\pi}| \to |\mathbb{C}\pi|$ results in a surjective filtered map*

$$\beta : \mathbb{C}\tilde{\mathcal{T}}(\bigcirc) \to |\mathbb{C}\pi|.$$

Proof. By Reidemeister's Theorem, framed knots in $\mathbb{C}\tilde{\mathcal{T}}(\bigcirc)$ are faithfully represented by knot diagrams in $D_p \times \{0\}$ – regular projections to the bottom with over/under information – modulo the Reidemeister moves (R2, R3). The bottom projection sends the Reidemeister moves for knots to the corresponding moves generating regular homotopies of immersed free loops, hence β is well-defined.

The projection is clearly surjective as any loop can be lifted to a knot by introducing arbitrary under/over information at the crossings.

The statement that β is filtered means that step k of the the Vassiliev t-filtration in $\mathbb{C}\tilde{\mathcal{T}}(\bigcirc)$ projects to step k of the filtration on $|\mathbb{C}\pi|$ induced by the I-adic filtration of π. Note that strand-strand double points and framing changes map to 0 under β, thus, we only have something to prove for knots with k strand-pole double points.

Let $\gamma_1, ..., \gamma_p$ denote the standard generators of π as in Figure 18. A knot $K \in \mathbb{C}\tilde{\mathcal{T}}(\bigcirc)$ maps to a free loop in $|\mathbb{C}\pi|$, whose conjugacy class in π is represented as a word in the generators γ_i. A pole-stand double point on pole j maps to a difference between two curves passing on either side of the j'th puncture (see Figure 19 for an example). Therefore, the words in π representing these curves differ in a single instance of $\gamma_j^{\pm 1}$. Thus, a knot with k pole-strand double points maps to a product with k factors of the form $\pm(\gamma_j^{\pm 1} - 1)$. This is by definition an element in $\mathcal{I}^k$. □

prop:kerbeta

Proposition 5.2. *The kernel of β is $\tilde{\mathcal{T}}^1(\bigcirc)$, and β descends to a filtered (with respect to the t-filtration) linear isomorphism $\beta : \tilde{\mathcal{T}}^{/1}(\bigcirc) \to |\mathbb{C}\pi|$.*

Proof. Two framed knots in $\mathbb{C}\tilde{\mathcal{T}}(\bigcirc)$ project to the same loop in $|\mathbb{C}\pi|$ if and only if they differ by framing changes and (strand-strand) crossing changes, which generate precisely the step 1 of the s-filtration, that is, $\tilde{\mathcal{T}}^1(\bigcirc)$. □

Recall from Corollary 4.23 that $\tilde{\mathcal{T}}^{/1}(\bigcirc) = \tilde{\mathcal{T}}_{\triangledown}^{/1}(\bigcirc)$. Hence, we get the following:

cor:loopsasknots

Corollary 5.3. *The map β descends to an isomorphism $\beta : \tilde{\mathcal{T}}_{\triangledown}^{/1}(\bigcirc) \to |\mathbb{C}\pi|$.*

Recall that $\tilde{\mathcal{A}}$ is the associated graded space of $\mathbb{C}\tilde{\mathcal{T}}$ with respect to the t-filtration, and $\mathbb{C}\tilde{\mathcal{T}}$ is also filtered by the s-filtration. Explicitly, $\tilde{\mathcal{A}}(\bigcirc)$ is the space of admissible chord diagrams on a circle skeleton as in Definition 4.6, $\tilde{\mathcal{A}}^{\geq i}(\bigcirc)$ is the s-degree i filtered component of $\tilde{\mathcal{A}}(\bigcirc)$, and $\tilde{\mathcal{A}}^{/i}(\bigcirc) = \tilde{\mathcal{A}}(\bigcirc)/\tilde{\mathcal{A}}^{\geq i}(\bigcirc)$. Recall from Section 3.2 that the associated graded vector space of $|\mathbb{C}\pi|$ is $|\,\mathrm{FA}\,|$, where $\mathrm{FA} = \mathrm{FA}\langle x_1, \cdots, x_p\rangle$ denotes the free associative algebra over $\mathbb{C}$, and the linear quotient $|\,\mathrm{FA}\,| = \mathrm{FA}/[\mathrm{FA}, \mathrm{FA}]$ is the $\mathbb{C}$-vector space generated by cyclic words in the letters $x_1, ..., x_p$.

Proposition 5.4. *The associated graded map $\mathrm{gr}\,\beta : \tilde{\mathcal{A}}(\bigcirc) \to |\,\mathrm{FA}\,|$ has kernel $\tilde{\mathcal{A}}^{\geq 1}(\bigcirc)$. Hence, $\mathrm{gr}\,\beta$ descends to an isomorphism $\mathrm{gr}\,\beta : \tilde{\mathcal{A}}^{/1}(\bigcirc) \to |\,\mathrm{FA}\,|$.*

Proof. The statement follows from applying the associated graded functor to the filtered short exact sequence

$$0 \longrightarrow \tilde{\mathcal{T}}^1(\bigcirc) \longrightarrow \tilde{\mathcal{T}}(\bigcirc) \xrightarrow{\beta} |\mathbb{C}\pi| \longrightarrow 0.$$

The filtrations on $\tilde{\mathcal{T}}^1(\bigcirc)$ and $|\mathbb{C}\pi|$ are induced from the filtration on $\tilde{\mathcal{T}}(\bigcirc)$, as in Lemma 2.3, so the associated graded sequence is also exact. □

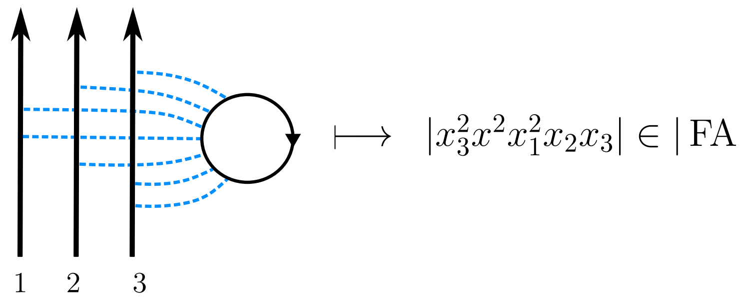


fig:CycWord

FIGURE 20. Chord diagrams with no strand-strand chords can be read as cyclic words.

Remark 5.5. In $\tilde{\mathcal{A}}^{/1}(\bigcirc)$ chord diagrams with any strand-strand chords are zero. Thus, non-zero elements of this space are represented as chord diagrams on poles and a single circle strand, with strand-pole chords only, as in Figure 20. Such a chord diagram corresponds naturally to a cyclic word by labelling the poles with $x_1, ..., x_p$ and reading the word along the circle skeleton, as shown. Indeed, this is the map $\operatorname{gr}\beta$.

We now derive the Goldman bracket from the stacking commutator on $\mathbb{C}\tilde{\mathcal{T}}$:

thm:bracketsnake

Theorem 5.6. *Let $\lambda_1 : \tilde{\mathcal{T}}_{\triangledown}^{/2}(\bigcirc) \otimes \tilde{\mathcal{T}}_{\triangledown}^{/2}(\bigcirc) \to \tilde{\mathcal{T}}_{\triangledown}^{/2}(\bigcirc)$ denote the stacking product. Let λ_2 denote the opposite product, that is, $\lambda_2(K_1, K_2) = K_2K_1$. Then $\lambda = \lambda_1 - \lambda_2$ induces the Goldman bracket on $|\mathbb{C}\pi|$: the diagram in Figure 21 is commutative and the induced homomorphism η agrees with the Goldman Bracket under the identification $\beta : \tilde{\mathcal{T}}^{/1}(\bigcirc) \to |\mathbb{C}\pi|$ as*

$$[-,-]_G = \beta \circ q_b \circ \eta \circ (\beta^{-1} \otimes \beta^{-1}).$$

Proof. For $K_1 \otimes K_2$ in $\tilde{\mathcal{T}}_{\triangledown}^{/2}(\bigcirc) \otimes \tilde{\mathcal{T}}_{\triangledown}^{/2}(\bigcirc)$, $\lambda(K_1 \otimes K_2) = K_1K_2 - K_2K_1$. Project K_1K_2 and K_2K_1 to the bottom to obtain link diagrams. Let a *mixed crossing* of such a diagram be a crossing where one strand belongs to K_1 and the other strand belongs to K_2. Notice that in K_2K_1 all mixed crossings are flipped compared to K_1K_2, while other crossings – those belonging to K_1 or K_2 only – are the same.

Using the double point notation, write positive mixed crossings in K_1K_2 as $\overcrossing = \undercrossing + \doublepoint$ and negative mixed crossings as $\undercrossing = \overcrossing - \doublepoint$, where each double point has one strand belongs to K_1 and the other belongs to K_2. Rewriting all the mixed crossings of K_1K_2 in this way yields a sum of tangles indexed by subsets of the mixed crossings. Denote the set of mixed crossings by M, and for a subset $X \subseteq M$, denote by L_X the singular link obtained by changing the crossings in X to double points, and flipping the other mixed crossings (those in $M \setminus X$). Also,

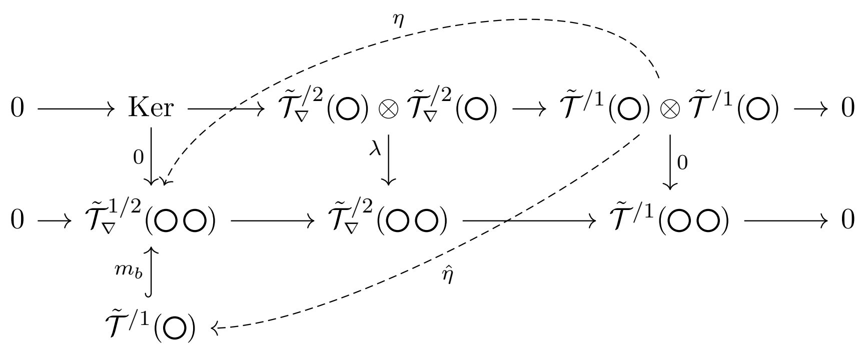


fig:Snakeforbracket

FIGURE 21. The nontrivial horizontal maps are the respective quotient and inclusion maps. The space Ker is the kernel of the projection map on the top right.

let ϵ_X be the product of the signs of all crossings in X. Then

eq:commutator

$$K_1K_2 = \sum_{X\subseteq M} \epsilon_X L_X. \tag{5.1}$$

Notice that $L_\emptyset = K_2K_1$, and if $|X| = i$ then $L_X \in \tilde{\mathcal{T}}^i_\triangledown(\bigcirc\bigcirc)$. Therefore, $\lambda(K_1K_2)$ is in $\tilde{\mathcal{T}}^1_\triangledown(\bigcirc\bigcirc)$, and therefore the right hand square commutes. Furthermore, we have

for N: rewrite as telescoping sum

eq:singletons

$$\lambda(K_1,K_2) = \sum_{X\subseteq M, |X|=1} L_X \in \tilde{\mathcal{T}}^{1/2}_\triangledown(\bigcirc\bigcirc). \tag{5.2}$$

Now for the left square, the kernel K of the projection map from $\tilde{\mathcal{T}}^{/2}_\triangledown(\bigcirc) \otimes \tilde{\mathcal{T}}^{/2}_\triangledown(\bigcirc) \to \tilde{\mathcal{T}}^{/1}_\triangledown(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}_\triangledown(\bigcirc)$ is generated by $\tilde{\mathcal{T}}^{1/2}_\triangledown(\bigcirc) \otimes \tilde{\mathcal{T}}^{/2}_\triangledown(\bigcirc)$ in $\tilde{\mathcal{T}}^{/2}_\triangledown(\bigcirc) \otimes \tilde{\mathcal{T}}^{/2}_\triangledown(\bigcirc)$. and $\tilde{\mathcal{T}}^{/2}_\triangledown(\bigcirc) \otimes \tilde{\mathcal{T}}^{1/2}_\triangledown(\bigcirc)$. Suppose that $K_1 \otimes K_2$ is in $\tilde{\mathcal{T}}^{1/2}_\triangledown(\bigcirc) \otimes \tilde{\mathcal{T}}^{/2}_\triangledown(\bigcirc)$, in other words, there is a double point in K_1. Then, by the same computation as in Equation 5.1, $\lambda(K_1 \otimes K_2)$ is in $\tilde{\mathcal{T}}^2_\triangledown(\bigcirc\bigcirc)$, as every term contains the pre-existing double point in K_1, and at least one additional mixed double point. Therefore, the left hand square commutes.

As in Section 2, then λ induces a unique well defined homomorphism $\eta : \tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc) \to \tilde{\mathcal{T}}^{1/2}_\triangledown(\bigcirc\bigcirc)$. We identify η as the Goldman bracket. We have that the isomorphism β gives $\tilde{\mathcal{T}}^{/1}(\bigcirc) \cong |\mathbb{C}\pi|$ (Proposition 5.2), identifying the domain of η with the domain of the Goldman bracket. We now argue that η has image in $\tilde{\mathcal{T}}^{/1}(\bigcirc) \cong |\mathbb{C}\pi|$.

By Equation (5.2), $\lambda(K_1,K_2)$ is a sum of terms, each with a single mixed double point. Applying the Conway relation to smooth each of these mixed double points changes the skeleton from two circles to one circle, and introduces a factor of b.

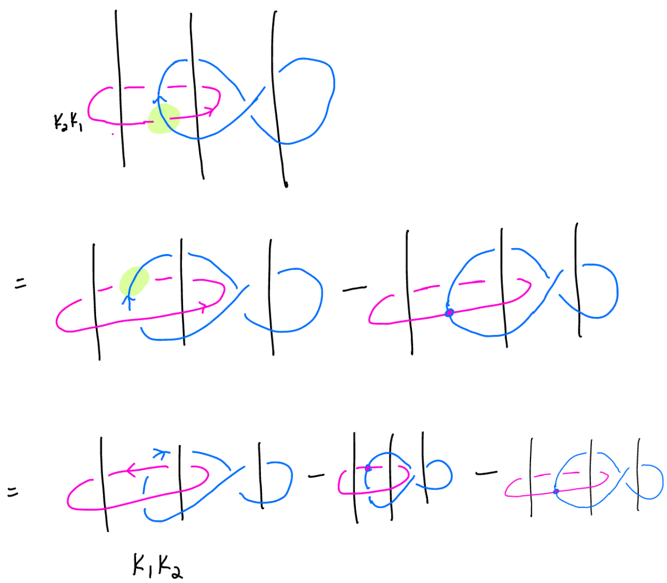


fig:combracket

FIGURE 22. Example commutator bracket computation.

In other words, $\lambda(K_1, K_2) \in b\tilde{\mathcal{T}}^{/2}_{\triangledown}(\bigcirc) \subseteq \tilde{\mathcal{T}}^{1/2}_{\triangledown}(\bigcirc\bigcirc)$. By Corollary 4.19, restricted to a circle skeleton, we know that $b\tilde{\mathcal{T}}^{/2}_{\triangledown}(\bigcirc) \cong \tilde{\mathcal{T}}^{/1}(\bigcirc)$ via the map q_b. (The map $\hat{\eta}$ in the diagram is $q_b \circ \eta$). In turn, $\tilde{\mathcal{T}}^{/1}(\bigcirc) \cong |\mathbb{C}\pi|$ again via the map β.

In summary, the map η is induced from λ in the following way. For curves $\gamma_1 \otimes \gamma_2 \in \tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc)$, let $K_1 \otimes K_2$ be an arbitrary vertical lift of $\gamma_1 \otimes \gamma_2$ to knots in $\tilde{\mathcal{T}}^{/2}_{\triangledown}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/2}_{\triangledown}(\bigcirc)$. Then

$$\eta(\gamma_1 \otimes \gamma_2) = \frac{\lambda(K_1 \otimes K_2)}{b} \in \tilde{\mathcal{T}}^{/1}(\bigcirc),$$

where we use the notation $\frac{1}{b}$ to mean composition with q_b. We need to show that this agrees with the Goldman bracket (Definition 3.2). This is clear from the definition: the Goldman bracket is a sum of smoothings of the mixed crossings of γ_1 and γ_2, exactly as above, and the signs in the sum match the signs of the crossings. See Figure 22 for an example calculation. □

The graded Goldman bracket is a map $[-,-]_{\operatorname{gr} G} : |\,\mathrm{FA}\,| \otimes |\,\mathrm{FA}\,| \to |\,\mathrm{FA}\,|$, as in Definition **??**. By taking the associated graded of the diagram in Figure 21 we arrive at the commutative diagram in Figure 23 and recover the associated graded Goldman bracket:

snakefor_gr_bracket

Corollary 5.7. *The diagram in Figure 23 commutes, the rows are exact,* $\operatorname{gr}\eta$ *is the induced connecting homomorphism. Therefore,* $\operatorname{gr}\hat{\eta}$ *is the associated graded Goldman bracket via the identification* $\mathcal{A}^{/1}(\bigcirc) \cong |\,\mathrm{FA}\,|$.

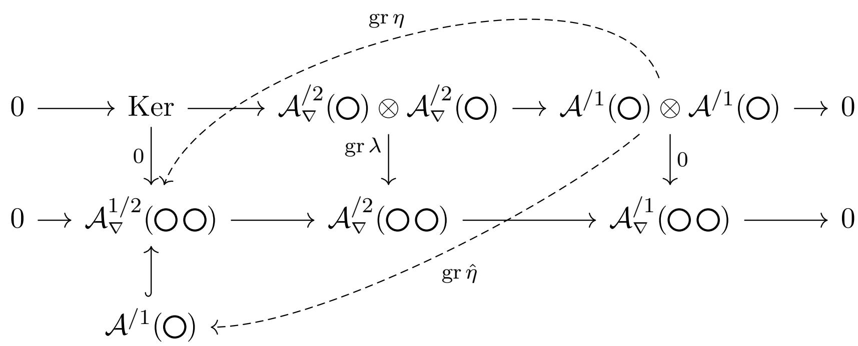


FIGURE 23. The associated graded commutative diagram of Figure 21.

Snakefor_gr_bracket

Proof. Corollary 2.4 states that applying gr to the commutative, exact diagram in Figure 21 gives the commutative and exact diagram in Figure 23, and guarantees that $\operatorname{gr}\eta$ is the unique induced connecting homomorphism. The graded Goldman bracket is the realized by

$$\operatorname{gr}[\cdot,\cdot]_G = \operatorname{gr}\beta \circ \operatorname{gr}\hat{\eta} \circ (\operatorname{gr}\beta^{-1} \otimes \operatorname{gr}\beta^{-1}).$$

□

thm:bracketsnake

Theorem 5.8. *The Kontsevich integral descends to a homomorphic expansion for the Goldman Bracket. That is, the outside square of the following diagram commutes:*

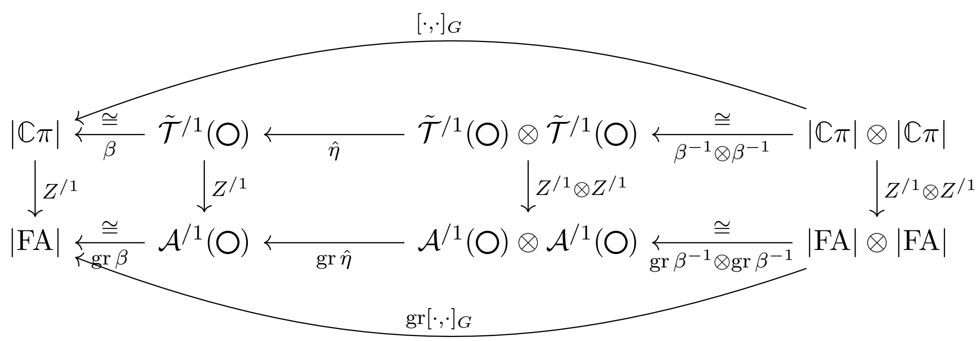


fig:Cube_for_bracket

Proof. The top and bottom squares commute by Theorem 5.8 and Corollary 5.7. All that needs to be shown is the commutativity of the middle square. This middle square occurs as the diagonal square of the multi-cube in Figure 24.

Using the construction in Section 2, we only need to show that the faces of the multi-cube in Figure 24 commute; this implies desired commutativity of the diagonal square. We have already established that the top and bottom faces commute

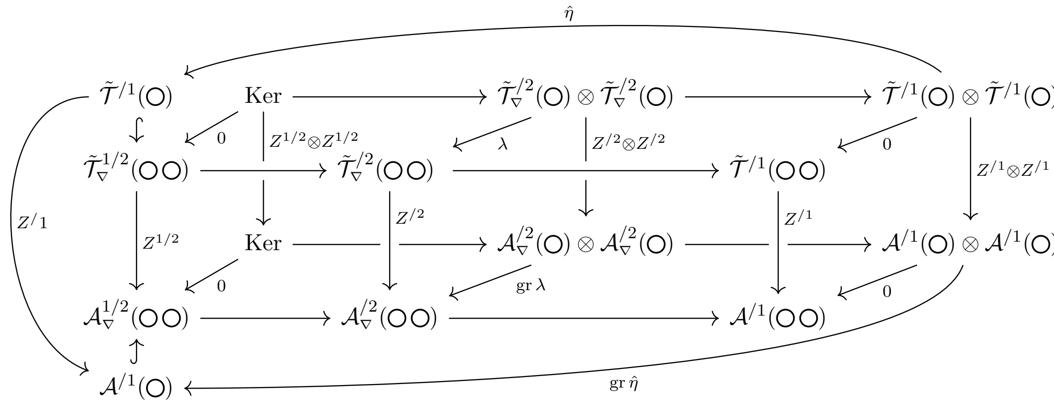
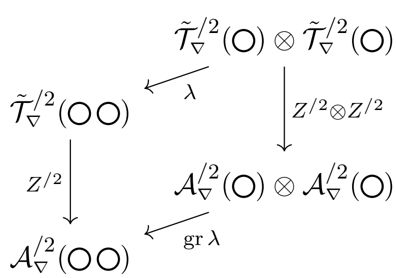


FIGURE 24. Commutative cube showing the formality of the Goldman bracket from the Kontsevich integral.

from Theorem 5.8 and Corollary 5.7. The front and back vertical faces commute because Z is a filtered map with respect to the s-filtration (Proposition 4.13). The left and right vertical sides commute trivially because of the zero maps.

The Kontsevich integral is homomorphic with respect to the stacking product (Proposition 4.10). Since λ is the difference between the stacking product and its opposite product, Z is homomorphic with respect to λ. In other words, the middle vertical face of Figure 24) commutes:



In summary, all faces of the multi-cube in diagram in Figure 24 commutes, and therefore so does the induced diagonal square, completing the proof. □

maybe we should revive the commented out example below, showing how the graded bracket works

sec:cobracketinCON

I can't find the example you are referring to.

5.2. **The Tureav co-bracket.**

In Section 3.2 we reviewed the definition of the Turaev cobracket on $|\mathbb{C}\pi|$ via the map $\mu : \mathbb{C}\tilde\pi \to |\mathbb{C}\pi| \otimes \mathbb{C}\pi$, which required choosing a rotation number $-1/2$ representative for curves in $\mathbb{C}\tilde\pi$. Our lift for the cobracket imitates this construction.

We start by interpreting $\mathbb{C}\tilde\pi$ in the context of tangles. Let $\frown$ denote an interval skeleton component where both endpoints are on the bottom $D_p \times \{0\}$. We call a tangle with skeleton $\frown$ a *bottom tangle*. We mark the endpoints of the interval

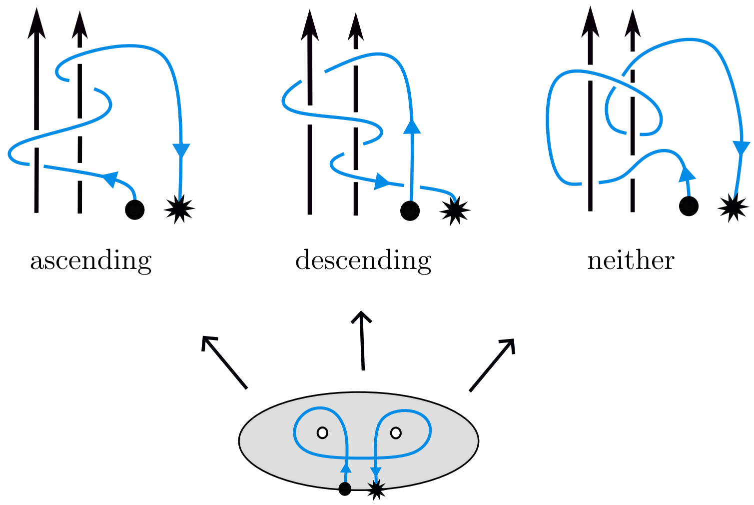


fig:ascending

FIGURE 25. An example curve in $\mathbb{C}\pi$ lifted to bottom tangles. The left lift is an ascending tangle, the middle lift is a descending tangle, and the last lift is neither ascending nor descending. All three tangles are equivalent in $\tilde{\mathcal{T}}^{/1}$, but distinct in $\tilde{\mathcal{T}}$.

by $\bullet$ and $*$, as in Figure 25. Furthermore, we denote by $\tilde{\mathcal{T}}(\bigcirc^k \frown^\ell)$ tangles with k circle skeleton components, and ℓ bottom intervals.

We extend the projection map β (Proposition 5.1) to such tangles to obtain an isomorphism similar to Corollary 5.3:

prop:ascispi

Proposition 5.9. *There is a well-defined natural bottom projection*

$$\beta : \mathbb{C}\tilde{\mathcal{T}}_\triangledown(\bigcirc^k \frown^\ell) \to |\mathbb{C}\pi|^{\otimes k} \otimes \mathbb{C}\pi^{\otimes \ell},$$

which descends to an isomorphism $\beta : \tilde{\mathcal{T}}^{/1}(\bigcirc^k \frown^\ell) \xrightarrow{\cong} |\mathbb{C}\pi|^{\otimes k} \otimes \mathbb{C}\pi^{\otimes \ell}$.

we probably need to say this is still filtered?

Proof. Identical to the proof of Proposition 5.1. □

p:qbonbottomtangles

Proposition 5.10. *The division by b map, q_b, descends to an isomorphism*

$$q_b : \tilde{\mathcal{T}}_\triangledown^{1/2}(\frown) \xrightarrow{\cong} \tilde{\mathcal{T}}_\triangledown^{/1}(\bigcirc\frown).$$

Do we even need a proof? If so, maybe restate as the mult by b map to follow like the Prop 4.18, none the less, the proof needs work or be deleted.

Proof. Thinking about the inverse map multiplication by b, m_b, an element of $\tilde{\mathcal{T}}_\triangledown^{/1}(\bigcirc\frown)$ The map q_b uses the Conway relation to smooth double points to get a two-component tangle, where one component has interval skeleton and the other component has circle skeleton. □

Next, we will recover μ as the connecting homomorphism induced from the difference between two ways to lift a bottom tangle.

compile error complaining about botskel in the caption def:asc+desc **Definition 5.11.** Let $\bullet$ and $*$ be two points on the boundary of D_p that are close together. An embedding

$$T : (I, \{0,1\}) \hookrightarrow (M_p, \{\bullet, *\})$$

representing a bottom tangle is called *ascending* if it "first winds upwards, and then goes *straight* down". More precisely, if (z, s) is a global coordinate system for $M_p = D_p \times I$, then T is an ascending tangle if there exists $c \in (0,1)$ such that when $t \in (0, c)$, the $\frac{d}{ds}$ component of $\dot{T}$ is positive, when $t \in (c+\epsilon, 1)$, $\dot{T}$ is a negative constant multiple of $\frac{d}{ds}$, and when $t \in (c, c+\epsilon)$, T smoothly transitions through a maximum (no sharp corner).

Likewise, such an embedding representing a bottom tangle T is *descending* if it "first goes straight up, and then winds downward". So there is $c \in (0,1)$ such that when $t \in (0, c)$, $\dot{T}$ is a positive constant multiple of $\frac{d}{ds}$ and when $t \in (c+\epsilon, 1)$ the $\frac{d}{ds}$ component of $\dot{T}$ is negative, and when $t \in (c, c+\epsilon)$, T smoothly transitions through a maximum.

Definition 5.12. An *ascending tangle* is a bottom tangle in M_p whose ambient isotopy class has an ascending embedding. Similarly, a *descending tangle* is a bottom tangle in M_p whose ambient isotopy class has an descending embedding. See Figure 25 for an example.

Given a curve K in $\mathbb{C}\pi$, through the isomorphism β, K can be lifted to a bottom tangle in $\tilde{\mathcal{T}}^{/1}(\frown)$. Because we are in the quotient by degree 1 terms, crossings can be changed at will to make the lifted tangle be ascending or descending. However, to lift K to a framed tangle takes some care. For any framed curve K in $\mathbb{C}\pi$, we can choose a homotopy class representative with rotation number 0 that is a sailing curve. A *sailing curve* is a curve whose tangent vector never points in a fixed specified direction. For this context, viewing $D_p \times 0$ as a subset of $\mathbb{C}$ a we fix the north direction $\vec{n}$ to be in the direction of i, and sailing curves never point north. For a curve to avoid pointing north when turning from west to east, (instead of tacking like a sailboat with your nose to the wind) a kink can be added to loop the curve back around through the south direction and then continue heading east (do a jib turn like a sailboat with your back to the wind). See Figure 26 for an example sailing curve. When taking a lift of a sailing curve K, there is an ascending lift of the curve where the north vector is never tangent to the curve. We will denote this lift as $\lambda_a(K)$. We can choose a framing at each point p on $\lambda_a(K)$ by taking the tangent vector $\dot{T}$ at p and the projection of $\vec{n}$ on to the plane normal to $\dot{T}$ (since $\dot{T}$ is never parallel to $\vec{n}$). Thus $\lambda_a(K)$ is a framed ascending bottom tangle. Similarly we can lift K to a framed descending bottom tangle denoted $\lambda_d(K)$. Finally, we define $\bar{\lambda} : \tilde{\mathcal{T}}^{/1}(\frown) \to \tilde{\mathcal{T}}^{/2}(\frown)$ by

$$\bar{\lambda}(K) = \lambda_a(K) - \lambda_b(K)$$

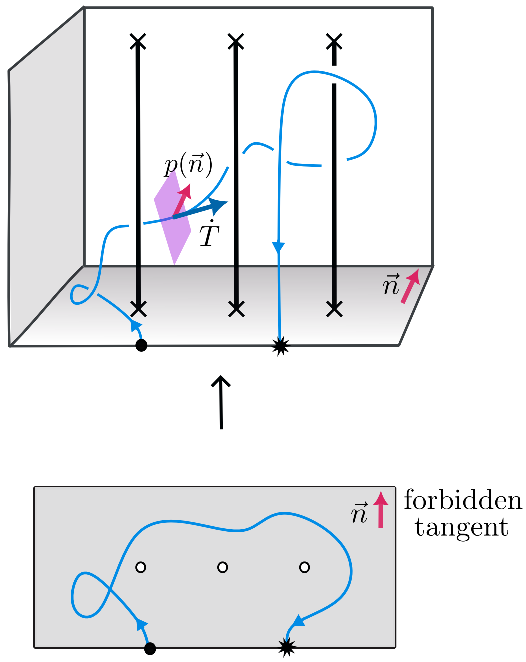


FIGURE 26. A rotation number 0 sailing curve in $\mathbb{C}\pi$ lifts to a framed bottom tangle in M_p. Here, $p(\vec{n})$ is the projection of $\vec{n}$ on to the plane normal to $\dot{T}$ at point p.

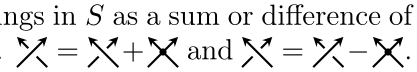
framing_from_sailing

to be the difference between the framed ascending bottom tangle and the framed descending bottom tangle. In $\tilde{\mathcal{T}}^{/2}(\frown)$, crossing changes matter so $\bar{\lambda}$ is not the zero map.

Notice that one can convert an ascending bottom tangle to a descending bottom tangle (and vice versa) by first identifying all strand-strand crossings, in all such crossings swap which strand is on top, and then re-adjust the height of the strands to make it descending.

Theorem 5.13. *The diagram in Figure 27 commutes and the unique induced map η, when composed with isomorphisms q_b and β, is the self intersection map μ, that is $\mu = \beta q_b \eta \beta^{-1}$.*

Proof. We need to show the right square commutes, which reduces to showing the right bottom triangle commutes. That is $\bar{\lambda}$ has image in the kernel of $\tilde{\mathcal{T}}^{/2}_{\triangledown}(\frown) \to \tilde{\mathcal{T}}^{/1}(\frown)$. Let T be a tangle in $\tilde{\mathcal{T}}^{/1}(\frown)$, and let T_a be a framed ascending bottom lift of T and T_b be a framed descending bottom lift of T. Then $\bar{\lambda}(T) = T_a - T_b$.

Starting with T_a, let S denote the set of all strand-strand crossings of T_a. Using double point notation, we can rewrite each crossings in S as a sum or difference of a double point and and the opposite crossing, i.e. .

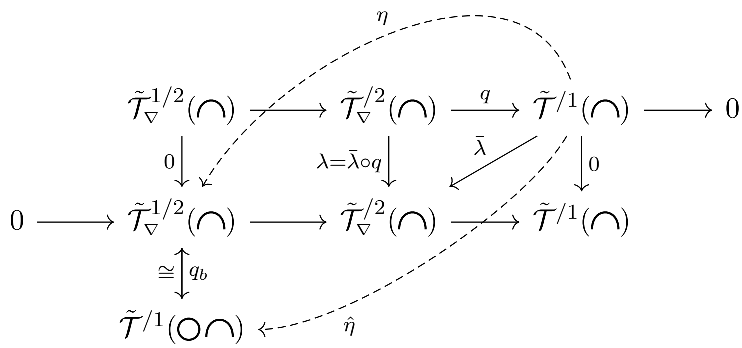


fig:Snakeformu

FIGURE 27. The nontrivial horizontal maps are the respective quotient maps, and q is one such quotient map.

As in the proof of Theorem 5.8, rewriting each s-crossing of T in this way yields a sum indexed by the subsets of S. For every subset X of S, let T_X denote the tangle where all crossings in X of T_a are changed to double points, and all other crossings in $S \setminus X$ are flipped. Letting ϵ_X be the product of the signs in of all the crossings in X, we get

eq:T_aAsASum

$$T_a = \sum_{X \subseteq S} \epsilon_X T_X. \tag{5.3}$$

Notice that $T_\emptyset = T_b$, as all of the strand-strand crossings in T_a have flipped, and if $|X| = i$ then $T_X \in \tilde{\mathcal{T}}_\triangledown^{/i}(\frown)$. Thus, we see that

$$\bar{\lambda}(T) = T_b - T_a = \sum_{X \subset S, |X|=1} \epsilon_X T_X \in \tilde{\mathcal{T}}_\triangledown^{1/2}(\frown)$$

which concludes that the right square of the diagram in Figure 27 commutes.

We need to show the left square commutes, which is that λ applied to anything in $\tilde{\mathcal{T}}_\triangledown^{1/2}(\frown)$ is in $\tilde{\mathcal{T}}_\triangledown^{1/2}(\frown)$. Starting with $T \in \tilde{\mathcal{T}}_\triangledown^{1/2}(\frown)$, T has a double point, and an ascending lift of T to T_a will also have a double point. By the same computation in Equation 5.3, $\lambda(T)$ is in $\tilde{\mathcal{T}}_\triangledown^{2}(\frown)$ as every term contains the pre-existing double point in T, and at least one additional double point. Therefore, the left hand square commutes.

The image of η actually lives in $\tilde{\mathcal{T}}^{/1}(\bigcirc\frown)$

As in Section 2, then λ induces a unique well defined homomorphism $\eta : \tilde{\mathcal{T}}^{/1}(\frown) \to \tilde{\mathcal{T}}_\triangledown^{/2}(\frown)$. We identify η as the self-intersecting map, μ, through some isomorphisms. We have that the isomorphism β gives $\tilde{\mathcal{T}}^{/1}(\frown) \cong \mathbb{C}\pi$ (Proposition 5.9), identifying the domain of η with the domain of μ. We now argue that η has image in $\tilde{\mathcal{T}}^{/1}(\bigcirc\frown)$.

By Equation (5.3), $\lambda(T)$ is a sum of terms, each with a single mixed double point. Applying the Conway relation to smooth each of these mixed double points changes the skeleton from one interval component skeleton (i.e. a bottom tangle) to get a two-component tangle, where one component has interval skeleton and the other component has circle skeleton, and introduces a factor of b. In other words, $\lambda(T) \in b\tilde{\mathcal{T}}_{\triangledown}^{/2}(\bigcirc/,\cap) \subseteq \tilde{\mathcal{T}}_{\triangledown}^{1/2}(\bigcirc\bigcirc)$. By Corollary 4.19, restricted to a circle skeleton, we know that $b\tilde{\mathcal{T}}_{\triangledown}^{/2}(\bigcirc) \cong \tilde{\mathcal{T}}^{/1}(\bigcirc)$ via the map q_b. (The map $\hat{\eta}$ in the diagram is $q_b \circ \eta$). In turn, $\tilde{\mathcal{T}}^{/1}(\bigcirc) \cong |\mathbb{C}\pi|$ again via the map β.

We need to show η is the self intersecting map, when composed with β's.

By passing to the quotient $\tilde{\mathcal{T}}_{\triangledown}^{1}/\tilde{\mathcal{T}}_{\triangledown}^{2}(\cap)$, only the terms that have a single double point remain, so $T - T^{f_b}$ becomes a sum over the s-crossings of T, where in each term the s-crossing is replaced by a double point. The map q_b uses the Conway relation to smooth these double points to get a two-component tangle, where one component has interval skeleton and the other component has circle skeleton. Thus we land in $\tilde{\mathcal{T}}_{\triangledown}^{/1}(\bigcirc\cap)$, which is isomorphic to $|\mathbb{C}\pi| \otimes \mathbb{C}\pi$ via β. □

For a bottom tangle, there is a closure map from $cl : \tilde{\mathcal{T}}(\cap) \to \tilde{\mathcal{T}}(\bigcirc)$ by connecting the endpoints of the bottom tangle, $\bullet$ and $*$, by a canonical path in the boundary of the disk. Recall from Section 3.2 that the cobracket δ is constructed from μ by post composing with the closure map and then antisymetrizing. In the context of tangle diagrams, this construction is shown in Figure 28. The closure map $cl : \tilde{\mathcal{T}}^{/1}(\bigcirc\cap) \to \tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc)$ orders the components by placing the closed bottom tangle in the second slot. The intermediate induced map after closing, but before antisymmtrizing, is denoted in the figure by $\hat{\delta}$ and is called the *ordered* Turaev cobracket. We will show the Kontsevich integral is homomorphic with respect to $\hat{\delta}$. The homomorphicity of δ with respect to Z follows from immediately the homomorphicity of $\hat{\delta}$ with respect to Z because $\mathrm{gr}(Alt) = Alt$.

Taking the associated graded of the diagram in Figure 21 we arrive at the diagram in Figure 29

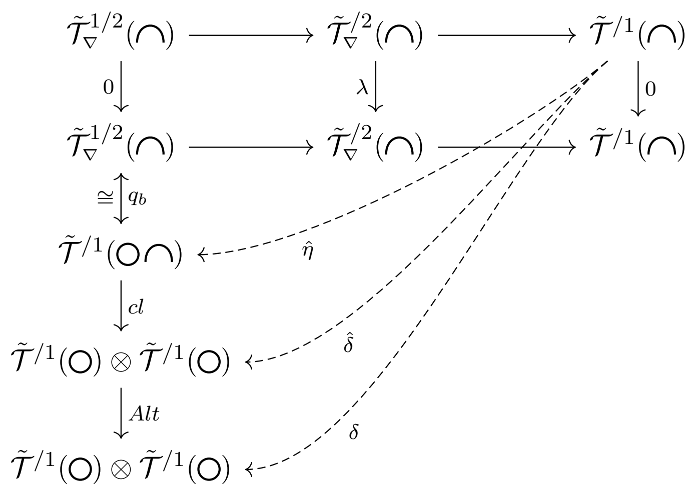
akefor_gr_cobracket **Theorem 5.14.** *The diagram in Figure 29 commutes and the induced map* $\mathrm{gr}\,\hat{\delta}$ *is the associated graded ordered Turaev cobracket.*

Proof. The maps in the diagram of Figure 28 are filtered maps, and therefore Figure 29 is obtained by applying the associated graded functor to it. As a result, the diagram of Figure 29 commutes, $\mathrm{gr}\,\mu$ is the induced map from the snake lemma for this diagram, and so $\mathrm{gr}\,\hat{\delta}$ coincides with the graded ordered Turaev cobracket. □

pcubesimplification **Lemma 5.15.** *There exists a map* $\rho : \tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc) \to \tilde{\mathcal{T}}_{\triangledown}^{/2}(\bigcirc)$ *that makes the diagram in Figure 30 commute.*

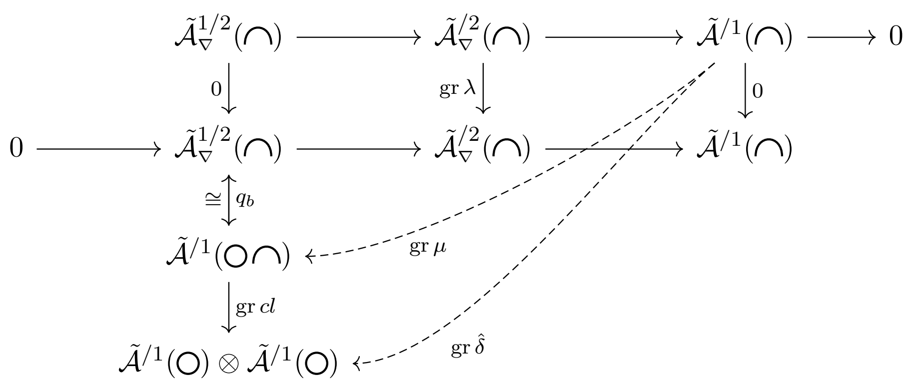
In this figure, do we need μ in it still?

Proof. There is an isomorphism from $\tilde{\mathcal{T}}^{/1}(\bigcirc)\otimes\tilde{\mathcal{T}}^{/1}(\bigcirc)$ to $\tilde{\mathcal{T}}^{/1}(\bigcirc\bigcirc)$ by combining the two tangles into a single tangle and forgetting the order of the components.



g:Snakeforcobracket

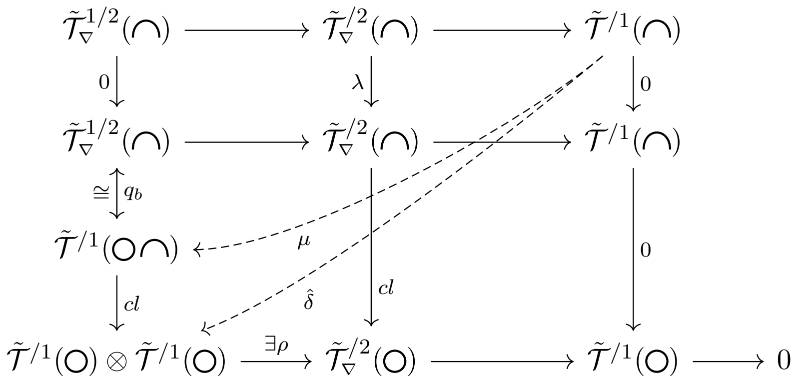
FIGURE 28. Constructing δ from $\hat{\eta}$.



akefor_gr_cobracket

FIGURE 29. Associated graded diagram constructing the graded ordered Turaev cobracket.

Since we are modding out by s degree 1, there is no notion of over or under, these are just curves in the disc.



opcubesimplification

FIGURE 30. Commutative diagram for Lemma 5.15.

The map $\rho : \tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc) \to \tilde{\mathcal{T}}^{/2}(\bigcirc)$ is defined to be the following composition of maps.

$$\tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc) \xrightarrow{forget} \tilde{\mathcal{T}}^{/1}(\bigcirc\bigcirc) \xleftrightarrow{q_b} \tilde{\mathcal{T}}^{1/2}_{\triangledown}(\bigcirc) \hookrightarrow \tilde{\mathcal{T}}^{/2}_{\triangledown}(\bigcirc) \qquad (\text{composite: } \rho)$$

Since the image of ρ in $\tilde{\mathcal{T}}^{/2}_{\triangledown}$ is all of $\tilde{\mathcal{T}}^{1/2}$ we get the following short exact sequence.

$$\tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc) \xrightarrow{\rho} \tilde{\mathcal{T}}^{/2}_{\triangledown}(\bigcirc) \longrightarrow \tilde{\mathcal{T}}^{/1}(\bigcirc) \longrightarrow 0$$

The commutativity of the diagram in Figure 30 relies finally on the commutativity of the bottom left square. We single this square out below and verify the commutativity.

$$\begin{array}{ccccccc}
\tilde{\mathcal{T}}^{1/2}_{\triangledown}(\bigcap) & & & \hookrightarrow & & & \tilde{\mathcal{T}}^{/2}_{\triangledown}(\bigcap) \\
\cong \updownarrow q_b & & & & & & \\
\tilde{\mathcal{T}}^{/1}(\bigcirc\bigcap) & & & & & & \downarrow cl \\
\downarrow cl & & & \overset{\rho}{\dashrightarrow} & & & \\
\tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc) & \longrightarrow & \tilde{\mathcal{T}}^{/1}(\bigcirc\bigcirc) & \underset{\cong}{\xleftrightarrow{q_b}} & \tilde{\mathcal{T}}^{1/2}_{\triangledown}(\bigcirc) & \hookrightarrow & \tilde{\mathcal{T}}^{/2}_{\triangledown}(\bigcirc)
\end{array}$$

$$\begin{array}{ccccccc} \tilde{\mathcal{T}}^{/1}(\bigcirc)\otimes\tilde{\mathcal{T}}^{/1}(\bigcirc) & \xrightarrow{forget} & \tilde{\mathcal{T}}^{/1}(\bigcirc\bigcirc) & \xleftrightarrow{q_b} & \tilde{\mathcal{T}}^{1/2}_{\triangledown}(\bigcirc) & \hookrightarrow & \tilde{\mathcal{T}}^{/2}_{\triangledown}(\bigcirc) \\ \downarrow{\scriptstyle Z^{/1}\otimes Z^{/1}} & & \downarrow{\scriptstyle Z^{/1}} & & \downarrow{\scriptstyle Z^{/1}} & & \downarrow{\scriptstyle Z^{/2}} \\ \tilde{\mathcal{A}}^{/1}(\bigcirc)\otimes\tilde{\mathcal{A}}^{/1}(\bigcirc) & \xrightarrow{forget} & \tilde{\mathcal{A}}^{/1}(\bigcirc\bigcirc) & \xleftrightarrow{\operatorname{gr} q_b} & \tilde{\mathcal{A}}^{1/2}_{\triangledown}(\bigcirc) & \hookrightarrow & \tilde{\mathcal{A}}^{/2}_{\triangledown}(\bigcirc) \end{array}$$

fig:frontlefthomom

FIGURE 31. Commutative diagram for Lemma 5.16

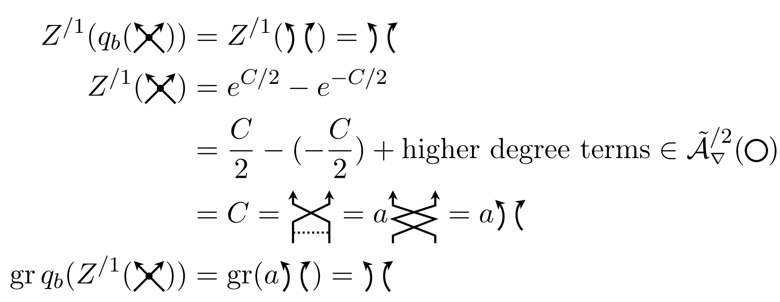
Let $T \in \tilde{\mathcal{T}}^{1/2}_{\triangledown}(\bigcap)$, then T is a bottom tangle with exactly one double point. Following along the top and right of the diagram in Figure 30, when T is closed, we get a closed loop with one double point inside $\tilde{\mathcal{T}}^{/2}_{\triangledown}(\bigcirc)$. Following along the right and bottom, $q_b(T)$ uses the Conway relation to snip off a loop of T to get a tangle in $\tilde{\mathcal{T}}^{/1}(\bigcirc\bigcap)$ with one closed loop and a bottom tangle, with no double points. Closing the bottom tangle and forgetting the order of the closed loops gives a tangle in $\tilde{\mathcal{T}}^{/1}(\bigcirc\bigcirc)$ with two closed loops and no double points. Reversing the Conway relation along q_b glues together the two closed loops to get a single closed loop with one double point then included into $\tilde{\mathcal{T}}^{/2}(\bigcirc)$. This arrives at the same closed loop with one double point as if we had closed T in the first place. □

lem:frontlefthomom

Lemma 5.16. *The diagram in Figure 31 commutes.*

Proof. The right square commutes because Z is a filtered map and respects filtered inclusions.

For the middle square, we use the map q_b from right to left and show commutativity on a double point.



For the left square, Z compatible with forgetful is because we land in $/1$, where there are no s-s chords. □

say more about left square

cobrackethomomorphic

Theorem 5.17. *The Kontsevich integral descends to a homomorphic expansion for the ordered Turaev cobracket. That is, the following square commutes:*

$$\begin{array}{ccc} \tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc) & \xleftarrow[\hat{\delta}]{} & \tilde{\mathcal{T}}^{/1}(\frown) \\ \downarrow{\scriptstyle Z^{/1}\otimes Z^{/1}} & & \downarrow{\scriptstyle Z^{/1}} \\ \mathcal{A}^{/1}(\bigcirc) \otimes \mathcal{A}^{/1}(\bigcirc) & \xleftarrow[\mathrm{gr}\,\hat{\delta}]{} & \mathcal{A}^{/1}(\frown) \end{array}$$

Proof. The diagram in Figure 32 is attained by taking the Kontsevich integral of the commutative diagram in Figure 30 (with the middle layers omitted). We have already established that the top and bottom faces commute by Lemma 5.15 and Theorem 5.14. The left and right vertical sides trivially commute because of the zero maps. The front-left vertical square commutes by Lemma 5.16. The front-right and back faces commute because Z respects the s-filtration and is homomorphic with respect to the inclusion and quotient maps of the filtered components.

The middle vertical face of Figure 32 is the following square.

$$\begin{array}{ccc} & & \tilde{\mathcal{T}}_{\triangledown}^{/2}(\frown) \\ \tilde{\mathcal{T}}_{\triangledown}^{/2}(\bigcirc) & \xleftarrow{cl\circ\lambda} & \downarrow{\scriptstyle Z^{/2}} \\ \downarrow{\scriptstyle Z^{/2}} & & \mathcal{A}_{\triangledown}^{/2}(\frown) \\ \mathcal{A}_{\triangledown}^{/2}(\bigcirc) & \xleftarrow{\mathrm{gr}(cl\circ\lambda)} & \end{array}$$

The Kontsevich integral is homomorphic with respect to the flip operation, as shown in Proposition 4.10. The map $cl \circ \lambda$ applied to a bottom tangle outputs the difference between the closed ascending lift and the closed descending lift. The closed descending lift is the flip of the closed ascending lift. So $cl \circ \lambda = (id - flip) \circ cl$ acting on ascending representatives. Z is homomorphic with respect to $(id - flip) \circ cl$.

This is not quite right, FIX ME!

The commutativity of all vertical faces of the cube diagram in Figure 32 implies that the induces diagonal square also commutes, which gives the desired formality of the theorem statement. □

where does conjugation come into play?? Something about flipping first then dragging the ends down and then closing.

remark–if we were doing this with μ is it wouldn't work because flip of a bottom tangle is not a bottom tangle. It is much cleaner to just pass to the closures.

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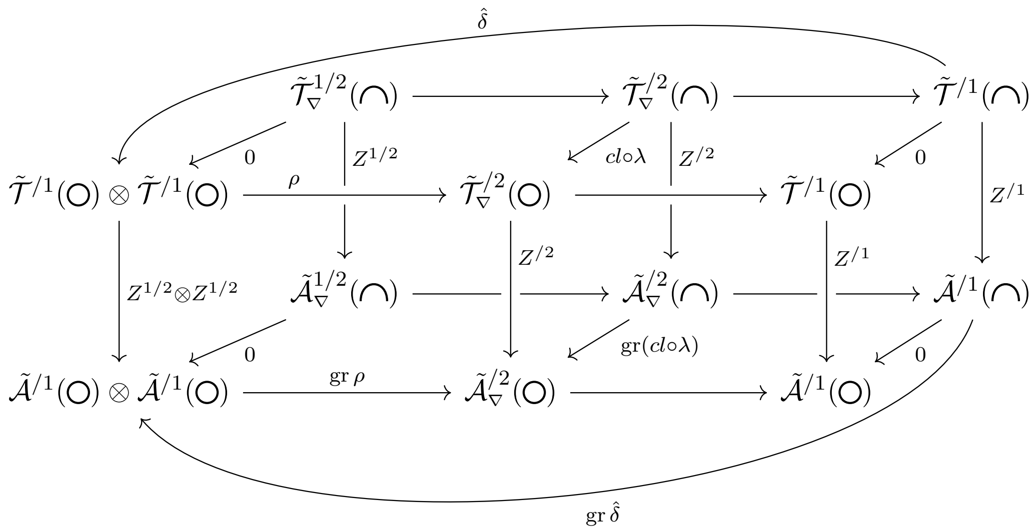

FIGURE 32. Commutative cube showing the formality of the ordered Turaev cobracket from the Kontsevich integral.

g:Cube_for_cobracket

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