# GOLDMAN-TURAEV FORMALITY FROM THE KONTSEVITCH INTEGRAL

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AND NANCY SCHERICH

Abstract. We present a three dimensional realisation of the Goldman-Turaev Lie biaglebra, and construct Goldman-Turaev homomorphic expansions from the Kontsevich integral.

## Contents

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*Key words and phrases.* knots, links in a handlebody, expansions, finite type invariants, Lie algebras .

To do list for Zsuzsi

1. (BIG COMMENT) Section 3.1, reconsider the depth for which we discuss the Kontsevich integral. Who is our audience?
2. Section 3.3, make 2 dummy figures–described in the side notes
3. Section 3.3, read over the added informal descriptions of the operations to tighten up.
4. Section 3.3, there is an old note from Jessica about signs. Do we need to keep that comment, or can we delete it?
5. Find the reference for Proposition 3.6– Quillen66? Or new reference for Magnus expansion.
6. I added a footnote for the Magnus expansion. Do we need it? Should we say more there?
7. add a reference for Proposition 3.8.
8. Section 4, make it clear where the proof for Theorem 4.9 ends.
9. Section 4, make dummy figure for chord diagram stacking
10. I reordered the intro section according to Dror's comments. Have you read it over? It probably needs proof reading again.

# 1. INTRODUCTION

In 1986, Goldman defined a Lie bracket [Gol86] on the space of homotopy classes of free loops on a compact oriented surface. Shortly after in 1991, Turaev defined a cobracket [Tur91] on the same space[1]. This bracket and cobracket make the space of free loops into a Lie bialgebra – known as the Goldman-Turaev (GoTu) Lie bialgebra – which forms the basis for the field of string topology [?] and has been an object of study from many perspectives.

add referemnces: chas-sullivan, kashiwara-vergne, AN, AT, Formality paper

In this paper we, describe a 3-dimensional lift of the Goldman-Turaev Lie bialgebra into a space of tangles in a handlebody. We recover the bracket and cobracket maps as projections of intuitive operations on tangles. We show the Kontsevich integral is homomorphic with respect to these tangle operations. Our main result is informally summarised as follows:

**Main Result.** *Let $\tilde{\mathcal{T}}$ denote the space of formal linear combinations of tangles in a punctured disc cross an interval $M_p = D_p \times I$. Projecting to the bottom $D_p \times 0$, one obtains curves on a punctured disc, and the Goldman–Turaev operations on these curves are induced*[2] *by the* stacking *and* flipping *operations on the tangles. The Kontsevich integral is a homomorphic expansion for tangles in $M_p$, and descends to a Goldman–Turaev homomorphic expansion on $D_p$.*

This result is parallel to Massuyeau's [Mas18], however, our approach to the cobracket is significantly different and simpler, hence, more likely to lead to give

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[1] Turaev's version required factoring out by the constant loop; there is a lift to the full space of homotopy classes of loops, given a framing on the surface [AKKN20].

[2] In a specific sense defined in Section 2

insight into the motivational application descried below. Another related result is [?], which constructs Goldman–Turaev expansions from the Khnizhnik-Zamolodchikov connection, a geometric incarnation of the Kontsevich integral.

In more detail, we describe a space $\widetilde{\mathcal{T}}$ of formal linear combintations of framed tangles in the handlebody $\mathcal{D}_p \times I$ and operations on this space, which induce the Goldman-Turaev operations in the bottom projection to $D_p \times \{0\}$. The Goldman bracket arises from the commutator associated to the stacking product in a Conway skein quotient of $\widetilde{\mathcal{T}}$, defined in Section 4.7, and the Turaev cobracked from taking the difference between a tangle and its vertical flip, again in a Conway quotient. We study the associated graded spaces and operations, and show that the Kontsevich integral is a homomorphic expansion for these tangles, in other words, intertwines the operations with their associated graded counterparts. We show that therefore, the Kontsevich integral descends to a homomorphic expansion for the Goldman-Turaev Lie bialgebra. For the flipping operation and the Turaev cobracket, the precise statements are subtle, and care needs to be taken with the technical details.

There are other papers by Turaev and Massuyeau-Turaev that are not mentioned here.

There are also some references that Yusuke mentioned that we should include

Turaev's paper- we can probably pull some of our lemmas from his paper, reference for relationship with HOMFLY, but he does not mention the free associative algebra at all. Our paper is not a subset of his. Skein algebra quantizes — symmetric lie algebra generated by the goldman lie algebra–you can get a poison algebra, These skien modules quantize that poisson algebra

1.1. **Motivation.** The Kashiwara–Vergne equations originally arose from the study of convolutions on Lie groups [?]. The equations were reformulated algebraically in terms of automorphisms of free Lie algebras [?], it this form they are a refinement of the Baker-Campbell-Hausdorff formula for products of exponentials of non-commuting variables.

Kashiwara–Vergne theory has multiple topological interpretations in which Kashiwara–Vergne solutions correspond to certain invariants – called *homomorphic expansions* – of topological objects. The existence of a homomorphic expansion is also called *formality* in the literature, this language is inspired by rational homotopy theory and group theory [?].

One of these topological interpretations is due to the first two authors [BND17], who showed that homomorphic expansions of welded foams – a class of 4-dimensional tangles – are in one to one correspondence with solutions to the KV equations. Recently, a series of papers by Alekseev, Kawazumi, Kuno and Naef [AKKN20,AKKN18b,AKKN18a] drew an analogous connection between KV solutions and homomorphic expansions for the Goldman-Turaev Lie bialgebra for the disc with two punctures (up to non-negligible differences in the technical details). This correspondence was used to generalise the Kashiwara–Vergne equations via considering different surfaces, including those of higher genus.

In other words, there is an intricate algebraic connection between four-dimensional welded foams and the GT Lie bi-algebra, which strongly suggests that there is a topological connection as well. In addition to the inherent interest in tangles in handlebodies, one goal for this paper is to work towards this connection between the two-dimensional Goldman–Turaev Lie bialgebra and four-dimesnional welded foams, by constructing a three-dimensional realisation of the Goldman-Turaev Lie bialgebra, with homomorphic expansions which descend to Goldman-Turaev expansions.

*The paper is organised as follows:* Section 2 gives a general algebraic framework for how the Goldman–Turaev operations are induced by tangle operations. In Section 3 we give a brief overview of the Kontsevich integral and the Goldman Turaev Lie bialgebra. In Section 4, we define tangles in handlebodies, relevant operations and Vassiliev filtrations. We identify the associated graded space of tangles as a space of chord diagrams, and introduce the Conway skein quotient. In Section **??**, we identify the GoTu Lie biaglebra in a low filtration degree, and prove the main theorem.

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`sec:conceptsum`

## 2. Conceptual summary

We induce the genus zero Goldman-Turaev operations from tangle operations, in the spirit of "connecting homomorphisms": this Section is a summary of the basic approach. We use the words *associated graded structures*, *homomorphic expansions*, and *Goldman-Turaev operations* without definition, only mentioning their basic properties which make this conceptual outline coherent; the definitions follow in Section 3.

In the diagram (2.1), top and bottom rows are exact and the right and left vertical maps are zero, and therefore, by minor diagram chasing, the middle vertical map $\lambda$ induces a unique map $\eta : C \to D$, a degenerate case of a connecting homomorphism. In our applications $\lambda$ is a difference of two maps $\lambda_1$ and $\lambda_2$, whose values differ in $E$ but coincide in a quotient $F$.

`eq:Snake`

$$\begin{tikzcd}
 & A \arrow[r] \arrow[d, "0"'] & B \arrow[r] \arrow[d, "\lambda=\lambda_1-\lambda_2"] & C \arrow[r] \arrow[d, "0"] \arrow[dll, dashed, bend right, "\eta"'] & 0 \\
0 \arrow[r] & D \arrow[r] & E \arrow[r] & F &
\end{tikzcd} \tag{2.1}$$

In Section 5 we present two constructions which produce the Goldman bracket and the Turaev cobracket, respectively, as induced homomorphisms $\eta$, from corresponding tangle operations $\lambda_0$ and $\lambda_1$. The following example is a schematic version of what will become the argument for the Goldman bracket:

These indices do not match the figure. Should it be 1 and 2?

**Example 2.1.** Let $A$ be an associative algebra, and let $\{L_i\}$ denote the lower central series of $A$. That is, $L_1 := A$, and $L_{i+1} := [L_i, A]$. Then the $L_i$ are Lie ideals, and let $M_i = AL_i$ denote the two-sided ideal generated by $L_i$. The quotient $A/M_1$ is the abelianisation of $A$, denoted by $A^{ab}$. Then we have the following diagram:

`eq:SnakeExample`

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, "0"'] & \frac{A}{M_2}\otimes\frac{A}{M_2} \arrow[r] \arrow[d, "{[\cdot,\cdot]}"] & A^{ab}\otimes A^{ab} \arrow[r] \arrow[d, "0"] \arrow[dll, dashed, bend right, "\eta"'] & 0 \\
0 \arrow[r] & \frac{M_1}{M_2} \arrow[r] & \frac{A}{M_2} \arrow[r] & A^{ab} \arrow[r] & 0
\end{tikzcd} \tag{2.2}$$

Here $\lambda$ is the algebra commutator, which is indeed the difference between two maps: the multiplication ($\lambda_1$) and the opposite multiplication ($\lambda_2$). The kernel $K$ of the projection to $A^{ab}\otimes A^{ab}$ is generated by the subalgebras $\left\{\frac{M_1}{M_2}\otimes\frac{A}{M_2}, \frac{A}{M_2}\otimes\frac{M_1}{M_2}\right\}$ in $\frac{A}{M_2}\otimes\frac{A}{M_2}$. The map $\eta$ is a well defined commutator map $A^{ab}\otimes A^{ab}\to\frac{M_1}{M_2}$, given by $\eta(x\otimes y) = [x, y] \bmod M_2$. □

opposite mult? or opp order mult? or mult in opp order?

The goal of this paper is to construct homomorphic expansions for the Goldman-Turaev Lie bialgebra from the Kontsevich integral. In outline, this follows from the naturality property of the construction above, under the associated graded functor. Namely, if all of the spaces in the diagram (2.1) are filtered and the maps are filtered maps, then the associated graded functor (denoted gr) produces an associated graded commutative diagram with the same properties. An expansion for an algebraic structure $X$ is a filtered homomorphism $Z : X \to \operatorname{gr} X$ (with special properties). Thus, if expansions exist for each of the spaces $A$ through $F$, we obtain a multi-cube:

Since we are using the word "formality" in the title, should we also use it here?

`eq:Cube`

$$\begin{tikzcd}
 & & A \arrow[rr] \arrow[dl] \arrow[dd, "Z_A"] & & B \arrow[rr] \arrow[dl, "\lambda"] \arrow[dd, "Z_B"] & & C \arrow[r] \arrow[dl] \arrow[dd, "Z_C"] \arrow[dlllll, dashed, bend right, "\eta"'] & 0 \\
0 \arrow[r] & D \arrow[rr] \arrow[dd, "Z_D"] & & E \arrow[rr] \arrow[dd, "Z_E"] & & F \arrow[dd, "Z_F"] & & \\
 & & \operatorname{gr} A \arrow[rr] \arrow[dl] & & \operatorname{gr} B \arrow[rr] \arrow[dl, "\operatorname{gr}\lambda"] & & \operatorname{gr} C \arrow[r] \arrow[dl] \arrow[dlllll, dashed, bend left, "\operatorname{gr}\eta"] & 0 \\
0 \arrow[r] & \operatorname{gr} D \arrow[rr] & & \operatorname{gr} E \arrow[rr] & & \operatorname{gr} F & &
\end{tikzcd} \tag{2.3}$$

If, in the multi-cube (2.3) all vertical faces commute, then so does the square:

$$\begin{tikzcd}
D \arrow[d, "Z_D"] & C \arrow[l, dashed, "\eta"'] \arrow[d, "Z_C"] \\
\operatorname{gr} D & \operatorname{gr} C \arrow[l, dashed, "\operatorname{gr}\eta"']
\end{tikzcd}$$

The commutativity of this square, where $\eta$ represents the Goldman bracket and the Turaev cobracket, respectively, is – by definition – the homomorphicity property of the expansion: our main result. The non-trivial vertical face of the multi-cube is the one containing $\lambda$, and the commutativity of this for each Goldman-Turaev operation will follow from the fact that the Kontsevich integral (standing in for $Z_B$ and $Z_E$) intertwines the appropriate tangle operations $\lambda_0$ and $\lambda_1$. This is the idea behind the proof of the main theorem.

## 3. Preliminaries: Homomorphic expansions and the Goldman-Turaev Lie bialgebra

sec:Prelims sec:KInt

Should we say formality instead of/in addition to homomorphic expansion?

3.1. **The Kontsevich Integral.** The Kontsevich Integral is the knot theoretic prototype of a *homomorphic expansion*. Homomorphic expansions (a.k.a. formality isomorphisms, universal finite type invariants) provide a connection between knot theory and quantum algebra/Lie theory. Many detailed expositions on the Kontsevich Integral exist in the literature, we recommend [CDM12, Section 8], or [Kon93, BN95, Dan10]. We briefly review the basics here from an algebraic perspective, which is outlined – in a slightly different, finitely presentated case – in [BND17, Section 2].

Let $\mathcal{K}$ denote the set of oriented knots in $\mathbb{R}^3$, and allow formal linear combinations of knots with coefficients in $\mathbb{C}$. There is a filtration on this infinite dimensional vector space called the Vassiliev filtration, which is defined in terms of resolutions of *double points*. Namely, a *double point* is defined to be the difference of an over and under crossing:

A knot with $k$ double points is a signed sum of $2^k$ knots. See Figure 1 for an example. The Vassiliev filtration is the decreasing filtration

$$\mathcal{K} = \mathcal{K}_0 \supseteq \mathcal{K}_1 \supseteq \mathcal{K}_2 \supseteq \dots$$

where $\mathcal{K}_i$ is linearly generated by knots with at least $i$ double points.

The degree completed associated graded space of $\mathcal{K}$ with respect to the Vassiliev filtration is defined as

$$\mathcal{A} := \prod_{n\geq 0} \mathcal{K}_n/\mathcal{K}_{n+1}.$$

Since $\mathcal{A}$ is a graded vector space, it lends itself naturally to recursive calculations and inductive arguments. An *expansion* $Z$ is a filtered linear map of knots taking

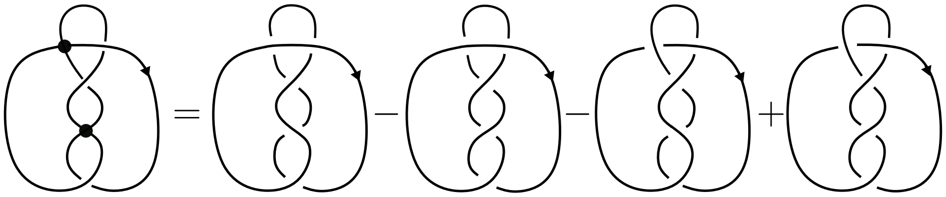

FIGURE 1. A knot with two double points written as a signed sum four knots.

`fig:pumpkins`

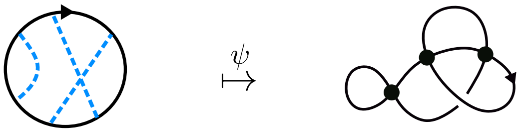

FIGURE 2. Example of $\psi$ mapping a chord diagram to a singular knot where the right-hand side is viewed as an element of $\mathcal{K}_3/\mathcal{K}_4$.

`fig:psionchord`

values in $\mathcal{A}$, which retains as much information as possible. Rigorously, this means that the associated graded map of $Z$ is the identity map of $\mathcal{A}$:

$$Z : \mathcal{K} \to \mathcal{A} \quad \text{such that} \quad \operatorname{gr} Z = id_{\mathcal{A}}.$$

An expansion is *homomorphic* with respect to some operations (such as connected sum) if it also intertwines these operations with their associated graded counterparts. This allows for a study of these operations via the associated graded space as well.

A crucial step towards making effective use of this machinery is to get a handle on the space $\mathcal{A}$ in concrete terms: namely, $\mathcal{A}$ has a combinatorial description as a space of *chord diagrams*. A chord diagram of degree $k$ on an oriented circle is a perfect matching[3] on a set of $2k$ points arranged around the circle, up to orientation preserving diffeomorphism. The circle which supports the chord diagram is called the *skeleton*. In other words, a chord diagram on a circle is a combinatorial object consisting of $2k$ cyclically ordered points, partitioned into pairs. In diagrams, each pair is indicated by a *chord*, as in the left of Figure 2.

There is a natural map $\psi$ from chord diagrams with $i$ chords to $\mathcal{K}_i/\mathcal{K}_{i+1}$, as shown in Figure 2. Namely, by contracting each chord into a double point, we obtain an $i$-singular knot. This is well-defined only up to crossing changes – as crossings other than double points may appear – however, the difference between the over/under choices for any additional crossing is in $\mathcal{K}_{i+1}$.

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[3] A perfect matching on a set is a partitioning of the set by 2-element subsets.

It is not difficult to establish that $\psi$ is surjective, and that there are two relations in its kernel: the 4-Term (4T) and Framing Independence (FI) relations, shown in Figure 3. In fact, these two relations generate the kernel, and $\psi$ descends to an isomorphism on the quotient; this, however, is significantly harder to prove.

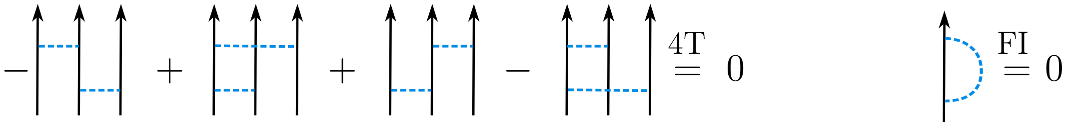

FIGURE 3. The 4T and FI relations, understood as local relations: the strand(s) are part(s) of the skeleton circle, and the skeleton may support additional chords outside the picture shown, which are the same throughout all terms of the relation.

`fig:4TFI`

The key technique is to construct an expansion as in the following Lemma, [BND17, Proposition 2.7]:

`lem:assocgradyoga`

**Lemma 3.1.** *[BND17] Let $\mathcal{K}$ be a filtered space of formal linear combinations of knotted objects*[4]*, and $\mathcal{A}$ the associated graded space of $\mathcal{K}$. Let $\mathcal{C}$ be a "candidate model" for $\mathcal{A}$: a graded linear space equipped with a surjective graded map $\psi : \mathcal{C} \to \mathcal{A}$. If there exists a filtered map $Z : \mathcal{K} \to \mathcal{C}$, such that $\psi \circ \operatorname{gr} Z = id_{\mathcal{A}}$, then $\psi$ is an isomorphism and $Z$ is an expansion for $\mathcal{K}$.*

$$\begin{tikzcd}
\mathcal{K} \arrow[r, "Z"] & \mathcal{C} \arrow[dd, "\psi"] & & \mathcal{A} \arrow[r, "\operatorname{gr} Z"] \arrow[ddr, dashed, "\psi\circ\operatorname{gr} Z=id_{\mathcal{A}}"'] & \mathcal{C} \arrow[dd, "\psi"] \\
& {} \arrow[rr, Mapsto, "\operatorname{gr}", phantom] & {} \arrow[r, Mapsto, "\operatorname{gr}"] & {} & \\
& \mathcal{A} & & & \mathcal{A}
\end{tikzcd}$$

In other words, once one finds a candidate model $\mathcal{C}$ for $\mathcal{A}$, finding an *expansion valued in* $\mathcal{C}$ also implies that $\psi$ is an isomorphism. In classical Vassiliev theory, $\mathcal{K}$ is the space of oriented knots, $\mathcal{C}$ is the space of chord diagrams, and a $\mathcal{C}$-valued expansion is the Kontsevich integral [Kon93].

For a detailed introduction to the Kontsevich integral we recommend [CDM12, Section 8]. The definition an explicit integral formula associated to a Morse representation of a knot or link in $\mathbb{C} \times \mathbb{R}$, as in Figure 4.

Note from Dror: reconsider depth of discussion of $Z$

**Definition 3.2.** Let $K$ be an oriented link in $\mathbb{R}^3 \cong \mathbb{R}_t \times \mathbb{C}$, where $t$ parametrises the vertical real dimension, and the embedding $K$ is Morse with respect to $t$: that

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[4] "Knotted objects" may mean knots, links, tangles, knotted graphs, etc, depending on context.

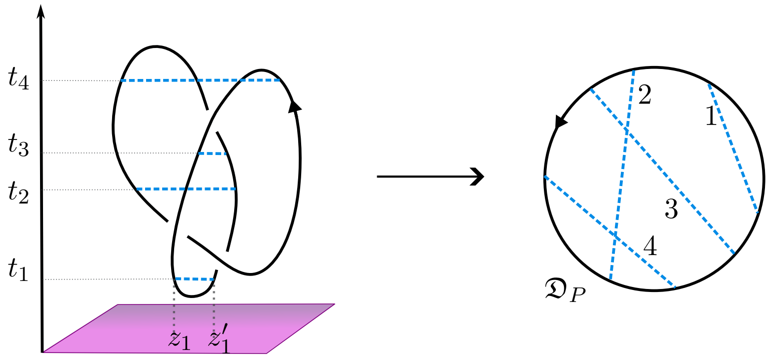


fig:Kint

FIGURE 4. The Kontsevich Integral is computed from a Morse embedding of the knot

is, critical points are cups or caps. The unnormalised Kontsevich integral $Z'(K)$ of $K$ is defined as:

eq:Kint

$$Z'(K) := \sum_{m=0}^{\infty} \int\limits_{\substack{t_{min}<t_m<\ldots<t_1<t_{max} \\ t_i \; non-critical}} \sum_{P=\{(z_i,z_i')\}_i} \frac{(-1)^{P_\downarrow}}{(2\pi i)^m} \mathfrak{D}_P \bigwedge_{i=1}^{m} \frac{dz_i - dz_i'}{z_i - z_i'}. \tag{3.1}$$

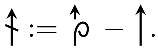
Here the values $t_{min}$ and $t_{max}$ denote the minimum and maximum heights of K, and each summand is an integral over an $m$-simplex determined by $t_m < \cdots < t_1$. The summation is over choices $P$ of "pairings" of two points on the knot of height $t_i$, each of which, when projected to the plane at $t = 0$, yields a complex pair $(z_i, z_i')$. We denote by $\mathfrak{D}_P$ the chord diagram given by interpreting the $m$ pairings $(z_i, z_i')$ as chords, as shown in Figure 4. Finally, $P_\downarrow$ is the number of points in $P$ where $(t_i, z_i)$ or $(t_i, z_i')$ is on a $t$-descending arc in $K$.

Kontsevich's famous result [Kon93] is that $Z(K) := \frac{Z'(K)}{Z'(\infty)^{c/2}}$ is an invariant of unframed links, where $c$ denotes the number of critical points – minima and maxima – in the Morse embedding of $K$. The Kontsevich integral takes values in the space of chord diagrams (for links, with chords on multiple circles) modulo the 4T and FI relations. The Kontsevich integral $Z$ satisfies $\psi \circ \operatorname{gr} Z = id_{\mathcal{C}}$. Therefore, $\psi$ is an isomorphism, and $Z$ is an expansion for unframed links. In light of this, we do not distinguish between $\mathcal{C}$ and $\mathcal{A}$, and use $\mathcal{A}$ to mean In addition, $Z$ has a number of good properties, for example, it is homomorphic with respect to knot connected sum.

subsec:FramedKon

3.2. **The framed Kontsevich Integral.** Kontsevich's original construction gives an invariant of unframed links. However, in this paper we work primarily with framed links and tangles, thus we briefly review the framed version; see also [CDM12, Sections 3.5 and 9.1] and [LM96].

First we need a framed version of the Vassiliev filtration. Let $\tilde{\mathcal{K}}$ denote the set of *framed* links in $\mathbb{R}^3$: that is, links along with a non-zero section of the normal bundle. A knot diagram is interpreted as a framed knot using the blackboard framing. The Reidemeister move R1 move changes the blackboard framing, and by ommitting it, one obtains a Reidemeister theory for framed links. In analogy with a double point, a *framing change* is defined to be the difference



The framed Vassiliev filtration is the descending filtration

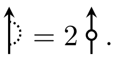
$$\tilde{\mathcal{K}} = \tilde{\mathcal{K}}_0 \supseteq \tilde{\mathcal{K}}_1 \supseteq \tilde{\mathcal{K}}_2 \supseteq \ldots$$

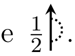
where $\tilde{\mathcal{K}}_i$ is linearly generated by knots with at least $i$ double points *or framing changes*. The degree completed associated graded space of $\tilde{\mathcal{K}}$ with respect to the framed Vassiliev filtration is

$$\tilde{\mathcal{A}} := \prod_{n \geq 0} \tilde{\mathcal{K}}_n / \tilde{\mathcal{K}}_{n+1}.$$

A natural first guess for a combinatorial description of $\tilde{\mathcal{A}}$ is in terms of chord diagrams with "framing change markings"  on the skeleton, graded by the number of chords and markings. There is a natural surjective graded map $\tilde{\psi}$ from marked chord diagrams onto $\tilde{\mathcal{A}}$, which is defined like $\psi$ for chords, and which replaces each marking  with a framing change . The kernel of $\tilde{\psi}$ includes the $4T$ relation as before.

In place of the $FI$ relation ( $=0$), a weaker relation arises from the equality  $-$  $=$  in $\tilde{\mathcal{K}}$. In fact,  $=$  $-$  $=$ (  $-$  ) $+$ (  $-$  ), and  $-$  $=$  $-$  modulo $\tilde{\mathcal{K}}_2$. In other words, the following relation is in the kernel of $\tilde{\psi}$:



Therefore, it is not necessary to have dedicated notation for the framing change markings, since . The candidate model for the associated graded space is simply chord diagrams modulo the $4T$ relation, and no $FI$ relation. We denote this space by $\tilde{\mathcal{C}}$.

To show that $\tilde{\psi} : \tilde{\mathcal{C}} \to \tilde{\mathcal{A}}$ is an isomorphism, the strategy is the same as before: construct a $\tilde{\mathcal{C}}$-valued expansion and use Lemma 3.1. This $\tilde{\mathcal{C}}$-valued expansion is the framed version $\tilde{Z}$ of the Kontsevich integral. The definition is similar to (3.1), the main issue is that in the absence of the $FI$ relation, the integral diverges at cups and caps. This is resolved with a renormalisation using the framing, for details see [CDM12, Section 9.1], or [LM96, Gor99].

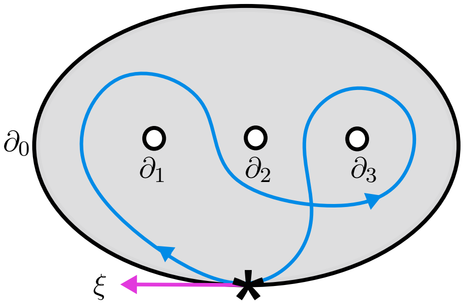


fig:DP

FIGURE 5. $D_3$ with an immersed loop based at $*$, with initial and terminal tangent vector $\xi$.

subsec:IntroGT

### 3.3. **The Goldman-Turaev Lie bialgebra.**

In order to define the Goldman-Turaev Lie bialgebra, we need to recall some basic definitions and notation.

Let $D_p$ denote $p$-punctured disc, with $p+1$ circle boundary components $\partial_0, \partial_1, ..., \partial_p$, embedded in the complex plane so that $\partial_0$ is the outer boundary, as in Figure 5. In particular, the plane-embedding specifies a framing on $D_p$, and thus loops in $D_p$ are equipped with a notion of rotation number. Let $\pi = \pi_1(D_p, *)$ denote the fundamental group of $D_p$ with basepoint $* \in \partial_0$. We denote by $\mathbb{C}\pi$ the group algebra of $\pi$, and by $\overline{\mathbb{C}\pi} = \mathbb{C}\pi/\mathbb{C}\mathbb{1}$ the linear quotient by the constant loop.

Let $\xi$ be a tangent vector to $\partial_0$ at $*$ and let $\tilde{\pi} = \tilde{\pi}_*$ denote the group of immersed curves $\gamma : ([0,1], 0, 1) \to (D_p, *, *)$ under regular homotopy, so that $\dot{\gamma}(0) = \dot{\gamma}(1) = \xi$, as shown in Figure 5. Note that rotation number is invariant under regular homotopy. Recall that $\tilde{\pi}$ is a group: the identity is a contractible loop of rotation number zero (figure eight), and inverses require the insertion of small twists as all curves in $\tilde{\pi}$ start in the same direction. Denote by $\mathbb{C}\tilde{\pi}$ the group algebra of $\tilde{\pi}$.

Zsuzsi, please draw dummy figure for me to include

For an algebra $A$ we denote by $|A|$ the *linear* quotient $A/[A, A]$ (not the abelianization), where $[A, A]$ denotes the subspace spanned by commutators $[x, y] = xy - yx$ for $x, y \in A$. We denote the quotient (trace) map by $|\cdot| : A \to |A|$. In our context, elements of $|\mathbb{C}\pi|$ have an explicit description as the $\mathbb{C}$-vector space generated by homotopy classes of free loops in $D_p$, and similarly, $|\mathbb{C}\tilde{\pi}|$ is comprised of free immersed loops.

The Goldman–Turaev Lie bialgebra comes in two flavours: *original* and *enhanced*. The original construction of the Goldman bracket is a Lie bracket on $|\mathbb{C}\pi|$. However, the original Turaev cobracket is only well-defined on $|\overline{\mathbb{C}\pi}| = |\mathbb{C}\pi|/\mathbb{C}\mathbb{1}$. The space $|\overline{\mathbb{C}\pi}|$ is a Lie bialgebra with this cobracket and the Goldman bracket, which descends from $|\mathbb{C}\pi|$.

fig:exbracket

FIGURE 6. Example calculation of the Goldman bracket.

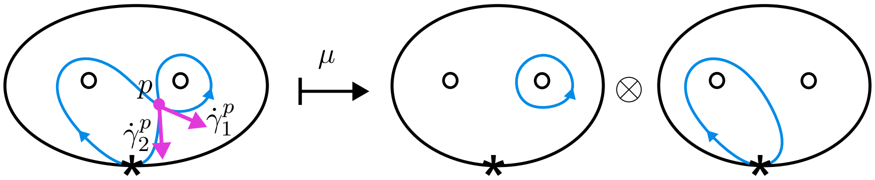


fig:defmu

FIGURE 7. Example of the self intersection map $\mu$ where $\epsilon_p = +1$.

There is an enhancement of the cobracket which promotes it to $|\mathbb{C}\pi|$, thereby making $|\mathbb{C}\pi|$ a Lie bialgebra under the Goldman bracket and the enhanced cobracket. In [AKKN18b] this enhancement was constructed in order to establish the relationship to Kashiwara–Vergne theory. To define the enhanced cobracket, a curve in $|\mathbb{C}\pi|$ is lifted to an immersed curve with rotation number zero. We work with this enhanced version.

Zsuzsi, please make figure draft for me to make pretty

sort of clunky

The Goldman Bracket, in summary, takes in two free loops and at each intersection between the two loops smooths the intersection to join the loops into one longer connected loop, then sums over each intersection. An example computation for the Goldman Bracket is shown in Figure 6. For a free loop $\alpha$ in $D_p$ and a point $q$ on $\alpha$, we denote by $\alpha_q$ be the loop $\alpha$ considered to be based at $q$.

def:bracket

**Definition 3.3** (The Goldman bracket)**.** Let $\alpha, \beta \in |\mathbb{C}\pi|$ be free loops with homotopy representatives chosen so that there are only finitely many transverse double intersections between $\alpha$ and $\beta$. The Goldman bracket $[\cdot,\cdot]_G : |\mathbb{C}\pi| \otimes |\mathbb{C}\pi| \to |\mathbb{C}\pi|$ is given by

$$[\alpha, \beta]_G \coloneqq - \sum_{q \in \alpha \cap \beta} \varepsilon_q |\alpha_q \beta_q|,$$

where $\varepsilon_q = \varepsilon(\dot\alpha_q, \dot\beta_q) \in \{\pm 1\}$ is the local intersection number of $\alpha$ and $\beta$ at $q$, $\alpha_q\beta_q$ is the concatenation of $\alpha_q$ and $\beta_q$, and the extension to $|\mathbb{C}\pi|$ is linear. Then $[\cdot,\cdot]_G$ is a Lie bracket on $|\mathbb{C}\pi|$.

The sign here (with the minus sign in front) matches with AKKN genus 0, but is the opposite of AKKN higher genus and Goldman's original definition. Our current multiplication and bracket matches the sign here, so if we change the sign then we should change the stacking order of our multiplication.

The original definition of the Turaev cobracket is similar, but uses self intersections of a curve in place of the intersections between two curves, but one has to take care for well definedness. We first construct the (enhanced) cobracket via a self-intersection map for *based* curves, as in [AKKN18b, Section 5.2]. This definition is suitable for direct comparison with the three-dimensional operations of Section 5. Figure 7 shows an example computation for the self intersection map $\mu$. For a based curve $\gamma$ in $\mathbb{C}\pi$, the idea behind $\mu$ is to "cut off" portions of $\gamma$ at self intersection points to get two curves, one that is based, one that is free, and then sum over all self intersections.

`def:mu`

**Definition 3.4** (The self-intersection map)**.** For $\gamma \in \mathbb{C}\pi$, let $\tilde{\gamma} : [0,1] \to D_p$ denote an immersed representative of $\gamma$ in $\mathbb{C}\tilde{\pi}$ with only transverse double points, and with rotation number $\mathrm{rot}(\tilde{\gamma}) = 0$. Let $\tilde{\gamma} \cap \tilde{\gamma}$ denote the set of double points. The self intersection map $\mu$ is defined as folows:

$$\mu : \mathbb{C}\pi \to |\mathbb{C}\pi| \otimes \mathbb{C}\pi$$

$$\mu(\gamma) = -\sum_{p \in \tilde{\gamma} \cap \tilde{\gamma}} \varepsilon_p |\tilde{\gamma}_{t_1^p t_2^p}| \otimes \tilde{\gamma}_{0 t_1^p} \tilde{\gamma}_{t_2^p 1},$$

where $t_1^p$ and $t_2^p$ are the first and second time parameter in $[0,1]$ where $\tilde{\gamma}$ goes through $p$; where $\tilde{\gamma}_{rs}$ denotes the path traced by $\tilde{\gamma}$ from $t = r$ to $t = s$; and $\varepsilon_p = \varepsilon\left(\dot{\tilde{\gamma}}(t_1^p), \dot{\tilde{\gamma}}(t_2^p)\right) \in \{\pm 1\}$ is the local self-intersection number, and the formula extends to $\mathbb{C}\pi$ liearly.

See Figure 7 for an example calculation of the self-intersection map.

**Definition 3.5** (The Turaev co-bracket)**.** The Turaev cobracket $\delta$ is the unique linear map which makes the following diagram commute, where $\mathrm{Alt}(x \otimes y) = x \otimes y - y \otimes x = x \wedge y$:

$$\begin{tikzcd}
\mathbb{C}\pi \arrow[r, "\mu"] \arrow[d, "|\cdot|"] & {|\mathbb{C}\pi| \otimes \mathbb{C}\pi} \arrow[r, "1 \otimes |\cdot|"] & {|\mathbb{C}\pi| \otimes |\mathbb{C}\pi|} \arrow[d, "\mathrm{Alt}"] \\
{|\mathbb{C}\pi|} \arrow[rr, dashed, "\delta"] & & {|\mathbb{C}\pi| \wedge |\mathbb{C}\pi|}
\end{tikzcd}$$

3.4. **Associated graded Goldman-Turaev Lie bialgebra.** There I-adic filtration on $\mathbb{C}\pi$ is the filtration by powers of the augmentation ideal $\mathcal{I} = \langle \{\alpha - 1\}_{\alpha \in \pi} \rangle$:

$$\mathbb{C}\pi = \mathcal{I}^0 \supseteq \mathcal{I} \supseteq \mathcal{I}^2 \supseteq \dots$$

This descends to a filtration on $|\mathbb{C}\pi|$:

$$|\mathbb{C}\pi| = |\mathcal{I}^0| \supseteq |\mathcal{I}| \supseteq |\mathcal{I}^2| \supseteq \dots$$

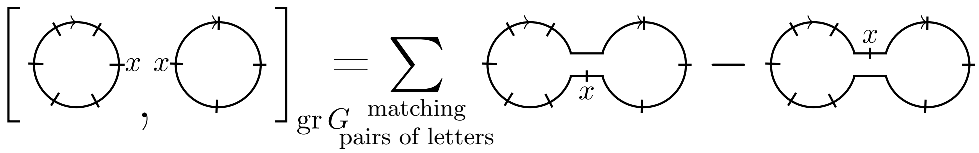
The completed associated graded algebra for $|\mathbb{C}\pi|$ with respect to this filtration

$$\mathrm{gr}\,|\mathbb{C}\pi| = \prod_{n=0}^{\infty} |\mathcal{I}^n| / |\mathcal{I}^{n+1}|$$

has an explicit description in terms of *cyclic words*, as follows. Let $\mathrm{FA} = \mathrm{FA}\langle x_1, \cdots, x_p \rangle$ denote the free associative algebra on $r$ generators. Then elements of $|\,\mathrm{FA}\,|$ are cyclic words in letters $x_1, \cdots, x_p$, that is, words modulo cyclic permutations of the letters. The following result is due to [].

complete with citation: Quillen66?

I couldn't find the reference. Can you give more info?



`fig:grbracket`

FIGURE 8. The graded Goldman bracket.

**Proposition 3.6.** *Given the set of standard generators* $\{\gamma_i\}_{i=1}^{p}$ *for* $\pi$*, the map* $\varphi(\gamma_i^{\pm 1}) = e^{\pm x_i}$ *defines an isomorphism*[5] *of algebras* $\operatorname{gr}|\mathbb{C}\pi| \to \mathrm{FA}$*, which descends to a linear isomorphism*

$$\operatorname{gr}|\mathbb{C}\pi| \cong |\,\mathrm{FA}\,|.$$

*Therefore,* $|\,\mathrm{FA}\,|$ *carries the structure of a Lie bialgebra under* $\operatorname{gr}[\cdot,\cdot]_G$ *and* $\operatorname{gr}\delta$*.*

Representing cyclic words diagrammatically as letters along a circle, the graded Goldman bracket sums over matching pairs of letters in $z$ and $w$, joins the circles at the matching letter, and takes the difference of the two ways of including only one copy of the letter in the new cyclic word, as in Figure 8.

`def:grbracket`

**Definition 3.7.** [The graded Goldman bracket] Let $z = z_1 \cdots z_l$ and $w = w_1 \cdots w_m$ be two cyclic words in $|\,\mathrm{FA}\,|$. The *graded Goldman bracket*

$$[-,-]_{\operatorname{gr}G} : |\,\mathrm{FA}\,| \otimes |\,\mathrm{FA}\,| \to |\,\mathrm{FA}\,|$$

of $z$ and $w$ is given by:

$$[z,w]_{\operatorname{gr}G} = \sum_{j,k} \delta_{z_j,w_k}(w_1 \dots w_{k-1} z_{j+1} \dots z_l z_1 \dots z_j w_{k+1} \dots w_m - \\ w_1 \dots w_{k-1} z_j \dots z_l z_1 \dots z_{j-1} w_{k+1} \dots w_m)$$

where $\delta_{z_j,w_k}$ is the Kronecker delta.

**Proposition 3.8.** *The graded Goldman bracket* $[-,-]_{\operatorname{gr}G}$ *is the associated graded map of the Goldman bracket under the filtration induced by the I-adic filtration.*

add reference or proof

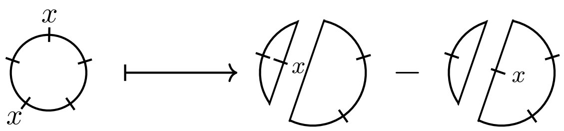
Using the same diagrammatic representation of cyclic words as letters along a circle, the graded Turaev co-bracket of a cyclic word is computed by a summation of *pairing cuts*, an example is shown in Figure 9. For a pairing cut, identify two matching letters in the cyclic word, call the letters $x$. Draw a chord between the two $x$'s and split the circle into to two separate circles along this chord. Remove the two $x$'s. Take the difference of the two words where an $x$ is placed on the left chord and an $x$ is placed on the right chord. (The cyclic word has an orientation which determines the 'left' and 'right'.)

**Definition 3.9** (The graded Turaev cobracket)**.** Let $w = w_1 \dots w_m \in |As_p|$. The graded Turaev cobracket for $w$ is given by

$$\delta_{\mathrm{gr}} : |\,\mathrm{FA}\,| \to |\,\mathrm{FA}\,| \wedge |\,\mathrm{FA}\,|$$

---

[5] $\varphi$ is the Magnus expansion



`fig:paircut`

Figure 9. An example pairing cut of a cyclic word as a term in the graded Turaev cobracket.

$$\delta_{\mathrm{gr}}(w) = \sum_{j<k} \delta_{w_j,w_k}(|w_j \dots w_{k-1}| \wedge |w_{k+1} \dots w_n w_1 \dots w_{j-1}| + \\ |w_k \dots w_n w_1 \dots w_{j-1}| \wedge |w_{j+1} \dots w_{k-1}|),$$

where $\delta_{w_j,w_k}$ is (unfortunately) the Kronecker delta.

**Proposition 3.10.** *The graded Turaev cobracket $\delta_{\mathrm{gr}}$ is the associated graded map of the Turaev cobracket under the filtration induced by the I-adic filtration.*

An diagrammatic example of a term in the definition of the graded cobracket is given in Figure 9.

`sec:TangleSetUp`

## 4. Expansions for tangles in handlebodies

4.1. **The space $\mathbb{C}\tilde{\mathcal{T}}$.** In this paper we consider the space $\mathbb{C}\tilde{\mathcal{T}}$ of framed, oriented tangles in a genus $p$ handlebody, and show that homomorphic expansions on this space descend to homomorphic expansions on the Goldman-Turaev Lie biagebra as defined in [AKKN20]. This section describes the space $\mathbb{C}\tilde{\mathcal{T}}$.

Let $M_p$ denote the manifold $D_p \times I$ where $D_p$ is a disc in the complex plane with $p$ points removed. While $M_p$ is not a compact manifold, knot theory in $M_p$ is equivalent to knot theory in a genus $p$ handlebody. For the purpose of the Kontsevich integral, we identify $D_p$ with a square in the complex plane with $p$ points removed, so $M_p$ can be drawn as a cube with $p$ vertical lines removed; we call these lines the *poles*, as shown in Figure 11. We refer to $D_p \times \{0\}$ as the "floor" or "bottom", and $D_p \times \{1\}$ as the "ceiling" or "top". The "back wall" is the north ($i \in \mathbb{C}$) edge of $D_p$ times $[0, 1]$.

`def:tangle`

**Definition 4.1.** An oriented tangle $T$ in $M_p$ is an embedding of an oriented compact 1-manifold

$$(S, \partial S) \hookrightarrow (M_p, D_p \times \{0\} \cup D_p \times \{1\}).$$

The interior of $S$ lies in the interior of $M_p$, and the boundary points of $S$ are mapped to the top or bottom. Oriented tangles in $M_p$ are considered up to ambient isotopy fixing the boundary. We denote the set of isotopy classes by $\mathcal{T}$.

**Definition 4.2.** A *framing* for an oriented tangle $T$ in $M_p$ is a continuous choice of unit normal vector at each point of $T$, which is fixed pointing in the north direction ($i \in \mathbb{C}$) at the boundary points. *Framed oriented tangles* in $M_p$ are

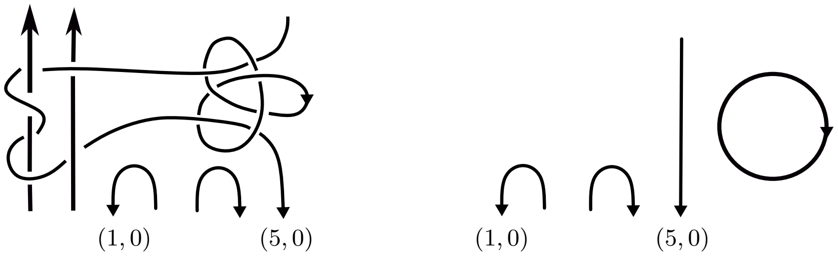


FIGURE 10. On the left is a tangle in $M_2$, and on the right is schematic diagram of the skeleton of the tangle. The skeleton of the tangle is the combinatorial data given by the following set of order pairs and the integer 1: $\{[((2,0),0),((1,0),0))],[((3,0),0),(4,0),0))],[((5,0),1),(5,0),0))],1\}$ .

fig:skeleton

considered up to ambient isotopy fixing the boundary. We denote the set of isotopy classes of framed oriented tangles by $\tilde{\mathcal{T}}$.

Henceforth, any tangle is assumed to be framed and oriented unless otherwise stated. The skeleton of a tangle is the underlying combinatorial information with the topology forgotten:

def:skeleton

**Definition 4.3.** The *skeleton* $\sigma(T)$ of a tangle $T = (S \hookrightarrow M_p)$ – see Figure 10 – is the set of tangle endpoints $P_{bot} \subseteq D_p \times \{0\}$ and $P_{top} \subseteq D_p \times \{1\}$, along with

(1) A partition of $P_{bot} \cup P_{top}$ into ordered pairs given by the oriented intervals of $S$.
(2) A non-negative integer $k$: the number of circles in $S$.

Maybe it would be better to define $P_{bot}, P_{top} \subseteq D_p$ and then say $P_{bot} \times \{0\}$ and $P_{\top} \times \{1\}$ are the tangle endpoints. Then it would make descriptions of tangle operations easier, as well as the info in figure 9.

The skeleton of a framed tangle is the same as the skeleton of the underlying unframed tangle. The set of framed tangles in $M_p$ with skeleton $S$ is denoted $\tilde{\mathcal{T}}(S)$. For example, $\tilde{\mathcal{T}}(\bigcirc)$ is the set of framed knots in $M_p$.

The linear extension of $\tilde{\mathcal{T}}(S)$, denoted $\mathbb{C}\tilde{\mathcal{T}}(S)$, is the vector space of $\mathbb{C}$-linear combinations of tangles in $\tilde{\mathcal{T}}(S)$. We denote by $\mathbb{C}\tilde{\mathcal{T}}$ the disjoint union $\sqcup_S \tilde{\mathcal{T}}(S)$ over all skeleta $S$, identified at 0. Tangles with different skeleta cannot be linearly combined.

We can look at tangles in $M_p$ using tangle diagrams in two different ways, by projecting either to the back wall of $M_p$ or to the floor.

If we project to the back wall, an $\ell$-component tangle in $M_p$ can be diagrammatically represented as a tangle diagram with $p$ straight vertical "poles", and $\ell$ tangle "strands" of circle and interval components. The strands pass over (in front of) and under (behind) the poles and other strands, as shown on the right in Figure 11. The poles are equipped with an orientation coming from the parametrization in $M_p \cong D_p \times I$, and in figures we draw them oriented upwards, unless otherwise stated. By Reidemeister's theorem, $\tilde{\mathcal{T}}$ is equivalent to such diagrams modulo the Reidemeister moves R2 and R3. (No R1, as the tangles are framed.)

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no its not–it is a refresh rate problem.

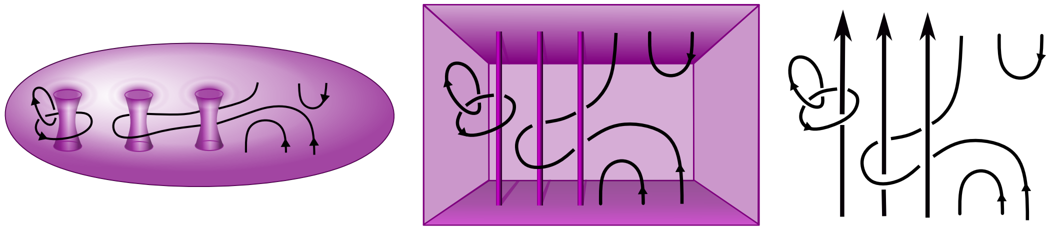


FIGURE 11. An example of a tangle in $M_3$, drawn first in a handlebody, secondly in a cube with poles, and lastly as a tangle diagram projected to the back wall of the cube.

`fig:polestudio`

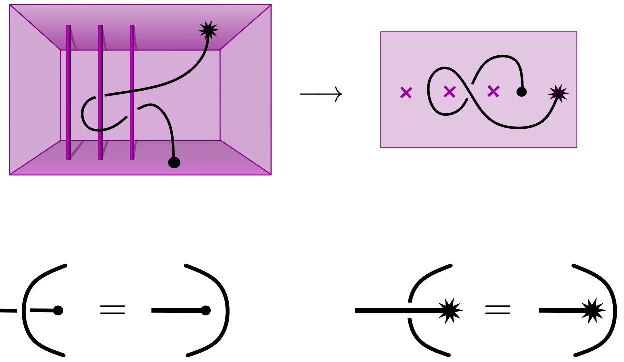


FIGURE 12. An example of a tangle in $M_3$ projected to the bottom floor of the cube. Strands of a tangle diagram can pass over bottom endpoints (dot) or under top endpoints (star).

`fig:BottomDiagram`

By projecting instead to the floor $D_p \times \{0\}$ of the cube, a tangle in $M_p$ is represented by a tangle diagram in $D_p$. The R2 and R3 moves continue to apply. The endpoints of the tangle are fixed: bottom endpoints are denoted by dots, top endpoints are denoted by stars. Strands of the tangle diagram can pass over bottom endpoints, or under top endpoints, as shown in Figure 12. However, the strands cannot pass over the punctures in $D_p$.

`sec:opsonT`

4.2. **Operations on $\tilde{\mathcal{T}}$.** There are several useful operations defined on $\tilde{\mathcal{T}}$. These operations extend naturally to $\mathbb{C}\tilde{\mathcal{T}}$, and are used in Section 5 to relate quotients of $\mathbb{C}\tilde{\mathcal{T}}$ to the Goldman-Turaev Lie bialgebra.

check this section reference after Section 4 is finalised

- *Stacking:* Given tangles $T_1, T_2 \in M_p$, if the top endpoints of $\sigma(T_1)$ match the bottom endpoints of $\sigma(T_2)$ in $D_p$, and the orientations on the strands of $T_1$ and $T_2$ agree at the matching endpoints, then we can stack $T_2$ on top of $T_1$ and shrink the height to get a new tangle $T_1T_2 \in M_p$.

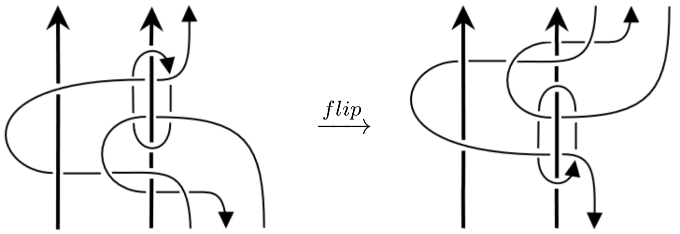


fig:flip

Figure 13. A tangle in $M_2$ and its flip

- *Strand addition:* The *strand addition* operation adds a non-interacting additional strand to a tangle $T$ at a point $q \in D_p$ to get a new tangle $T \sqcup_q \uparrow$. More precisely, pick a contractible $U \subseteq D_p$ such that $T$ is contained entirely in $U \times [0,1]$ and a point $q \in D_p$ outside of $U$. The tangle $T \sqcup_q \uparrow$ is $T$ together with an upward-oriented vertical strand $q \times I$ at $q$.
- *Strand orientation switch:* This operation reverses the orientation of a given strand of the tangle.
- *Flip:* Given a tangle $T$ in $M_p$, the flip of a tangle $T$ in $M_p$, denoted $\overline{T}$, is the mirror image of $T$ with respect to the ceiling, as shown in Figure 13. When $T$ is flipped, each top boundary point $(q,1)$ becomes a bottom boundary point $(q,0)$, and vice versa. The orientations and framing of the strands of $T$ are reflected along with the strands. However, the orientations of the poles remain ascending. Equivalently, we can define the flip operation as reversing the parametrisation of $I$ in $M_p \cong D_p \times I$. This, in effect, flips the orientation of the poles but changes nothing else.

In section 5.1, we relate commutator of tangles with respect to stacking, given by $[T_1, T_2] = T_1T_2 - T_2T_1$, to the Goldman bracket, and in section 5.2 we relate the flip operation to the Turaev cobracket.

sec:t-filtration

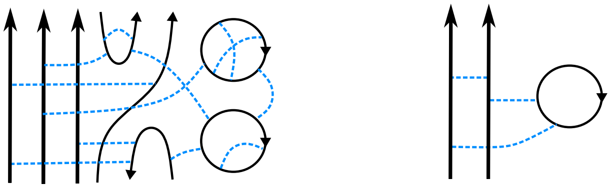
### 4.3. The $t$-filtration on $\tilde{\mathcal{T}}$ and the associated graded $\tilde{\mathcal{A}}$.

There are different filtrations on the space $\mathbb{C}\tilde{\mathcal{T}}$ that one might consider in setting up a Vassiliev theory. In line with classical notation of Vassiliev invariants, we denote by a double point the difference between an over-crossing and an under-crossing:

In the context of tangles in $M_p$, double points come in two varieties: *pole-strand*, if the crossing occurs between a pole and a tangle strand, and *strand-strand*, if the crossing occurs between two tangle strands.

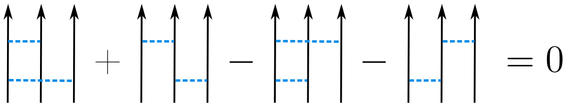
The main filtration we consider on $\mathbb{C}\tilde{\mathcal{T}}$ is the filtration by the total number of double points of either type, as well as strand framing changes (as described in Section 3.2). We call this the *t̲otal filtration*, or simply $t$-filtration, and write it as

$$\mathbb{C}\tilde{\mathcal{T}} = \tilde{\mathcal{T}}_0 \supseteq \tilde{\mathcal{T}}_1 \supseteq \tilde{\mathcal{T}}_2 \supseteq \tilde{\mathcal{T}}_3 \supseteq \cdots$$



(A) Two chord diagrams: an admissible one (left) that doesn't contain any pole-pole chords, and non-admissible one (right) that does contain a pole-pole chord.

ssibleNonAdmissible



(B) The 4T relation, which is admissible if at most one of the three skeleton components is a pole.

fig:Admissible 4T

FIGURE 14. Examples of admissible and non-admissible chord diagrams, and the 4T relation

fig:admissible

where $\tilde{\mathcal{T}}_t$ is the set of linear combinations of framed tangle diagrams with at least $t$ total double points and strand framing changes.

**Definition 4.4.** The associated graded space of $\mathbb{C}\tilde{\mathcal{T}}$ with respect to the total filtration is

$$\tilde{\mathcal{A}} := \operatorname{gr} \mathbb{C}\tilde{\mathcal{T}} = \prod_{t \geq 0} \tilde{\mathcal{T}}_t / \tilde{\mathcal{T}}_{t+1}.$$

The degree $t$ component of $\tilde{\mathcal{A}}$ is $\tilde{\mathcal{A}}_t := \tilde{\mathcal{T}}_t / \tilde{\mathcal{T}}_{t+1}$.

rem:2frame=double

*Remark* 4.5. Modulo $\tilde{\mathcal{T}}_2$,  $=$  $-$  $=$  $-$ . As a result, in $\tilde{\mathcal{A}}$, a framing change can always be represented as $\frac{1}{2}$ a double point as

 $=$  $-$  $= ($  $-$  $) + ($  $-$  $) = 2$ .

As in classical Vassiliev theory (cf. section 3.2), the associated graded space $\tilde{\mathcal{A}}$ has a combinatorial description in terms of *chord diagrams*.

**Definition 4.6.** A *chord diagram* on a tangle skeleton is an even number of marked points on the poles and skeleton strands, up to orientation preserving diffeomorphism, along with a perfect matching on the marked points – that is, a partition of marked points into unordered pairs. In diagrams, the pairs are connected by a *chord*, indicated by a dotted line, as in Figure 14(A).

def:admissible

**Definition 4.7.** A chord diagram is *admissible* if all chords connect strands to strands, or strands to poles. That is, there are no pole-pole chords in an admissible diagram, see Figure 14(A) for an example.

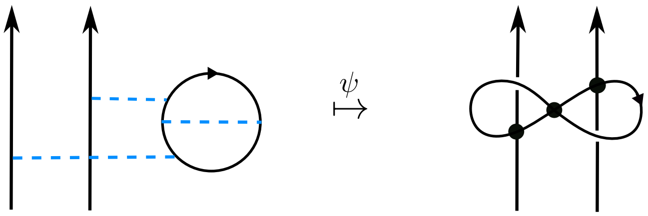


FIGURE 15. Example of $\psi$ with the right hand side viewed as an element of $\tilde{\mathcal{T}}_3/\tilde{\mathcal{T}}_4$. Different choices of over or under crossings with the poles only differ by elements of $\tilde{\mathcal{T}}_4$.

fig:psi

def:cdspace

**Definition 4.8.** The space $\mathcal{D}(S)$ of *admissible chord diagrams on a diagram* $S$ is the space of $\mathbb{C}$-linear combinations of admissible chord diagrams on the skeleton $S$ factored out by *admissible* $4T$ relations, shown in Figure **??**. Admissible $4T$ relations are $4T$ relations in the classical sense, subject to the condition that all four terms are admissible[6]. That is,

Do we need the concept of "admissible 4T"? Since 4T is a relation, so just saying "admissible chord diagrams mod 4T" would only apply 4T to admissible diagrams?

$$\mathcal{D}(S) = \frac{\{\text{linear combinations of admissible chord diagrams on } S\}}{\{\text{admissible } 4T \text{ relations}\}}$$

The space $\mathcal{D}(S)$ is a graded vector space, where the degree is given by the number of chords. Denote the degree $t$ component of $\mathcal{D}(S)$ by $D_t(S)$. Let $\mathcal{D}$ be the disjoint union $\sqcup_S \mathcal{D}(S)$, identified at 0. We denote the degree $t$ component of $\mathcal{D}$ by $\mathcal{D}_t = \sqcup_S \mathcal{D}_t(S)$.

There is a familiar isomorphism from classical finite type theory

$$\psi : \mathcal{D} \to \tilde{\mathcal{A}}.$$

In degree $t$, $\psi_t : \mathcal{D}_t \to \tilde{\mathcal{T}}_t/\tilde{\mathcal{T}}_{t+1}$, is defined as before by contracting chords to double points, as shown in Figure 15. This may create other crossings, but modulo $\tilde{\mathcal{T}}_{t+1}$ it does not matter which skeleton component is over or under at these crossings.

add reference for theorem?

thm:tassocgraded

**Theorem 4.9.** *The map $\psi : \mathcal{D} \to \tilde{\mathcal{A}}$ is an isomorphism.*

We prove that $\psi$ is an isomorphism by showing that it is well-defined and surjective, then using lemma 3.1 to show that it is an isomorphism.

**Lemma 4.10.** *The map $\psi$ is well-defined and surjective.*

*Proof.* To show $\psi$ is well-defined, it suffices to show that admissible $4T$ relations in $\mathcal{D}_t$ are in the kernel of $\psi$. This is shown in Figure 16. For surjectivity, a framing change in $\tilde{\mathcal{A}}$ can always be written as one half a double point, as described in Remark 4.5. So all framing changes are in the image of $\psi$, and $\psi$ is surjective. □

---

[6] Equivalently, a 4T relation is admissible if at most one of the three skeleton components involved is a pole.

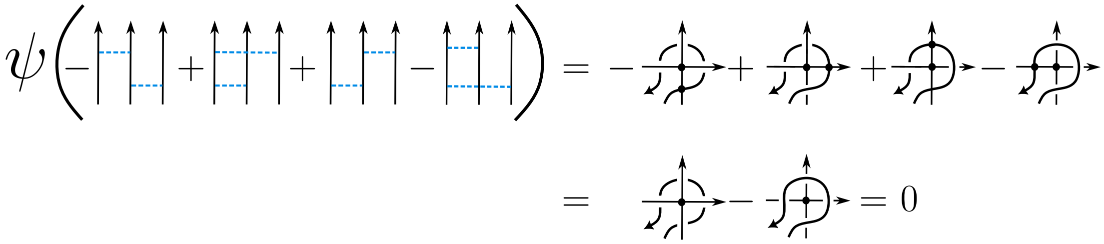


FIGURE 16. The proof that $\psi : \mathcal{D} \to \tilde{\mathcal{A}}$ is well defined. The figure is understood locally: If the figure is a map in the degree $t$ component, then the chord diagrams have $t-2$ other chords that are not shown but in the same position throughout all four terms, and similarly, the tangles have $t-2$ other double points that are not shown, but in the same positions throughout all the terms.

`fig:psicomputation`

`thm:Zwelldefined`

**Lemma 4.11.** *The Kontsevich integral $Z$ is a well-defined filtered map from $\mathbb{C}\tilde{\mathcal{T}}$ to $\mathcal{D}$ such that $\psi \circ \operatorname{gr} Z = \operatorname{id}_{\tilde{\mathcal{A}}}$.*

*Proof.* The image of $Z$ on an element in $\mathbb{C}\tilde{\mathcal{T}}$ will be a chord diagram on a skeleton with $p$ poles and some number of circles. Since the poles in $M_p$ are parallel, any pair of points $(z_i, z_i')$ on the poles will be constant, the form $dz_i - dz_i' = 0$, and the contribution to the integral will be zero. Therefore chord diagrams in the image of $Z$ don't contain pole-pole chords, so they are always admissible. So $Z$ indeed always lands in $\mathcal{D}$.

It remains to show that $\psi \circ \operatorname{gr} Z = id_{\tilde{\mathcal{A}}}$.

$$\begin{tikzcd}
\mathbb{C}\tilde{\mathcal{T}} \arrow[r, "Z"] & \mathcal{D} \arrow[d, "\psi"] & & \tilde{\mathcal{A}} \arrow[r, "\operatorname{gr} Z"] \arrow[dr, dashed, "\psi\circ\operatorname{gr} Z = id_{\tilde{\mathcal{A}}}"'] & \mathcal{D} \arrow[d, "\psi"] \\
& \tilde{\mathcal{A}} \arrow[r, phantom, "\overset{\operatorname{gr}}{\Longmapsto}"] & & & \tilde{\mathcal{A}}
\end{tikzcd}$$

Recall that for a filtered map $f : A \to B$, the associated graded $\operatorname{gr} f : \operatorname{gr} A \to \operatorname{gr} B$ is defined on graded components by $[a] \in A_t/A_{t+1} \mapsto [f(a)] \in B_t/B_{t+1}$. We consider $\operatorname{gr} Z : \tilde{\mathcal{A}} \to \mathcal{D}$. Let $[T] \in \tilde{\mathcal{T}}_t/\tilde{\mathcal{T}}_{t+1}$ so that is $T$ is a tangle in $M_p$ with at least $t$ double points. Note that it's always possible to pick such a representative, since a framing change can be written as $\frac{1}{2}$ times a double point in $\tilde{\mathcal{T}}_t/\tilde{\mathcal{T}}_{t+1}$. Then $Z(T)$ is a sum of chord diagrams with $e^{\frac{C}{2}} - e^{-\frac{C}{2}}$ at each chord $C$ corresponding to each double point in $T$. All terms with degree less than $t$ are zero, so the value of $\operatorname{gr} Z(T)$ depends only on the degree $t$ term of $Z(T)$. The degree $t$ term is a single chord diagram with a single chord for each double point, so applying $\psi$ to

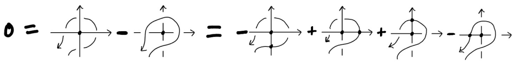


FIGURE 17. DUMMY IMAGE!!! placeholder for picture of chord diagram stacking and flip

`chorddiagoperations`

this turns all the chords back to double points, which up to crossing changes in $\tilde{\mathcal{T}}_{t+1}$, is just $[T]$. Therefore $\psi\,\mathrm{gr}\,Z = \mathrm{id}_{\tilde{\mathcal{A}}}$. Since $\psi\,\mathrm{gr}\,Z = id_{\tilde{\mathcal{A}}}$. □

The next corollary is immediate from lemma 3.1.

**Corollary 4.12.** *The map $\psi : \mathcal{D} \to \tilde{\mathcal{A}}$ is an isomorphism and $Z$ is an expansion for $\tilde{\mathcal{T}}$.*

Now it is established that $\tilde{\mathcal{A}}$ can be identified with the space of admissible chord diagrams $\mathcal{D}$. For a skeleton $S$, define $\tilde{\mathcal{A}}(S)$ to be the space of admissible chord diagrams on the skeleton $S$, so that $\tilde{\mathcal{A}}(S)$ is the associated graded of $\mathbb{C}\tilde{\mathcal{T}}(S)$. For example, $\tilde{\mathcal{A}}(\bigcirc)$ is the associated graded of $\mathbb{C}\tilde{\mathcal{T}}(\bigcirc)$, the space of knots in $M_p$.

4.4. **Operations on $\tilde{\mathcal{A}}$.** The operations *stacking* and *flip* on $\mathcal{T}$ induce operations by the same names on $\tilde{\mathcal{A}}$. In view of Theorem 4.9, we give descriptions of these operations using chord diagrams.

The operation *stacking* is given by stacking $D_1$ on top of $D_2$ by concatenating the the top ends of the poles in $D_2$ to the bottom ends of the poles in $D_2$ to get $D_1D_2$, see Figure 17. It is clear from the definition of $\psi$ that this is the correct chord diagram description of stacking, and as in $\mathcal{T}$, is only defined when the endpoints of $D_1$ and $D_2$ match appropriately.

The operation *flip* reflects a chord diagram with respect to a "mirror on the ceiling", reverses the orientations of the poles so that they are the same as they were originally, and adds a factor of $(-1)^m$, where $m$ is the total number of marked points on the poles. The factor of $(-1)^m$ comes from the fact that reversing the orientation of one strand at a double point is the same as multiplying by a factor of $-1$. See Figure 17.

describe the associated graded operations of all the tangle operations

`prop:Zhomom`

**Proposition 4.13.** *The Kontsevich integral $Z$ is homomorphic with respect to stacking, strand additions and flips.*

*Proof.* It is clear for stacking and strand addition. When the orientation of the poles are reversed, every chord diagram $D_P$ in the output of the Kontsevich integral will be multiplied get $(-1)^m$, where $m$ is the total number of chord endings on poles, because $m$ points in $P$ will change whether they are on a descending arc or not, so $P_{\downarrow}$ will change by $m$ mod 2. □

`sec:s-sfiltration`

4.5. **The $s$-filtration on $\tilde{\mathcal{T}}$ and $\tilde{\mathcal{A}}$.** As described in Section 4.3, the space $\mathbb{C}\tilde{\mathcal{T}}$ (and therefore $\tilde{\mathcal{A}}$) has a total filtration given by strand framing changes and double points of either type, strand-pole and strand-strand. In this section we look at a second filtration on $\mathbb{C}\tilde{\mathcal{T}}$ and $\tilde{\mathcal{A}}$, where we still look at strand framing changes, but only consider the number of strand-strand double points. This filtration will be called the *strand filtration*, or simply $s$-filtration. The $s$-filtration is given by

$$\mathbb{C}\tilde{\mathcal{T}} = \tilde{\mathcal{T}}^0 \supseteq \tilde{\mathcal{T}}^1 \supseteq \tilde{\mathcal{T}}^2 \supseteq \tilde{\mathcal{T}}^3 \supseteq \cdots$$

where $\tilde{\mathcal{T}}^s \subseteq \mathbb{C}\tilde{\mathcal{T}}$ are linear combinations of link diagrams with at least $s$ strand framing changes and strand double points.

*Remark* 4.14. We do *not* consider the full associated graded of $\mathbb{C}\tilde{\mathcal{T}}$ with respect to the $s$-filtration, but instead use it to identify the Goldman-Turaev spaces in low degrees in Section 5. The associated graded of $\mathbb{C}\tilde{\mathcal{T}}$ with respect to the $s$-filtration has been studied by Habiro and Massuyeau in [HM21], where they consider "bottom tangles". Note the language – if we project to the "bottom" instead of the "back wall", then all double points are of type strand-strand, so the $s$-filtration is just the usual Vassiliev filtration in the bottom projection.

The $s$-filtration also induces a filtration on $\tilde{\mathcal{A}}$ as follows. Combining the notations for the $t$- and $s$-filtrations, let $\tilde{\mathcal{T}}_t^s$ denote the set of linear combinations of tangle diagrams in $\mathbb{C}\tilde{\mathcal{T}}$ that have at least $t$ double points, at least $s$ of which are strand-strand.

ionQuotientNotation

**Definition 4.15.** The $s$-filtered component of $\tilde{\mathcal{A}}$ denoted $\tilde{\mathcal{A}}^{\geq s} := \prod \tilde{\mathcal{T}}_t^s/\tilde{\mathcal{T}}_{t+1}^s$ is the set of linear combinations of chord diagrams with at least $s$ strand-strand chords, or rather at least $s$ chords between the non-pole skeleton components.

Note that the number of $s$ chords is not a grading on $\tilde{\mathcal{A}}$ because the 4T relation is not homogeneous with respect to strand-strand chords.

*Remark* 4.16. The Kontsevich integral respects the $s$-filtration. This follows immediately from Theorem 4.11– as $Z$ is an expansion with respect to the total filtration, so respects the $s$-filtration.

sec:notation

maybe this is a (trivial) proposition

I changed it to a remark. is that better?

4.6. **Notation conventions.** Throughout this paper we consider the $t$ and $s$ filtrations on $\mathbb{C}\tilde{\mathcal{T}}$ and $\tilde{\mathcal{A}}$, as well as on their various quotients and subspaces. We summarize the notation in the section below:

- $\mathbb{C}\tilde{\mathcal{T}}$ is the space of $\mathbb{C}$-linear combinations of framed tangles in $M_p$
- $\mathbb{C}\tilde{\mathcal{T}}(\bigcirc)$ is the space of $C$-linear combinations of framed knots in $M_p$
- $\tilde{\mathcal{T}}_t$ is the $t$'th filtered component of $\mathbb{C}\tilde{\mathcal{T}}$ with respect to the $t$-filtration, which contains all linear combinations of framed tangles in $M_p$ with at least $t$ double points(both strand-strand and strand-pole types) and framing changes.
- $\tilde{\mathcal{T}}^s$ is the $s$'th filtered component of $\mathbb{C}\tilde{\mathcal{T}}$ with respect to the $s$-filtration, which contains all linear combinations of framed tangles in $M_p$ with at least $s$ strand-strand double points and framing changes.

- $\tilde{\mathcal{T}}^s_t := \tilde{\mathcal{T}}_t \cap \tilde{\mathcal{T}}^s$, which is the set of elements of $\mathbb{C}\tilde{\mathcal{T}}$ with at least $s$ framing changes and strand-strand double points, and at least $t$ framing changes and double points of any type.
- $\tilde{\mathcal{T}}^{/s} := \mathbb{C}\tilde{\mathcal{T}}/\tilde{\mathcal{T}}^s$, is the quotient of $\mathbb{C}\tilde{\mathcal{T}}$ where diagrams with more than $s$ strand-strand double points or framing changes are in the kernel.
- $\tilde{\mathcal{T}}^{1/2} := \tilde{\mathcal{T}}^1/\tilde{\mathcal{T}}^2$, is the quotient of $\mathbb{C}\tilde{\mathcal{T}}$ where diagrams with 0 or greater than 1 strand-strand double point or framing change are in the kernel.
- $\tilde{\mathcal{A}}$ is the associated graded space of $\mathbb{C}\tilde{\mathcal{T}}$ under the $t$-filtration, and is the space of admissible chord diagrams modulo admissible $4T$ relations.
- $\tilde{\mathcal{A}}_t := \tilde{\mathcal{T}}_t/\tilde{\mathcal{T}}_{t+1}$ is the degree $t$ component of $\tilde{\mathcal{A}}$ which consists of all admissible chord diagrams in $\tilde{\mathcal{A}}$ with exactly $t$ chords of any type.
- $\tilde{\mathcal{A}}^{\geq s} := \prod_t \tilde{\mathcal{T}}^s_t/\tilde{\mathcal{T}}^s_{t+1}$ is the $s$'th filtered component of $\tilde{\mathcal{A}}$
- $\tilde{\mathcal{A}}^{/s} := \tilde{\mathcal{A}}/\tilde{\mathcal{A}}^{\geq s}$

not sure if we use the $\tilde{\mathcal{T}}^{/s}$ and $\tilde{\mathcal{A}}^{/s}$ notations enough to justify having them

Theses notations are extended to subspaces and quotients of $\mathbb{C}\tilde{\mathcal{T}}$ and $\tilde{\mathcal{A}}$ in the natural way.

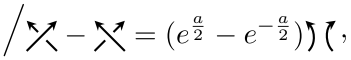
Not sure if $\tilde{\mathcal{T}}^{/s}$ and $\tilde{\mathcal{A}}^{/s}$ are relevant enough to be included here. I think they are only used for $s = 1$. We also sometimes use for example $\tilde{\mathcal{T}}^2/\tilde{\mathcal{T}}^1$, which doesn't have a shorthand, so maybe $\tilde{\mathcal{T}}^{/s}$ should be $\tilde{\mathcal{T}}^s/\tilde{\mathcal{T}}^{s+1}$ or something (i.e. degree $s$ component of the $s$-associated graded)

`sec:Conway`

### 4.7. The Conway quotient.

In this section we introduce the Conway quotient of $\mathbb{C}\tilde{\mathcal{T}}$: essentially, a Conway skein module of tangles in $M_p$ without fixing the value of the unknot. The Conway relation respects the $t$ and $s$ filtrations and the Kontsevich integral descends to the Conway quotient.

**Definition 4.17.** The Conway quotient of $\mathbb{C}\tilde{\mathcal{T}}$ is defined as

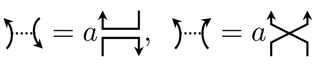
$$\mathbb{C}\tilde{\mathcal{T}}_\triangledown := \mathbb{C}\tilde{\mathcal{T}}[\![a]\!] \Big/$$


where $a$ is a formal variable that has $t$ and $s$ degree 1 ~~so that the skein relation preserves both $t$ and $s$ filtrations.~~ The skein relation is applied only to strand-strand crossings, not strand-pole crossings. We will use the variable $b$ as a shorthand for $b = e^{\frac{a}{2}} - e^{-\frac{a}{2}}$.

I don't think this has a meaning.

The $t$ and $s$ filtrations on $\mathbb{C}\tilde{\mathcal{T}}$ induce filtrations on $\mathbb{C}\tilde{\mathcal{T}}_\triangledown$. Following the notation conventions in Section 4.6, let $\tilde{\mathcal{T}}_{\triangledown,t}$ denote the $t$'th filtered component of $\mathbb{C}\tilde{\mathcal{T}}_\triangledown$ and $\tilde{\mathcal{A}}_\triangledown := \mathrm{gr}_t\, \mathbb{C}\tilde{\mathcal{T}}_\triangledown = \prod \tilde{\mathcal{T}}_{\triangledown,t}/\tilde{\mathcal{T}}_{\triangledown,t+1}$ denote the associated graded algebra of $\mathbb{C}\tilde{\mathcal{T}}_\triangledown$ with respect the total filtration. We now show that $\tilde{\mathcal{A}}_\triangledown$ has a diagrammatic description similar to $\tilde{\mathcal{A}}$, where $\tilde{\mathcal{A}} \cong \mathcal{D}$ as in Theorem 4.9.

**Definition 4.18.** Let

$$\mathcal{D}_\triangledown := \mathcal{D}[\![a]\!] \Big/$$


where $a$ is a formal variable of degree 1 as above, and the relations locally apply only when all skeleton components involved are strands, not poles.

Note that the quotient relations in $\mathcal{D}_\triangledown$ preserve the $t$-grading on $\mathcal{D}$ and the grading descends to $\mathcal{D}_\triangledown$. The next theorem shows that $\tilde{\mathcal{A}}_\triangledown \cong \mathcal{D}_\triangledown$. This theorem essentially follows from the results of [LM95], and we present a brief direct proof.

`thm:Z_conway`

**Theorem 4.19.** *The Kontsevich integral descends to an expansion* $Z_\nabla : \mathbb{C}\tilde{\mathcal{T}}_\nabla \to \mathcal{D}_\nabla$ *and* $\tilde{\mathcal{A}}_\nabla \cong \mathcal{D}_\nabla$*.*

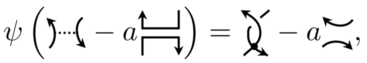
This proof uses R1, so I don't know how a framed analogue works exactly, and also not sure that we need it. I commented it out for now.

I believe this theorem is correct with framing changes. Please double check.

*Proof.* This proof follows the general schema introduced in Section 3.1, in particular Lemma 3.1 and the map $\psi$, which assigns singular tangles to chord diagrams.

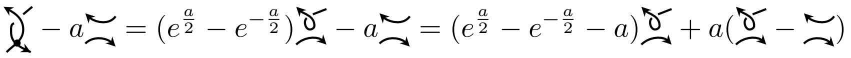
First we show that $\psi$ descends to a graded surjection $\psi : \mathcal{D}_\nabla \to \tilde{\mathcal{A}}_\nabla$. To show that $\psi$ is well-defined, we need to show that the Conway relation in $\mathcal{D}_\nabla$ is in the kernel. Locally,

We can use  and R1 disappears,



and denote the (global) total degree on both sides by $t$. In other words, the (global) right hand side is interpreted as an element of $\tilde{\mathcal{T}}_{H,t}/\tilde{\mathcal{T}}_{H,t+1}$. Using the Conway skein relation in $\tilde{\mathcal{A}}_\nabla$, the right had side can be simplified

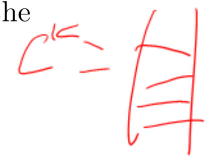
∇

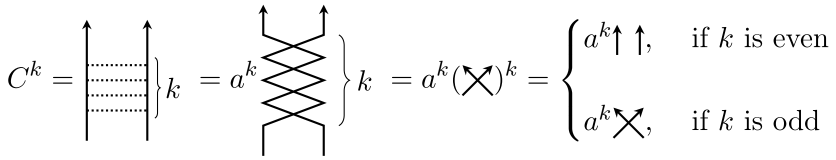


Observe that $a(\;-\;)$ and the lowest degree term of $e^{\frac{a}{2}} - e^{-\frac{a}{2}} - a$ are both of degree 2, hence $(\;-a\;) \in \tilde{\mathcal{T}}_{H,t+1}$, and therefore is zero in $\tilde{\mathcal{T}}_{H,t}/\tilde{\mathcal{T}}_{H,t+1}$.

But Z isn't defined on tangles?

We now verify that the Kontsevich integral $Z$ descends to the quotient $\mathbb{C}\tilde{\mathcal{T}}_\nabla$ by checking the relations in $\mathbb{C}\tilde{\mathcal{T}}_\nabla$ directly. Recall* that $Z(\nearrow\!\!\!\!\nwarrow) = (e^{\frac{C}{2}})\times$ and $Z(\nwarrow\!\!\!\!\nearrow) = (e^{-\frac{C}{2}})\times$, where $C$ denotes a chord, the exponential is interpreted formally as a power series, and $C^k$ denotes stacking $k$ chords as ~~shown below~~ on the right. Using the Conway relation, we compute:

$C^k =$ 



* Strictly speaking, Z is defined on whole tangles and not on fragments like , or . Yet these fragments can be made arbitrarily small and then their contributions are localized and are given by the formulas in the text.

Now applying $Z$ to the left hand side of the Conway relation, we obtain

$$\begin{aligned} Z(\slashoverback) - Z(\backoverslash) &= (e^{\frac{C}{2}} - e^{\frac{-C}{2}})\doublepoint \\ &= \sum_{k=0}^{\infty}\left(\frac{C^k}{2^k k!} - \frac{(-1)^k C^k}{2^k k!}\right)\doublepoint \\ &= \sum_{k=0}^{\infty}\frac{C^{2k+1}}{2^{2k}(2k+1)!}\doublepoint \\ &= \sum_{k=0}^{\infty}\frac{a^{2k+1}\doublepoint}{2^{2k}(2k+1)!}\doublepoint \\ &= \sum_{k=0}^{\infty}\frac{a^{2k+1}}{2^{2k}(2k+1)!}\uparrow\uparrow \\ &= (e^{\frac{a}{2}} - e^{-\frac{a}{2}})\uparrow\uparrow \\ &= Z\left((e^{\frac{a}{2}} - e^{-\frac{a}{2}})\,)\,(\right). \end{aligned}$$

→ Thus, $Z$ is ~~well-defined on~~ descends to the Conway quotient $\mathbb{C}\tilde{\mathcal{T}}_\triangledown$.

Therefore, by Lemma 3.1, $Z$ is a homomorphic expansion for $\mathbb{C}\tilde{\mathcal{T}}_\triangledown$ and $\psi : \mathcal{D}_\triangledown \to \tilde{\mathcal{A}}_\triangledown$ is an isomorphism. □

While our main focus is the $t$-filtration on $\mathbb{C}\tilde{\mathcal{T}}_\triangledown$ and its associated graded space $\tilde{\mathcal{A}}_\triangledown$, the low degree components of the associated graded of $\mathbb{C}\tilde{\mathcal{T}}_\triangledown$ with respect to the $s$-filtration also show up. In particular, there is a well-defined "division by $b$" map $q_b : \tilde{\mathcal{T}}_\triangledown^{/2} \to \tilde{\mathcal{T}}_\triangledown^{/1}$ which restricts to an isomorphism on $\tilde{\mathcal{T}}_\triangledown^1/\tilde{\mathcal{T}}_\triangledown^2$. We now show that this map exists by defining it explicitly.

much better to phrase this as "the multiplication by $b$ map, $b: \hat{\mathcal{T}}^{/1} \to \hat{\mathcal{T}}^{/2}$ is an isomorphism.

There is confusion here between tangles and tangle diagrams

**Proposition 4.20.** *For a tangle $T$ and a crossing $x$ of $T$, let $\epsilon(x) \in \{\pm 1\}$ be the sign of $x$, and $T|_{x\to )(}$ be the tangle $T$ with $x$ replaced by a smoothing. There is a well defined map $q_b : \tilde{\mathcal{T}}_\triangledown^{/2} \to \tilde{\mathcal{T}}_\triangledown^{/1}$ given by the linear extension of the following:*

$$bT \overset{q_b}{\mapsto} T$$

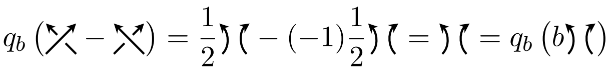
$$T \overset{q_b}{\mapsto} \frac{1}{2}\sum_{x \text{ crossing of } T} \epsilon(x)T|_{x\to )(}$$

It's straightforward to check R moves so I don't check it explicitly. Is this fine?

not clear.

*Proof.* It is straightforward to check that Reidemeister moves are preserved. We also need to check that $\tilde{\mathcal{T}}_\triangledown^2$ and the Conway relation are in the kernel. For $b^kT \in \tilde{\mathcal{T}}_\triangledown^2$, if $k = 1$, then $T \in \tilde{\mathcal{T}}_\triangledown^1$, so $q_b(bT) = 0$. If instead $k = 0$, then $T$ has at least two double points. Replacing a crossing by a smoothing only changes the crossing that is replaced, so other crossings (and therefore double points) remain unchanged. Therefore $q_b(T)$ can be written as a sum where each term has at least one double point, so $q_b(T) = 0$ as well.

To show that the Conway relation also vanishes, note that the terms in $q_b(\doublepoint) = q_b(\slashoverback - \backoverslash)$ come from either smoothing a crossing that is a part of the double point or smoothing a crossing that is not. In the latter, the double point remains as before, so those terms are in $\tilde{\mathcal{T}}^1_\nabla$. The only remaining terms are those where the crossings forming the double point are smoothed, so we get



so $q_b$ is well-defined. $\square$

`cor:divbyb`

**Corollary 4.21.** *The map $q_b$ restricts to an isomorphism $q_b : \tilde{\mathcal{T}}^{1/2}_\nabla \to \tilde{\mathcal{T}}^{/1}_\nabla$.*

*Proof.* It is clearly surjective. To show it's injective, note that the restriction is simply given by $bT \mapsto T$, and if $T \in \tilde{\mathcal{T}}^1_\nabla$, then $bT \in \tilde{\mathcal{T}}^2_\nabla$. $\square$

`rem:grdivbyb`

*Remark* 4.22. The associated graded of $q_b$ is an isomorphism $\operatorname{gr} q_B : \tilde{\mathcal{A}}^{1/2} \to \tilde{\mathcal{A}}^{/1}$ given by drawing the chord diagram as with one $s$-$s$ chord, smoothing that chord using $\nabla$ and getting a factor of $b$ with no remaining $s$-$s$ chords, and then diving off the $b$.

not clear.

must we ever write that?

Clean up

In general, the Conway relation changes the skeleton of a diagram. So when we write $\tilde{\mathcal{T}}_\nabla(\bigcirc)$ this is the set of diagrams for which there is a representative expressed in terms of knots and no factors of $b$– the grading comes from the actual skeleton, not from factors of $b$. Similarly, $\tilde{\mathcal{A}}^{/r}_\nabla(S)$ represents the associated graded space of $\tilde{\mathcal{T}}^{/r}_\nabla(S)$, and a nontrivial chord diagram in $\tilde{\mathcal{A}}^{/r}_\nabla(S)$ has a representative that with a chord diagram on skeleton $S$ with at most $r$ strand-strand chords.

not clear.

In the quotient $\tilde{\mathcal{T}}^{/1}$ by the $s$-degree 1 component, crossing changes yield a double point and thus are killed in the quotient. So, tangles in $\tilde{\mathcal{T}}^{/1}$ are only defined up to crossing changes. Similarly, in $\tilde{\mathcal{T}}^{/1}_\nabla$ the Conway quotient only applies to crossing changes, (the Conway relation occurs in degree 1), so further quotienting by the Conway relation has no effect on $\tilde{\mathcal{T}}^{/1}$, and so $\tilde{\mathcal{T}}^{/1}_\nabla \cong \tilde{\mathcal{T}}^{/1}$.

Fuzzy

`prop:/1conway`

**Proposition 4.23.** $\tilde{\mathcal{T}}^{/1}(\bigcirc) \cong \tilde{\mathcal{T}}^{/1}_\nabla(\bigcirc)$.

`cor:gr/1conway`

**Corollary 4.24.** $\tilde{\mathcal{A}}^{/1}(\bigcirc) \cong \tilde{\mathcal{A}}^{/1}_\nabla(\bigcirc)$.

However, Proposition 4.23 and Corollary 4.24 are not true for higher degree quotients because of the skeleton changing issue induced by the Conway relation.

`:IdentifyingGTinCON`

## 5. IDENTIFYING THE GOLDMAN-TURAEV LIE BIALGEBRA

In this section we establish our main results: we identify the Goldman-Turaev Lie bialgebra in the low $s$-filtered degree quotients of $\mathbb{C}\tilde{\mathcal{T}}$, and show that the Kontsevich integral induces a homomorphic expansion on this space with respect to the $s$-filtration. Appealing to the principle summarized in Section 2 we present diagrams like (2.1), where the induced map $\eta$ is the Goldman bracket and the self intersection map $\mu$, repsectively. We deduce the homomorphicity of the expansion from the naturality of the construction as in (2.3).

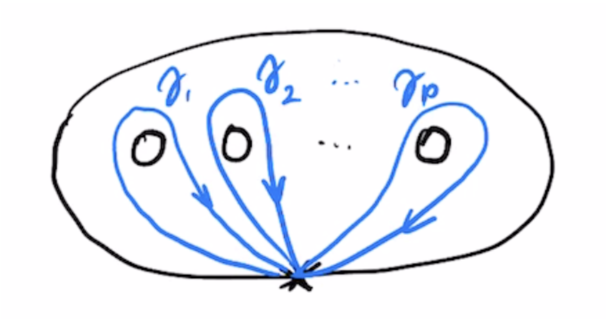


fig:GenCurves

FIGURE 18. The standard generating curves of $\pi$.

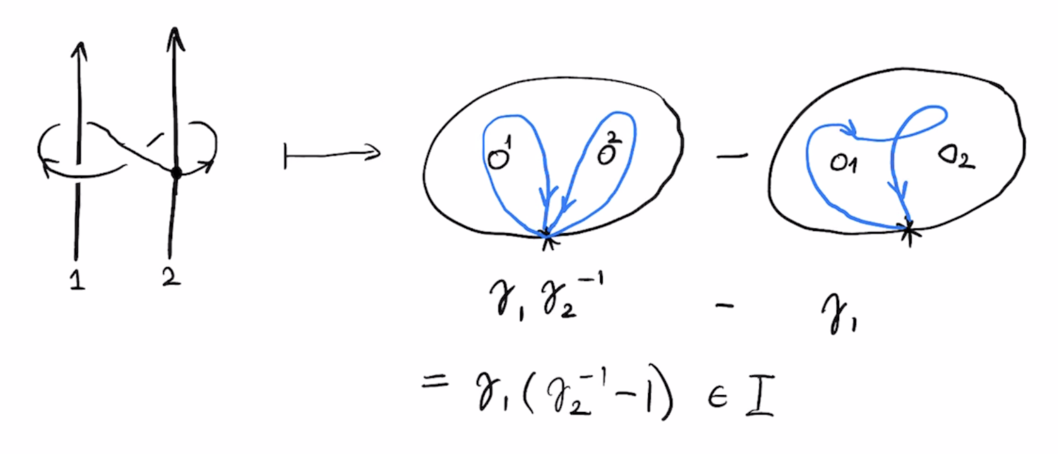


FIGURE 19. Example calculation demonstrating that $\beta$ is a filtered map.

fig:BetaFiltered

dentifybracketinCON

5.1. **The Goldman Bracket.** Recall from Section 3.3 that $D_p$ denotes the $p$-punctured disc, $\pi$ is its fundamental group, and $|\mathbb{C}\pi|$ is the linear quotient $|\mathbb{C}\pi| := \mathbb{C}\pi/[\mathbb{C}\pi, \mathbb{C}\pi]$, which is linearly generated by homotopy classes of free loops in $D_p$. The Goldman bracket (Definition 3.3) is a lie bracket $[\cdot,\cdot]_G : |\mathbb{C}\pi| \otimes |\mathbb{C}\pi| \to |\mathbb{C}\pi|$. Recall from Section 4.7 the space $\mathbb{C}\tilde{\mathcal{T}}(\bigcirc)$ is the vector space of $\mathbb{C}$-linear combinations of framed knots in $M_p = D_p \times I$.

prop:BotProj

**Proposition 5.1.** *The bottom projection $M_p \to D_p \times \{0\}$ induces a surjective map $\mathbb{C}\tilde{\mathcal{T}}(\bigcirc) \to |\mathbb{C}\tilde{\pi}|$. Post-composing this with the projection $|\mathbb{C}\tilde{\pi}| \to |\mathbb{C}\pi|$ results in a surjective filtered map*

$$\beta : \mathbb{C}\tilde{\mathcal{T}}(\bigcirc) \to |\mathbb{C}\pi|.$$

Framed?

*Proof.* ~~By Reidemeister's Theorem,~~ knots in $\mathbb{C}\tilde{\mathcal{T}}(\bigcirc)$ are faithfully represented by knot diagrams in $D_p \times \{0\}$ – regular projections to the bottom with over/under information – modulo the Reidemeister moves (R2, R3). The bottom projection sends the Reidemeister moves for knots to the corresponding moves generating regular homotopies of immersed free loops, hence $\beta$ is well-defined. The projection is clearly surjective as any loop can be lifted to a knot by introducing arbitrary under/over information at the crossings.

The statement that $\beta$ is filtered means that step $k$ of the the Vassiliev $t$-filtration in $\mathbb{C}\tilde{\mathcal{T}}(\bigcirc)$ projects to step $k$ of the filtration on $|\mathbb{C}\pi|$ induced by the I-adic filtration of $\pi$. Note that strand-strand double points and framing changes map to 0 under $\beta$, thus, we only have something to prove for knots with $k$ strand-pole double points.

Let $\gamma_1, ..., \gamma_p$ denote the standard generators of $\pi$ as in Figure 18. A knot $K \in \mathbb{C}\tilde{\mathcal{T}}(\bigcirc)$ maps to a free loop in $|\mathbb{C}\pi|$, whose conjugacy class in $\pi$ is represented as a word in the generators $\gamma_i$. A pole-stand double point on pole $j$ maps to a difference between two curves passing on either side of the $j$'th puncture, ~~as shown in Figure 19~~ therefore, the words in $\pi$ representing these curves differ in a single instance of $\gamma_j^{\pm 1}$. Thus, a knot with $k$ pole-strand double points maps to a product with $k$ factors of the form $\pm(\gamma_j^{\pm 1} - 1)$. This is by definition an element in $\mathcal{I}^k$. □

(See an example in Figure 19]

Can we make the two figures for this proof into real figures? In the second figure the arrow should have $\beta$ on it.

`prop:kerbeta`

**Proposition 5.2.** *The kernel of $\beta$ is $\tilde{\mathcal{T}}^1(\bigcirc)$, and $\beta$ descends to a filtered (with respect to the t-filtration) linear isomorphism $\beta : \tilde{\mathcal{T}}^{/1}(\bigcirc) \to |\mathbb{C}\pi|$.*

*Proof.* Two framed knots in $\mathbb{C}\tilde{\mathcal{T}}(\bigcirc)$ project to the same loop in $|\mathbb{C}\pi|$ if and only if they differ by framing changes and (strand-strand) crossing changes, which generate precisely the step 1 of the $s$-filtration, that is, $\tilde{\mathcal{T}}^1(\bigcirc)$. □

Recall from Proposition 4.23 that $\tilde{\mathcal{T}}^{/1}(\bigcirc) = \tilde{\mathcal{T}}_{\triangledown}^{/1}(\bigcirc)$. Hence, we get the following:

`cor:loopsasknots`

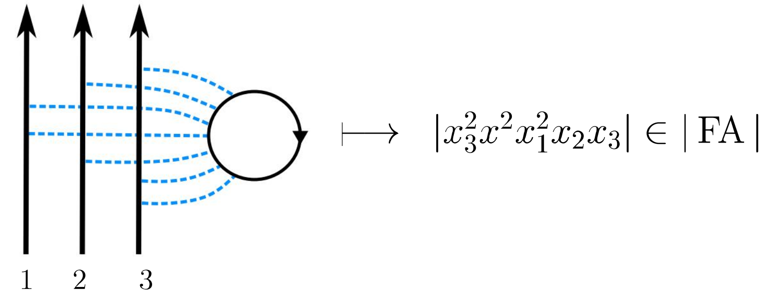
**Corollary 5.3.** *The map $\beta$ descends to an isomorphism $\beta : \tilde{\mathcal{T}}_{\triangledown}^{/1}(\bigcirc) \to |\mathbb{C}\pi|$.*

Recall that $\tilde{\mathcal{A}}$ is the associated graded space of $\mathbb{C}\tilde{\mathcal{T}}$ with respect to the $t$-filtration, and $\mathbb{C}\tilde{\mathcal{T}}$ is also filtered by the $s$-filtration. Explicitly, $\tilde{\mathcal{A}}(\bigcirc)$ is the space of admissible chord diagrams on a circle skeleton as in Definition 4.8, $\tilde{\mathcal{A}}^{\geq i}(\bigcirc)$ is the $s$-degree $i$ filtered component of $\tilde{\mathcal{A}}(\bigcirc)$, and $\tilde{\mathcal{A}}^{/i}(\bigcirc) = \tilde{\mathcal{A}}(\bigcirc)/\tilde{\mathcal{A}}^{\geq i}(\bigcirc)$. Recall from Section 3.3 that the associated graded vector space of $|\mathbb{C}\pi|$ is $|\,\mathrm{FA}\,|$, where $\mathrm{FA} = \mathrm{FA}\langle x_1, \cdots, x_p\rangle$ denotes the free associative algebra over $\mathbb{C}$, and the linear quotient $|\,\mathrm{FA}\,| = \mathrm{FA}\,/[\mathrm{FA}, \mathrm{FA}]$ is the $\mathbb{C}$-vector space generated by cyclic words in the letters $x_1, ..., x_p$.

**Proposition 5.4.** *The associated graded map $\operatorname{gr}\beta : \tilde{\mathcal{A}}(\bigcirc) \to |\,\mathrm{FA}\,|$ has kernel $\tilde{\mathcal{A}}^{\geq 1}(\bigcirc)$. Hence, $\operatorname{gr}\beta$ descends to an isomorphism $\operatorname{gr}\beta : \tilde{\mathcal{A}}^{/1}(\bigcirc) \to |\,\mathrm{FA}\,|$.*

*Proof.* The statement follows from applying the associated graded functor to the filtered short exact sequence

I don't think gr preserves exact sequences?
Yet even if it did, I'd rather see a direct proof.



`fig:CycWord`

FIGURE 20. Chord diagrams with no strand-strand chords can be read as cyclic words.

$$\begin{tikzcd}
 & 0 \arrow[r] & \mathrm{Ker} \arrow[r] \arrow[d, "0"'] & \tilde{\mathcal{T}}_{\nabla}^{/2}(\bigcirc) \otimes \tilde{\mathcal{T}}_{\nabla}^{/2}(\bigcirc) \arrow[r] \arrow[d, "\lambda"] & \tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc) \arrow[r] \arrow[d, "0"] \arrow[ddll, dashed, "\eta"] \arrow[dll, bend right, "\eta"'] & 0 \\
 & 0 \arrow[r] & \tilde{\mathcal{T}}_{\nabla}^{1/2}(\bigcirc\bigcirc) \arrow[r] & \tilde{\mathcal{T}}_{\nabla}^{/2}(\bigcirc\bigcirc) \arrow[r] & \tilde{\mathcal{T}}^{/1}(\bigcirc\bigcirc) \arrow[r] & 0 \\
 & |\mathbb{C}\pi| \arrow[r, dash, "\sim", "\beta"'] & \tilde{\mathcal{T}}^{/1}(\bigcirc) \arrow[u, "\hat{b}"] & & &
\end{tikzcd}$$

I still prefer the version w/ two arrows:

FIGURE 21. The nontrivial horizontal maps are the respective quotient and inclusion maps. The space Ker is the kernel of the projection map on the top right.

`fig:Snakeforbracket`

*Remark* 5.5. In $\tilde{\mathcal{A}}^{/1}(\bigcirc)$ chord diagrams with any strand-strand chords are zero. Thus, non-zero elements of this space are represented as chord diagrams on poles and a single circle strand, with strand-pole chords only, as in Figure 20. Such a chord diagram corresponds naturally to a cyclic word by labelling the poles with $x_1, ..., x_p$ and reading the word along the circle skeleton, as shown. Indeed, this is the map $\operatorname{gr}\beta$.

You have to check that the order of the chords along the poles doesn't matter!

We are now ready to derive the Goldman bracket from the stacking commutator on $\mathbb{C}\tilde{\mathcal{T}}$:

`thm:bracketsnake`

**Theorem 5.6.** *Let* $\lambda_1 : \tilde{\mathcal{T}}_{\nabla}^{/2}(\bigcirc) \otimes \tilde{\mathcal{T}}_{\nabla}^{/2}(\bigcirc) \to \tilde{\mathcal{T}}_{\nabla}^{/2}(\bigcirc)$ *denote the stacking product. Let* $\lambda_2$ *denote the opposite product, that is,* $\lambda_2(K_1, K_2) = K_2K_1$. *Then* $\lambda = \lambda_1 - \lambda_2$ *induces the Goldman bracket on* $|\mathbb{C}\pi|$*: the commutative diagram in Figure 21 the induced homomorphism* $\eta$ *agrees with the Goldman Bracket under the identification* $\beta : \tilde{\mathcal{T}}^{/1}(\bigcirc) \to |\mathbb{C}\pi|$.

$\hat{b}$ is missing here.

*Proof.* For $K_1 \otimes K_2$ in $\tilde{\mathcal{T}}_{\nabla}^{/2}(\bigcirc) \otimes \tilde{\mathcal{T}}_{\nabla}^{/2}(\bigcirc)$, $\lambda(K_1 \otimes K_2) = K_1K_2 - K_2K_1$. Project $K_1K_2$ and $K_2K_1$ to the bottom to obtain link diagrams. Let a *mixed crossing* of

$K_1K_2 = L_0 \quad L_1 \quad L_2 \quad L_3 \qquad\qquad L_{n-1} \quad L_n = K_2K_1$

such a diagram be a crossing where one strand belongs to $K_1$ and the other strand belongs to $K_2$. Notice that in $K_2K_1$ all mixed crossings are flipped compared to $K_1K_2$, while other crossings – those belonging to $K_1$ or $K_2$ only – are the same.

Using the double point notation, write positive mixed crossings in $K_1K_2$ as  and negative mixed crossings as , where each double point has one strand belongs to $K_1$ and the other belongs to $K_2$. Rewriting all the mixed crossings of $K_1K_2$ in this way yields a sum of tangles indexed by subsets of the mixed crossings. Denote the set of mixed crossings by $M$, and for a subset $X \subseteq M$, denote by $L_X$ the singular link obtained by changing the crossings in $X$ to double points, and flipping the other mixed crossings (those in $M \setminus X$). Also, let $\epsilon_X$ be the product of the signs of all crossings in $X$. Then

`eq:commutator`
$$K_1K_2 = \sum_{X\subseteq M} \epsilon_X L_X. \tag{5.1}$$

Notice that $L_\emptyset = K_2K_1$, and if $|X| = i$ then $L_X \in \tilde{\mathcal{T}}^{i}_{\nabla}(\bigcirc\bigcirc)$. Therefore, $\lambda(K_1K_2)$ is in $\tilde{\mathcal{T}}^{1}_{\nabla}(\bigcirc\bigcirc)$, and therefore the right hand square commutes. Furthermore, we have

`eq:singletons`
$$\lambda(K_1, K_2) = \sum_{X\subseteq M, |X|=1} L_X \in \tilde{\mathcal{T}}^{1/2}_{\nabla}(\bigcirc\bigcirc). \tag{5.2}$$

This may not be true, but it's too complicated.

Now for the left square, the kernel $K$ of the projection map from $\tilde{\mathcal{T}}^{/2}_{\nabla}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/2}_{\nabla}(\bigcirc) \to \tilde{\mathcal{T}}^{/1}_{\nabla}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}_{\nabla}(\bigcirc)$ is generated by $\tilde{\mathcal{T}}^{1/2}_{\nabla}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/2}_{\nabla}(\bigcirc)$ in $\tilde{\mathcal{T}}^{/2}_{\nabla}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/2}_{\nabla}(\bigcirc)$. and $\tilde{\mathcal{T}}^{/2}_{\nabla}(\bigcirc) \otimes \tilde{\mathcal{T}}^{1/2}_{\nabla}(\bigcirc)$. Suppose that $K_1 \otimes K_2$ is in $\tilde{\mathcal{T}}^{1/2}_{\nabla}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/2}_{\nabla}(\bigcirc)$, in other words, there is a double point in $K_1$. Then, by the same computation as in Equation 5.1, $\lambda(K_1 \otimes K_2)$ is in $\tilde{\mathcal{T}}^{2}_{\nabla}(\bigcirc\bigcirc)$, as every term contains the pre-existing duble point in $K_1$, and at least one additional mixed double point. Therefore, the left hand square commutes.

As in Section 2, then $\lambda$ induces a unique well defined homomorphism $\eta : \tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc) \to \tilde{\mathcal{T}}^{1/2}_{\nabla}(\bigcirc\bigcirc)$. We need to identify $\eta$ as the Goldman bracket. We have that $\tilde{\mathcal{T}}^{/1}(\bigcirc) \cong |\mathbb{C}\pi|$ (Proposition 5.2), identifying the domain of $\eta$ with the domain of the Goldman bracket. We now argue that $\eta$ has image in $\tilde{\mathcal{T}}^{/1}(\bigcirc) \cong |\mathbb{C}\pi|$.

By Equation (5.2), $\lambda(K_1, K_2)$ is a sum of terms, each with a single mixed double point. Applying the Conway relation to smooth each of these mixed double points changes the skeleton from two circles to one circle, and introduces a factor of $b$. In other words, $\lambda(K_1, K_2) \in b\tilde{\mathcal{T}}^{/2}_{\nabla}(\bigcirc) \subseteq \tilde{\mathcal{T}}^{1/2}_{\nabla}(\bigcirc\bigcirc)$. By Corollary 4.21, restricted to a circle skeleton, we know that $b\tilde{\mathcal{T}}^{/2}_{\nabla}(\bigcirc) \cong \tilde{\mathcal{T}}^{/1}(\bigcirc)$ via the map $q_b$. In turn, $\tilde{\mathcal{T}}^{/1}(\bigcirc) \cong |\mathbb{C}\pi|$ again via the map $\beta$.

In summary, the map $\eta$ is induced from $\lambda$ in the following way. For curves $\gamma_1 \otimes \gamma_2 \in \tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc)$, let $K_1 \otimes K_2$ be an arbitrary vertical lift of $\gamma_1 \otimes \gamma_2$ to

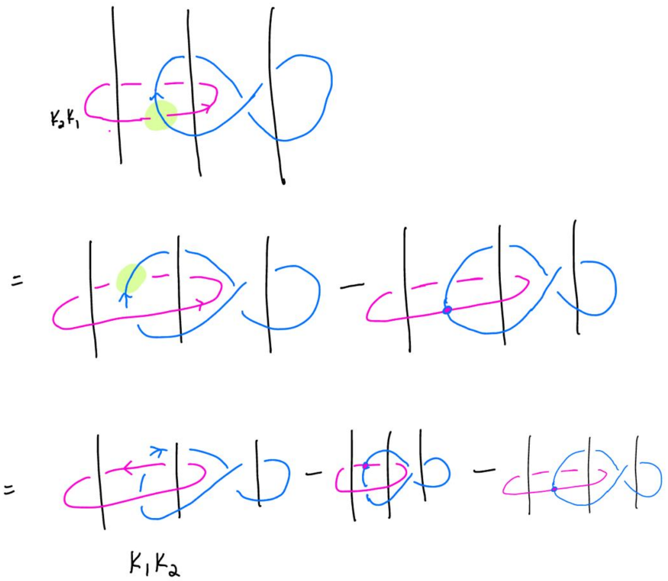


FIGURE 22. Example commutator bracket computation.

fig:combracket

knots in $\tilde{\mathcal{T}}_{\triangledown}^{/2}(\bigcirc) \otimes \tilde{\mathcal{T}}_{\triangledown}^{/2}(\bigcirc)$. Then

$$\eta(\gamma_1 \otimes \gamma_2) = \frac{\lambda(K_1 \otimes K_2)}{b} \in \tilde{\mathcal{T}}^{/1}(\bigcirc),$$

where we use the notation $\frac{1}{b}$ to mean composition with $q_b$. We need to show that this agrees with the Goldman bracket (Definition 3.3). This is clear from the definition: the Goldman bracket is a sum of smoothings of the mixed crossings of $\gamma_1$ and $\gamma_2$, exactly as above, and the signs in the sum match the signs of the crossings. See Figure 22 for an example calculation. □

I'm not so sure.

The graded Goldman bracket is a map $[-,-]_{\mathrm{gr}\, G} : |\,\mathrm{FA}\,| \otimes |\,\mathrm{FA}\,| \to |\,\mathrm{FA}\,|$, as in Definition 3.7. By taking the associated graded of the diagram in Figure 21 we arrive at the commutative diagram in Figure 23 and recover the associated graded Goldman bracket:

Usually people write commutative diagrams as formulas, not as figures.

snakefor_gr_bracket

**Corollary 5.7.** *The diagram in Figure 23 commutes and* $\mathrm{gr}\,\eta$ *is the associated graded Goldman bracket via the identification* $\mathcal{A}^{/1}(\bigcirc) \cong |\,\mathrm{FA}\,|$.

no proof is required here, in my opinion

thm:bracketsnake

**Theorem 5.8.** *The Kontsevich integral descends to a homomorphic expansion for the Goldman Bracket. That is, the following square commutes:*

$$\begin{tikzcd}
\tilde{\mathcal{T}}^{/1}(\bigcirc) \arrow[d, "Z^{/1}"] & \tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc) \arrow[l, "{\eta=[\cdot,\cdot]_G}"'] \arrow[d, "Z^{/1}\otimes Z^{/1}"] \\
\mathcal{A}^{/1}(\bigcirc) & \mathcal{A}^{/1}(\bigcirc) \otimes \mathcal{A}^{/1}(\bigcirc) \arrow[l, "\mathrm{gr}\,\eta"']
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker} \arrow[r] \arrow[d, "0"'] & \mathcal{A}_{\triangledown}^{/2}(\bigcirc) \otimes \mathcal{A}_{\triangledown}^{/2}(\bigcirc) \arrow[r] \arrow[d, "\mathrm{gr}\,\lambda"'] & \mathcal{A}^{/1}(\bigcirc) \otimes \mathcal{A}^{/1}(\bigcirc) \arrow[r] \arrow[d, "0"] \arrow[dddll, dashed, bend left, "\mathrm{gr}\,\eta"] & 0 \\
0 \arrow[r] & \mathcal{A}_{\triangledown}^{1/2}(\bigcirc\bigcirc) \arrow[r] & \mathcal{A}_{\triangledown}^{/2}(\bigcirc\bigcirc) \arrow[r] & \mathcal{A}_{\triangledown}^{/1}(\bigcirc\bigcirc) \arrow[r] & 0 \\
& & & & \\
& \mathcal{A}^{/1}(\bigcirc) \arrow[uu, hook] & & &
\end{tikzcd}$$

FIGURE 23. The associated graded commutative diagram of Figure 21.

Snakefor_gr_bracket

$$\begin{tikzcd}
\tilde{\mathcal{T}}^{/1}(\bigcirc) \arrow[d, hook] \arrow[dddd, bend right, "Z^{/1}"'] & \mathrm{Ker} \arrow[rr] \arrow[dl, "0"] \arrow[dd, "Z^{1/2}\otimes Z^{1/2}"] & & \tilde{\mathcal{T}}_{\triangledown}^{/2}(\bigcirc) \otimes \tilde{\mathcal{T}}_{\triangledown}^{/2}(\bigcirc) \arrow[rr] \arrow[dl, "\lambda"] \arrow[dd, "Z^{/2}\otimes Z^{/2}"] & & \tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc) \arrow[dl, "0"] \arrow[dd, "Z^{/1}\otimes Z^{/1}"] \arrow[lllll, bend right, "\eta"'] \\
\tilde{\mathcal{T}}_{\triangledown}^{1/2}(\bigcirc\bigcirc) \arrow[rr] \arrow[dd, "Z^{1/2}"] & & \tilde{\mathcal{T}}_{\triangledown}^{/2}(\bigcirc\bigcirc) \arrow[rr] \arrow[dd, "Z^{/2}"] & & \tilde{\mathcal{T}}^{/1}(\bigcirc\bigcirc) \arrow[dd, "Z^{/1}"] & \\
& \mathrm{Ker} \arrow[rr] \arrow[dl, "0"] & & \mathcal{A}_{\triangledown}^{/2}(\bigcirc) \otimes \mathcal{A}_{\triangledown}^{/2}(\bigcirc) \arrow[rr] \arrow[dl, "\mathrm{gr}\,\lambda"] & & \mathcal{A}^{/1}(\bigcirc) \otimes \mathcal{A}^{/1}(\bigcirc) \arrow[dl, "0"] \arrow[ddlllll, bend left, "\mathrm{gr}\,\eta"] \\
\mathcal{A}_{\triangledown}^{1/2}(\bigcirc\bigcirc) \arrow[rr] & & \mathcal{A}_{\triangledown}^{/2}(\bigcirc\bigcirc) \arrow[rr] & & \mathcal{A}^{/1}(\bigcirc\bigcirc) & \\
\mathcal{A}^{/1}(\bigcirc) \arrow[u, hook] & & & & &
\end{tikzcd}$$

FIGURE 24. Commutative cube showing the formality of the Goldman bracket from the Kontsevich integral.

ig:Cube_for_bracket

*Proof.* Taking the Kontsevich integral of the diagram in Figure 21 we get the cube in Figure 24. We have already established that the top and bottom faces all commute from Theorem 5.8 and Corollary 5.7. The front and back vertical faces commute because $Z$ respects the $s$-filtration and is homomorphic with respect to the inclusions and quotient maps of the filtered component. The left and right vertical sides trivially commute because of the zero maps.

add K and gr $K$

say this better!

The Kontsevich integral is homomorphic with respect to diagram stacking, as proved in Proposition 4.13. Since $\lambda$ is the difference between two orderings of diagram stacking, $Z$ is homomorphic with respect to $\lambda$ and the following square commutes (which is the middle vertical face of Figure 24).

$$\begin{tikzcd}
 & \tilde{\mathcal{T}}_\nabla^{/2}(\bigcirc) \otimes \tilde{\mathcal{T}}_\nabla^{/2}(\bigcirc) \arrow[dl, "\lambda"'] \arrow[dd, "Z^{/2}\otimes Z^{/2}"] \\
\tilde{\mathcal{T}}_\nabla^{/2}(\bigcirc\bigcirc) \arrow[dd, "Z^{/2}"'] & \\
 & \mathcal{A}_\nabla^{/2}(\bigcirc) \otimes \mathcal{A}_\nabla^{/2}(\bigcirc) \arrow[dl, "\operatorname{gr}\lambda"] \\
\mathcal{A}_\nabla^{/2}(\bigcirc\bigcirc) &
\end{tikzcd}$$

The commutativity of all vertical faces of the cube diagram in Figure 24 implies that the induced diagonal square also commutes.

Technically we get that this square commutes. But the square we want has the inclusion maps and a different descension of $Z$. I wasn't sure if we need to make this distinction, or maybe add to this diagram? OR only write the simpler diagram as in the statement of the theorem

$$\begin{tikzcd}
\tilde{\mathcal{T}}^{1/2}(\bigcirc\bigcirc) \arrow[d, "Z^{1/2}"] & \tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc) \arrow[l, "{\eta=[\cdot,\cdot]_G}"'] \arrow[d, "Z^{/1}\otimes Z^{/1}"] \\
\mathcal{A}^{1/2}(\bigcirc\bigcirc) & \mathcal{A}^{/1}(\bigcirc) \otimes \mathcal{A}^{/1}(\bigcirc) \arrow[l, "\operatorname{gr}\eta"]
\end{tikzcd}$$

□

`sec:cobracketinCON`

Using $\mathbb{C}\tilde{\pi}$ to mean "sailing curves" i.e. those that "never look north", but not introducing it here because it seems like it should be defined already

5.2. **The Tureav co-bracket.** Recall from section **??** that the Turaev cobracket on $|\mathbb{C}\pi|$ can be defined using the map $\mu : \mathbb{C}\tilde{\pi} \to |\mathbb{C}\pi| \otimes \mathbb{C}\pi$. Our knot-theoretic definition of the cobracket imitates this construction, and we interpret the domain $\mathbb{C}\tilde{\pi}$ and codomain $|\mathbb{C}\pi| \otimes \mathbb{C}\pi$ of $\mu$ in the context of tangles.

Let $\frown$ denote an interval skeleton component where both endpoints are on the bottom $D_p \times \{0\}$. We will call a tangle whose endpoints are all on the bottom a *bottom tangle*. In diagrams, the endpoints of the tangle will be drawn in the bottom right corner. As before, the beginning of each interval section will be marked by a $\bullet$, and the end will be marked by $*$, see figure **??**.

drawing conventions are random for now

We can extend the projection map $\beta$ from proposition 5.1 to bottom tangles to get an isomorphism similar to corollary 5.3. We state the map $\beta$ and corresponding isomorphism in proposition 5.9, but omit the proof as it is the same as before.

`prop:ascispi`

**Proposition 5.9.** *There is a well-defined natural bottom projection*

$$\beta : \tilde{\mathcal{T}}_\nabla(\bigcirc^k \frown^\ell) \to |\mathbb{C}\pi|^{\otimes k} \otimes \mathbb{C}\pi^{\otimes \ell}$$

*that descends to an isomorphism* $\beta : \tilde{\mathcal{T}}^{/1}(\bigcirc^k \frown^\ell) \xrightarrow{\cong} |\mathbb{C}\pi|^{\otimes k} \otimes \mathbb{C}\pi^{\otimes \ell}$.

`p:qbonbottomtangles`

**Proposition 5.10.** *The division by $b$ map, $q_b$, descends to an isomorphism*

$$q_b : \tilde{\mathcal{T}}_\nabla^{1/2}(\frown) \xrightarrow{\cong} \tilde{\mathcal{T}}_\nabla^{/1}(\bigcirc\frown).$$

*Proof.* The map $q_b$ uses the Conway relation to smooth double points to get a two-component tangle, where one component has interval skeleton and the other component has circle skeleton. □

This proof needs work.

`fig:ascending`

placeholder figure of an ascending tangle with an ascending embedding and a non-ascending embedding

`def:asc+desc`

**Definition 5.11.** Let $\bullet$ and $*$ be two points on the boundary of $D_p$ that are close together. An embedding

$$T : (I, \{0,1\}) \hookrightarrow (M_p, \{\bullet, *\})$$

representing a bottom tangle is called *ascending* if it "first goes up, and then goes *straight* down". More precisely, if $(z, s)$ is a global coordinate system for $M_p = D_p \times I$, then $T$ is an ascending tangle if there exists $c \in (0,1)$ such that when $t \in (0,c)$, the $\frac{d}{ds}$ component of $\dot{T}$ is positive, and when $t \in (c,1)$, $\dot{T}$ is a negative constant multiple of $\frac{d}{ds}$.

Likewise, such an embedding representing a bottom tangle $T$ is *descending* if it "first goes straight up, and then goes down". So there is $c \in (0,1)$ such that when $t \in (0,c)$, $\dot{T}$ is a positive constant multiple of $\frac{d}{ds}$ and when $t \in (c,1)$ the $\frac{d}{ds}$ component of $\dot{T}$ is negative.

**Definition 5.12.** An *ascending tangle* is a bottom tangle in $M_p$ whose ambient isotopy class has an ascending embedding. See figure **??** for an example.

TODO: add figure

Given a curve $K$ in $\mathbb{C}\pi$, through the isomorphism $\beta$, $K$ can be lifted to a bottom tangle in $\tilde{\mathcal{T}}^{/1}(\frown)$. Because we are in the quotient by degree 1 terms, crossings can be changed at will to make the lifted tangle be ascending or descending. However, to lift $K$ to a framed tangle takes some care. For any framed curve $K$ in $\mathbb{C}\pi$, we can choose a homotopy class representative that with rotation number 0 that is a sailing curve. A *sailing curve* is a curve whose tangent vector never points north. When taking a lift of a sailing $K$, there is an ascending lift of the curve where the north vector is never tangent to the curve. We will denote this lift as $\lambda_a(K)$ We can choose a framing at each point $p$ on $\lambda_a(K)$ by taking the tangent vector $\dot{T}$ at $p$ and the projection of $\vec{n}$ on to the plane normal to $\dot{T}$. Thus $\lambda_a(K)$ is a framed ascending bottom tangle. Similarly we will let $\lambda_d(K)$ be a framed descending bottom tangle. Finally, we define $\bar{\lambda} : \tilde{\mathcal{T}}^{/1}(\frown) \to \tilde{\mathcal{T}}^{/2}(\frown)$ by

make rigorous the notion of north in the disk. Add diagram of sailing trick to avoid north

$$\bar{\lambda}(K) = \lambda_a(K) - \lambda_b(K)$$

to be the difference between the framed ascending bottom tangle and the framed descending bottom tangle. In $\tilde{\mathcal{T}}^{/2}(\frown)$, crossing changes matter so $\bar{\lambda}$ is not the zero map.

**Theorem 5.13.** *The diagram in Figure 25 commutes and the unique induced map $\eta$ is the self intersection map $\mu$.*

*Proof.* Let $\gamma \in |\mathbb{C}\pi|$ and let $T = \lambda_a(\gamma)$ be an ascending lift. We can rewrite each strand-strand crossings ($s$-crossing for short) of $T$ as a sum or difference of a double point and its counterpart in $T^{f_b}$. As in the proof of **??**, rewriting each $s$-crossing of $T$ in this way yields a sum indexed by the subsets of its $s$-crossings.

old proof, not yet updated

$$\begin{tikzcd}
 & \tilde{\mathcal{T}}_{\nabla}^{1/2}(\frown) \arrow[r] \arrow[d, "0"'] & \tilde{\mathcal{T}}_{\nabla}^{/2}(\frown) \arrow[r, "q"] \arrow[d, "\lambda=\bar{\lambda}\circ q"'] & \tilde{\mathcal{T}}^{/1}(\frown) \arrow[r] \arrow[dl, "\bar{\lambda}"'] \arrow[d, "0"] \arrow[ddll, dashed, bend left, "\mu"] & 0 \\
0 \arrow[r] & \tilde{\mathcal{T}}_{\nabla}^{1/2}(\frown) \arrow[r] \arrow[d, leftrightarrow, "\cong"', "q_b"] & \tilde{\mathcal{T}}_{\nabla}^{/2}(\frown) \arrow[r] & \tilde{\mathcal{T}}^{/1}(\frown) & \\
 & \tilde{\mathcal{T}}^{/1}(\bigcirc\frown) & & &
\end{tikzcd}$$

FIGURE 25. The nontrivial horizontal maps are the respective quotient maps.

fig:Snakeformu

For every subset $X$ of $s$-crossings, there is a term in the sum where the $s$-crossings in $X$ are replaced by double points, and those not in $X$ are replaced with their counterparts in $T^{f_b}$. In particular, the term corresponding to $X = \emptyset$ is exactly $T^{f_b}$, so $T - T^{f_b}$ lives in $\tilde{\mathcal{T}}^1(\frown)$.

By passing to the quotient $\tilde{\mathcal{T}}_{\nabla}^1/\tilde{\mathcal{T}}_{\nabla}^2(\frown)$, only the terms that have a single double point remain, so $T - T^{f_b}$ becomes a sum over the $s$-crossings of $T$, where in each term the $s$-crossing is replaced by a double point. The map $q_b$ uses the Conway relation to smooth these double points to get a two-component tangle, where one component has interval skeleton and the other component has circle skeleton. Thus we land in $\tilde{\mathcal{T}}_{\nabla}^{/1}(\bigcirc\frown)$, which is isomorphic to $|\mathbb{C}\pi| \otimes \mathbb{C}\pi$ via $\beta$. $\square$

For a bottom tangle, there is a closure map from $cl : \tilde{\mathcal{T}}(\frown) \to \tilde{\mathcal{T}}(\bigcirc)$ by connecting the endpoints of the bottom tangle, $\bullet$ and $*$, by a canonical path in the boundary of the disk. Recall from Section 3.3 that the cobracket $\delta$ is constructed from $\mu$ by post composing with the closure map and then antisymetrizing. In the context of tangle diagrams, this construction is shown in Figure 26. The closure map $cl : \tilde{\mathcal{T}}^{/1}(\bigcirc\frown) \to \tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc)$ orders the components by placing the closed bottom tangle in the second slot. The intermediate induced map after closing, but before antisymmtrizing, is denoted in the figure by $\hat{\delta}$ and is called the *ordered* Turaev cobracket. We will show the Kontsevich integral is homomorphic with respect to $\hat{\delta}$. The homomorphicity of $\delta$ with respect to $Z$ follows from immediately the homomorphicity of $\hat{\delta}$ with respect to $Z$ because $\operatorname{gr}(Alt) = Alt$.

Taking the associated graded of the diagram in Figure 21 we arrive at the diagram in Figure 27

akefor_gr_cobracket

**Theorem 5.14.** *The diagram in Figure 27 commutes and the induced map* $\operatorname{gr}\hat{\delta}$ *is the associated graded ordered Turaev cobracket.*

*Proof.* The maps in the diagram of Figure 26 are filtered maps, and therefore Figure 27 is obtained by applying the associated graded functor to it. As a result, the diagram of Figure 27 commutes, $\operatorname{gr}\mu$ is the induced map from the

$$\begin{tikzcd}
\tilde{\mathcal{T}}_{\triangledown}^{1/2}(\frown) \arrow[r] \arrow[d, "0"'] & \tilde{\mathcal{T}}_{\triangledown}^{/2}(\frown) \arrow[r] \arrow[d, "\lambda"'] & \tilde{\mathcal{T}}^{/1}(\frown) \arrow[d, "0"] \arrow[ddll, dashed, "\mu"] \arrow[dddll, dashed, "\hat{\delta}"] \arrow[ddddll, dashed, "\delta"] \\
\tilde{\mathcal{T}}_{\triangledown}^{1/2}(\frown) \arrow[r] \arrow[d, leftrightarrow, "\cong"', "q_b"] & \tilde{\mathcal{T}}_{\triangledown}^{/2}(\frown) \arrow[r] & \tilde{\mathcal{T}}^{/1}(\frown) \\
\tilde{\mathcal{T}}^{/1}(\bigcirc\frown) \arrow[d, "cl"] & & \\
\tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc) \arrow[d, "Alt"] & & \\
\tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc) & &
\end{tikzcd}$$

g:Snakeforcobracket

FIGURE 26. Constructing $\delta$ from $\mu$.

$$\begin{tikzcd}
 & \tilde{\mathcal{A}}_{\triangledown}^{1/2}(\frown) \arrow[r] \arrow[d, "0"'] & \tilde{\mathcal{A}}_{\triangledown}^{/2}(\frown) \arrow[r] \arrow[d, "\operatorname{gr}\lambda"'] & \tilde{\mathcal{A}}^{/1}(\frown) \arrow[r] \arrow[d, "0"] \arrow[ddll, dashed, "\operatorname{gr}\mu"] \arrow[dddll, dashed, "\operatorname{gr}\hat{\delta}"] & 0 \\
0 \arrow[r] & \tilde{\mathcal{A}}_{\triangledown}^{1/2}(\frown) \arrow[r] \arrow[d, leftrightarrow, "\cong"', "q_b"] & \tilde{\mathcal{A}}_{\triangledown}^{/2}(\frown) \arrow[r] & \tilde{\mathcal{A}}^{/1}(\frown) & \\
 & \tilde{\mathcal{A}}^{/1}(\bigcirc\frown) \arrow[d, "\operatorname{gr} cl"] & & & \\
 & \tilde{\mathcal{A}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{A}}^{/1}(\bigcirc) & & &
\end{tikzcd}$$

akefor_gr_cobracket

FIGURE 27. Associated graded diagram constructing the graded ordered Turaev cobracket.

snake lemma for this diagram, and so $\operatorname{gr}\hat{\delta}$ coincides with the graded ordered Turaev cobracket. □

pcubesimplification

**Lemma 5.15.** *There exists a map* $\rho : \tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc) \to \tilde{\mathcal{T}}_{\triangledown}^{/2}(\bigcirc)$ *that makes the diagram in Figure 28 commute.*

In this figure, do we need $\mu$ in it still?

*Proof.* There is an isomorphism from $\tilde{\mathcal{T}}^{/1}(\bigcirc)\otimes\tilde{\mathcal{T}}^{/1}(\bigcirc)$ to $\tilde{\mathcal{T}}^{/1}(\bigcirc\bigcirc)$ by combining the two tangles into a single tangle and forgetting the order of the components.

$$\begin{tikzcd}
\tilde{\mathcal{T}}^{1/2}_{\triangledown}(\cap) \arrow[r] \arrow[d, "0"'] & \tilde{\mathcal{T}}^{/2}_{\triangledown}(\cap) \arrow[r] \arrow[d, "\lambda"'] & \tilde{\mathcal{T}}^{/1}(\cap) \arrow[d, "0"] \arrow[ddll, dashed, "\mu"] \arrow[dddll, dashed, "\hat{\delta}"] & \\
\tilde{\mathcal{T}}^{1/2}_{\triangledown}(\cap) \arrow[r] \arrow[d, leftrightarrow, "\cong"', "q_b"] & \tilde{\mathcal{T}}^{/2}_{\triangledown}(\cap) \arrow[r] \arrow[dd, "cl"] & \tilde{\mathcal{T}}^{/1}(\cap) \arrow[dd, "0"] & \\
\tilde{\mathcal{T}}^{/1}(\bigcirc\cap) \arrow[d, "cl"] & & & \\
\tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc) \arrow[r, "\exists\rho"] & \tilde{\mathcal{T}}^{/2}_{\triangledown}(\bigcirc) \arrow[r] & \tilde{\mathcal{T}}^{/1}(\bigcirc) \arrow[r] & 0
\end{tikzcd}$$

`pcubesimplification`

Figure 28. Commutative diagram for Lemma 5.15.

Since we are modding out by $s$ degree 1, there is no notion of over or under, these are just curves in the disc.

The map $\rho : \tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc) \to \tilde{\mathcal{T}}^{/2}(\bigcirc)$ is defined to be the following composition of maps.

$$\begin{tikzcd}
\tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc) \arrow[r, "forget"] \arrow[rrr, bend left, "\rho"] & \tilde{\mathcal{T}}^{/1}(\bigcirc\bigcirc) \arrow[r, leftrightarrow, "q_b"] & \tilde{\mathcal{T}}^{1/2}_{\triangledown}(\bigcirc) \arrow[r, hook] & \tilde{\mathcal{T}}^{/2}_{\triangledown}(\bigcirc)
\end{tikzcd}$$

Since the image of $\rho$ in $\tilde{\mathcal{T}}^{/2}_{\triangledown}$ is all of $\tilde{\mathcal{T}}^{1/2}$ we get the following short exact sequence.

$$\begin{tikzcd}
\tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc) \arrow[r, "\rho"] & \tilde{\mathcal{T}}^{/2}_{\triangledown}(\bigcirc) \arrow[r] & \tilde{\mathcal{T}}^{/1}(\bigcirc) \arrow[r] & 0
\end{tikzcd}$$

The commutativity of the diagram in Figure 28 relies finally on the commutativity of the bottom left square. We single this square out below and verify the commutativity.

$$\begin{tikzcd}
\tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc) \arrow[r, "forget"] \arrow[d, "Z^{/1}\otimes Z^{/1}"] \arrow[rrr, bend left, "\rho"] & \tilde{\mathcal{T}}^{/1}(\bigcirc\bigcirc) \arrow[r, leftrightarrow, "q_b"] \arrow[d, "Z^{/1}"] & \tilde{\mathcal{T}}^{1/2}_{\triangledown}(\bigcirc) \arrow[r, hook] \arrow[d, "Z^{/1}"] & \tilde{\mathcal{T}}^{/2}_{\triangledown}(\bigcirc) \arrow[d, "Z^{/2}"] \\
\tilde{\mathcal{A}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{A}}^{/1}(\bigcirc) \arrow[r, "forget"] \arrow[rrr, bend right, "\operatorname{gr}\rho"'] & \tilde{\mathcal{A}}^{/1}(\bigcirc\bigcirc) \arrow[r, leftrightarrow, "\operatorname{gr} q_b"] & \tilde{\mathcal{A}}^{1/2}_{\triangledown}(\bigcirc) \arrow[r, hook] & \tilde{\mathcal{A}}^{/2}_{\triangledown}(\bigcirc)
\end{tikzcd}$$

`fig:frontlefthomom`

FIGURE 29. Commutative diagram for Lemma 5.16

$$\begin{tikzcd}
\tilde{\mathcal{T}}^{1/2}_{\triangledown}(\cap) \arrow[rrr, hook] \arrow[d, leftrightarrow, "q_b", "\cong"'] & & & \tilde{\mathcal{T}}^{/2}_{\triangledown}(\cap) \arrow[dd, "cl"] \\
\tilde{\mathcal{T}}^{/1}(\bigcirc\cap) \arrow[d, "cl"] & & & \\
\tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc) \arrow[r] \arrow[rrr, dashed, bend left, "\rho"] & \tilde{\mathcal{T}}^{/1}(\bigcirc\bigcirc) \arrow[r, leftrightarrow, "q_b", "\cong"'] & \tilde{\mathcal{T}}^{1/2}_{\triangledown}(\bigcirc) \arrow[r, hook] & \tilde{\mathcal{T}}^{/2}_{\triangledown}(\bigcirc)
\end{tikzcd}$$

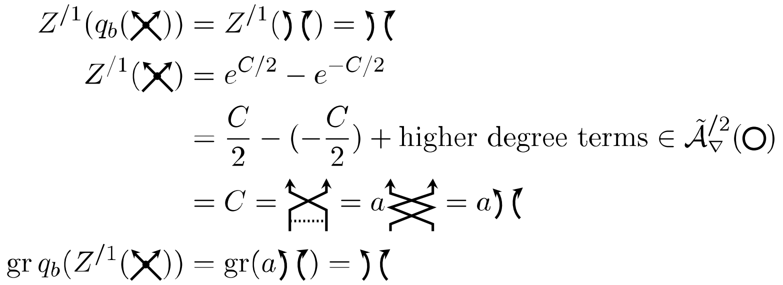
Let $T \in \tilde{\mathcal{T}}^{1/2}_{\triangledown}(\cap)$, then $T$ is a bottom tangle with exactly one double point. Following along the top and right of the diagram in Figure 28, when $T$ is closed, we get a closed loop with one double point inside $\tilde{\mathcal{T}}^{/2}_{\triangledown}(\bigcirc)$. Following along the right and bottom, $q_b(T)$ uses the Conway relation to snip off a loop of $T$ to get a tangle in $\tilde{\mathcal{T}}^{/1}(\bigcirc\cap)$ with one closed loop and a bottom tangle, with no double points. Closing the bottom tangle and forgetting the order of the closed loops gives a tangle in $\tilde{\mathcal{T}}^{/1}(\bigcirc\bigcirc)$ with two closed loops and no double points. Reversing the Conway relation along $q_b$ glues together the two closed loops to get a single closed loop with one double point then included into $\tilde{\mathcal{T}}^{/2}(\bigcirc)$. This arrives at the same closed loop with one double point as if we had closed $T$ in the first place. □

`lem:frontlefthomom`

**Lemma 5.16.** *The diagram in Figure 29 commutes.*

*Proof.* The right square commutes because $Z$ is a filtered map and respects filtered inclusions.

For the middle square, we use the map $q_b$ from right to left and show commutativity on a double point.



$$\begin{aligned} Z^{/1}(q_b(\times)) &= Z^{/1}()() = )( \\ Z^{/1}(\times) &= e^{C/2} - e^{-C/2} \\ &= \frac{C}{2} - (-\frac{C}{2}) + \text{higher degree terms} \in \tilde{\mathcal{A}}^{/2}_{\triangledown}(\bigcirc) \\ &= C = \times = a\times = a)( \\ \operatorname{gr} q_b(Z^{/1}(\times)) &= \operatorname{gr}(a)() = )( \end{aligned}$$

For the left square, $Z$ compatible with forgetful is because we land in $/1$, where there are no s-s chords. ☐

say more about left square

cobrackethomomorphic

**Theorem 5.17.** *The Kontsevich integral descends to a homomorphic expansion for the ordered Turaev cobracket. That is, the following square commutes:*

$$\begin{tikzcd} \tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc) & \tilde{\mathcal{T}}^{/1}(\cap) \arrow[l, "\hat{\delta}"'] \arrow[d, "Z^{/1}"] \\ \mathcal{A}^{/1}(\bigcirc) \otimes \mathcal{A}^{/1}(\bigcirc) \arrow[from=u, "Z^{/1}\otimes Z^{/1}"] & \mathcal{A}^{/1}(\cap) \arrow[l, "\operatorname{gr}\hat{\delta}"] \end{tikzcd}$$

*Proof.* The diagram in Figure 30 is attained by taking the Kontsevich integral of the commutative diagram in Figure 28 (with the middle layers omitted). We have already established that the top and bottom faces commute by Lemma 5.15 and Theorem 5.14. The left and right vertical sides trivially commute because of the zero maps. The front-left vertical square commutes by Lemma 5.16. The front-right and back faces commute because $Z$ respects the $s$-filtration and is homomorphic with respect to the inclusion and quotient maps of the filtered components.

The middle vertical face of Figure 30 is the following square.

$$\begin{tikzcd} & \tilde{\mathcal{T}}^{/2}_{\triangledown}(\cap) \arrow[dl, "cl\circ\lambda"] \arrow[d, "Z^{/2}"] \\ \tilde{\mathcal{T}}^{/2}_{\triangledown}(\bigcirc) \arrow[d, "Z^{/2}"'] & \mathcal{A}^{/2}_{\triangledown}(\cap) \arrow[dl, "\operatorname{gr}(cl\circ\lambda)"] \\ \mathcal{A}^{/2}_{\triangledown}(\bigcirc) & \end{tikzcd}$$

The Kontsevich integral is homomorphic with respect to the flip operation, as shown in Proposition 4.13. The map $cl \circ \lambda$ applied to a bottom tangle outputs the difference between the closed ascending lift and the closed descending lift. The closed descending lift is the flip of the closed ascending lift. So $cl \circ \lambda =$

This is not quite right, FIX ME!

$$\begin{tikzcd}
 & \tilde{\mathcal{T}}^{1/2}_{\triangledown}(\cap) \arrow[rr] \arrow[dl, "0"'] \arrow[dd, "Z^{1/2}"] & & \tilde{\mathcal{T}}^{/2}_{\triangledown}(\cap) \arrow[rr] \arrow[dl, "cl\circ\lambda"'] \arrow[dd, "Z^{/2}"] & & \tilde{\mathcal{T}}^{/1}(\cap) \arrow[dl, "0"'] \arrow[dd, "Z^{/1}"] \arrow[lllll, bend right, "\hat{\delta}"'] \\
\tilde{\mathcal{T}}^{/1}(\bigcirc)\otimes\tilde{\mathcal{T}}^{/1}(\bigcirc) \arrow[rr, "\rho"] \arrow[dd, "Z^{1/2}\otimes Z^{1/2}"] & & \tilde{\mathcal{T}}^{/2}_{\triangledown}(\bigcirc) \arrow[rr] \arrow[dd, "Z^{/2}"] & & \tilde{\mathcal{T}}^{/1}(\bigcirc) \arrow[dd, "Z^{/1}"] & \\
 & \tilde{\mathcal{A}}^{1/2}_{\triangledown}(\cap) \arrow[rr] \arrow[dl, "0"'] & & \tilde{\mathcal{A}}^{/2}_{\triangledown}(\cap) \arrow[rr] \arrow[dl, "\mathrm{gr}(cl\circ\lambda)"] & & \tilde{\mathcal{A}}^{/1}(\cap) \arrow[dl, "0"'] \arrow[lllll, bend left, "\mathrm{gr}\,\hat{\delta}"] \\
\tilde{\mathcal{A}}^{/1}(\bigcirc)\otimes\tilde{\mathcal{A}}^{/1}(\bigcirc) \arrow[rr, "\mathrm{gr}\,\rho"] & & \tilde{\mathcal{A}}^{/2}_{\triangledown}(\bigcirc) \arrow[rr] & & \tilde{\mathcal{A}}^{/1}(\bigcirc) &
\end{tikzcd}$$

FIGURE 30. Commutative cube showing the formality of the ordered Turaev cobracket from the Kontsevich integral.

`g:Cube_for_cobracket`

$(id - flip) \circ cl$ acting on ascending representatives. $Z$ is homomorphic with respect to $(id - flip) \circ cl$.

The commutativity of all vertical faces of the cube diagram in Figure 30 implies that the induces diagonal square also commutes, which gives the desired formality of the theorem statement. □

where does conjugation come into play?? Something about flipping first then dragging the ends down and then closing.

remark–if we were doing this with $\mu$ is it wouldn't work because flip of a bottom tangle is not a bottom tangle. It is much cleaner to just pass to the closures.

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$$\begin{tikzcd}
 & \tilde{\mathcal{T}}^{1/2}_{\triangledown}(\cap) \arrow[rr] \arrow[dl, "0"] \arrow[dd, "Z^{1/2}"] & & \tilde{\mathcal{T}}^{/2}_{\triangledown}(\cap) \arrow[rr] \arrow[dl, "cl\circ\lambda"'] \arrow[dd, "Z^{/2}"] & & \tilde{\mathcal{T}}^{/1}(\cap) \arrow[dl, "0"] \arrow[dd, "Z^{/1}"] \arrow[lllll, bend right, "\hat{\delta}"'] \\
\tilde{\mathcal{T}}^{/1}(\bigcirc)\otimes\tilde{\mathcal{T}}^{/1}(\bigcirc) \arrow[rr, "\rho"] \arrow[dd, "Z^{1/2}\otimes Z^{1/2}"] & & \tilde{\mathcal{T}}^{/2}_{\triangledown}(\bigcirc) \arrow[rr] \arrow[dd, "Z^{/2}"] & & \tilde{\mathcal{T}}^{/1}(\bigcirc) \arrow[dd, "Z^{/1}"] & \\
 & \tilde{\mathcal{A}}^{1/2}_{\triangledown}(\cap) \arrow[rr] \arrow[dl, "0"] & & \tilde{\mathcal{A}}^{/2}_{\triangledown}(\cap) \arrow[rr] \arrow[dl, "\mathrm{gr}(cl\circ\lambda)"] & & \tilde{\mathcal{A}}^{/1}(\cap) \arrow[dl, "0"] \arrow[dlllll, bend left, "\mathrm{gr}\,\hat{\delta}"] \\
\tilde{\mathcal{A}}^{/1}(\bigcirc)\otimes\tilde{\mathcal{A}}^{/1}(\bigcirc) \arrow[rr, "\mathrm{gr}\,\rho"] & & \tilde{\mathcal{A}}^{/2}_{\triangledown}(\bigcirc) \arrow[rr] & & \tilde{\mathcal{A}}^{/1}(\bigcirc) &
\end{tikzcd}$$

g:Cube_for_cobracket

FIGURE 30. Commutative cube showing the formality of the ordered Turaev cobracket from the Kontsevich integral.

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