GOLDMAN-TURAEV FORMALITY FROM THE KONTSEVITCH INTEGRAL

DROR BAR-NATAN, ZSUZSANNA DANCSO, TAMARA HOGAN, JESSICA LIU, AND NANCY SCHERICH

Abstract. We present a three dimensional realisation of the Goldman-Turaev Lie biaglebra, and construct Goldman-Turaev homomorphic expansions from the Kontsevich integral.

Contents

Key words and phrases. knots, links in a handlebody, expansions, finite type invariants, Lie algebras .

1. Introduction

In 1986, Goldman defined a Lie bracket [Gol86] on the space of homotopy classes of free loops on a compact oriented surface. Shortly after in 1991, Turaev defined a cobracket [Tur91] on the same space[1]. This bracket and cobracket make the space of free loops into a Lie bialgebra – known as the Goldman-Turaev (GoTu) Lie bialgebra – which forms the basis for the field of string topology [?] and has been an object of study from many perspectives.

add referemnces: chas-sullivan, kashiwara-vergne, AN, AT, Formality paper

In this paper we, describe a 3-dimensional lift of the Goldman-Turaev Lie bialgebra into a space of tangles in a handlebody. We recover the bracket and cobracket maps as projections of intuitive operations on tangles. We show the Kontsevich integral is homomorphic with respect to these tangle operations. Our main result is informally summarised as follows:

Main Result. *Let $\tilde{\mathcal{T}}$ denote the space of formal linear combinations of tangles in a punctured disc cross an interval $M_p = D_p \times I$. Projecting to the bottom $D_p \times 0$, one obtains curves on a punctured disc, and the Goldman–Turaev operations on these curves are induced[2] by the* stacking *and* flipping *operations on the tangles. The Kontsevich integral is a homomorphic expansion for tangles in M_p, and descends to a Goldman–Turaev homomorphic expansion on D_p.*

This result is parallel to Massuyeau's [Mas18], however, our approach to the cobracket is significantly different and simpler, hence, more likely to lead to give insight into the motivational application descried below. Another related result is [?], which constructs Goldman–Turaev expansions from the Khnizhnik-Zamolodchikov connection, a geometric incarnation of the Kontsevich integral.

In more detail, we describe a space $\tilde{\mathcal{T}}$ of formal linear combintations of framed tangles in the handlebody $\mathcal{D}_p \times I$ and operations on this space, which induce the Goldman-Turaev operations in the bottom projection to $D_p \times \{0\}$. The Goldman bracket arises from the commutator associated to the stacking product in a Conway skein quotient of $\tilde{\mathcal{T}}$, defined in Section 4.7, and the Turaev cobracked from taking the difference between a tangle and its vertical flip, again in a Conway quotient. We study the associated graded spaces and operations, and show that the Kontsevich integral is a homomorphic expansion for these tangles, in other words, intertwines the operations with their associated graded counterparts. We show that therefore, the Kontsevich integral descends to a homomorphic expansion for the Goldman-Turaev Lie bialgebra. For the flipping operation and the Turaev cobracket, the precise statements are subtle, and care needs to be taken with the technical details.

There are other papers by Turaev and Massuyeau-Turaev that are not mentioned here.

There are also some references that Yusuke mentioned that we should include

Turaev's paper- we can probably pull some of our lemmas from his paper, reference for relationship with HOMFLY, but he does not mention the free associative algebra at all. Our paper is not a subset of his. Skein algebra quantizes — symmetric lie algebra generated by the goldman lie algebra–you can get a

[1]Turaev's version required factoring out by the constant loop; there is a lift to the full space of homotopy classes of loops, given a framing on the surface [AKKN20].

[2]In a specific sense defined in Section 2

1.1. **Motivation.** The Kashiwara–Vergne equations originally arose from the study of convolutions on Lie groups [**?**]. The equations were reformulated algebraically in terms of automorphisms of free Lie algebras [**?**], it this form they are a refinement of the Baker-Campbell-Hausdorff formula for products of exponentials of non-commuting variables.

Kashiwara–Vergne theory has multiple topological interpretations in which Kashiwara–Vergne solutions correspond to certain invariants – called *homomorphic expansions* – of topological objects. The existence of a homomorphic expansion is also called *formality* in the literature, this language is inspired by rational homotopy theory and group theory [**?**].

One of these topological interpretations is due to the first two authors [BND17], who showed that homomorphic expansions of welded foams – a class of 4-dimensional tangles – are in one to one correspondence with solutions to the KV equations. Recently, a series of papers by Alekseev, Kawazumi, Kuno and Naef [AKKN20,AKKN18b,AKKN18a] drew an analogous connection between KV solutions and homomorphic expansions for the Goldman-Turaev Lie bialgebra for the disc with two punctures (up to non-negligible differences in the technical details). This correspondence was used to generalise the Kashiwara–Vergne equations via considering different surfaces, including those of higher genus.

In other words, there is an intricate algebraic connection between four-dimensional welded foams and the GT Lie bi-algebra, which strongly suggests that there is a topological connection as well. In addition to the inherent interest in tangles in handlebodies, one goal for this paper is to work towards this connection between the two-dimensional Goldman–Turaev Lie bialgebra and four-dimesnional welded foams, by constructing a three-dimensional realisation of the Goldman-Turaev Lie bialgebra, with homomorphic expansions which descend to Goldman-Turaev expansions.

The paper is organised as follows: Section 2 gives a general algebraic framework for how the Goldman–Turaev operations are induced by tangle operations. In Section 3 we give a brief overview of the Kontsevich integral and the Goldman Turaev Lie bialgebra. In Section 4, we define tangles in handlebodies, relevant operations and Vassiliev filtrations. We identify the associated graded space of tangles as a space of chord diagrams, and introduce the Conway skein quotient. In Section 5, we identify the GoTu Lie biaglebra in a low filtration degree, and prove the main theorem.

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sec:conceptsum

2. Conceptual summary

We induce the genus zero Goldman-Turaev operations from tangle operations, in the spirit of "connecting homomorphisms": this Section is a summary of the basic approach. We use the words *associated graded structures*, *homomorphic expansions*, and *Goldman-Turaev operations* without definition, only mentioning their basic properties which make this conceptual outline coherent; the definitions follow in Section 3.

In the diagram (2.1), top and bottom rows are exact and the right and left vertical maps are zero, and therefore, by minor diagram chasing, the middle vertical map λ induces a unique map $\eta : C \to D$, a degenerate case of a connecting homomorphism. In our applications λ is a difference of two maps λ_0 and λ_1, whose values differ in E but coincide in a quotient F.

eq:Snake (2.1)

$$\begin{array}{ccccccccc} & & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle \lambda=\lambda_0-\lambda_1} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & D & \longrightarrow & E & \longrightarrow & F & & \end{array} \qquad \eta : C \to D \tag{2.1}$$

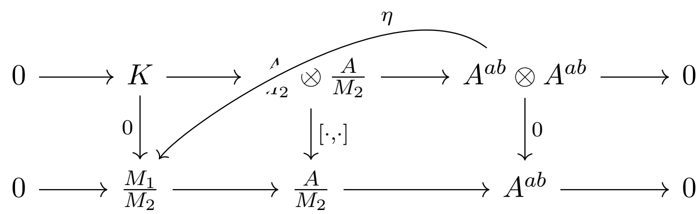
Just for Dror (temporary) here is the simplest possible example, the induced map η is the identity on $\mathbb{Q}$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Q} & \xrightarrow{1\mapsto x} & \mathbb{Q}[x]/x^2 & \xrightarrow{x\mapsto 0} & \mathbb{Q} & \longrightarrow & 0 \\ & & {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle \lambda=\cdot x} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & \mathbb{Q} & \xrightarrow{1\mapsto x} & \mathbb{Q}[x]/x^2 & \xrightarrow{x\mapsto 0} & \mathbb{Q} & \longrightarrow & 0 \end{array}$$

In Section 5 we present two constructions which produce the Goldman bracket and the Turaev cobracket, respectively, as induced homomorphisms η, from corresponding tangle operations λ_0 and λ_1. The following example is a schematic version of what will become the argument for the Goldman bracket:

Example 2.1. Let A be an associative algebra, and let $\{L_i\}$ denote the lower central series of A. That is, $L_1 := A$, and $L_{i+1} := [L_i, A]$. Then the L_i are Lie ideals, and let $M_i = AL_i$ denote the two-sided ideal generated by L_i. The quotient A/M_1 is the abelianisation of A, denoted by A^{ab}. Then we have the following diagram:

eq:SnakeExample (2.2)

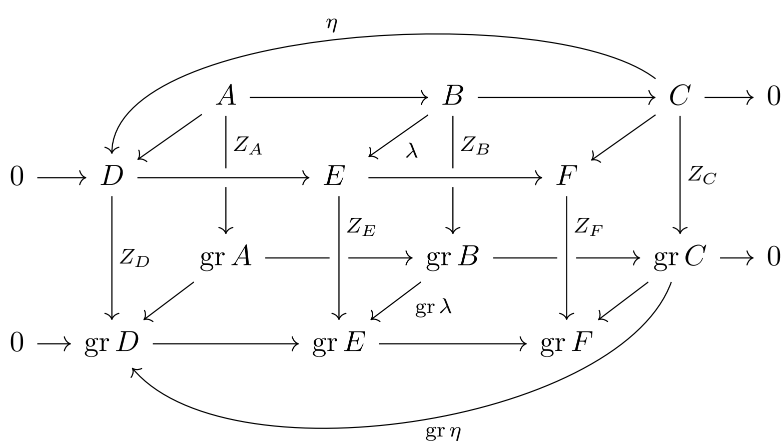


Here λ is the algebra commutator, which is indeed the difference between two maps: the multiplication (λ_0) and the opposite multiplication (λ_1). The kernel K of the projection to $A^{ab} \otimes A^{ab}$ is generated by the subalgebras $\left\{ \frac{M_1}{M_2} \otimes \frac{A}{M_2}, \frac{A}{M_2} \otimes \frac{M_1}{M_2} \right\}$ in $\frac{A}{M_2} \otimes \frac{A}{M_2}$. The map η is a well defined commutator map $A^{ab} \otimes A^{ab} \to \frac{M_1}{M_2}$, given by $\eta(x \otimes y) = [x, y] \bmod M_2$. □

Nancy, can you fix the positioning of the η map?

The goal of this paper is to construct homomorphic expansions for the Goldman-Turaev Lie bialgebra from the Kontsevich integral. In outline, this follows from the naturality property of the construction above, under the associated graded functor. Namely, if all of the spaces in the diagram (2.1) are filtered and the maps are filtered maps, then the associated graded functor (denoted gr) produces an associated graded commutative diagram with the same properties. An expansion for an algebraic structure X is a filtered homomorphism $Z : X \to \operatorname{gr} X$ (with special properties). Thus, if expansions exist for each of the spaces A through F, we obtain a multi-cube:

eq:Cube (2.3)



If, in the multi-cube (2.3) all vertical faces commute, then so does the square:

$$\begin{array}{ccc} D & \xleftarrow{\ \eta\ } & C \\ \downarrow{\scriptstyle Z_D} & & \downarrow{\scriptstyle Z_C} \\ \operatorname{gr} D & \xleftarrow{\ \operatorname{gr}\eta\ } & \operatorname{gr} C \end{array}$$

The commutativity of this square, where η represents the Goldman bracket and the Turaev cobracket, respectively, is – by definition – the homomorphicity property of the expansion: our main result. The non-trivial vertical face of the multi-cube is the one containing λ, and the commutativity of this for each Goldman-Turaev operation will follow from the fact that the Kontsevich integral (standing in for Z_B and Z_E) intertwines the appropriate tangle operations λ_0 and λ_1. This is the idea behind the proof of the main theorem.

3. Preliminaries: Homomorphic expansions and the Goldman-Turaev Lie bialgebra

3.1. **The Kontsevich Integral.** The Kontsevich Integral is the knot theoretic prototype of a *homomorphic expansion.* Homomorphic expansions (a.k.a. formality isomorphisms, universal finite type invariants) provide a connection between knot theory and quantum algebra / Lie theory. Many detailed expositions on the Kontsevich Integral exist in the literature, we recommend [CDM12, Section 8], or [Kon93, BN95, Dan10]. We briefly review the basics here from an algebraic perspective, which is outlined – in a slightly different, finitely presentated case – in [BND17, Section 2].

Let $\mathcal{K}$ denote the set of oriented knots in $\mathbb{R}^3$, and allow formal linear combinations of knots with coefficients in $\mathbb{C}$. There is a filtration on this infinite dimensional vector space called the Vassiliev filtration, which is defined in terms of resolutions of *double points.* Namely, a *double point* is defined to be the difference of an over and under crossing:

A knot with k double points is a signed sum of 2^k knots. See Figure 1 for an example. The Vassiliev filtration is the decreasing filtration

$$\mathcal{K} = \mathcal{K}_0 \supseteq \mathcal{K}_1 \supseteq \mathcal{K}_2 \supseteq ...$$

where $\mathcal{K}_i$ is linearly generated by knots with at least i double points.

The degree completed associated graded space of $\mathcal{K}$ with respect to the Vassiliev filtration is defined as

$$\mathcal{A} := \prod_{n\geq 0} \mathcal{K}_n/\mathcal{K}_{n+1}.$$

Since $\mathcal{A}$ is a graded vector space, it lends itself naturally to recursive calculations and inductive arguments. The key idea of Goussarov and Vassiliev was to study knots (the space $\mathcal{K}$) via invariants which take values in $\mathcal{A}$. An *expansion* Z is a filtered linear map of knots taking values in $\mathcal{A}$, which retains as much information

I'm worried that we start with knots, too far from our target, tangles in a PDS.

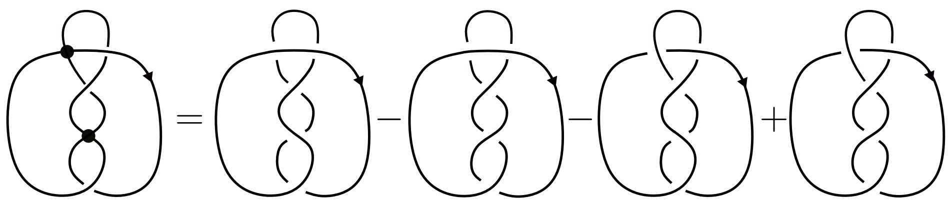


FIGURE 1. A knot with two double points written as a signed sum four knots.

fig:pumpkins

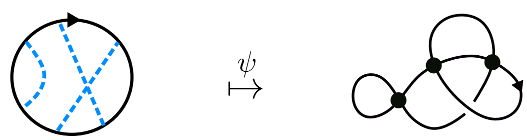


FIGURE 2. Example of ψ mapping a chord diagram to a singular knot where the right-hand side is viewed as an element of $\mathcal{K}_3/\mathcal{K}_4$.

fig:psionchord

as possible. Rigorously, this means that the associated graded map of Z is the identity map of $\mathcal{A}$:

$$Z : \mathcal{K} \to \mathcal{A} \quad \text{such that} \quad \operatorname{gr} Z = id_{\mathcal{A}}.$$

An expansion is *homomorphic* if it also intertwines ~~knot~~ these operations (~~such as connected sum~~) with their associated graded counterparts. This allows for a study of these operations via the associated graded space as well.

A: with respect to some operations (e.g., the connected sum operation)

A crucial step towards making effective use of this machinery is to get a handle on the space $\mathcal{A}$ in concrete terms: namely, $\mathcal{A}$ has a combinatorial description as a space of *chord diagrams*. A chord diagram of degree k on an oriented circle is a perfect matching[3] on a set of $2k$ points arranged around the circle, up to orientation preserving diffeomorphism. The circle which supports the chord diagram is called the *skeleton*. In other words, a chord diagram on a circle is a combinatorial object consisting of $2k$ cyclically ordered points, partitioned into pairs. In diagrams, each pair is indicated by a *chord*, as in the left of Figure 2.

There is a natural map ψ from chord diagrams with i chords to $\mathcal{K}_i/\mathcal{K}_{i+1}$, as shown in Figure 2. Namely, by contracting each chord into a double point, we obtain an i-singular knot. This is well-defined only up to crossing changes – as crossings other than double points may appear – however, the difference between the over/under choices for any additional crossing is in $\mathcal{K}_{i+1}$.

It is not difficult to establish that ψ is surjective, and that there are two relations in its kernel: the 4-Term (4T) and Framing Independence (FI) relations,

[3]A perfect matching on a set is a partitioning of the set by 2-element subsets.

shown in Figure 3. In fact, these two relations generate the kernel, and ψ descends to an isomorphism on the quotient; this, however, is significantly harder to prove.

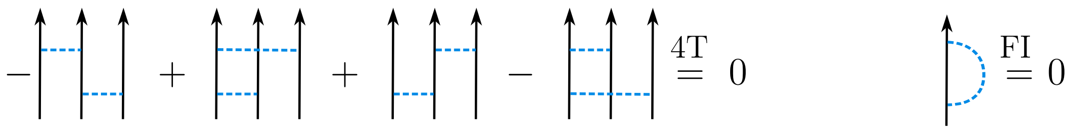


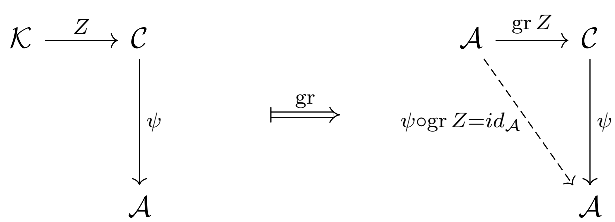
fig:4TFI

FIGURE 3. The 4T and FI relations, understood as local relations: the strand(s) are part(s) of the skeleton circle, and the skeleton may support additional chords outside the picture shown, which are the same throughout all terms of the relation.

The key technique is to construct an expansion as in the following Lemma, [BND17, Proposition 2.7]:

lem:assocgradyoga

Lemma 3.1. *[BND17] Let $\mathcal{K}$ be a filtered space of formal linear combinations of knotted objects*[4]*, and $\mathcal{A}$ the associated graded space of $\mathcal{K}$. Let $\mathcal{C}$ be a "candidate model" for $\mathcal{A}$: a graded linear space equipped with a surjective graded map $\psi : \mathcal{C} \to \mathcal{A}$. If there exists a filtered map $Z : \mathcal{K} \to \mathcal{C}$, such that $\psi \circ \operatorname{gr} Z = id_{\mathcal{A}}$, then ψ is an isomorphism and Z is an expansion for $\mathcal{K}$.*



In other words, once one finds a candidate model $\mathcal{C}$ for $\mathcal{A}$, finding an *expansion valued in* $\mathcal{C}$ also implies that ψ is an isomorphism. In classical Vassiliev theory, $\mathcal{K}$ is the space of oriented knots, $\mathcal{C}$ is the space of chord diagrams, and ~~the~~ a $\mathcal{C}$-valued expansion is the Kontsevich integral [Kon93].

For a detailed introduction to the Kontsevich integral we recommend [CDM12, Section 8]. The definition an explicit integral formula associated to a Morse representation of a knot or link in $\mathbb{C} \times \mathbb{R}$, as in Figure 4.

Definition 3.2. Let K be an oriented link in $\mathbb{R}^3 \cong \mathbb{R}_t \times \mathbb{C}$, where t parametrises the vertical real dimension, and the embedding K is Morse with respect to t: that is, ~~critical points occur at distinct heights~~. The unnormalised Kontsevich integral

I think "Morse" is only the statement that the critical points are quadratic.

[4]"Knotted objects" may mean knots, links, tangles, knotted graphs, etc, depending on context.

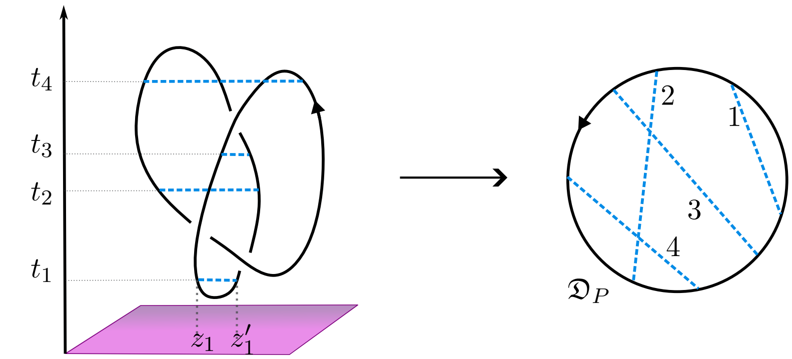


FIGURE 4. The Kontsevich Integral is computed from a Morse embedding of the knot

fig:Kint

$Z'(K)$ of K is defined as:

eq:Kint

$$Z'(K) := \sum_{m=0}^{\infty} \int_{\substack{t_{min}<t_m<\dots<t_1<t_{max} \\ t_i \text{ non-critical}}} \sum_{P=\{(z_i,z_i')\}_i} \frac{(-1)^{P_\downarrow}}{(2\pi i)^m} \mathfrak{D}_P \bigwedge_{i=1}^{m} \frac{dz_i - dz_i'}{z_i - z_i'}. \tag{3.1}$$

Here the values t_{min} and t_{max} denote the minimum and maximum heights of K, and each summand is an integral over an m-simplex determined by $t_m < \dots < t_1$. The summation is over choices P of "pairings" of two points on the knot of height t_i, each of which, when projected to the plane at $t = 0$, yields a complex pair (z_i, z_i'). We denote by $\mathfrak{D}_P$ the chord diagram given by interpreting the m pairings (z_i, z_i') as chords, as shown in Figure 4. Finally, $P_\downarrow$ is the number of points in P where (t_i, z_i) or (t_i, z_i') is on a t-descending arc in K.

Kontsevich's famous result [Kon93] is that $Z(K) := \frac{Z'(K)}{Z'(\infty)^{c/2}}$ is an invariant of unframed links, where c denotes the number of critical points – minima and maxima – in the Morse embedding of K. The Kontsevich integral takes values in the space of chord diagrams (for links, with chords on multiple circles) modulo the 4T and FI relations. The Kontsevich integral Z satisfies $\psi \circ \operatorname{gr} Z = id_{\mathcal{C}}$. Therefore, ψ is an isomorphism, and Z is an expansion for unframed links. In light of this, we do not distinguish between $\mathcal{C}$ and $\mathcal{A}$, and use $\mathcal{A}$ to mean In addition, Z has a number of good properties, for example, it is homomorphic with respect to knot connected sum.

subsec:FramedKon

3.2. **The framed Kontsevich Integral.** Kontsevich's original construction gives an invariant of unframed links. However, in this paper we work primarily with framed links and tangles, thus we briefly review the framed version; see also [CDM12, Sections 3.5 and 9.1] and [LM96].

First we need a framed version of the Vassiliev filtration. Let $\tilde{\mathcal{K}}$ denote the set of *framed* links in $\mathbb{R}^3$: that is, links along with a non-zero section of the normal bundle. A knot diagram is interpreted as a framed knot using the blackboard framing. The Reidemeister move R1 move changes the blackboard framing, and by ommitting it, one obtains a Reidemeister theory for framed links. In analogy with a double point, a *framing change* is defined to be the difference

$$\uparrow := \uparrow - \uparrow .$$

} too small.

The framed Vassiliev filtration is the descending filtration

$$\tilde{\mathcal{K}} = \tilde{\mathcal{K}}_0 \supseteq \tilde{\mathcal{K}}_1 \supseteq \tilde{\mathcal{K}}_2 \supseteq \ldots$$

where $\tilde{\mathcal{K}}_i$ is linearly generated by knots with at least i double points *or framing changes*. The degree completed associated graded space of $\tilde{\mathcal{K}}$ with respect to the framed Vassiliev filtration is

$$\tilde{\mathcal{A}} := \prod_{n\geq 0} \tilde{\mathcal{K}}_n / \tilde{\mathcal{K}}_{n+1}.$$

A natural first guess for a combinatorial description of $\tilde{\mathcal{A}}$ is in terms of chord diagrams with "framing change markings" $\uparrow$ on the skeleton, graded by the number of chords and markings. There is a natural surjective graded map $\tilde{\psi}$ from marked chord diagrams onto $\tilde{\mathcal{A}}$, which is defined like ψ for chords, and which replaces each marking $\uparrow$ with a framing change $\uparrow$. The kernel of $\tilde{\psi}$ includes the $4T$ relation as before. spacing

In place of the FI relation ($\uparrow = 0$), a weaker relation arises from the equality $\uparrow - \uparrow = \uparrow$ in $\tilde{\mathcal{K}}$. In fact, $\uparrow = \uparrow - \uparrow = (\uparrow - \uparrow) + (\uparrow - \uparrow)$, and $\uparrow - \uparrow = \uparrow - \uparrow$ modulo $\tilde{\mathcal{K}}_2$. In other words, the following relation is in the kernel of $\tilde{\psi}$:

spacing

$$\uparrow = 2\uparrow.$$

spacing

Therefore, it is not necessary to have dedicated notation for the framing change markings, since $\frac{1}{2}\uparrow$. The candidate model for the associated graded space is simply chord diagrams modulo the $4T$ relation, and no FI relation. We denote this space by $\tilde{\mathcal{C}}$.

To show that $\tilde{\psi} : \tilde{\mathcal{C}} \to \tilde{\mathcal{A}}$ is an isomorphism, the strategy is the same as before: construct a $\tilde{\mathcal{C}}$-valued expansion and use Lemma 3.1. This $\tilde{\mathcal{C}}$-valued expansion is the framed version $\tilde{Z}$ of the Kontsevich integral. The definition is similar to (3.1), the main issue is that in the absence of the FI relation, the integral diverges at cups and caps. This is resolved with a renormalisation using the framing, for details see [CDM12, Section 9.1], or [LM96, Gor99].

subsec:IntroGT

3.3. **The Goldman-Turaev Lie bialgebra.** Let D_p be the disc with $p+1$ disc with boundary components $\partial_0, \partial_1, \ldots, \partial_p$, with ∂_0 a distinguished as an "outer" boundary component, as in Figure 5. Let $\pi = \pi_1(D_p, *)$ denote the fundamental group of D_p with basepoint $* \in \partial_0$. We denote by $\mathbb{C}\pi$ the group algebra of π, and by $\overline{\mathbb{C}\pi} = \mathbb{C}\pi / \mathbb{C}\mathbb{1}$ the linear quotient by the constant loop.

this paragraph should move, or possibly be deleted

Let $\tilde{\pi} = \tilde{\pi}_*$ denote the group of immersed curves $\gamma : ([0,1], 0, 1) \to (D_p, *, *)$ under regular homotopy, so that $\dot{\gamma}(0) = \dot{\gamma}(1) = \xi$, as shown in Figure 5. Note

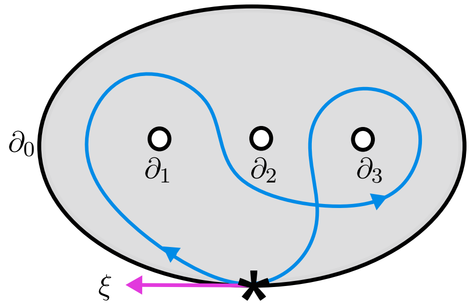


fig:DP

FIGURE 5. D_3 with an immersed loop based at $*$ on ∂_0, with initial and terminal tangent vector ξ.

that $\tilde{\pi}$ is a group, although there are mild technicalities around the identity and inverses. Denote by $\mathbb{C}\tilde{\pi}$ the group algebra of $\tilde{\pi}$.

For an algebra A we denote by $|A|$ the linear quotient $A/[A, A]$, where $[A, A]$ denotes the subspace spanned by commutators $[x, y] = xy - yx$ for $x, y \in A$. We denote the quotient (trace) map by $|\cdot| : A \to |A|$.

Are these technicalities because of the tangent vector? I feel like this sentence begs more questions than it answers.

We are particularly interested in the vector spaces $|\mathbb{C}\pi|$ and $|\overline{\mathbb{C}\pi}| = |\mathbb{C}\pi|/\mathbb{C}\mathbb{1}$. The latter has the structure of a Lie bialgebra given by the *Goldman bracket* $[,]_G$ and *Turaev co-bracket* δ. Given a framing on D_p, this structure lifts to $|\mathbb{C}\pi|$, with the bracket denoted the same (as it is given by the same formula), and cobracket $\tilde{\delta}$ [AKKN18b]. Note that $|\mathbb{C}\pi|$ has an explicit description as the $\mathbb{C}$-vector space generated by homotopy classes of free loops in D_p. For a free loop α in D_p and a point q on α, we denote by α_q be the loop α considered to be based at q.

def:bracket

Definition 3.3 (The Goldman bracket)**.** Let α, β be immersed representatives of free loops in $|\mathbb{C}\pi|$, with only transverse double intersections. The Goldman bracket $[\cdot,\cdot]_G : |\mathbb{C}\pi| \otimes |\mathbb{C}\pi| \to |\mathbb{C}\pi|$ is given by

$$[\alpha, \beta]_G := -\sum_{q \in \alpha \cap \beta} \varepsilon_q |\alpha_q \beta_q|$$

where $\varepsilon_q = \varepsilon(\alpha_q, \beta_q) \in \{\pm 1\}$ is the *local intersection number* of α and β at q, and $\alpha_q\beta_q$ is the concatenation of α_q and β_q. This definition extends linearly to a Lie bracket $|\mathbb{C}\pi|$, and also descends to a Lie bracket $[\cdot,\cdot]_G : |\overline{\mathbb{C}\pi}| \to |\overline{\mathbb{C}\pi}| \otimes |\overline{\mathbb{C}\pi}|$.

The naive definition of the Turaev cobracket is analogous, except for using self intersections of a curve rather than the intersections between two curves. This is not well defined on $|\mathbb{C}\pi|$, with respect to the Reidemeister 1 relation in particular. It is well-defined on $|\overline{\mathbb{C}\pi}|$, and can be lifted to $|\mathbb{C}\pi|$ by a correction term which uses a framing on D_p (a homotopy class of a trivialisation of the tangent bundle), and the notion of rotation number for curves induced by the framing. Here we review a slightly different, equivalent definition of δ, using self-intersections of

The sign here (with the minus sign in front) matches with AKKN genus 0, but is the opposite of AKKN higher genus and Goldman's original definition. Our current multiplication and bracket matches the sign here, so if we change the sign then we should change the stacking order of our multiplication.

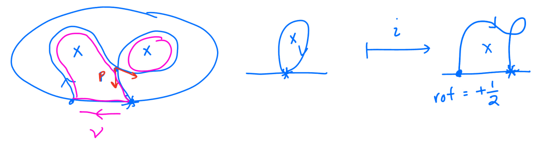


fig:enter-label

FIGURE 6. Example of Goldman bracket. Dummy figure inserted–this is not what figure we want here.

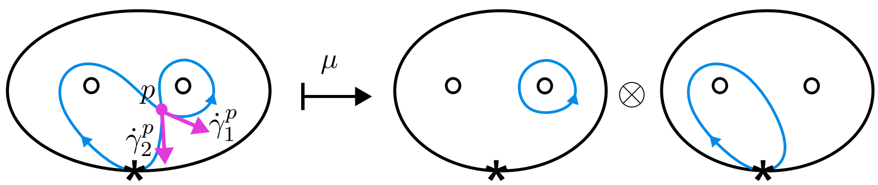


fig:defmu

FIGURE 7. Example of the self intersection map μ where $\epsilon_p = +1$.

based curves, as in [AKKN18b, Section 5.2]. This definition is easier to directly compare with the three-dimensional construction of δ and $\tilde{\delta}$ in Section 5.

def:mu

Definition 3.4 (The self-intersection map μ)**.** Let $\gamma : [0,1] \to D_p$ be an immersed representative of a loop (based at $*$) in $\overline{\mathbb{C}\pi}$, with only transverse double points, and let $\gamma \cap \gamma$ denote the set of such double points. The self intersection map is:

$$\mu : \overline{\mathbb{C}\pi} \to |\overline{\mathbb{C}\pi}| \otimes \overline{\mathbb{C}\pi}$$

$$\mu(\gamma) = - \sum_{p \in \gamma \cap \gamma} \varepsilon_p |\gamma_{t_1^p t_2^p}| \otimes \gamma_{0 t_1^p} \gamma_{t_2^p 1},$$

where $t_1^p, t_2^p \in [0,1]$ are the first and second time point where γ goes through p; γ_{rs} denotes the part of γ from time r to s; and $\varepsilon_p = \varepsilon\left(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^p)\right) \in \{\pm 1\}$ is the local self-intersection number.

Definition 3.5 (The Turaev co-bracket δ)**.** The Turaev cobracket is defined as the unique linear map which makes the following diagram commute:

$$\begin{array}{ccccc} \overline{\mathbb{C}\pi} & \xrightarrow{\mu} & |\overline{\mathbb{C}\pi}| \otimes \overline{\mathbb{C}\pi} & \xrightarrow{1 \otimes |\cdot|} & |\overline{\mathbb{C}\pi}| \otimes |\overline{\mathbb{C}\pi}| \\ \downarrow{\scriptstyle |\cdot|} & & & & \downarrow{\scriptstyle \mathrm{Alt}} \\ |\overline{\mathbb{C}\pi}| & & \overset{\delta}{\dashrightarrow} & & |\overline{\mathbb{C}\pi}| \wedge |\overline{\mathbb{C}\pi}| \end{array}$$

Here $\mathrm{Alt}(x \otimes y) = x \otimes y - y \otimes x = x \wedge y$.

The framed version is probably not needed. If needed, we should change it to single base point. For now I commented it out.

Given a framing on D_p, there is a well-defined rotation number rot : $\tilde{\pi} \to \mathbb{Z}$, which descends to $|\tilde{\pi}|$. This allows for a lift of δ to $|\mathbb{C}\pi|$. The following definition is phrased differently but is equivalent to the one in [AKKN18b, Section 5.2].

There is a natural quotient map $q : \mathbb{C}\pi \to \overline{\mathbb{C}\pi}$, with a section $s : \overline{\mathbb{C}\pi} \to \mathbb{C}\pi$, such that the coefficient of the constant loop $\mathbb{1}$ in $s(x)$ is 0 for $x \in \overline{\mathbb{C}\pi}$. Both the quotient map and s descend naturally to $|\mathbb{C}\pi|$ and $|\overline{\mathbb{C}\pi}|$, also denoted q and s.

Definition 3.6 (The enhanced Turaev cobracket $\tilde{\delta}$)**.** The enhanced Turaev cobracket $\tilde{\delta} : |\mathbb{C}\pi| \to |\mathbb{C}\pi| \wedge |\mathbb{C}\pi|$ is defined by $\tilde{\delta}(\alpha) := s(\delta(q(\alpha))) + \mathrm{rot}(\alpha) \cdot \mathbb{1} \wedge \alpha$.

3.4. **Associated graded Goldman-Turaev Lie bialgebra.** There is a filtration on $\mathbb{C}\pi$ by powers of the augmentation ideal $\mathcal{I} = \langle \alpha - 1 \rangle$:

$$\mathbb{C}\pi = \mathcal{I}^0 \supseteq \mathcal{I} \supseteq \mathcal{I}^2 \supseteq ...$$

which descends to a filtration on $|\mathbb{C}\pi|$:

$$|\mathbb{C}\pi| = |\mathcal{I}^0| \supseteq |\mathcal{I}| \supseteq |\mathcal{I}^2| \supseteq ...$$

The completed associated graded algebra for $|\mathbb{C}\pi|$ with respect to this filtration, $\mathrm{gr}\,|\mathbb{C}\pi| = \prod_{n=0}^{\infty} |\mathcal{I}^n|/|\mathcal{I}^{n+1}|$, has an explicit description in terms of "cyclic words". Let $\mathrm{Asc} = \mathrm{Asc}\langle x_1, \cdots, x_p \rangle$ be the free associative algebra on r generators. Note that elements $|\,\mathrm{Asc}\,|$ can be described as "cyclic words" in letters $x_1, \cdots, x_p$, that is, words modulo cyclic permutation of the letters. The following result is due to [].

complete with citation: Quillen66?

I couldnt find the reference. Can you give more info?

Proposition 3.7. *There is an isomorphism of vector spaces*

$$\mathrm{gr}\,|\mathbb{C}\pi| \cong |\,\mathrm{Asc}\,|$$

which becomes an isomorphism of Lie bialgebras when $|\,\mathrm{Asc}\,|$ *is equipped with an appropriate bracket and cobracket structure, which we call the (graded) Goldman bracket and Turaev cobracket.*

def:grbracket

Definition 3.8. [The graded Goldman bracket]Let $z = z_1 \cdots z_l$ and $w = w_1 \cdots w_m$ be two cyclic words in $|\,\mathrm{Asc}\,|$. The *graded Goldman bracket*

$$[-,-]_{\mathrm{gr}\,G} : |\,\mathrm{Asc}\,| \otimes |\,\mathrm{Asc}\,| \to |\,\mathrm{Asc}\,|$$

of z and w is given by:

$$[z,w]_{\mathrm{gr}\,G} = \sum_{j,k} \delta_{z_j,w_k}(w_1 \dots w_{k-1} z_{j+1} \dots z_l z_1 \dots z_j w_{k+1} \dots w_m - \\ w_1 \dots w_{k-1} z_j \dots z_l z_1 \dots z_{j-1} w_{k+1} \dots w_m)$$

where δ_{z_j,w_k} is the Kronecker delta.

By representing cyclic words diagrammatically as letters along a circle, the graded Goldman bracket sums over matching pairs of letters in z and w, joins the circles at the matching letter, and takes the difference of the two ways of replacing one copy of the letter in the new cyclic word. This is shown in Figure 8.

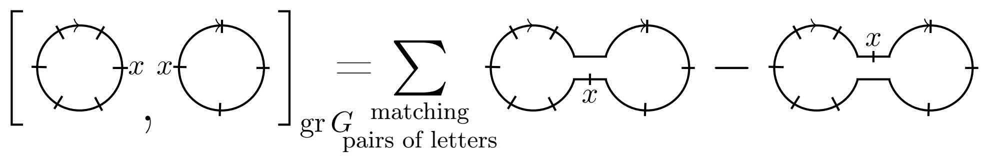


fig:grbracket

FIGURE 8. The graded Goldman bracket.

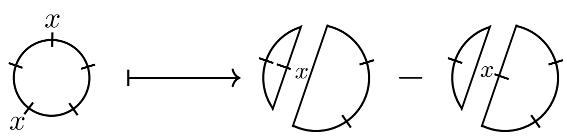


fig:paircut

FIGURE 9. An example pairing cut.

Definition 3.9 (The graded Turaev cobracket)**.** Let $w = w_1 \dots w_m \in |As_p|$. The graded Turaev cobracket

$$\delta_{gr} : |\operatorname{Asc}| \to |\operatorname{Asc}| \wedge |\operatorname{Asc}|$$

on w is given by

$$\delta_{\text{gr}}(w) = \sum_{j<k} \delta_{w_j,w_k}(|w_j \dots w_{k-1}| \wedge |w_{k+1} \dots w_n w_1 \dots w_{j-1}| + |w_k \dots w_n w_1 \dots w_{j-1}| \wedge |w_{j+1} \dots w_{k-1}|).$$

Diagrammatically, the cobracket can be computed by a summation of *pairing cuts*, an example of which is given in Figure 9. The sum of wedge products in the definition of the cobracket is given by the sum of taking the cut with one element of a matching pair at the top of the circle, then taking the cut with the other element at the top of the circle.

sec:TangleSetUp

4. Expansions for tangles in handlebodies

4.1. **The space $\mathbb{C}\tilde{\mathcal{T}}$.** In this paper we consider the space $\mathbb{C}\tilde{\mathcal{T}}$ of framed, oriented tangles in a genus p handlebody, and show that homomorphic expansions on this space descend to homomorphic expansions on the Goldman-Turaev Lie biagebra as defined in [AKKN20]. This section describes the space $\mathbb{C}\tilde{\mathcal{T}}$.

Let M_p denote the manifold $D_p \times I$ where D_p is a disc with p points removed. While M_p is not a compact manifold, knot theory in M_p is equivalent to knot theory in a genus p handlebody. For the purpose of the Kontsevich integral, we identify D_p with a square in the complex plane with p points removed, so M_p can be drawn as a cube with p vertical lines removed; we call these lines the *poles*, as shown in Figure 11. We refer to $D_p \times \{0\}$ as the "floor" or "bottom", and $D_p \times \{1\}$ as the "ceiling" or "top". The "back wall" is the north ($i \in \mathbb{C}$) edge of D_p times $[0, 1]$.

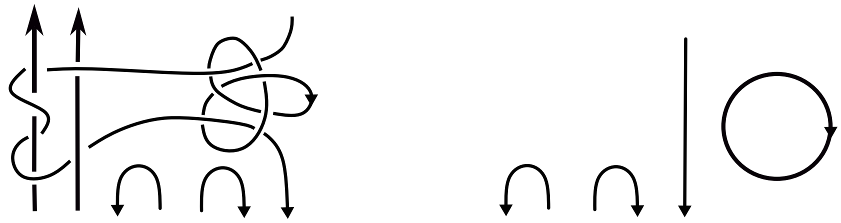


FIGURE 10. On the left is a tangle in M_2, and on the right is schematic diagram of the skeleton of the tangle. The skeleton of the tangle is the combinatorial data given by the following set of order pairs and the integer 1: $\{[((2,0),0),((1,0),0))],[((3,0),0),(4,0),0))],[((5,0),1),(5,0),0))],1\}$
.

fig:skeleton

def:tangle

Definition 4.1. An oriented tangle T in M_p is an embedding of an oriented compact 1-manifold

$$(S, \partial S) \hookrightarrow (M_p, D_p \times \{0\} \cup D_p \times \{1\}).$$

The interior of S lies in the interior of M_p, and the boundary points of S are mapped to the top or bottom. Oriented tangles in M_p are considered up to ambient isotopy fixing the boundary. We denote the set of isotopy classes by $\mathcal{T}$.

Definition 4.2. A *framing* for an oriented tangle T in M_p is a continuous choice of unit normal vector at each point of T, which is fixed pointing ~~in~~ the north to ($i \in \mathbb{C}$) direction at the boundary points. *Framed oriented tangles* in M_p are considered up to ambient isotopy fixing the boundary. We denote the set of isotopy classes of framed oriented tangles by $\tilde{\mathcal{T}}$.

Henceforth, any tangle is assumed to be framed and oriented unless otherwise stated. The skeleton of a tangle is the underlying combinatorial information with the topology forgotten:

def:skeleton

Definition 4.3. The *skeleton* $\sigma(T)$ of a tangle $T = (S \hookrightarrow M_p)$ – see Figure 10 – is the set of tangle endpoints $P_{bot} \subseteq D_p \times \{0\}$ and $P_{top} \subseteq D_p \times \{1\}$, along with

(1) A partition of $P_{bot} \cup P_{top}$ into ordered pairs given by the oriented intervals of S.
(2) A non-negative integer k: the number of circles in S.

The skeleton of a framed tangle is the same as the skeleton of the underlying unframed tangle. The set of framed tangles in M_p ~~on~~ with skeleton S is denoted $\tilde{\mathcal{T}}(S)$. For example, $\tilde{\mathcal{T}}(\bigcirc)$ is the set of framed knots in M_p.

Maybe it would be better to define $P_{bot}, P_{top} \subseteq D_p$ and then say $P_{bot} \times \{0\}$ and $P_\top \times \{1\}$ are the tangle endpoints. Then it would make descriptions of tangle operations easier, as well as the info in figure 9.

The linear extension of $\tilde{\mathcal{T}}(S)$, denoted $\mathbb{C}\tilde{\mathcal{T}}(S)$, is the vector space of $\mathbb{C}$-linear combinations of tangles in $\tilde{\mathcal{T}}(S)$. We denote by $\mathbb{C}\tilde{\mathcal{T}}$ the disjoint union $\sqcup_S \tilde{\mathcal{T}}(S)$ over all skeleta S, identified at 0. Tangles with different skeleta cannot be linearly combined.

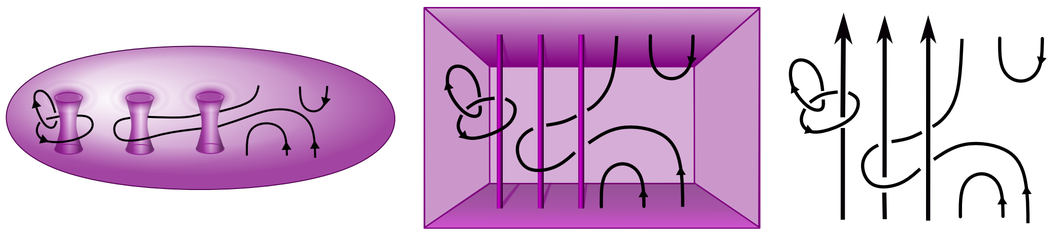


FIGURE 11. An example of a tangle in M_3, drawn first in a handlebody, secondly in a cube with poles, and lastly as a tangle diagram projected to the back wall of the cube.

fig:polestudio

We can look at tangles in M_p using tangle diagrams in two different ways, by projecting either to the back wall of M_p or to the floor.

If we project to the back wall, an ℓ-component tangle in M_p can be diagrammatically represented as a tangle diagram with p straight vertical "poles", and ℓ tangle "strands" of circle and interval components. The strands pass over (in front of) and under (behind) the poles and other strands, as shown on the right in Figure 11. The poles are equipped with an orientation coming from the parametrization in $M_p \cong D_p \times I$, and in figures we draw them oriented upwards, unless otherwise stated. By Reidemeister's theorem, $\tilde{\mathcal{T}}$ is equivalent to such diagrams modulo the Reidemeister moves R2 and R3. (No R1, as the tangles are framed.)

on the the left hand side of this figure something is missing

By projecting instead to the floor $D_p \times \{0\}$ of the cube, a tangle in M_p is represented by a tangle diagram in D_p. The R2 and R3 moves continue to apply. The endpoints of the tangle are fixed: bottom endpoints are denoted by dots, top endpoints are denoted by stars. Strands of the tangle diagram can pass over bottom endpoints, or under top endpoints, as shown in Figure 12. However, the strands cannot pass over the punctures in D_p.

sec:opsonT

4.2. **Operations on $\tilde{\mathcal{T}}$.** There are several useful operations defined on $\tilde{\mathcal{T}}$. These operations extend naturally to $\mathbb{C}\tilde{\mathcal{T}}$, and are used in section 5 to relate quotients of $\mathbb{C}\tilde{\mathcal{T}}$ to the Goldman-Turaev Lie bialgebra.:

check this section reference after Section 4 is finalised

- *Stacking:* Given tangles $T_1, T_2 \in M_p$, if the top endpoints of $\sigma(T_1)$ match the bottom endpoints of $\sigma(T_2)$ in D_p, and the orientations on the strands of T_1 and T_2 agree at the matching endpoints, then we can stack T_2 on top of T_1 and shrink the height to get a new tangle $T_1T_2 \in M_p$.
- *Strand addition:* The *strand addition* operation adds a non-interacting additional strand to a tangle T at a point $q \in D_p$ to get a new tangle $T\sqcup_q \uparrow$. More precisely, pick a contractible $U \subseteq D_p$ such that T is contained entirely in $U \times [0,1]$ and a point $q \in D_p$ outside of U. The tangle $T\sqcup_q \uparrow$ is T together with an upward-oriented vertical strand $q \times I$ at q.
- *Strand orientation switch:* This operation reverses the orientation of a given strand of the tangle.

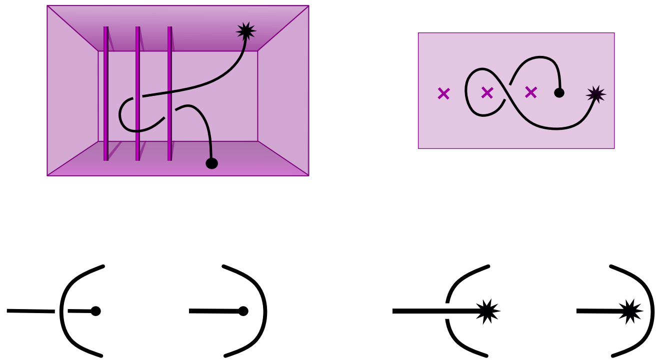


fig:BottomDiagram

FIGURE 12. An example of a tangle in M_3 projected to the floor. Strands of a tangle diagram can pass over bottom endpoints (dot) or under top endpoints (star).need to add arrow and equal signs

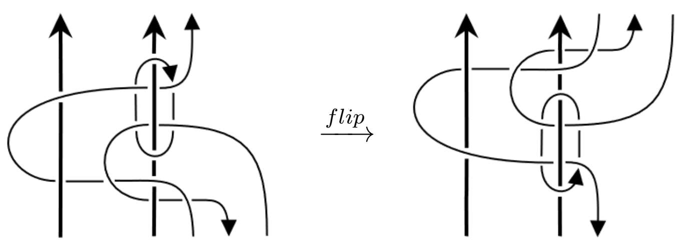


fig:flip

FIGURE 13. A tangle in M_2 and its flip

- *Flip:* Given a tangle T in M_p, the flip of a tangle T in M_p, denoted $\overline{T}$, is the mirror image of T with respect to the ceiling, as shown in Figure 13. When T is flipped, each top boundary point $(q, 1)$ becomes a bottom boundary point $(q, 0)$, and vice versa. The orientations and framing of the strands of T are reflected along with the strands. However, the orientations of the poles remain ascending. Equivalently, we can define the flip operation as reversing the parametrisation of I in $M_p \cong D_p \times I$. This, in effect, flips the orientation of the poles but changes nothing else.

In section **??**, we relate commutator of tangles with respect to stacking, given by $[T_1, T_2] = T_1T_2 - T_2T_1$, to the Goldman bracket, and in section **??** we relate the flip operation to the Turaev cobracket.

sec:t-filtration

4.3. **The t-filtration on $\tilde{\mathcal{T}}$ and the associated graded $\tilde{\mathcal{A}}$.** There are different filtrations on the space $\mathbb{C}\tilde{\mathcal{T}}$ that one might consider in setting up a Vassiliev theory. In line with classical notation of Vassiliev invariants, we denote by a double point the difference between an over-crossing and an under-crossing:

In the context of tangles in M_p, double points come in two varieties: *pole-strand*, if the crossing occurs between a pole and a tangle strand, and *strand-strand*, if the crossing occurs between two tangle strands.

The main filtration we consider on $\mathbb{C}\tilde{\mathcal{T}}$ is the filtration by the total number of double points of either type, as well as strand framing changes (as described in Section 3.2). We call this the *total filtration*, or simply t-filtration, and write it as

$$\mathbb{C}\tilde{\mathcal{T}} = \tilde{\mathcal{T}}_0 \supseteq \tilde{\mathcal{T}}_1 \supseteq \tilde{\mathcal{T}}_2 \supseteq \tilde{\mathcal{T}}_3 \supseteq \cdots$$

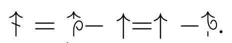
where $\tilde{\mathcal{T}}_t$ is the set of linear combinations of framed tangle diagrams with at least t total double points and strand framing changes. ~~Note that while the filtration is defined using tangle diagrams, the filtration respects R2 and R3 moves, and hence descends to tangles; that is, different diagrams for the same tangle live in the same filtered component.~~

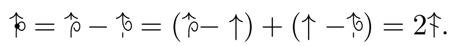
Definition 4.4. The associated graded space of $\mathbb{C}\tilde{\mathcal{T}}$ with respect to the total filtration is

$$\tilde{\mathcal{A}} := \operatorname{gr}\mathbb{C}\tilde{\mathcal{T}} = \prod_{t\geq 0} \tilde{\mathcal{T}}_t/\tilde{\mathcal{T}}_{t+1}.$$

The degree t component of $\tilde{\mathcal{A}}$ is $\tilde{\mathcal{A}}_t := \tilde{\mathcal{T}}_t/\tilde{\mathcal{T}}_{t+1}$.

rem:2frame=double

Remark 4.5. Modulo $\tilde{\mathcal{T}}_2$,  As a result, in $\tilde{\mathcal{A}}$, a framing change can always be represented as $\frac{1}{2}$ a double point as



As in classical Vassiliev theory (cf. section 3.2), the associated graded space $\tilde{\mathcal{A}}$ has a combinatorial description in terms of *chord diagrams*.

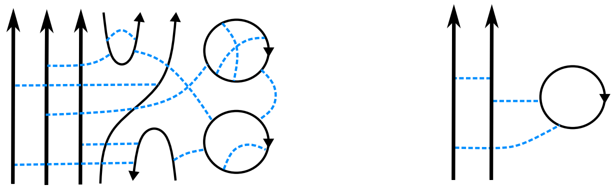
Definition 4.6. A *chord diagram* on a tangle skeleton is an even number of marked points on the poles and skeleton strands, up to orientation preserving diffeomorphism, along with a perfect matching on the marked points – that is, a partition of marked points into unordered pairs. In diagrams, the pairs are connected by a *chord*, indicated by a dotted line, as in Figure 14(A).

def:admissible

Definition 4.7. A chord diagram is *admissible* if all chords connect strands to strands, or strands to poles. That is, there are no pole-pole chords in an admissible diagram, see Figure 14(A) for an example.

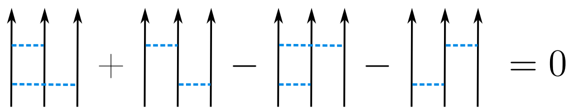
def:cdspace

Definition 4.8. The space $\mathcal{D}(S)$ of *admissible chord diagrams on a diagram S* is the space of $\mathbb{C}$-linear combinations of admissible chord diagrams on the skeleton S factored out by *admissible* $4T$ relations, shown in Figure **??**. Admissible $4T$



(A) Two chord diagrams: an admissible one (left) that doesn't contain any pole-pole chords, and non-admissible one (right) that does contain a pole-pole chord.

ssibleNonAdmissible



(B) The 4T relation, which is admissible if at most one of the three skeleton components is a pole.

fig:Admissible 4T

FIGURE 14. Examples of admissible and non-admissible chord diagrams, and the 4T relation

fig:admissible

relations are $4T$ relations in the classical sense, subject to the condition that all four terms are admissible[5]. That is,

$$\mathcal{D}(S) = \frac{\{\text{linear combinations of admissible chord diagrams on } S\}}{\{\text{admissible } 4T \text{ relations}\}}$$

Do we need the concept of "admissible 4T"? Since 4T is a relation, so just saying "admissible chord diagrams mod 4T" would only apply 4T to admissible diagrams?

The space $\mathcal{D}(S)$ is a graded vector space, where the degree is given by the number of chords. Denote the degree t component of $\mathcal{D}(S)$ by $D_t(S)$. Let $\mathcal{D}$ be the disjoint union $\sqcup_S \mathcal{D}(S)$, identified at 0. We denote the degree t component of $\mathcal{D}$ by $\mathcal{D}_t = \sqcup_S \mathcal{D}_t(S)$.

thm:tassocgraded

Theorem 4.9. *There is a canonical isomorphism $\mathcal{D} \cong \tilde{\mathcal{A}}$.*

what is "canonical"?
should be: The map Ψ is an isomorphism
not well posed.

To prove this Theorem, we use the isomorphism familiar from classical finite type theory

$$\psi : \mathcal{D} \to \tilde{\mathcal{A}}$$

In degree t, $\psi_t : \mathcal{D}_t \to \tilde{\mathcal{T}}_t/\tilde{\mathcal{T}}_{t+1}$, is defined as before by contracting chords to double points, as shown in Figure 15. This may create other crossings, but modulo $\tilde{\mathcal{T}}_{t+1}$ it does not matter which skeleton component is over or under at these crossings.

We prove that ψ is an isomorphism by showing that it's well-defined and surjective, then using lemma 3.1 to show that it's an isomorphism.

it is
it is

Lemma 4.10. *The map ψ is well-defined and surjective.*

Proof. To show ψ is well-defined, it suffices to show that admissible $4T$ relations in $\mathcal{D}_t$ are in the kernel of ψ. This is shown in Figure 16. For surjectivity, a framing

[5]Equivalently, a 4T relation is admissible if at most one of the three skeleton components involved is a pole.

where is the proof of theorem 4.9?

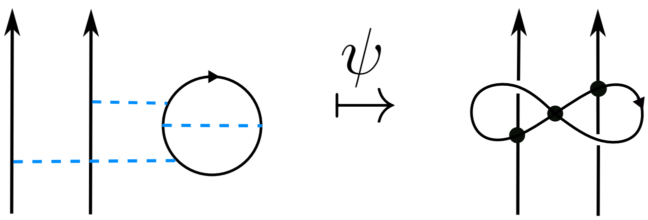


FIGURE 15. Example of ψ with the right hand side viewed as an element of $\tilde{\mathcal{T}}_3/\tilde{\mathcal{T}}_4$. Different choices of over or under crossings with the poles only differ by elements of $\tilde{\mathcal{T}}_4$.

fig:psi

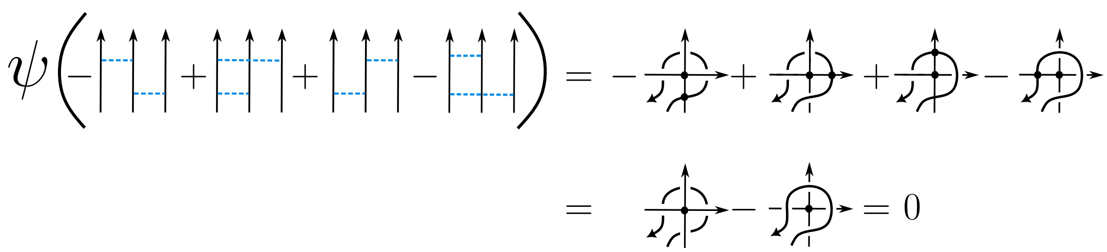


FIGURE 16. The proof that $\psi : \mathcal{D} \to \tilde{\mathcal{A}}$ is well defined. The figure is understood locally: If the figure is a map in the degree t component, then the chord diagrams have $t-2$ other chords that are not shown but in the same position throughout all four terms, and similarly, the tangles have $t-2$ other double points that are not shown, but in the same positions throughout all the terms.

fig:psicomputation

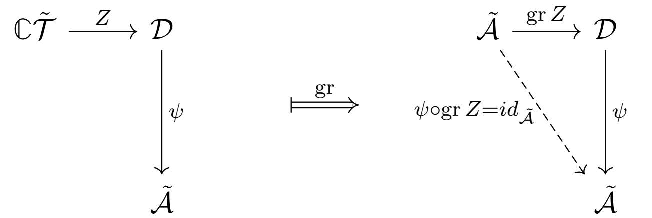
change in $\tilde{\mathcal{A}}$ can always be written as one half a double point, as described in Remark 4.5. So all framing changes are in the image of ψ, and ψ is surjective. □

thm:Zwelldefined

Lemma 4.11. *The Kontsevich integral Z is a well-defined filtered map from $\mathbb{C}\tilde{\mathcal{T}}$ to $\mathcal{D}$ such that $\psi \circ \operatorname{gr} Z = \operatorname{id}_{\tilde{\mathcal{A}}}$.*

Proof. The image of Z on an element in $\mathbb{C}\tilde{\mathcal{T}}$ will be a chord diagram on a skeleton with p poles and some number of circles. Since the poles in M_p are parallel, any pair of points (z_i, z_i') on the poles will be constant, the form $dz_i - dz_i' = 0$, and the contribution to the integral will be zero. Therefore chord diagrams in the image of Z don't contain pole-pole chords, so they are always admissible. So Z indeed always lands in $\mathcal{D}$.

It remains to show that $\psi \circ \operatorname{gr} Z = id_{\tilde{\mathcal{A}}}$.



Recall that for a filtered map $f : A \to B$, the associated graded $\operatorname{gr} f : \operatorname{gr} A \to \operatorname{gr} B$ is defined on graded components by $[a] \in A_t/A_{t+1} \mapsto [f(a)] \in B_t/B_{t+1}$. We consider $\operatorname{gr} Z : \tilde{\mathcal{A}} \to \mathcal{D}$. Let $[T] \in \tilde{\mathcal{T}}_t/\tilde{\mathcal{T}}_{t+1}$ so that is T is a tangle in M_p with at least t double points. Note that it's always possible to pick such a representative, since a framing change can be written as $\frac{1}{2}$ times a double point in $\tilde{\mathcal{T}}_t/\tilde{\mathcal{T}}_{t+1}$. Then $Z(T)$ is a sum of chord diagrams with $e^{\frac{C}{2}} - e^{-\frac{C}{2}}$ at each chord C corresponding to each double point in T. All terms with degree less than t are zero, so the value of $\operatorname{gr} Z(T)$ depends only on the degree t term of $Z(T)$. The degree t term is a single chord diagram with a single chord for each double point, so applying ψ to this turns all the chords back to double points, which up to crossing changes in $\tilde{\mathcal{T}}_{t+1}$, is just $[T]$. Therefore $\psi \operatorname{gr} Z = \mathrm{id}_{\tilde{\mathcal{A}}}$. Since $\psi \operatorname{gr} Z = id_{\tilde{\mathcal{A}}}$. □

The next corollary is immediate from lemma 3.1.

Corollary 4.12. *The map $\psi : \mathcal{D} \to \tilde{\mathcal{A}}$ is an isomorphism and Z is an expansion for $\tilde{\mathcal{T}}$.*

Now it is established that $\tilde{\mathcal{A}}$ can be identified with the space of admissible chord diagrams $\mathcal{D}$. For a skeleton S, define $\tilde{\mathcal{A}}(S)$ to be the space of admissible chord diagrams on the skeleton S, so that $\tilde{\mathcal{A}}(S)$ is the associated graded of $\mathbb{C}\tilde{\mathcal{T}}(S)$. For example, $\tilde{\mathcal{A}}(\bigcirc)$ is the associated graded of $\mathbb{C}\tilde{\mathcal{T}}(\bigcirc)$, the space of knots in M_p.

4.4. **Operations on $\tilde{\mathcal{A}}$.** The operations *stacking*, *increase*, and *flip* on $\mathcal{T}$ induce operations by the same names on $\tilde{\mathcal{A}}$. In view of theorem 4.9, we give descriptions of these operations using chord diagrams.

The operation *stacking* is given by stacking D_1 on top of D_2 to get D_1D_2, and the operation *strand addition* on a diagram D adds a new skeleton strand without any marked points at a specified point p to give $D \sqcup_p \uparrow$. It is clear from the definition of ψ that these are indeed the correct chord diagram descriptions of these operations, and as in $\mathcal{T}$, they are only defined when the endpoints of D_1 and D_2 match appropriately, and when p is an appropriate point to pick.

The operation *flip* reflects a chord diagram with respect to a "mirror on the ceiling", reverses the orientations of the poles so that they are the same as they were originally, and adds a factor of $(-1)^m$, where m is the total number of marked points on the poles. The factor of $(-1)^m$ comes from the fact that reversing the orientation of one strand at a double point is the same as multiplying by a factor of -1. See figure ....[include figure]

For those who know, this is obvious. For the rest?

Explain here about the general notation of t'th component of filtration?

I added this paragraph because we need to define $A(\bigcirc)$ somewhere. This definition has to come after the theorem that $\mathcal{D} \cong \tilde{\mathcal{A}}$, but it feels weird to have it right here. It also needs to be improved. Suggestions?

We could define $\tilde{\mathcal{A}}(S)$ as the associated graded of $\mathbb{C}\tilde{\mathcal{T}}(S)$ and put it where we define associated graded. Then we can define $\mathcal{D}(S)$ and make theorem 3.7 say $\mathcal{D}(S) \cong \tilde{\mathcal{A}}(S)$ which would be a bit more clear as well?

describe the associated graded operations of all the tangle operations prop:Zhomom

Proposition 4.13. *The Kontsevich integral Z is homomorphic with respect to stacking, strand additions and flips.*

Proof. It is clear for stacking and strand addition. When the orientation of the poles are reversed, every chord diagram D_P in the output of the Kontsevich integral will be multiplied get $(-1)^m$, where m is the total number of chord endings on poles, because m points in P will change whether they are on a descending arc or not, so $P_\downarrow$ will change by m mod 2. □

sec:s-sfiltration

4.5. **The s-filtration on $\tilde{\mathcal{T}}$ and $\tilde{\mathcal{A}}$.** As described in Section 4.3, the space $\mathbb{C}\tilde{\mathcal{T}}$ (and therefore $\tilde{\mathcal{A}}$) has a total filtration given by strand framing changes and double points of either type, strand-pole and strand-strand. In this section we look at a second filtration on $\mathbb{C}\tilde{\mathcal{T}}$ and $\tilde{\mathcal{A}}$, where we still look at strand framing changes, but only consider the number of strand-strand double points. This filtration will be called the *strand filtration*, or simply s-filtration. The s-filtration is given by

$$\mathbb{C}\tilde{\mathcal{T}} = \tilde{\mathcal{T}}^0 \supseteq \tilde{\mathcal{T}}^1 \supseteq \tilde{\mathcal{T}}^2 \supseteq \tilde{\mathcal{T}}^3 \supseteq \cdots$$

where $\tilde{\mathcal{T}}^s \subseteq \mathbb{C}\tilde{\mathcal{T}}$ are linear combinations of link diagrams with at least s strand framing changes and strand double points.

Remark 4.14. We do *not* consider the full associated graded of $\mathbb{C}\tilde{\mathcal{T}}$ with respect to the s-filtration, but instead use it to identify the Goldman-Turaev spaces in low degrees in section **??**. The associated graded of $\mathbb{C}\tilde{\mathcal{T}}$ with respect to the s-filtration has been studied by Habiro and Massuyeau in [HM21], where they consider "bottom tangles". Note the language – if we project to the "bottom" instead of the "back wall", then all double points are of type strand-strand, so the s-filtration is just the usual Vassiliev filtration if we project to the bottom.

The s-filtration also induces a filtration on $\tilde{\mathcal{A}}$ as follows. Combining the notations for the t- and s-filtrations, let $\tilde{\mathcal{T}}_t^s$ denote the set of linear combinations of tangle diagrams in $\mathbb{C}\tilde{\mathcal{T}}$ that have at least t double points, at least s of which are strand-strand.

ionQuotientNotation

Definition 4.15. The s-filtered component of $\tilde{\mathcal{A}}$ denoted $\tilde{\mathcal{A}}^{\geq s} := \prod \tilde{\mathcal{T}}_t^s / \tilde{\mathcal{T}}_{t+1}^s$ is the set of linear combinations of chord diagrams with at least s strand-strand chords, or rather at least s chords between the non-pole skeleton components. Let $\tilde{\mathcal{T}}^{/s} := \mathbb{C}\tilde{\mathcal{T}}/\tilde{\mathcal{T}}^s$ Let $\tilde{\mathcal{A}}^{/s} := \tilde{\mathcal{A}}/\tilde{\mathcal{A}}^{\geq s}$.

maybe this is a (trivial) proposition

Proposition 4.16. *The Kontsevich integral Z respects the s-filtration.*

Proof. Does this follow immediately from Theorem 4.11– as Z is an expansion with respect to the total filtration, so it certainly plays nice with the s-s filtration. □

write this proof in a non-questioning way.

definition? remark? better not to say it at all?

sec:notation

4.6. **Notation conventions.**

Throughout this paper we consider the t and s filtrations on $\mathbb{C}\tilde{\mathcal{T}}$ and $\tilde{\mathcal{A}}$, as well as on their various quotients and subspaces. We summarize the notation from this section below:

- $\mathbb{C}\tilde{\mathcal{T}}$ is the space of $\mathbb{C}$-linear combinations of framed tangles in M_p
- $\mathbb{C}\tilde{\mathcal{T}}(\bigcirc)$ is the space of C-linear combinations of of framed knots in M_p
- $\tilde{\mathcal{T}}_t$ is the t'th filtered component of $\mathbb{C}\tilde{\mathcal{T}}$ with respect to the t-filtration, which contains all linear combinations of framed tangles in M_p with at least t double points(both strand-strand and strand-pole types) and framing changes
- $\tilde{\mathcal{T}}^s$ is the s'th filtered component of $\mathbb{C}\tilde{\mathcal{T}}$ with respect to the s-filtration, which contains all linear combinations of framed tangles in M_p with at least s strand-strand double points and framing changes
- $\tilde{\mathcal{T}}^s_t := \tilde{\mathcal{T}}_t \cap \tilde{\mathcal{T}}^s$, which is the set of elements of $\mathbb{C}\tilde{\mathcal{T}}$ with at least s framing changes and strand-strand double points, and at least t framing changes and double points of any type.
- $\tilde{\mathcal{T}}^{/s} := \mathbb{C}\tilde{\mathcal{T}}/\tilde{\mathcal{T}}^s$, which is the set of elements in $\mathbb{C}\tilde{\mathcal{T}}$ with at most s strand-strand double points and framing changes — or s-1?
- $\tilde{\mathcal{T}}^{1/2} := \tilde{\mathcal{T}}^1/\tilde{\mathcal{T}}^2$, which is the set of elements in $\mathbb{C}\tilde{\mathcal{T}}$ with at exactly 1 strand-strand double point or framing change
- $\tilde{\mathcal{A}}$ is the associated graded space of $\mathbb{C}\tilde{\mathcal{T}}$ under the t-filtration, and is the space of admissible chord diagrams modulo admissible $4T$ relations
- $\tilde{\mathcal{A}}_t := \tilde{\mathcal{T}}_t/\tilde{\mathcal{T}}_{t+1}$ is the degree t component of $\tilde{\mathcal{A}}$ which consists of all admissible chord diagrams in $\tilde{\mathcal{A}}$ with exactly t chords
- $\tilde{\mathcal{A}}^{\geq s} := \prod_t \tilde{\mathcal{T}}^s_t/\tilde{\mathcal{T}}^s_{t+1}$ is the s'th filtered component of $\tilde{\mathcal{A}}$ — is this correct?
- $\tilde{\mathcal{A}}^{/s} := \tilde{\mathcal{A}}/\tilde{\mathcal{A}}^{\geq s}$

Theses notations are extended to subspaces and quotients of $\mathbb{C}\tilde{\mathcal{T}}$ and $\tilde{\mathcal{A}}$ in the natural way.

not sure if we use the $\tilde{\mathcal{T}}^{/s}$ and $\tilde{\mathcal{A}}^{/s}$ notations enough to justify having them

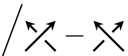
Not sure if $\tilde{\mathcal{T}}^{/s}$ and $\tilde{\mathcal{A}}^{/s}$ are relevant enough to be included here. I think they are only used for $s=1$. We also sometimes use for example $\tilde{\mathcal{T}}^2/\tilde{\mathcal{T}}^1$, which doesn't have a shorthand, so maybe $\tilde{\mathcal{T}}^{/s}$ should be $\tilde{\mathcal{T}}^s/\tilde{\mathcal{T}}^{s+1}$ or something (i.e. degree s component of the s-associated graded)

sec:Conway

4.7. **The Conway quotient.**

In this section we introduce the Conway quotient of $\mathbb{C}\tilde{\mathcal{T}}$: essentially, a Conway skein module of tangles in M_p without fixing the value of the unknot. The Conway relation respects the t and s filtrations and the Kontsevich integral descends to the Conway quotient.

Definition 4.17. The Conway quotient of $\mathbb{C}\tilde{\mathcal{T}}$ is defined as

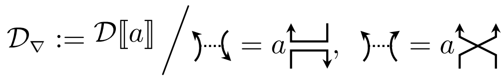
$$\mathbb{C}\tilde{\mathcal{T}}_\nabla := \mathbb{C}\tilde{\mathcal{T}}[\![a]\!] \Big/$$  $= (e^{\frac{a}{2}} - e^{-\frac{a}{2}})$ ,

where a is a formal variable that has t and s degree 1 so that the skein relation preserves both t and s filtrations. The skein relation is applied only to strand-strand crossings, not strand-pole crossings. We will use the variable b as a shorthand for $b = e^{\frac{a}{2}} - e^{-\frac{a}{2}}$.

The t and s filtrations on $\mathbb{C}\tilde{\mathcal{T}}$ induce filtrations on $\mathbb{C}\tilde{\mathcal{T}}_\nabla$. Following the notation conventions in Section 4.6, let $\tilde{\mathcal{T}}_{\nabla,t}$ denote the t'th filtered component of $\mathbb{C}\tilde{\mathcal{T}}_\nabla$ and

$\tilde{\mathcal{A}}_\triangledown := \mathrm{gr}_t\, \mathbb{C}\tilde{\mathcal{T}}_\triangledown = \prod \tilde{\mathcal{T}}_{\triangledown,t}/\tilde{\mathcal{T}}_{\triangledown,t+1}$ denote the associated graded algebra of $\mathbb{C}\tilde{\mathcal{T}}_\triangledown$ with respect the total filtration. We now show that $\tilde{\mathcal{A}}_\triangledown$ has a diagrammatic description similar to $\tilde{\mathcal{A}}$, where $\tilde{\mathcal{A}} \cong \mathcal{D}$ as in Theorem 4.9.

Definition 4.18. Let



where a is a formal variable of degree 1 as above, and the relations locally apply only when all skeleton components involved are strands, not poles.

Note that the quotient relations in $\mathcal{D}_\triangledown$ preserve the t-grading on $\mathcal{D}$ and the grading descends to $\mathcal{D}_\triangledown$. The next theorem shows that $\tilde{\mathcal{A}}_\triangledown \cong \mathcal{D}_\triangledown$. This theorem essentially follows from the results of [LM95], and we present a brief direct proof.

thm:Z_conway

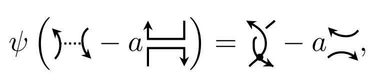
Theorem 4.19. *The Kontsevich integral descends to an expansion* $Z_\triangledown : \mathbb{C}\tilde{\mathcal{T}}_\triangledown \to \mathcal{D}_\triangledown$ *and* $\tilde{\mathcal{A}}_\triangledown \cong \mathcal{D}_\triangledown$.

This proof uses R1, so I don't know how a framed analogue works exactly, and also not sure that we need it. I commented it out for now.

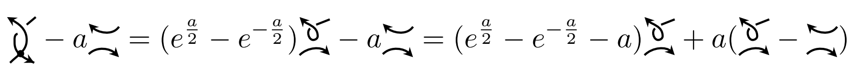
I believe this theorem is correct with framing changes. Please double check.

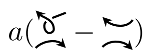
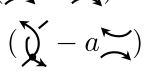
Proof. This proof follows the general schema introduced in Section 3.1, in particular Lemma 3.1 and the map ψ, which assigns singular tangles to chord diagrams.

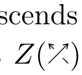
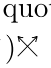
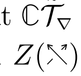
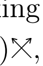
First we show that ψ descends to a graded surjection $\psi : \mathcal{D}_\triangledown \to \tilde{\mathcal{A}}_\triangledown$. To show that ψ is well-defined, we need to show that the Conway relation in $\mathcal{D}_\triangledown$ is in the kernel. Locally,

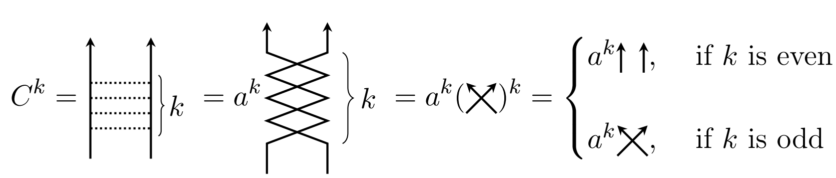


and denote the (global) total degree on both sides by t. In other words, the (global) right hand side is interpreted as an element of $\tilde{\mathcal{T}}_{H,t}/\tilde{\mathcal{T}}_{H,t+1}$. Using the Conway skein relation in $\tilde{\mathcal{A}}_\triangledown$, the right had side can be simplified



Observe that  and the lowest degree term of $e^{\frac{a}{2}} - e^{-\frac{a}{2}} - a$ are both of degree 2, hence  $\in \tilde{\mathcal{T}}_{H,t+1}$, and therefore is zero in $\tilde{\mathcal{T}}_{H,t}/\tilde{\mathcal{T}}_{H,t+1}$.

We now verify that the Kontsevich integral Z descends to the quotient $\mathbb{C}\tilde{\mathcal{T}}_\triangledown$ by checking the relations in $\mathbb{C}\tilde{\mathcal{T}}_\triangledown$ directly. Recall that  $= (e^{\frac{C}{2}})$ and  $= (e^{-\frac{C}{2}})$, where C denotes a chord, the exponential is interpreted formally as a power series, and C^k denotes stacking k chords as shown below. Using the Conway relation, we compute:



Now applying Z to the left hand side of the Conway relation, we obtain

$$\begin{aligned} Z(\nwarrow\!\!\!\!\nearrow) - Z(\nearrow\!\!\!\!\nwarrow) &= (e^{\frac{C}{2}} - e^{\frac{-C}{2}})\nwarrow\!\!\nearrow \\ &= \sum_{k=0}^{\infty} \left(\frac{C^k}{2^k k!} - \frac{(-1)^k C^k}{2^k k!} \right) \nwarrow\!\!\nearrow \\ &= \sum_{k=0}^{\infty} \frac{C^{2k+1}}{2^{2k}(2k+1)!} \nwarrow\!\!\nearrow \\ &= \sum_{k=0}^{\infty} \frac{a^{2k+1} \nwarrow\!\!\nearrow}{2^{2k}(2k+1)!} \nwarrow\!\!\nearrow \\ &= \sum_{k=0}^{\infty} \frac{a^{2k+1}}{2^{2k}(2k+1)!} \uparrow\uparrow \\ &= (e^{\frac{a}{2}} - e^{-\frac{a}{2}}) \uparrow\uparrow \\ &= Z\left((e^{\frac{a}{2}} - e^{-\frac{a}{2}}))(\right). \end{aligned}$$

Thus, Z is well-defined on the Conway quotient $\mathbb{C}\tilde{\mathcal{T}}_\triangledown$.

Therefore, by Lemma 3.1, Z is a homomorphic expansion for $\mathbb{C}\tilde{\mathcal{T}}_\triangledown$ and $\psi : \mathcal{D}_\triangledown \to \tilde{\mathcal{A}}_\triangledown$ is an isomorphism. ☐

While our main focus is the t-filtration on $\mathbb{C}\tilde{\mathcal{T}}_\triangledown$ and its associated graded space $\tilde{\mathcal{A}}_\triangledown$, the low degree components of the associated graded of $\mathbb{C}\tilde{\mathcal{T}}_\triangledown$ with respect to the s-filtration also show up. In particular, there is a well-defined "division by b" map $q_b : \tilde{\mathcal{T}}_\triangledown^{/2} \to \tilde{\mathcal{T}}_\triangledown^{/1}$ which restricts to an isomorphism on $\tilde{\mathcal{T}}_\triangledown^1/\tilde{\mathcal{T}}_\triangledown^2$. We now show that this map exists by defining it explicitly.

Proposition 4.20. *For a tangle T and a crossing x of T, let $\epsilon(x) \in \{\pm 1\}$ be the sign of x, and $T|_{x \to)(}$ be the tangle T with x replaced by a smoothing. There is a well defined map $q_b : \tilde{\mathcal{T}}_\triangledown^{/2} \to \tilde{\mathcal{T}}_\triangledown^{/1}$ given by the linear extension of the following:*

$$bT \overset{q_b}{\mapsto} T$$

$$T \overset{q_b}{\mapsto} \frac{1}{2} \sum_{x \text{ crossing of } T} \epsilon(x) T|_{x \to)(}$$

Proof. It is straightforward to check that Reidemeister moves are preserved.We also need to check that $\tilde{\mathcal{T}}_\triangledown^2$ and the Conway relation are in the kernel. For $b^k T \in \tilde{\mathcal{T}}_\triangledown^2$, if $k = 1$, then $T \in \tilde{\mathcal{T}}_\triangledown^1$, so $q_b(bT) = 0$. If instead $k = 0$, then T has at least two double points. Replacing a crossing by a smoothing only changes the crossing that is replaced, so other crossings (and therefore double points) remain unchanged. Therefore $q_b(T)$ can be written as a sum where each term has at least one double point, so $q_b(T) = 0$ as well.

To show that the Conway relation also vanishes, note that the terms in $q_b(\times) = q_b(\nwarrow\!\!\nearrow - \nearrow\!\!\nwarrow)$ come from either smoothing a crossing that is a part of the double point,

It's straightforward to check R moves so I don't check it explicitly. Is this fine?

or smoothing a crossing that is not. In the latter, the double point remains as before, so those terms are in $\tilde{\mathcal{T}}^1_\nabla$. The only remaining terms are those where the crossings forming the double point are smoothed, so we get

$$q_b\left(\nwarrow\!\!\!\nearrow - \nearrow\!\!\!\nwarrow\right) = \frac{1}{2}\,)(\, - (-1)\frac{1}{2}\,)(\, = \,)(\, = q_b\left(b\,)(\,\right)$$

so q_b is well-defined. ☐

cor:divbyb

do we need this proof?

Corollary 4.21. *The map q_b restricts to an isomorphism $q_b : \tilde{\mathcal{T}}^{1/2}_\nabla \to \tilde{\mathcal{T}}^{/1}_\nabla$.*

Proof. It is clearly surjective. To show it's injective, note that the restriction is simply given by $bT \mapsto T$, and if $T \in \tilde{\mathcal{T}}^1_\nabla$, then $bT \in \tilde{\mathcal{T}}^2_\nabla$. ☐

cor:grdivbyb

Corollary 4.22. *The associated graded of q_b is an isomorphism $\operatorname{gr} q_B$"$\tilde{\mathcal{A}}^{1/2} \to \tilde{\mathcal{A}}^{/1}$ given by*

drawing the chord diagram as with one s-s chord, smoothing that chord using ∇ and getting a factor of b with no remaining s-s chords, and then diving off the b

Clean up

add the following discussion

We should add a discussion here about how the Conway relation occurs in degree 1, so $\tilde{\mathcal{T}}^{/1}_\nabla \cong \tilde{\mathcal{T}}^{/1}$ but this is not true for higher degree quotients. Also, we should discuss how the Conway relation changes the skeleton, and set up what ever specific scenario we need to describe the Goldman Bracket in the next section.

Specify that when we say $\tilde{\mathcal{T}}_\nabla(\bigcirc)$ we there is a representative which is expressed in terms of knots and no factors of b– the greading comes from the actual skeleton, not from factors of b

:IdentifyingGTinHOM

5. Identifying the Goldman-Turaev Lie bialgebra

We identify the Goldman-Turaev Lie bialgebra in low s-degree quotients of the Conway quotient, and show that the Kontsevich integral induces a homomorphic expansion on this space with respect to the s-filtration. Appealing to the schema summarized in Section 2 the approach presented here is to create diagrams like the one in Equation 2.1 where the induced maps η's are the Goldman bracket and the self intersection map μ from which the cobracket is constructed.

5.1. The Goldman Bracket.

Recall from Section 3.3 that D_p is the p-punctured disc, π is its fundamental group, and $|\mathbb{C}\pi|$ is the linear quotient $|\mathbb{C}\pi| := \mathbb{C}\pi/[\mathbb{C}\pi, \mathbb{C}\pi]$. As a vector space over $\mathbb{C}$, $|\mathbb{C}\pi|$ is generated by homotopy classes of free loops in D_p. The Goldman bracket is a map $|\mathbb{C}\pi| \otimes |\mathbb{C}\pi| \to |\mathbb{C}\pi|$ with formula given in Definition 3.3. Recall from Section 4.7 the space $\mathbb{C}\tilde{\mathcal{T}}(\bigcirc)$ is the vector space of $\mathbb{C}$-linear combinations of framed knots in $M_p = D_p \times I$.

prop:BotProj

Proposition 5.1. *The bottom projection in $M_p \to D_p \times \{0\}$ induces a filtered map*

$$\beta : \mathbb{C}\tilde{\mathcal{T}}(\bigcirc) \to |\mathbb{C}\pi|$$

by projecting to free loops in $D_p \times \{0\}$.

Proof. By the Reidemeister Theorem, knots in $\mathbb{C}\tilde{\mathcal{T}}(\bigcirc)$ are faithfully represented by knot diagrams in $D_p \times \{0\}$ – that is, regular projections to the bottom with over/under information – modulo the Reidemeister moves (R2, R3). The map β maps the Reidemeister moves for knots to the corresponding moves generating homotopies between framed free loops. The map β also forgets the framing information: that is, further quotients by the R1 move for curves. The map β is clearly surjective as any loop can be lifted to a knot by introducing arbitrary under/over information at the crossings.

The statement that β is filtered means that step k of the the Vassiliev t-filtration in $\mathbb{C}\tilde{\mathcal{T}}_{\triangledown}(\bigcirc)$ projects into step k of the filtration on $|\mathbb{C}\pi|$ induced by the I-adic filtration of Π. Note that strand-strand double points and framing changes map to 0 in $|\mathbb{C}\pi|$, thus we only have something to prove for knots with k strand-pole double points.

What do we mean by Π here? do we mean $\Pi_k I^k$?

Let $\gamma_1, ..., \gamma_p$ denote the generators of π, that is, γ_i is a simple curve ∂_i, starting and ending at the base point. A knot in $\mathbb{C}\tilde{\mathcal{T}}(\bigcirc)$ maps to a free loop, whose conjugacy class in π is represented as a product of the generators γ_i in π. A pole-stand double point on pole j maps to a difference between two curves passing on either side of ∂_j, and the words in π representing these curves differ in a single instance of γ_j. Thus, knot with k pole-strand double points maps to a product with k factors of the form $\pm(\gamma_j - 1)$, while the other factors are single generators. This is by definition an element in $\mathcal{I}^k$. □

prop:kerbeta

Proposition 5.2. *The kernel of the projection* β *is* $\tilde{\mathcal{T}}^1(\bigcirc)$*, and* β *descends to an isomorphism* $\beta : \tilde{\mathcal{T}}^{/1}(\bigcirc) \to |\mathbb{C}\pi|$.

Proof. Two framed knots in $\mathbb{C}\tilde{\mathcal{T}}(\bigcirc)$ project to the same loop in $|\mathbb{C}\pi|$ if and only if they differ by framing changes and (strand) crossing changes, which generate precisely the step 1 of the s-filtration, that is, $\tilde{\mathcal{T}}^1(\bigcirc)$. □

cor:loopsasknots

Corollary 5.3. *The map* β *descends to an isomorphism* $\beta : \tilde{\mathcal{T}}_{\triangledown}^{/1}(\bigcirc) \to |\mathbb{C}\pi|$.

Proof. $\tilde{\mathcal{T}}^{/1}(\bigcirc) \cong \tilde{\mathcal{T}}_{\triangledown}^{/1}(\bigcirc)$ as the Conway relations occur in degree 1. □

Corollary 5.3 successfully identifies $|\mathbb{C}\pi|$ as a quotient of knots. We now look to the associated graded space $\tilde{\mathcal{A}}$ to find algebraic cyclic words $|\operatorname{Asc}|$. As a reminder, $\tilde{\mathcal{A}}$ is the associated graded space of $\mathbb{C}\tilde{\mathcal{T}}$ with respect to the t-filtration, $\tilde{\mathcal{A}} \cong gr_t\mathbb{C}\tilde{\mathcal{T}}$, and $\tilde{\mathcal{A}}$ is further filtered by the s-filtration. Recalling definitions 4.15 and 4.8, $\tilde{\mathcal{A}}(\bigcirc)$ is the space of admissible chord diagrams on a circle skeleton, $\tilde{\mathcal{A}}^1(\bigcirc)$ is the s-degree 1 filtered component of $\tilde{\mathcal{A}}(\bigcirc)$, and $\tilde{\mathcal{A}}^{/1}(\bigcirc)$ is its quotient of $\tilde{\mathcal{A}}(\bigcirc)$ by $\tilde{\mathcal{A}}^1(\bigcirc)$. Recall from Section 3.3 that $\operatorname{Asc} = \operatorname{Asc}\langle x_1, \cdots, x_p\rangle$ denotes the free associative algebra over $\mathbb{C}$, and its linear quotient $|\operatorname{Asc}| = \operatorname{Asc}/[\operatorname{Asc}, \operatorname{Asc}]$ is the $\mathbb{C}$-vector space generated by cyclic words in p letters. Since $|\operatorname{Asc}|$ is the associated graded of $|\mathbb{C}\pi|$, we take the associated graded (with respect to the t-filtration) of the projection β above to find $|\operatorname{Asc}|$.

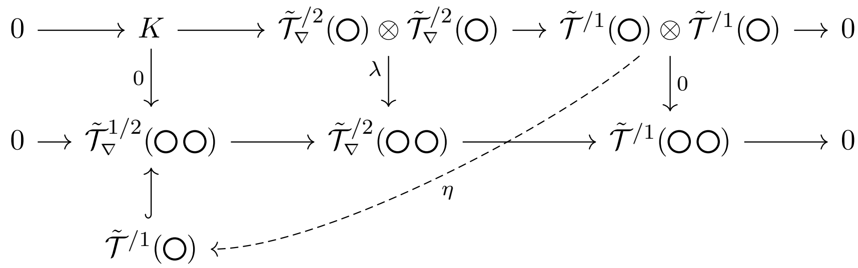


FIGURE 17. The nontrivial horizontal maps are the respective quotient and inclusion maps. The space K is the kernel of the projection map in the top of the right square. Inside $\tilde{\mathcal{T}}_\triangledown^{/2}(\bigcirc)\otimes\tilde{\mathcal{T}}_\triangledown^{/2}(\bigcirc)$, K is generated by $\tilde{\mathcal{T}}_\triangledown^{1/2}(\bigcirc) \otimes \tilde{\mathcal{T}}_\triangledown^{/2}(\bigcirc)$ and $\tilde{\mathcal{T}}_\triangledown^{/2}(\bigcirc) \otimes \tilde{\mathcal{T}}_\triangledown^{1/2}(\bigcirc)$.

fig:Snakeforbracket

Proposition 5.4. *The associated graded map* $\operatorname{gr}\beta : \tilde{\mathcal{A}}(\bigcirc) \to |\operatorname{Asc}|$ *has kernel* $\tilde{\mathcal{A}}^1(\bigcirc)$*. Hence,* $\operatorname{gr}\beta$ *descends to an isomorphism* $\operatorname{gr}\beta : \tilde{\mathcal{A}}^{/1}(\bigcirc) \to |\operatorname{Asc}|$.

Proof. This follows from the fact that β is a filtered map, and fits into the filtered short exact sequence

$$0 \to \tilde{\mathcal{T}}^1(\bigcirc) \to \tilde{\mathcal{T}}(\bigcirc) \to |\mathbb{C}\pi| \to 0.$$

The statement follows from applying the associated graded functor to this sequence. □

put β on the arrow

Add remark about how you can read off a cyclic word from a chord diagram with no s-s chords. This is not needed for any proofs, but is interesting for intuition.

We define two maps $\lambda_1, \lambda_2 : \tilde{\mathcal{T}}_\triangledown^{/2}(\bigcirc) \otimes \tilde{\mathcal{T}}_\triangledown^{/2}(\bigcirc) \to \tilde{\mathcal{T}}_\triangledown^{/2}(\bigcirc) \otimes \tilde{\mathcal{T}}_\triangledown^{/2}(\bigcirc)$ given by $\lambda_1(K_1 \otimes K_2) = K_1K_2$ and $\lambda_2(K_1 \otimes K_2) = K_2K_1$ (stacking the diagrams in different orders). Then lastly $\lambda = \lambda_1 - \lambda_2$ is essentially the commutator on $\tilde{\mathcal{T}}_\triangledown^{/2}(\bigcirc) \otimes \tilde{\mathcal{T}}_\triangledown^{/2}(\bigcirc)$

thm:bracketsnake

Theorem 5.5. *The diagram in 17 commutes. Moreover, the induced connecting homomorphism* η *is the Goldman Bracket.*

Proof. For $K_1 \otimes K_2$ in $\tilde{\mathcal{T}}_\triangledown^{/2}(\bigcirc) \otimes \tilde{\mathcal{T}}_\triangledown^{/2}(\bigcirc)$,

$$\lambda(K_1 \otimes K_2) = \lambda_1(K_1 \otimes K_2) - \lambda_2(K_1 \otimes K_2) = K_1K_2 - K_2K_1.$$

Let a tangle diagram for K_1K_2 given by projection to the bottom. Let a *mixed crossing* of such a diagram be a crossing where one strand belongs to K_1 and the other strand belongs to K_2. Since the product in $\mathbb{C}\tilde{\mathcal{T}}$ is vertical diagram stacking, we can change K_1K_2 to K_2K_1 by passing all the strands of K_2 through the strands of K_1, that is, by flipping every mixed crossing. Using the double point notation, we can write positive mixed crossings as + and negative mixed crossings as −, where each double point has one strand belongs to K_1 and the other belongs to K_2. Rewriting all the mixed crossings of K_1K_2 in this way yields a sum of tangles indexed by the subsets of the mixed crossings. More precisely, for every subset X of the mixed crossings, there is a term where the crossings in X are replaced by double points, and the crossings not in X are replaced by their opposite so that they agree with their counterparts in K_2K_1. In particular, the term corresponding to $X = \emptyset$ is exactly K_2K_1. Thus $\lambda(K_1 \otimes K_2) = K_1K_2 - K_2K_1$ can be written as a sum of terms with at least one double point, so it lives in $\tilde{\mathcal{T}}^1$. Thus, $\lambda(K_1 \otimes K_2)$ is in the kernel of the quotient map $\tilde{\mathcal{T}}_\triangledown^{/2}(\bigcirc\bigcirc) \to \tilde{\mathcal{T}}_\triangledown^{/1}(\bigcirc\bigcirc)$, and the right square of Figure 17 commutes.

Now for the left square, the kernel K of the projection map from $\tilde{\mathcal{T}}_\triangledown^{/2}(\bigcirc) \otimes \tilde{\mathcal{T}}_\triangledown^{/2}(\bigcirc) \to \tilde{\mathcal{T}}_\triangledown^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}_\triangledown^{/1}(\bigcirc)$ is generated by $\tilde{\mathcal{T}}_\triangledown^{1/2}(\bigcirc) \otimes \tilde{\mathcal{T}}_\triangledown^{/2}(\bigcirc)$ and $\tilde{\mathcal{T}}_\triangledown^{/2}(\bigcirc) \otimes \tilde{\mathcal{T}}_\triangledown^{1/2}(\bigcirc)$. Suppose that $K_1 \otimes K_2$ is in $\tilde{\mathcal{T}}_\triangledown^{1/2}(\bigcirc) \otimes \tilde{\mathcal{T}}_\triangledown^{/2}(\bigcirc)$, so K_1 has exactly degree 1. Then, by the same computation as above by $\lambda(K_1 \otimes K_2)$ is a sum of terms each with two or more double points, one coming from the double point in K_1 and the other coming from a mixed crossing. Thus $\lambda(K_1 \otimes K_2) = 0$ in $\tilde{\mathcal{T}}_\triangledown^{/2}(\bigcirc\bigcirc)$ and the left square commutes.

This argument needs work.

By Section 2, λ induces a well defined connecting homomorphism $\eta : \tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc) \to \tilde{\mathcal{T}}_\triangledown^{1/2}(\bigcirc\bigcirc)$. From Proposition 5.2, we know that $\tilde{\mathcal{T}}_\triangledown^{/1}(\bigcirc) \cong |\mathbb{C}\pi|$. We need to argue that image of η lands in $\tilde{\mathcal{T}}_\triangledown^{1/2}(\bigcirc)$.

After taking the quotient to $\tilde{\mathcal{T}}_\triangledown^1/\tilde{\mathcal{T}}_\triangledown^2$, only the terms that have a single double point remain, so $[K_1, K_2]$ becomes a sum over the mixed crossings, where in each term the mixed crossing is replaced by a double point. The map q_b uses the Conway relation to smooth each of these double point in by pulling out a factor of b. Each smoothing merges the two circle components into one circle component, so after applying q_b, we land in $\tilde{\mathcal{T}}^{/1}(\bigcirc) \subseteq \tilde{\mathcal{T}}^{/1}$, which is isomorphic to $|\mathbb{C}\pi|$ through β. The result is a signed sum obtained by smoothing each mixed crossing.

On the other hand, the mixed crossings in K_1K_2 correspond to intersections between $\beta(K_1)$ and $\beta(K_2)$ in the overlap of their projected images. The Goldman bracket $[\beta(K_1), \beta(K_2)]_G$ is a signed sum of the result of smoothing each mixed intersection. Since the double points that come from mixed crossings correspond with intersections between different components, to show the diagram commutes, it only remains to check that the signs on each term agree.

If the tangent vectors of $\beta(K_1)$ and $\beta(K_2)$ make a positive basis at the intersection point, the term in the Goldman bracket has a negative sign. Since K_1K_2 stacks of K_2 on top of K_1, if $\beta(K_1)$ and $\beta(K_2)$ form a positive basis, then the mixed crossing is negative in K_1K_2 and positive in K_2K_1, thus the resulting double point also has a negative sign. Thus we can conclude that $\eta(K_1 \otimes K_2) = [K_1, K_2]$, completing the proof.

□

Recall that the graded Goldman bracket is a map $[-,-]_{\mathrm{gr}\, G} : |\operatorname{Asc}| \otimes |\operatorname{Asc}| \to |\operatorname{Asc}|$ with formal given in Definition 3.8. By taking the associated graded of

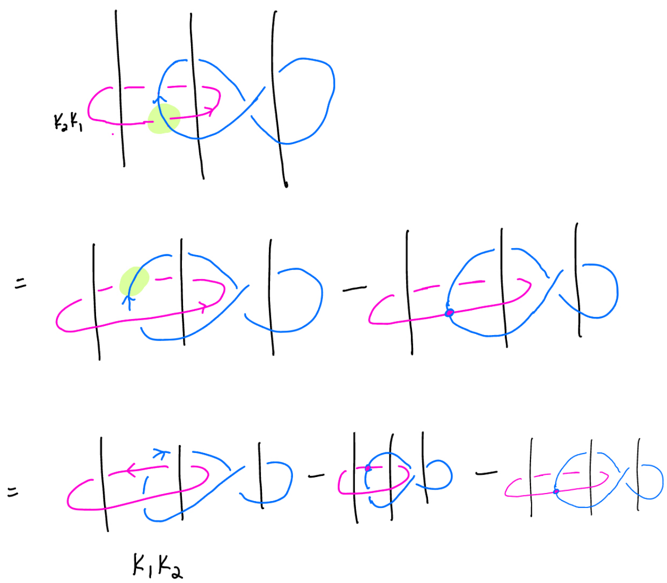
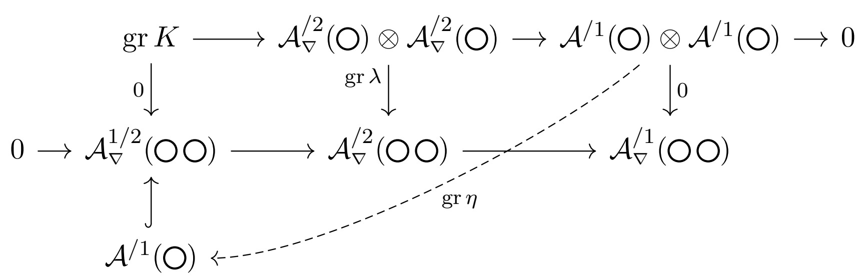


fig:combracket

FIGURE 18. Example commutator bracket computation.



Snakefor_gr_bracket

FIGURE 19. Commutative diagram that is the associated graded of the diagram in Figure 17. The nontrivial horizontal maps are the respective quotient maps.

the diagram in Figure 17 we arrive at the commutative diagram in Figure 19 and recover the associated graded Goldman bracket.

add discussion of gr K

snakefor_gr_bracket

Corollary 5.6. *The diagram in Figure 19 commutes and* $\mathrm{gr}\,\eta$ *is the associated graded Goldman bracket.*

Proof. The maps in the diagram of Figure 17 are filtered maps, and therefore Figure 19 is obtained by applying the associated graded functor to it. As a result, the diagram of Figure 19 commutes, $\mathrm{gr}\,\eta$ is the induced map from the snake lemma for this diagram, and coincides with the graded Goldman bracket. □

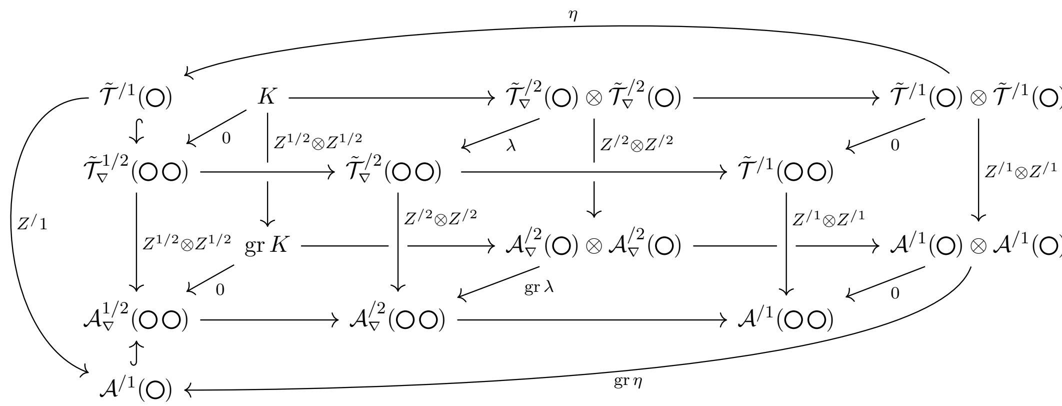


Figure 20. Commutative cube showing the formality of the Goldman bracket from the Kontsevich integral.

`thm:bracketsnake`

Theorem 5.7. *The Kontsevich integral descends to a homomorphic expansion for the Goldman Bracket. That is, the following square commutes:*

$$\begin{array}{ccc} \tilde{\mathcal{T}}^{/1}(\bigcirc) & \xleftarrow[\eta=[\cdot,\cdot]_G]{} & \tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc) \\ \big\downarrow Z^{/1} & & \big\downarrow Z^{/1}\otimes Z^{/1} \\ \mathcal{A}^{/1}(\bigcirc) & \xleftarrow[\mathrm{gr}\,\eta]{} & \mathcal{A}^{/1}(\bigcirc) \otimes \mathcal{A}^{/1}(\bigcirc) \end{array}$$

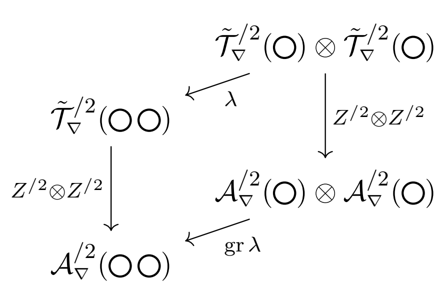
`ig:Cube_for_bracket`

Proof. Taking the Kontsevich integral of the diagram in Figure 17 we get the cube in Figure 20. We have already established that the top and bottom faces all commute from Theorem 5.7 and Corollary 5.6. The front and back vertical faces commute because Z respects the s-filtration and is homomorphic with respect to the inclusions and quotient maps of the filtered component. The left and right vertical sides trivially commute because of the zero maps.

add K and gr K

say this better!

The Kontsevich integral is homomorphic with respect to diagram stacking, as proved in Proposition 4.13. Since λ is the difference between two orderings of diagram stacking, Z is homomorphic with respect to λ and the following square commutes (which is the middle vertical face of Figure 20).



The commutativity of all vertical faces of the cube diagram in Figure 20 implies that the induced diagonal square also commutes.

Technically we get that this square commutes. But the square we want has the inclusion maps and a different descension of Z. I wasn't sure if we need to make this distinction, or maybe add to this diagram? OR only write the simpler diagram as in the statement of the theorem

$$\begin{array}{ccc} \tilde{\mathcal{T}}^{1/2}(\bigcirc) \otimes \tilde{\mathcal{T}}^{1/2}(\bigcirc) & \xleftarrow[\eta=[\cdot,\cdot]_G]{} & \tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc) \\ \downarrow Z^{1/2}\otimes Z^{1/2} & & \downarrow Z^{/1}\otimes Z^{/1} \\ \mathcal{A}^{1/2}(\bigcirc) \otimes \mathcal{A}^{1/2}(\bigcirc) & \xleftarrow[\operatorname{gr}\eta]{} & \mathcal{A}^{/1}(\bigcirc) \otimes \mathcal{A}^{/1}(\bigcirc) \end{array}$$

□

Using $\mathbb{C}\tilde{\pi}$ to mean "sailing curves" i.e. those that "never look north", but not introducing it here because it seems like it should be defined already

5.2. **The Tureav co-bracket.** Recall from section **??** that the Turaev cobracket on $|\mathbb{C}\pi|$ can be defined using the map $\mu : \mathbb{C}\tilde{\pi} \to |\mathbb{C}\pi| \otimes \mathbb{C}\pi$. Our knot-theoretic definition of the cobracket imitates this construction, and we interpret the domain $\mathbb{C}\tilde{\pi}$ and codomain $|\mathbb{C}\pi| \otimes \mathbb{C}\pi$ of μ in the context of tangles.

Let $\frown$ denote an interval skeleton component where both endpoints are on the bottom $D_p \times \{0\}$. We will call a tangle whose endpoints are all on the bottom a *bottom tangle*. In diagrams, the endpoints of the tangle will be drawn in the bottom right corner. As before, the beginning of each interval section will be marked by a $\bullet$, and the end will be marked by $*$, see figure **??**.

drawing conventions are random for now

We can extend the projection map β from proposition 5.1 to bottom tangles to get an isomorphism similar to corollary 5.3. We state the map β and corresponding isomorphism in proposition 5.8, but omit the proof as it is the same as before.

prop:ascispi

Proposition 5.8. *There is a well-defined natural bottom projection*

$$\beta : \tilde{\mathcal{T}}_{\triangledown}(\bigcirc^k \frown^\ell) \to |\mathbb{C}\pi|^{\otimes k} \otimes \mathbb{C}\pi^{\otimes \ell}$$

that descends to an isomorphism $\beta : \tilde{\mathcal{T}}^{/1}(\bigcirc^k \frown^\ell) \xrightarrow{\cong} |\mathbb{C}\pi|^{\otimes k} \otimes \mathbb{C}\pi^{\otimes \ell}$.

p:qbonbottomtangles

Proposition 5.9. *The division by b map, q_b, descends to an isomorphism*

$$q_b : \tilde{\mathcal{T}}_{\triangledown}^{1/2}(\frown) \xrightarrow{\cong} \tilde{\mathcal{T}}_{\triangledown}^{/1}(\bigcirc\frown).$$

Proof. The map q_b uses the Conway relation to smooth double points to get a two-component tangle, where one component has interval skeleton and the other component has circle skeleton. □

This proof needs work.

def:asc+desc

Definition 5.10. Let $\bullet$ and $*$ be two points on the boundary of D_p that are close together. An embedding

$$T : (I, \{0,1\}) \hookrightarrow (M_p, \{\bullet, *\})$$

representing a bottom tangle is called *ascending* if it "first goes up, and then goes *straight* down". More precisely, if (z, s) is a global coordinate system for $M_p = D_p \times I$, then T is an ascending tangle if there exists $c \in (0,1)$ such that when $t \in (0,c)$, the $\frac{d}{ds}$ component of $\dot{T}$ is positive, and when $t \in (c,1)$, $\dot{T}$ is a negative constant multiple of $\frac{d}{ds}$.

fig:ascending

placeholder figure of an ascending tangle with an ascending embedding and a non-ascending embedding

Likewise, such an embedding representing a bottom tangle T is *descending* if it "first goes straight up, and then goes down". So there is $c \in (0,1)$ such that when $t \in (0,c)$, $\dot{T}$ is a positive constant multiple of $\frac{d}{ds}$ and when $t \in (c,1)$ the $\frac{d}{ds}$ component of $\dot{T}$ is negative.

Definition 5.11. An *ascending tangle* is a bottom tangle in M_p whose ambient isotopy class has an ascending embedding. See figure **??** for an example.

TODO: add figure

Given a curve K in $\mathbb{C}\pi$, through the isomorphism β, K can be lifted to a bottom tangle in $\tilde{\mathcal{T}}^{/1}(\frown)$. Because we are in the quotient by degree 1 terms, crossings can be changed at will to make the lifted tangle be ascending or descending. However, to lift K to a framed tangle takes some care. For any framed curve K in $\mathbb{C}\pi$, we can choose a homotopy class representative that with rotation number 0 that is a sailing curve. A *sailing curve* is a curve whose tangent vector never points north. When taking a lift of a sailing K, there is an ascending lift of the curve where the north vector is never tangent to the curve. We will denote this lift as $\lambda_a(K)$ We can choose a framing at each point p on $\lambda_a(K)$ by taking the tangent vector $\dot{T}$ at p and the projection of $\vec{n}$ on to the plane normal to $\dot{T}$. Thus $\lambda_a(K)$ is a framed ascending bottom tangle. Similarly we will let $\lambda_d(K)$ be a framed descending bottom tangle. Finally, we define $\bar{\lambda} : \tilde{\mathcal{T}}^{/1}(\frown) \to \tilde{\mathcal{T}}^{/2}(\frown)$ by

make rigorous the notion of north in the disk. Add diagram of sailing trick to avoid north

$$\bar{\lambda}(K) = \lambda_a(K) - \lambda_b(K)$$

to be the difference between the framed ascending bottom tangle and the framed descending bottom tangle. In $\tilde{\mathcal{T}}^{/2}(\frown)$, crossing changes matter so $\bar{\lambda}$ is not the zero map.

Theorem 5.12. *The diagram in Figure 21 commutes and the unique induced map η is the self intersection map μ.*

Proof. Let $\gamma \in |\mathbb{C}\pi|$ and let $T = \lambda_a(\gamma)$ be an ascending lift. We can rewrite each strand-strand crossings (s-crossing for short) of T as a sum or difference of a double point and its counterpart in T^{f_b}. As in the proof of **??**, rewriting each s-crossing of T in this way yields a sum indexed by the subsets of its s-crossings. For every subset X of s-crossings, there is a term in the sum where the s-crossings in X are replaced by double points, and those not in X are replaced with their counterparts in T^{f_b}. In particular, the term corresponding to $X = \emptyset$ is exactly T^{f_b}, so $T - T^{f_b}$ lives in $\tilde{\mathcal{T}}^1(\frown)$.

old proof, not yet updated

By passing to the quotient $\tilde{\mathcal{T}}^1_\nabla / \tilde{\mathcal{T}}^2_\nabla(\frown)$, only the terms that have a single double point remain, so $T - T^{f_b}$ becomes a sum over the s-crossings of T, where in each term the s-crossing is replaced by a double point. The map q_b uses the Conway relation to smooth these double points to get a two-component tangle, where one component has interval skeleton and the other component has circle skeleton. Thus we land in $\tilde{\mathcal{T}}^{/1}_\nabla(\bigcirc\frown)$, which is isomorphic to $|\mathbb{C}\pi| \otimes \mathbb{C}\pi$ via β. $\square$

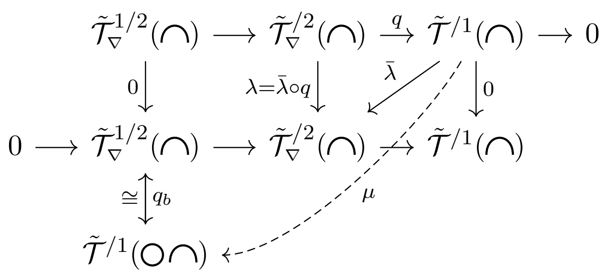
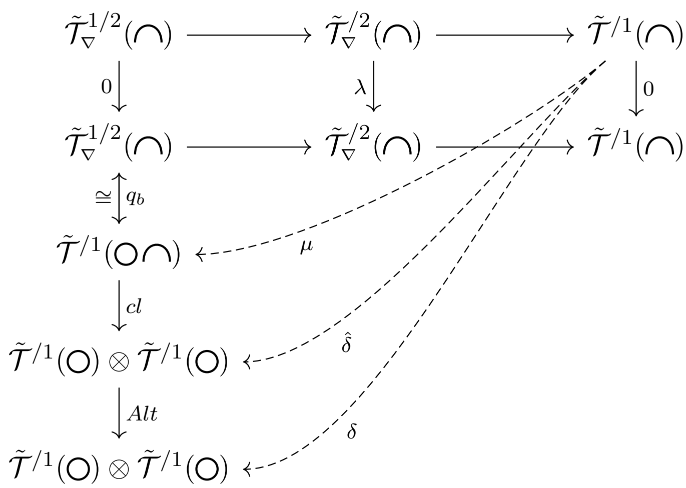


FIGURE 21. The nontrivial horizontal maps are the respective quotient maps.

fig:Snakeformu



g:Snakeforcobracket

FIGURE 22. Constructing δ from μ.

For a bottom tangle, there is a closure map from $cl : \tilde{\mathcal{T}}(\bigcap) \to \tilde{\mathcal{T}}(\bigcirc)$ by connecting the endpoints of the bottom tangle, $\bullet$ and $*$, by a canonical path in the boundary of the disk. Recall from Section 3.3 that the cobracket δ is constructed from μ by post composing with the closure map and then antisymetrizing. In the context of tangle diagrams, this construction is shown in Figure 22. The closure map $cl : \tilde{\mathcal{T}}^{/1}(\bigcirc\bigcap) \to \tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc)$ orders the components by placing the closed bottom tangle in the second slot. The intermediate induced map after closing, but before antisymmtrizing, is denoted in the figure by $\hat{\delta}$ and is called the *ordered* Turaev cobracket. We will show the Kontsevich integral is homomorphic with respect to $\hat{\delta}$. The homomorphicity of δ with respect to Z follows from immediately the homomorphicity of $\hat{\delta}$ with respect to Z because $\mathrm{gr}(Alt) = Alt$.

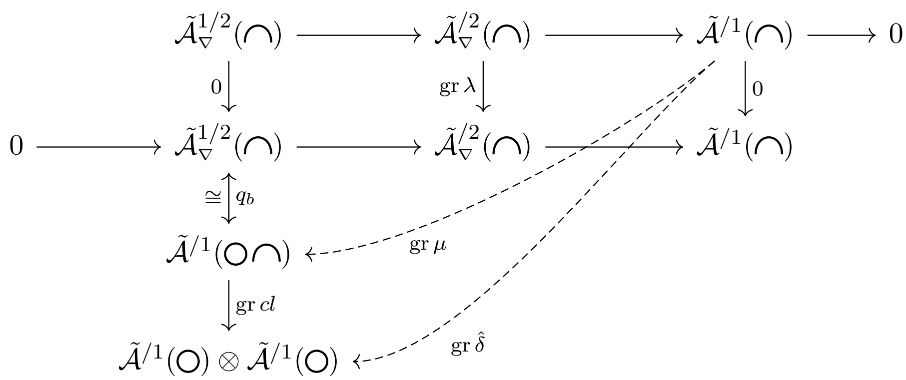


FIGURE 23. Associated graded diagram constructing the graded ordered Turaev cobracket.

akefor_gr_cobracket

Taking the associated graded of the diagram in Figure 17 we arrive at the diagram in Figure 23

akefor_gr_cobracket

Theorem 5.13. *The diagram in Figure 23 commutes and the induced map* $\operatorname{gr}\hat{\delta}$ *is the associated graded ordered Turaev cobracket.*

Proof. The maps in the diagram of Figure 22 are filtered maps, and therefore Figure 23 is obtained by applying the associated graded functor to it. As a result, the diagram of Figure 23 commutes, $\operatorname{gr}\mu$ is the induced map from the snake lemma for this diagram, and so $\operatorname{gr}\hat{\delta}$ coincides with the graded ordered Turaev cobracket. □

pcubesimplification

Lemma 5.14. *There exists a map* $\rho : \tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc) \to \tilde{\mathcal{T}}_{\triangledown}^{/2}(\bigcirc)$ *that makes the diagram in Figure 24 commute.*

In this figure, do we need μ in it still?

Proof. There is an isomorphism from $\tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc)$ to $\tilde{\mathcal{T}}^{/1}(\bigcirc\bigcirc)$ by combining the two tangles into a single tangle and forgetting the order of the components. Since we are modding out by s degree 1, there is no notion of over or under, these are just curves in the disc.

The map $\rho : \tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc) \to \tilde{\mathcal{T}}^{/2}(\bigcirc)$ is defined to be the following composition of maps.

$$\rho:\ \tilde{\mathcal{T}}^{/1}(\bigcirc) \otimes \tilde{\mathcal{T}}^{/1}(\bigcirc) \xrightarrow{forget} \tilde{\mathcal{T}}^{/1}(\bigcirc\bigcirc) \xleftrightarrow{q_b} \tilde{\mathcal{T}}_{\triangledown}^{1/2}(\bigcirc) \hookrightarrow \tilde{\mathcal{T}}_{\triangledown}^{/2}(\bigcirc)$$

Since the image of ρ in $\tilde{\mathcal{T}}_{\triangledown}^{/2}$ is all of $\tilde{\mathcal{T}}^{1/2}$ we get the following short exact sequence.

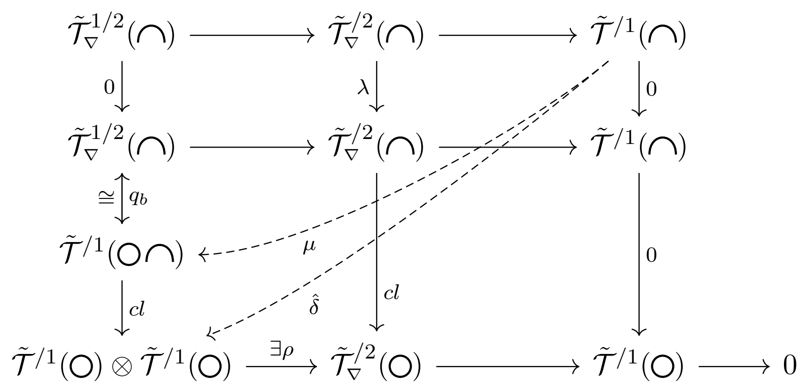
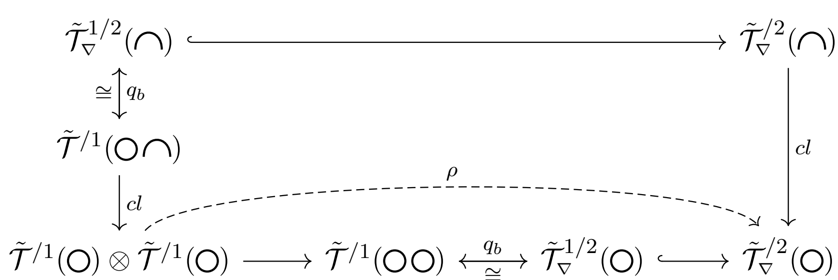


FIGURE 24. Commutative diagram for Lemma 5.14.

pcubesimplification

$$\tilde{\mathcal{T}}^{/1}(\bigcirc)\otimes\tilde{\mathcal{T}}^{/1}(\bigcirc) \xrightarrow{\rho} \tilde{\mathcal{T}}_{\triangledown}^{/2}(\bigcirc) \longrightarrow \tilde{\mathcal{T}}^{/1}(\bigcirc) \longrightarrow 0$$

The commutativity of the diagram in Figure 24 relies finally on the commutativity of the bottom left square. We single this square out below and verify the commutativity.



Let $T \in \tilde{\mathcal{T}}_{\triangledown}^{1/2}(\frown)$, then T is a bottom tangle with exactly one double point. Following along the top and right of the diagram in Figure 24, when T is closed, we get a closed loop with one double point inside $\tilde{\mathcal{T}}_{\triangledown}^{/2}(\bigcirc)$. Following along the right and bottom, $q_b(T)$ uses the Conway relation to snip off a loop of T to get a tangle in $\tilde{\mathcal{T}}^{/1}(\bigcirc\frown)$ with one closed loop and a bottom tangle, with no double points. Closing the bottom tangle and forgetting the order of the closed loops gives a tangle in $\tilde{\mathcal{T}}^{/1}(\bigcirc\bigcirc)$ with two closed loops and no double points. Reversing the Conway relation along q_b glues together the two closed loops to get a single

$$\begin{array}{ccccccc}
\tilde{\mathcal{T}}^{/1}(\bigcirc)\otimes\tilde{\mathcal{T}}^{/1}(\bigcirc) & \xrightarrow{forget} & \tilde{\mathcal{T}}^{/1}(\bigcirc\bigcirc) & \xleftrightarrow{q_b} & \tilde{\mathcal{T}}^{1/2}_{\triangledown}(\bigcirc) & \hookrightarrow & \tilde{\mathcal{T}}^{/2}_{\triangledown}(\bigcirc) \\
\downarrow{\scriptstyle Z^{/1}\otimes Z^{/1}} & & \downarrow{\scriptstyle Z^{/1}} & & \downarrow{\scriptstyle Z^{/1}} & & \downarrow{\scriptstyle Z^{/2}} \\
\tilde{\mathcal{A}}^{/1}(\bigcirc)\otimes\tilde{\mathcal{A}}^{/1}(\bigcirc) & \xrightarrow{forget} & \tilde{\mathcal{A}}^{/1}(\bigcirc\bigcirc) & \xleftrightarrow{\operatorname{gr} q_b} & \tilde{\mathcal{A}}^{1/2}_{\triangledown}(\bigcirc) & \hookrightarrow & \tilde{\mathcal{A}}^{/2}_{\triangledown}(\bigcirc)
\end{array}$$

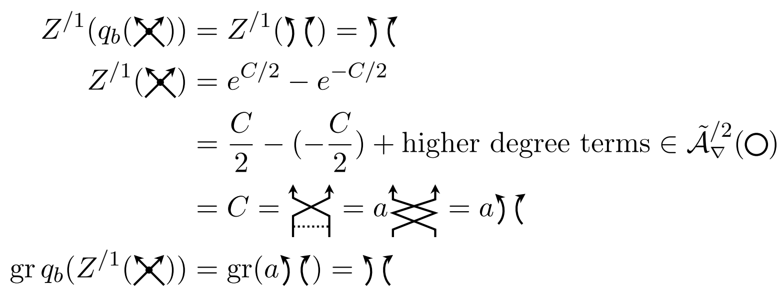
`fig:frontlefthomom` FIGURE 25. Commutative diagram for Lemma 5.15

closed loop with one double point then included into $\tilde{\mathcal{T}}^{/2}(\bigcirc)$. This arrives at the same closed loop with one double point as if we had closed T in the first place. □

`lem:frontlefthomom` **Lemma 5.15.** *The diagram in Figure 25 commutes.*

Proof. The right square commutes because Z is a filtered map and respects filtered inclusions.

For the middle square, we use the map q_b from right to left and show commutativity on a double point.



For the left square, Z compatible with forgetful is because we land in $/1$, where there are no s-s chords. □

Make new diagrams for power chord and power swap to have 1 chord and 1 and 2 swaps

say more about left square

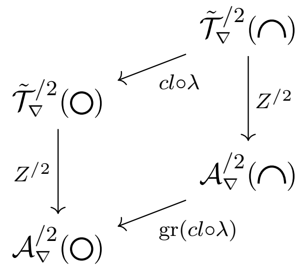
`cobrackethomomorphic` **Theorem 5.16.** *The Kontsevich integral descends to a homomorphic expansion for the ordered Turaev cobracket. That is, the following square commutes:*

$$\begin{array}{ccc}
\tilde{\mathcal{T}}^{/1}(\bigcirc)\otimes\tilde{\mathcal{T}}^{/1}(\bigcirc) & \xleftarrow[\hat{\delta}]{} & \tilde{\mathcal{T}}^{/1}(\frown) \\
\downarrow{\scriptstyle Z^{/1}\otimes Z^{/1}} & & \downarrow{\scriptstyle Z^{/1}} \\
\mathcal{A}^{/1}(\bigcirc)\otimes\mathcal{A}^{/1}(\bigcirc) & \xleftarrow[\operatorname{gr}\hat{\delta}]{} & \mathcal{A}^{/1}(\frown)
\end{array}$$

Proof. The diagram in Figure 26 is attained by taking the Kontsevich integral of the commutative diagram in Figure 24 (with the middle layers omitted). We have already established that the top and bottom faces commute by Lemma 5.14

and Theorem 5.13. The left and right vertical sides trivially commute because of the zero maps. The front-left vertical square commutes by Lemma 5.15. The front-right and back faces commute because Z respects the s-filtration and is homomorphic with respect to the inclusion and quotient maps of the filtered components.

The middle vertical face of Figure 26 is the following square.



This is not quite right, FIX ME!

The Kontsevich integral is homomorphic with respect to the flip operation, as shown in Proposition 4.13. The map $cl \circ \lambda$ applied to a bottom tangle outputs the difference between the closed ascending lift and the closed descending lift. The closed descending lift is the flip of the closed ascending lift. So $cl \circ \lambda = (id - flip) \circ cl$ acting on ascending representatives. Z is homomorphic with respect to $(id - flip) \circ cl$.

where does conjugation come into play?? Something about flipping first then dragging the ends down and then closing.

The commutativity of all vertical faces of the cube diagram in Figure 26 implies that the induces diagonal square also commutes, which gives the desired formality of the theorem statement. □

remark–if we were doing this with μ is it wouldn't work because flip of a bottom tangle is not a bottom tangle. It is much cleaner to just pass to the closures.

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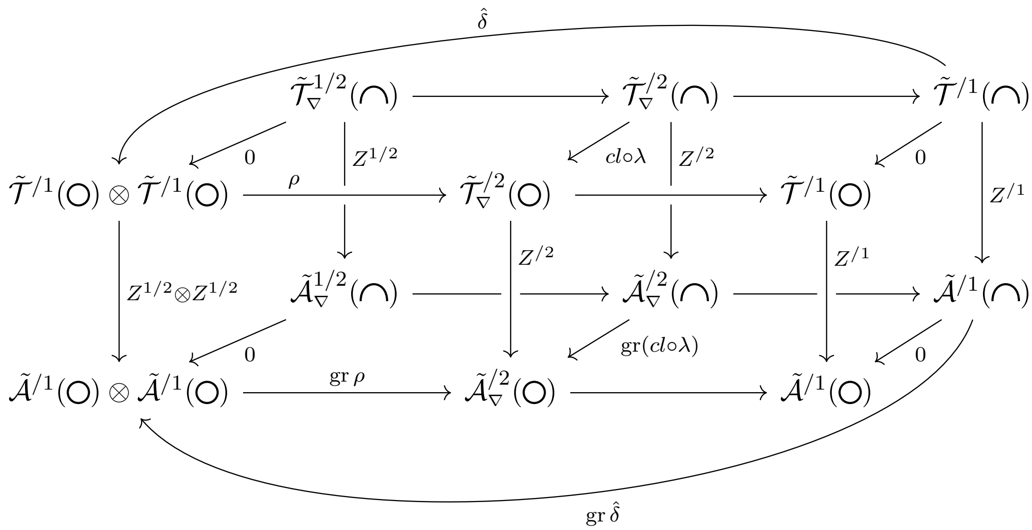

g:Cube_for_cobracket

FIGURE 26. Commutative cube showing the formality of the ordered Turaev cobracket from the Kontsevich integral.

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