

Pensieve header: Finding the most general R in abc notation.

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SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\OneCo-1604";
<< abc.m
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Finding R

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VerifyR3[t_, expr_] := Module[{lhs, rhs},
  lhs = expr // R[t, 1, 2] // R[t, 1, 3] // R[t, 2, 3] // S;
  rhs = expr // R[t, 2, 3] // R[t, 1, 3] // R[t, 1, 2] // S;
  expr -> S[lhs - rhs] == 0
];
VerifyR3[expr_] := VerifyR3[1, expr];

ρ0[j_, k_] := Total[MapIndexed[ (#1 /. ff -> ff#2[[1]]) &,
  DeleteCases[FormalBasis[{j, k}, ff], _β | _a]]];
R[t_, j_, k_][x_] := Expand[
  x // Ad[a[t, j, k]] // (# + t B[ρ0[j, k], #]) &
];
R[j_, k_][x_] := R[1, j, k][x];

ρ0[j, k]
c[ff1[bj, bk], j] + c[ff2[bj, bk], k] + ca[ff7[bj, bk], j, j, j] + ca[ff8[bj, bk], j, j, k] +
ca[ff9[bj, bk], j, k, j] + ca[ff10[bj, bk], j, k, k] + ca[ff11[bj, bk], k, j, j] +
ca[ff12[bj, bk], k, j, k] + ca[ff13[bj, bk], k, k, j] + ca[ff14[bj, bk], k, k, k] +
δa[ff3[bj, bk], j, j] + δa[ff4[bj, bk], j, k] + δa[ff5[bj, bk], k, j] +
δa[ff6[bj, bk], k, k] + δaa[ff15[bj, bk], j, j, j, j] + δaa[ff16[bj, bk], j, j, j, k] +
δaa[ff17[bj, bk], j, j, k, j] + δaa[ff18[bj, bk], j, j, k, k] +
δaa[ff19[bj, bk], j, k, j, k] + δaa[ff20[bj, bk], j, k, k, k] +
δaa[ff21[bj, bk], k, j, k, j] + δaa[ff22[bj, bk], k, j, k, k] + δaa[ff23[bj, bk], k, k, k, k]

VerifyR3[a[f[b1, b2, b3, b4], 1, 4]]
a[f[b1, b2, b3, b4], 1, 4] -> c[-(-1 + e^b2) f[b1, b2, b3, b4] b1 b3^2 ff22[b1, b3], 4] +
ca[e^-b1-b2 f[b1, b2, b3, b4] (e^b1 (-1 + e^b2) b2 ff13[b1, b2] -
e^b1 (-1 + e^b2) b3 ff13[b1, b3] - 2 (-1 + e^b1) b2^2 ff21[b1, b2]), 4, 1, 3] +
ca[-e^-b1-b2 f[b1, b2, b3, b4] (e^b1 (-1 + e^b2) b1 (ff8[b1, b2] - ff8[b1, b3]) -
2 (-1 + e^b1) b2 (ff9[b1, b2] + b2 ff21[b1, b2])), 3, 1, 4] +
ca[-2 (1 - e^-b1) f[b1, b2, b3, b4] b1 b3 ff21[b1, b3], 4, 3, 2] +
ca[2 (-1 + e^2 b2) f[b1, b2, b3, b4] b1 b3 ff21[b1, b3], 4, 3, 1] +
ca[2 e^-b1 (-1 + e^b1) f[b1, b2, b3, b4] b3^2 ff21[b1, b3], 4, 1, 2] +
ca[- 2 (-1 + e^2 b2) f[b1, b2, b3, b4] b1 b3^2 ff21[b1, b3], 4, 2, 1] +
ca[2 (1 - e^-b1) f[b1, b2, b3, b4] b1 (ff9[b1, b3] + b3 ff21[b1, b3]), 2, 3, 4] +
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$$\begin{aligned}
& ca[-2 e^{-b_1} (-1 + e^{b_1}) f[b_1, b_2, b_3, b_4] b_3 (ff_9[b_1, b_3] + b_3 ff_{21}[b_1, b_3]), 2, 1, 4] + \\
& ca[-2 (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_1 (ff_9[b_1, b_3] + (1 + e^{b_2}) b_3 ff_{21}[b_1, b_3]), 1, 3, 4] + \\
& ca\left[\frac{1}{b_2} 2 (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_1 b_3 (ff_9[b_1, b_3] + (1 + e^{b_2}) b_3 ff_{21}[b_1, b_3]), 1, 2, 4\right] + \\
& ca[-(-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_1 b_3 ff_{22}[b_1, b_3], 3, 3, 4] + \\
& ca[(-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_1 b_3 ff_{22}[b_1, b_3], 4, 3, 3] + \\
& ca\left[\frac{1}{b_2} e^{-b_1-b_2} f[b_1, b_2, b_3, b_4] b_1 (e^{b_1} (-1 + e^{b_2}) b_1 (ff_8[b_1, b_2] - ff_8[b_1, b_3]) - 2 (-1 + e^{b_1})\right. \\
& \quad \left. b_2 ff_9[b_1, b_2] - 2 (-1 + e^{b_1}) b_2^2 ff_{21}[b_1, b_2] + e^{b_1+b_2} (-1 + e^{b_2}) b_3^2 ff_{22}[b_1, b_3]),\right. \\
& \quad \left. 3, 2, 4\right] + ca\left[\frac{1}{b_2} e^{-b_1-b_2} f[b_1, b_2, b_3, b_4] b_1 (-e^{b_1} (-1 + e^{b_2}) b_2 ff_{13}[b_1, b_2] +\right. \\
& \quad \left. 2 (-1 + e^{b_1}) b_2^2 ff_{21}[b_1, b_2] - e^{b_1} (-1 + e^{b_2}) b_3 (-ff_{13}[b_1, b_3] + e^{b_2} b_3 ff_{22}[b_1, b_3])\right), \\
& \quad \left. 4, 2, 3\right] + \delta a[-(-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_1 (ff_5[b_1, b_3] + b_3 ff_{21}[b_1, b_3]), 3, 4] + \\
& \delta a\left[\frac{1}{b_2} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_1 b_3 (ff_5[b_1, b_3] + b_3 (ff_{21}[b_1, b_3] + ff_{22}[b_1, b_3])),\right. \\
& \quad \left. 2, 4\right] + \delta aa\left[-\frac{1}{b_2} e^{-b_1-b_2} f[b_1, b_2, b_3, b_4] (e^{b_1} (-1 + e^{b_2}) b_1 (ff_8[b_1, b_2] - ff_8[b_1, b_3]) +\right. \\
& \quad \left. b_2 (-2 (-1 + e^{b_1}) ff_9[b_1, b_2] + e^{b_1} (-1 + e^{b_2}) ff_{13}[b_1, b_2]) -\right. \\
& \quad \left. e^{b_1} (-1 + e^{b_2}) b_3 ff_{13}[b_1, b_3] - 4 (-1 + e^{b_1}) b_2^2 ff_{21}[b_1, b_2]), 1, 3, 2, 4\right] + \\
& \delta aa\left[\frac{1}{b_2^2} e^{-b_1-b_2} f[b_1, b_2, b_3, b_4] b_1 (e^{b_1} (-1 + e^{b_2}) b_2 ff_{13}[b_1, b_2] +\right. \\
& \quad \left. e^{b_1} (-1 + e^{b_2})^2 b_3 ff_{13}[b_1, b_3] - 2 (-1 + e^{b_1}) b_2^2 ff_{21}[b_1, b_2]), 2, 3, 2, 4\right] + \\
& \delta aa\left[\frac{1}{b_1} e^{-b_1-b_2} f[b_1, b_2, b_3, b_4] (e^{b_1} (-1 + e^{b_2}) b_1 (ff_8[b_1, b_2] - ff_8[b_1, b_3]) -\right. \\
& \quad \left. 2 (-1 + e^{b_1}) b_2 (ff_9[b_1, b_2] + b_2 ff_{21}[b_1, b_2]), 1, 3, 1, 4\right] + \\
& \delta aa[2 (1 - e^{-b_1}) f[b_1, b_2, b_3, b_4] b_1 ff_{21}[b_1, b_3], 3, 2, 3, 4] + \\
& \delta aa[-2 (-1 + e^{2 b_2}) f[b_1, b_2, b_3, b_4] b_1 ff_{21}[b_1, b_3], 3, 1, 3, 4] + \\
& \delta aa\left[\frac{4 e^{b_2} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_1 b_3 ff_{21}[b_1, b_3]}{b_2}, 2, 1, 3, 4\right] + \\
& \delta aa\left[-\frac{2 (-1 + e^{b_2})^2 f[b_1, b_2, b_3, b_4] b_1 b_3^2 ff_{21}[b_1, b_3]}{b_2^2}, 2, 1, 2, 4\right] + \\
& \delta aa\left[\frac{1}{b_1} 2 e^{-b_1} (-1 + e^{b_1}) f[b_1, b_2, b_3, b_4] b_3 (ff_9[b_1, b_3] + b_3 ff_{21}[b_1, b_3]), 1, 2, 1, 4\right] + \\
& \delta aa[-2 e^{-b_1} (-1 + e^{b_1}) f[b_1, b_2, b_3, b_4] (ff_9[b_1, b_3] + 2 b_3 ff_{21}[b_1, b_3]), 1, 2, 3, 4] + \\
& \delta aa[(-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] (ff_9[b_1, b_3] - b_1 ff_{17}[b_1, b_3] + 2 b_3 ff_{21}[b_1, b_3]), \\
& \quad 1, 1, 3, 4] + \\
& \delta aa\left[-\frac{1}{b_2} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_3 (ff_9[b_1, b_3] - b_1 ff_{17}[b_1, b_3] + 2 b_3 ff_{21}[b_1, b_3]),\right. \\
& \quad \left. 1, 1, 2, 4\right] + \delta aa[-(-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_1 ff_{22}[b_1, b_3], 3, 3, 3, 4] + \\
& \delta aa\left[-\frac{1}{b_2} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_1 (ff_{13}[b_1, b_3] - b_3 ff_{22}[b_1, b_3]), 2, 3, 3, 4\right] = 0
\end{aligned}$$

$$ff_{22}[_] = 0; \quad ff_{21}[_] = 0;$$

VerifyR3[a[f[b₁, b₂, b₃, b₄], 1, 4]]

a[f[b₁, b₂, b₃, b₄], 1, 4] →

$$\begin{aligned}
& ca[-e^{-b_1-b_2} f[b_1, b_2, b_3, b_4] (e^{b_1} (-1 + e^{b_2}) b_1 (ff_8[b_1, b_2] - ff_8[b_1, b_3]) - \\
& \quad 2 (-1 + e^{b_1}) b_2 ff_9[b_1, b_2]), 3, 1, 4] + ca\left[\frac{1}{b_2} e^{-b_1-b_2} f[b_1, b_2, b_3, b_4] b_1 \right. \\
& \quad \left. (e^{b_1} (-1 + e^{b_2}) b_1 (ff_8[b_1, b_2] - ff_8[b_1, b_3]) - 2 (-1 + e^{b_1}) b_2 ff_9[b_1, b_2]), 3, 2, 4\right] + \\
& ca[2 (1 - e^{-b_1}) f[b_1, b_2, b_3, b_4] b_1 ff_9[b_1, b_3], 2, 3, 4] + \\
& ca[-2 (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_1 ff_9[b_1, b_3], 1, 3, 4] + \\
& ca[-2 e^{-b_1} (-1 + e^{b_1}) f[b_1, b_2, b_3, b_4] b_3 ff_9[b_1, b_3], 2, 1, 4] + \\
& ca\left[\frac{2 (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_1 b_3 ff_9[b_1, b_3]}{b_2}, 1, 2, 4\right] + \\
& ca[e^{-b_2} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] (b_2 ff_{13}[b_1, b_2] - b_3 ff_{13}[b_1, b_3]), 4, 1, 3] + \\
& ca\left[-\frac{1}{b_2} e^{-b_2} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_1 (b_2 ff_{13}[b_1, b_2] - b_3 ff_{13}[b_1, b_3]), 4, 2, 3\right] + \\
& \delta a[-(-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_1 ff_5[b_1, b_3], 3, 4] + \\
& \delta a\left[\frac{(-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_1 b_3 ff_5[b_1, b_3]}{b_2}, 2, 4\right] + \delta aa\left[\frac{1}{b_1} e^{-b_1-b_2} f[b_1, b_2, b_3, b_4] \right. \\
& \quad \left. (e^{b_1} (-1 + e^{b_2}) b_1 (ff_8[b_1, b_2] - ff_8[b_1, b_3]) - 2 (-1 + e^{b_1}) b_2 ff_9[b_1, b_2]), 1, 3, 1, 4\right] + \\
& \delta aa[2 (-1 + e^{-b_1}) f[b_1, b_2, b_3, b_4] ff_9[b_1, b_3], 1, 2, 3, 4] + \\
& \delta aa\left[\frac{2 e^{-b_1} (-1 + e^{b_1}) f[b_1, b_2, b_3, b_4] b_3 ff_9[b_1, b_3]}{b_1}, 1, 2, 1, 4\right] + \\
& \delta aa\left[-\frac{(-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_1 ff_{13}[b_1, b_3]}{b_2}, 2, 3, 3, 4\right] + \\
& \delta aa\left[\frac{1}{b_2^2} e^{-b_2} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_1 (b_2 ff_{13}[b_1, b_2] + (-1 + e^{b_2}) b_3 ff_{13}[b_1, b_3]), 2, \right. \\
& \quad \left. 3, 2, 4\right] + \delta aa\left[-\frac{1}{b_2} e^{-b_1-b_2} f[b_1, b_2, b_3, b_4] (e^{b_1} (-1 + e^{b_2}) b_1 (ff_8[b_1, b_2] - ff_8[b_1, b_3]) + \right. \\
& \quad \left. b_2 (-2 (-1 + e^{b_1}) ff_9[b_1, b_2] + e^{b_1} (-1 + e^{b_2}) ff_{13}[b_1, b_2]) - \right. \\
& \quad \left. e^{b_1} (-1 + e^{b_2}) b_3 ff_{13}[b_1, b_3]), 1, 3, 2, 4\right] + \\
& \delta aa[(-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] (ff_9[b_1, b_3] - b_1 ff_{17}[b_1, b_3]), 1, 1, 3, 4] + \\
& \delta aa\left[-\frac{1}{b_2} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_3 (ff_9[b_1, b_3] - b_1 ff_{17}[b_1, b_3]), 1, 1, 2, 4\right] = 0
\end{aligned}$$

ff₉[__] = 0; ff₅[__] = 0;

VerifyR3[a[f[b₁, b₂, b₃, b₄], 1, 4]]

a[f[b₁, b₂, b₃, b₄], 1, 4] →

$$\begin{aligned}
& \text{ca} \left[-e^{-b_2} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_1 (ff_8[b_1, b_2] - ff_8[b_1, b_3]), 3, 1, 4 \right] + \\
& \text{ca} \left[\frac{1}{b_2} e^{-b_2} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_1^2 (ff_8[b_1, b_2] - ff_8[b_1, b_3]), 3, 2, 4 \right] + \\
& \text{ca} \left[e^{-b_2} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] (b_2 ff_{13}[b_1, b_2] - b_3 ff_{13}[b_1, b_3]), 4, 1, 3 \right] + \\
& \text{ca} \left[-\frac{1}{b_2} e^{-b_2} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_1 (b_2 ff_{13}[b_1, b_2] - b_3 ff_{13}[b_1, b_3]), 4, 2, 3 \right] + \\
& \delta_{aa} \left[e^{-b_2} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] (ff_8[b_1, b_2] - ff_8[b_1, b_3]), 1, 3, 1, 4 \right] + \\
& \delta_{aa} \left[-\frac{(-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_1 ff_{13}[b_1, b_3]}{b_2}, 2, 3, 3, 4 \right] + \\
& \delta_{aa} \left[-\frac{1}{b_2} e^{-b_2} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] \right. \\
& \quad \left. (b_1 (ff_8[b_1, b_2] - ff_8[b_1, b_3]) + b_2 ff_{13}[b_1, b_2] - b_3 ff_{13}[b_1, b_3]), 1, 3, 2, 4 \right] + \\
& \delta_{aa} \left[\frac{1}{b_2^2} e^{-b_2} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_1 (b_2 ff_{13}[b_1, b_2] + (-1 + e^{b_2}) b_3 ff_{13}[b_1, b_3]), \right. \\
& \quad \left. 2, 3, 2, 4 \right] + \delta_{aa} \left[-(-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_1 ff_{17}[b_1, b_3], 1, 1, 3, 4 \right] + \\
& \delta_{aa} \left[\frac{(-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_1 b_3 ff_{17}[b_1, b_3]}{b_2}, 1, 1, 2, 4 \right] = 0
\end{aligned}$$

ff₁₃[__] = 0; ff₁₇[__] = 0;

VerifyR3[a[f[b₁, b₂, b₃, b₄], 1, 4]]

a[f[b₁, b₂, b₃, b₄], 1, 4] →

$$\begin{aligned}
& \text{ca} \left[-e^{-b_2} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_1 (ff_8[b_1, b_2] - ff_8[b_1, b_3]), 3, 1, 4 \right] + \\
& \text{ca} \left[\frac{1}{b_2} e^{-b_2} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_1^2 (ff_8[b_1, b_2] - ff_8[b_1, b_3]), 3, 2, 4 \right] + \\
& \delta_{aa} \left[e^{-b_2} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] (ff_8[b_1, b_2] - ff_8[b_1, b_3]), 1, 3, 1, 4 \right] + \\
& \delta_{aa} \left[-\frac{1}{b_2} e^{-b_2} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_1 (ff_8[b_1, b_2] - ff_8[b_1, b_3]), 1, 3, 2, 4 \right] = 0
\end{aligned}$$

ff₈[x_, y_] := gg₁[x];

VerifyR3[a[f[b₁, b₂, b₃, b₄], 1, 4]]

a[f[b₁, b₂, b₃, b₄], 1, 4] → True

VerifyR3[a[f[b₁, b₂, b₃, b₄], 2, 4]]

$$\begin{aligned}
& a[f[b_1, b_2, b_3, b_4], 2, 4] \rightarrow c[-(-1 + e^{b_1}) f[b_1, b_2, b_3, b_4] b_2 \\
& \quad (-ff_1[b_1, b_3] + ff_1[b_2, b_3] + b_2 ff_7[b_2, b_3] - b_3 ff_{10}[b_1, b_3] + b_3 ff_{10}[b_2, b_3]), 4] + \\
& \quad ca[-(-1 + e^{b_1}) f[b_1, b_2, b_3, b_4] b_2 ff_7[b_1, b_3], 4, 1, 1] + \\
& \quad ca[-(-1 + e^{b_1}) f[b_1, b_2, b_3, b_4] b_2 ff_7[b_2, b_3], 2, 2, 4] + \\
& \quad ca[-\frac{(-1 + e^{b_1}) f[b_1, b_2, b_3, b_4] b_2^2 ff_7[b_2, b_3]}{b_1}, 4, 1, 2] + \\
& \quad ca[\frac{(-1 + e^{b_1}) f[b_1, b_2, b_3, b_4] b_2^2 ff_7[b_2, b_3]}{b_1}, 2, 1, 4] + \\
& \quad ca[-(-1 + e^{b_1}) f[b_1, b_2, b_3, b_4] b_2 (ff_{10}[b_1, b_3] - ff_{10}[b_2, b_3]), 3, 3, 4] + \\
& \quad ca[2 e^{b_1 - b_2} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_1 (ff_{12}[b_1, b_2] + b_1 ff_{19}[b_1, b_2]), 4, 2, 3] + \\
& \quad ca[-\frac{1}{b_1} e^{-b_2} f[b_1, b_2, b_3, b_4] b_2 \\
& \quad \quad (-e^{b_2} (-1 + e^{b_1}) b_3 (ff_{10}[b_1, b_3] - ff_{10}[b_2, b_3]) + 2 e^{b_1} (-1 + e^{b_2}) b_1^2 ff_{19}[b_1, b_2] + \\
& \quad \quad b_1 (2 e^{b_1} (-1 + e^{b_2}) ff_{12}[b_1, b_2] - e^{b_2} (-1 + e^{b_1}) gg_1[b_1]) + e^{b_2} (-1 + e^{b_1}) b_2 gg_1[b_2]), \\
& \quad 4, 1, 3] + ca[-\frac{1}{b_1} e^{-b_2} f[b_1, b_2, b_3, b_4] (-2 (-1 + e^{b_1}) (-1 + e^{b_2}) + 2 e^{b_1} (-1 + e^{b_2}) b_1^2 ff_{19}[\\
& \quad \quad b_1, b_2] - (-1 + e^{b_1}) b_1^2 gg_1[b_1] + b_1 ((1 + e^{b_1}) (-1 + e^{b_2}) + e^{b_2} (-1 + e^{b_1}) b_2 gg_1[b_2])), \\
& \quad 3, 2, 4] + ca[\frac{1}{b_1^2} e^{-b_2} f[b_1, b_2, b_3, b_4] b_2 (-2 (-1 + e^{b_1}) (-1 + e^{b_2}) + \\
& \quad \quad 2 e^{b_1} (-1 + e^{b_2}) b_1^2 ff_{19}[b_1, b_2] - (-1 + e^{b_1}) b_1^2 gg_1[b_1] + b_1 ((1 + e^{b_1}) (-1 + e^{b_2}) - \\
& \quad \quad e^{b_2} (-1 + e^{b_1}) b_3 (ff_{10}[b_1, b_3] - ff_{10}[b_2, b_3]) + e^{b_2} (-1 + e^{b_1}) b_2 gg_1[b_2])), \\
& \quad 3, 1, 4] + \delta a[\frac{1}{b_1} (-1 + e^{b_1}) f[b_1, b_2, b_3, b_4] b_2 (-ff_1[b_1, b_3] + ff_1[b_2, b_3] + \\
& \quad \quad b_2 ff_7[b_2, b_3] - b_3 ff_{10}[b_1, b_3] + b_3 ff_{10}[b_2, b_3]), 1, 4] + \\
& \quad \delta aa[-\frac{(-1 + e^{b_1}) f[b_1, b_2, b_3, b_4] b_2 ff_7[b_1, b_3]}{b_1}, 1, 1, 1, 4] + \\
& \quad \delta aa[\frac{(-1 + e^{b_1}) f[b_1, b_2, b_3, b_4] b_2 ff_7[b_2, b_3]}{b_1}, 1, 2, 2, 4] + \\
& \quad \delta aa[-\frac{1}{b_1} (-1 + e^{b_1}) f[b_1, b_2, b_3, b_4] b_2 (ff_{10}[b_1, b_3] - ff_{10}[b_2, b_3]), 1, 3, 3, 4] + \\
& \quad \delta aa[-\frac{1}{b_2} 2 e^{b_1 - b_2} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_1 (ff_{12}[b_1, b_2] + b_1 ff_{19}[b_1, b_2]), \\
& \quad 2, 3, 2, 4] + \delta aa[-\frac{1}{b_1^3} e^{-b_2} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_2 \\
& \quad \quad (2 - 2 e^{b_1} + (1 + e^{b_1}) b_1 + 2 e^{b_1} b_1^3 ff_{19}[b_1, b_2] + (-1 + e^{b_1}) b_1^2 gg_1[b_1]), 1, 3, 1, 4] + \\
& \quad \delta aa[\frac{1}{b_1^2} e^{-b_2} f[b_1, b_2, b_3, b_4] (-2 (-1 + e^{b_1}) (-1 + e^{b_2}) + 4 e^{b_1} (-1 + e^{b_2}) b_1^3 ff_{19}[b_1, b_2] + \\
& \quad \quad b_1^2 (2 e^{b_1} (-1 + e^{b_2}) ff_{12}[b_1, b_2] - (-1 + e^{b_1}) gg_1[b_1]) + \\
& \quad \quad b_1 ((1 + e^{b_1}) (-1 + e^{b_2}) + e^{b_2} (-1 + e^{b_1}) b_2 gg_1[b_2]), 1, 3, 2, 4] = 0
\end{aligned}$$

ff₇[__] = 0;

VerifyR3[a[f[b₁, b₂, b₃, b₄], 2, 4]]

a[f[b₁, b₂, b₃, b₄], 2, 4] →

$$\begin{aligned}
& c \left[(-1 + e^{b_1}) f[b_1, b_2, b_3, b_4] b_2 (ff_1[b_1, b_3] - ff_1[b_2, b_3] + b_3 (ff_{10}[b_1, b_3] - ff_{10}[b_2, b_3])), \right. \\
& \quad 4] + ca \left[(-1 + e^{b_1}) f[b_1, b_2, b_3, b_4] b_2 (ff_{10}[b_1, b_3] - ff_{10}[b_2, b_3]), 3, 3, 4 \right] + \\
& \quad ca \left[2 e^{b_1 - b_2} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_1 (ff_{12}[b_1, b_2] + b_1 ff_{19}[b_1, b_2]), 4, 2, 3 \right] + \\
& \quad ca \left[-\frac{1}{b_1} e^{-b_2} f[b_1, b_2, b_3, b_4] b_2 \right. \\
& \quad \quad \left. (-e^{b_2} (-1 + e^{b_1}) b_3 (ff_{10}[b_1, b_3] - ff_{10}[b_2, b_3]) + 2 e^{b_1} (-1 + e^{b_2}) b_1^2 ff_{19}[b_1, b_2] + \right. \\
& \quad \quad \left. b_1 (2 e^{b_1} (-1 + e^{b_2}) ff_{12}[b_1, b_2] - e^{b_2} (-1 + e^{b_1}) gg_1[b_1]) + e^{b_2} (-1 + e^{b_1}) b_2 gg_1[b_2]), \right. \\
& \quad 4, 1, 3] + ca \left[-\frac{1}{b_1} e^{-b_2} f[b_1, b_2, b_3, b_4] (-2 (-1 + e^{b_1}) (-1 + e^{b_2}) + 2 e^{b_1} (-1 + e^{b_2}) b_1^3 ff_{19}[\right. \\
& \quad \quad \left. b_1, b_2] - (-1 + e^{b_1}) b_1^2 gg_1[b_1] + b_1 ((1 + e^{b_1}) (-1 + e^{b_2}) + e^{b_2} (-1 + e^{b_1}) b_2 gg_1[b_2])), \right. \\
& \quad 3, 2, 4] + ca \left[\frac{1}{b_1^2} e^{-b_2} f[b_1, b_2, b_3, b_4] b_2 (-2 (-1 + e^{b_1}) (-1 + e^{b_2}) + \right. \\
& \quad \quad \left. 2 e^{b_1} (-1 + e^{b_2}) b_1^3 ff_{19}[b_1, b_2] - (-1 + e^{b_1}) b_1^2 gg_1[b_1] + b_1 ((1 + e^{b_1}) (-1 + e^{b_2}) - \right. \\
& \quad \quad \left. e^{b_2} (-1 + e^{b_1}) b_3 (ff_{10}[b_1, b_3] - ff_{10}[b_2, b_3]) + e^{b_2} (-1 + e^{b_1}) b_2 gg_1[b_2])), \right. \\
& \quad 3, 1, 4] + \delta a \left[-\frac{1}{b_1} (-1 + e^{b_1}) f[b_1, b_2, b_3, b_4] b_2 \right. \\
& \quad \quad \left. (ff_1[b_1, b_3] - ff_1[b_2, b_3] + b_3 (ff_{10}[b_1, b_3] - ff_{10}[b_2, b_3])), 1, 4 \right] + \\
& \quad \delta aa \left[-\frac{1}{b_1} (-1 + e^{b_1}) f[b_1, b_2, b_3, b_4] b_2 (ff_{10}[b_1, b_3] - ff_{10}[b_2, b_3]), 1, 3, 3, 4 \right] + \\
& \quad \delta aa \left[-\frac{1}{b_2} 2 e^{b_1 - b_2} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_1 (ff_{12}[b_1, b_2] + b_1 ff_{19}[b_1, b_2]), \right. \\
& \quad 2, 3, 2, 4] + \delta aa \left[-\frac{1}{b_1^3} e^{-b_2} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_2 \right. \\
& \quad \quad \left. (2 - 2 e^{b_1} + (1 + e^{b_1}) b_1 + 2 e^{b_1} b_1^3 ff_{19}[b_1, b_2] + (-1 + e^{b_1}) b_1^2 gg_1[b_1]), 1, 3, 1, 4 \right] + \\
& \quad \delta aa \left[\frac{1}{b_1^2} e^{-b_2} f[b_1, b_2, b_3, b_4] (-2 (-1 + e^{b_1}) (-1 + e^{b_2}) + 4 e^{b_1} (-1 + e^{b_2}) b_1^3 ff_{19}[b_1, b_2] + \right. \\
& \quad \quad \left. b_1^2 (2 e^{b_1} (-1 + e^{b_2}) ff_{12}[b_1, b_2] - (-1 + e^{b_1}) gg_1[b_1]) + \right. \\
& \quad \quad \left. b_1 ((1 + e^{b_1}) (-1 + e^{b_2}) + e^{b_2} (-1 + e^{b_1}) b_2 gg_1[b_2]), 1, 3, 2, 4 \right] = 0
\end{aligned}$$

ff₁₀[x_, y_] := gg₂[y];

VerifyR3[a[f[b₁, b₂, b₃, b₄], 2, 4]]

$$\begin{aligned}
& a[f[b_1, b_2, b_3, b_4], 2, 4] \rightarrow c[(-1 + e^{b_1}) f[b_1, b_2, b_3, b_4] b_2 (ff_1[b_1, b_3] - ff_1[b_2, b_3]), 4] + \\
& ca[2 e^{b_1-b_2} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_1 (ff_{12}[b_1, b_2] + b_1 ff_{19}[b_1, b_2]), 4, 2, 3] + \\
& ca[-\frac{1}{b_1} e^{-b_2} f[b_1, b_2, b_3, b_4] b_2 (2 e^{b_1} (-1 + e^{b_2}) b_1^2 ff_{19}[b_1, b_2] + b_1 (2 e^{b_1} (-1 + e^{b_2}) \\
& \quad ff_{12}[b_1, b_2] - e^{b_2} (-1 + e^{b_1}) gg_1[b_1]) + e^{b_2} (-1 + e^{b_1}) b_2 gg_1[b_2]), 4, 1, 3] + \\
& ca[-\frac{1}{b_1} e^{-b_2} f[b_1, b_2, b_3, b_4] (-2 (-1 + e^{b_1}) (-1 + e^{b_2}) + 2 e^{b_1} (-1 + e^{b_2}) b_1^2 ff_{19}[b_1, b_2] - \\
& \quad (-1 + e^{b_1}) b_1^2 gg_1[b_1] + b_1 ((1 + e^{b_1}) (-1 + e^{b_2}) + e^{b_2} (-1 + e^{b_1}) b_2 gg_1[b_2])), 3, 2, 4] + \\
& ca[\frac{1}{b_1^2} e^{-b_2} f[b_1, b_2, b_3, b_4] b_2 (-2 (-1 + e^{b_1}) (-1 + e^{b_2}) + 2 e^{b_1} (-1 + e^{b_2}) b_1^2 ff_{19}[b_1, b_2] - \\
& \quad (-1 + e^{b_1}) b_1^2 gg_1[b_1] + b_1 ((1 + e^{b_1}) (-1 + e^{b_2}) + e^{b_2} (-1 + e^{b_1}) b_2 gg_1[b_2])), 3, 1, 4] + \\
& \delta a[-\frac{1}{b_1} (-1 + e^{b_1}) f[b_1, b_2, b_3, b_4] b_2 (ff_1[b_1, b_3] - ff_1[b_2, b_3]), 1, 4] + \\
& \delta aa[-\frac{1}{b_2} 2 e^{b_1-b_2} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_1 (ff_{12}[b_1, b_2] + b_1 ff_{19}[b_1, b_2]), \\
& \quad 2, 3, 2, 4] + \delta aa[-\frac{1}{b_1^3} e^{-b_2} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_2 \\
& \quad (2 - 2 e^{b_1} + (1 + e^{b_1}) b_1 + 2 e^{b_1} b_1^3 ff_{19}[b_1, b_2] + (-1 + e^{b_1}) b_1^2 gg_1[b_1]), 1, 3, 1, 4] + \\
& \delta aa[\frac{1}{b_1^2} e^{-b_2} f[b_1, b_2, b_3, b_4] (-2 (-1 + e^{b_1}) (-1 + e^{b_2}) + 4 e^{b_1} (-1 + e^{b_2}) b_1^3 ff_{19}[b_1, b_2] + \\
& \quad b_1^2 (2 e^{b_1} (-1 + e^{b_2}) ff_{12}[b_1, b_2] - (-1 + e^{b_1}) gg_1[b_1]) + \\
& \quad b_1 ((1 + e^{b_1}) (-1 + e^{b_2}) + e^{b_2} (-1 + e^{b_1}) b_2 gg_1[b_2]), 1, 3, 2, 4] = 0
\end{aligned}$$

ff₁[x_, y_] := gg₃[y];

VerifyR3[a[f[b₁, b₂, b₃, b₄], 2, 4]]

$$\begin{aligned}
& a[f[b_1, b_2, b_3, b_4], 2, 4] \rightarrow \\
& ca[2 e^{b_1-b_2} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_1 (ff_{12}[b_1, b_2] + b_1 ff_{19}[b_1, b_2]), 4, 2, 3] + \\
& ca\left[-\frac{1}{b_1} e^{-b_2} f[b_1, b_2, b_3, b_4] b_2 (2 e^{b_1} (-1 + e^{b_2}) b_1^2 ff_{19}[b_1, b_2] + b_1 (2 e^{b_1} (-1 + e^{b_2}) \right. \\
& \quad \left. ff_{12}[b_1, b_2] - e^{b_2} (-1 + e^{b_1}) gg_1[b_1]) + e^{b_2} (-1 + e^{b_1}) b_2 gg_1[b_2]), 4, 1, 3] + \\
& ca\left[-\frac{1}{b_1} e^{-b_2} f[b_1, b_2, b_3, b_4] (-2 (-1 + e^{b_1}) (-1 + e^{b_2}) + 2 e^{b_1} (-1 + e^{b_2}) b_1^3 ff_{19}[b_1, b_2] - \right. \\
& \quad \left. (-1 + e^{b_1}) b_1^2 gg_1[b_1] + b_1 ((1 + e^{b_1}) (-1 + e^{b_2}) + e^{b_2} (-1 + e^{b_1}) b_2 gg_1[b_2]))], 3, 2, 4] + \\
& ca\left[\frac{1}{b_1^2} e^{-b_2} f[b_1, b_2, b_3, b_4] b_2 (-2 (-1 + e^{b_1}) (-1 + e^{b_2}) + 2 e^{b_1} (-1 + e^{b_2}) b_1^3 ff_{19}[b_1, b_2] - \right. \\
& \quad \left. (-1 + e^{b_1}) b_1^2 gg_1[b_1] + b_1 ((1 + e^{b_1}) (-1 + e^{b_2}) + e^{b_2} (-1 + e^{b_1}) b_2 gg_1[b_2]))], 3, 1, 4] + \\
& \delta aa\left[-\frac{1}{b_2} 2 e^{b_1-b_2} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_1 (ff_{12}[b_1, b_2] + b_1 ff_{19}[b_1, b_2]), \right. \\
& \quad \left. 2, 3, 2, 4] + \delta aa\left[-\frac{1}{b_1^3} e^{-b_2} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_2 \right. \right. \\
& \quad \left. \left. (2 - 2 e^{b_1} + (1 + e^{b_1}) b_1 + 2 e^{b_1} b_1^3 ff_{19}[b_1, b_2] + (-1 + e^{b_1}) b_1^2 gg_1[b_1]), 1, 3, 1, 4] + \right. \right. \\
& \delta aa\left[\frac{1}{b_1^2} e^{-b_2} f[b_1, b_2, b_3, b_4] (-2 (-1 + e^{b_1}) (-1 + e^{b_2}) + 4 e^{b_1} (-1 + e^{b_2}) b_1^3 ff_{19}[b_1, b_2] + \right. \\
& \quad \left. b_1^2 (2 e^{b_1} (-1 + e^{b_2}) ff_{12}[b_1, b_2] - (-1 + e^{b_1}) gg_1[b_1]) + \right. \\
& \quad \left. b_1 ((1 + e^{b_1}) (-1 + e^{b_2}) + e^{b_2} (-1 + e^{b_1}) b_2 gg_1[b_2]), 1, 3, 2, 4] = 0
\end{aligned}$$

ff₁₂[b_j, b_k] := -b_j ff₁₉[b_j, b_k];

VerifyR3[a[f[b₁, b₂, b₃, b₄], 2, 4]]

$$\begin{aligned}
& a[f[b_1, b_2, b_3, b_4], 2, 4] \rightarrow \\
& ca\left[\frac{(-1 + e^{b_1}) f[b_1, b_2, b_3, b_4] b_2 (b_1 gg_1[b_1] - b_2 gg_1[b_2])}{b_1}, 4, 1, 3] + \\
& ca\left[-\frac{1}{b_1} e^{-b_2} f[b_1, b_2, b_3, b_4] (-2 (-1 + e^{b_1}) (-1 + e^{b_2}) + 2 e^{b_1} (-1 + e^{b_2}) b_1^3 ff_{19}[b_1, b_2] - \right. \\
& \quad \left. (-1 + e^{b_1}) b_1^2 gg_1[b_1] + b_1 ((1 + e^{b_1}) (-1 + e^{b_2}) + e^{b_2} (-1 + e^{b_1}) b_2 gg_1[b_2]))], 3, 2, 4] + \\
& ca\left[\frac{1}{b_1^2} e^{-b_2} f[b_1, b_2, b_3, b_4] b_2 (-2 (-1 + e^{b_1}) (-1 + e^{b_2}) + 2 e^{b_1} (-1 + e^{b_2}) b_1^3 ff_{19}[b_1, b_2] - \right. \\
& \quad \left. (-1 + e^{b_1}) b_1^2 gg_1[b_1] + b_1 ((1 + e^{b_1}) (-1 + e^{b_2}) + e^{b_2} (-1 + e^{b_1}) b_2 gg_1[b_2]))], 3, 1, 4] + \\
& \delta aa\left[-\frac{1}{b_1^3} e^{-b_2} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_2 (2 - 2 e^{b_1} + (1 + e^{b_1}) b_1 + 2 e^{b_1} b_1^3 ff_{19}[b_1, b_2] + \right. \\
& \quad \left. (-1 + e^{b_1}) b_1^2 gg_1[b_1]), 1, 3, 1, 4] + \delta aa\left[\frac{1}{b_1^2} e^{-b_2} f[b_1, b_2, b_3, b_4] \right. \right. \\
& \quad \left. \left. (-2 (-1 + e^{b_1}) (-1 + e^{b_2}) + 2 e^{b_1} (-1 + e^{b_2}) b_1^3 ff_{19}[b_1, b_2] - (-1 + e^{b_1}) b_1^2 gg_1[b_1] + \right. \right. \\
& \quad \left. \left. b_1 ((1 + e^{b_1}) (-1 + e^{b_2}) + e^{b_2} (-1 + e^{b_1}) b_2 gg_1[b_2]), 1, 3, 2, 4] = 0
\end{aligned}$$

gg₁[x_] := 0;

VerifyR3[a[f[b₁, b₂, b₃, b₄], 2, 4]]

a[f[b₁, b₂, b₃, b₄], 2, 4] →

$$\begin{aligned} & ca \left[-\frac{1}{b_1} e^{-b_2} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] (2 - 2 e^{b_1} + (1 + e^{b_1}) b_1 + 2 e^{b_1} b_1^3 ff_{19}[b_1, b_2]), \right. \\ & \quad \left. 3, 2, 4 \right] + ca \left[\frac{1}{b_1^2} e^{-b_2} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_2 \right. \\ & \quad \left. (2 - 2 e^{b_1} + (1 + e^{b_1}) b_1 + 2 e^{b_1} b_1^3 ff_{19}[b_1, b_2]), 3, 1, 4 \right] + \delta aa \left[\frac{1}{b_1^2} \right. \\ & \quad \left. e^{-b_2} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] (2 - 2 e^{b_1} + (1 + e^{b_1}) b_1 + 2 e^{b_1} b_1^3 ff_{19}[b_1, b_2]), 1, 3, 2, 4 \right] + \\ & \delta aa \left[-\frac{1}{b_1^3} e^{-b_2} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_2 (2 - 2 e^{b_1} + (1 + e^{b_1}) b_1 + 2 e^{b_1} b_1^3 ff_{19}[b_1, b_2]), \right. \\ & \quad \left. 1, 3, 1, 4 \right] = 0 \end{aligned}$$

$$ff_{19}[b_{j-}, b_{k-}] := -\frac{e^{-b_j} (2 - 2 e^{b_j} + b_j + e^{b_j} b_j)}{2 b_j^3};$$

VerifyR3[a[f[b₁, b₂, b₃, b₄], 2, 4]]

a[f[b₁, b₂, b₃, b₄], 2, 4] → True

VerifyR3[a[f[b₁, b₂, b₃, b₄], 3, 4]]

a[f[b₁, b₂, b₃, b₄], 3, 4] →

$$\begin{aligned} & c \left[f[b_1, b_2, b_3, b_4] b_3 (-e^{b_1} (-1 + e^{b_2}) ff_2[b_1, b_2] + e^{b_1} (-1 + e^{b_2}) ff_2[b_1, b_3] + \right. \\ & \quad e^{b_2} ff_2[b_2, b_3] - e^{b_1+b_2} ff_2[b_2, b_3] + e^{b_2} b_2 ff_{11}[b_2, b_3] - e^{b_1+b_2} b_2 ff_{11}[b_2, b_3] - \\ & \quad e^{b_1} b_3 ff_{14}[b_1, b_3] + e^{b_1+b_2} b_3 ff_{14}[b_1, b_3] + e^{b_2} b_3 ff_{14}[b_2, b_3] - \\ & \quad e^{b_1+b_2} b_3 ff_{14}[b_2, b_3] - e^{b_1} b_1 b_2 ff_{20}[b_1, b_2] + e^{b_1+b_2} b_1 b_2 ff_{20}[b_1, b_2] + \\ & \quad \left. b_2 gg_2[b_2] - e^{b_1} b_2 gg_2[b_2] + gg_3[b_2] - e^{b_1} gg_3[b_2]), 4 \right] + \\ & ca \left[e^{b_1} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_3 (-ff_{11}[b_1, b_2] + ff_{11}[b_1, b_3]), 4, 1, 1 \right] + \\ & ca \left[f[b_1, b_2, b_3, b_4] b_3 (e^{b_1} (-1 + e^{b_2}) ff_{14}[b_1, b_3] - e^{b_2} (-1 + e^{b_1}) ff_{14}[b_2, b_3]), 3, 3, 4 \right] + \\ & ca \left[-e^{b_1} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_3 (ff_{14}[b_1, b_2] + b_1 ff_{20}[b_1, b_2]), 4, 2, 2 \right] + \\ & ca \left[-\frac{1}{b_2} e^{b_1} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_3^2 (ff_{14}[b_1, b_3] + b_1 ff_{20}[b_1, b_3]), 3, 2, 4 \right] + \\ & ca \left[\frac{1}{b_2} e^{b_1} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_3^2 (ff_{14}[b_1, b_3] + b_1 ff_{20}[b_1, b_3]), 4, 2, 3 \right] + \\ & ca \left[-\frac{1}{b_1} f[b_1, b_2, b_3, b_4] b_3^2 (e^{b_2} (-1 + e^{b_1}) ff_{14}[b_2, b_3] + e^{b_1} (-1 + e^{b_2}) b_1 ff_{20}[b_1, b_3]), \right. \\ & \quad \left. 4, 1, 3 \right] + ca \left[\frac{1}{b_1} \right. \\ & \quad \left. f[b_1, b_2, b_3, b_4] b_3^2 (e^{b_2} (-1 + e^{b_1}) ff_{14}[b_2, b_3] + e^{b_1} (-1 + e^{b_2}) b_1 ff_{20}[b_1, b_3]), 3, 1, 4 \right] + \\ & ca \left[\frac{1}{2 b_1 b_2} f[b_1, b_2, b_3, b_4] b_3 (-2 (-1 + e^{b_1}) (-1 + e^{b_2}) + (-1 + e^{b_1}) (1 + e^{b_2}) b_2 - \right. \\ & \quad \left. 2 b_2^2 (e^{b_2} (-1 + e^{b_1}) ff_{11}[b_2, b_3] - e^{b_1} (-1 + e^{b_2}) b_1 ff_{20}[b_1, b_2] + (-1 + e^{b_1}) gg_2[b_2]), \right. \\ & \quad \left. 4, 1, 2 \right] + ca \left[-\frac{1}{2 b_2^2} f[b_1, b_2, b_3, b_4] b_3 (2 (-1 + e^{b_1}) (-1 + e^{b_2}) - (-1 + e^{b_1}) (1 + e^{b_2}) b_2 + \right. \\ & \quad \left. 2 b_2^2 (e^{b_2} (-1 + e^{b_1}) ff_{11}[b_2, b_3] - e^{b_1} (-1 + e^{b_2}) b_1 ff_{20}[b_1, b_2] + (-1 + e^{b_1}) gg_2[b_2]), \right. \end{aligned}$$

$$\begin{aligned}
 & 2, 2, 4] + ca \left[\frac{1}{2 b_1 b_2} f[b_1, b_2, b_3, b_4] b_3 \left(2 (-1 + e^{b_1}) (-1 + e^{b_2}) - (-1 + e^{b_1}) (1 + e^{b_2}) b_2 + \right. \right. \\
 & \quad \left. \left. 2 b_2^2 (e^{b_2} (-1 + e^{b_1}) ff_{11}[b_2, b_3] - e^{b_1} (-1 + e^{b_2}) b_1 ff_{20}[b_1, b_2] + (-1 + e^{b_1}) gg_2[b_2]) \right) \right], \\
 & 2, 1, 4] + \delta a \left[-\frac{1}{b_2} e^{b_1} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_3 (-ff_2[b_1, b_2] + ff_2[b_1, b_3] - \right. \\
 & \quad \left. b_1 ff_4[b_1, b_2] + b_1 ff_4[b_1, b_3] + b_3 ff_{14}[b_1, b_3] + b_1 b_3 ff_{20}[b_1, b_3]) \right], 2, 4] + \\
 & \delta a \left[\frac{1}{b_1} f[b_1, b_2, b_3, b_4] b_3 (e^{b_2} (-1 + e^{b_1}) ff_2[b_2, b_3] - e^{b_1} (-1 + e^{b_2}) b_1 \right. \\
 & \quad (ff_4[b_1, b_2] - ff_4[b_1, b_3] + b_2 ff_{20}[b_1, b_2] - b_3 ff_{20}[b_1, b_3]) + \\
 & \quad \left. (-1 + e^{b_1}) (e^{b_2} b_3 ff_{14}[b_2, b_3] + b_2 (e^{b_2} ff_{11}[b_2, b_3] + gg_2[b_2]) + gg_3[b_2]) \right), \\
 & 1, 4] + \delta aa \left[\frac{1}{b_2} e^{b_1} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_3 \right. \\
 & \quad \left. (ff_{11}[b_1, b_2] - ff_{11}[b_1, b_3] + b_1 (ff_{16}[b_1, b_2] - ff_{16}[b_1, b_3])) \right], 1, 1, 2, 4] + \\
 & \delta aa \left[e^{b_1} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_3 (-ff_{16}[b_1, b_2] + ff_{16}[b_1, b_3]) \right], 1, 1, 1, 4] + \\
 & \delta aa \left[\frac{1}{b_2} e^{b_1} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_3 (ff_{14}[b_1, b_2] + b_1 ff_{20}[b_1, b_2]) \right], 2, 2, 2, 4] + \\
 & \delta aa \left[-\frac{1}{b_2} e^{b_1} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_3 (ff_{14}[b_1, b_3] + b_1 ff_{20}[b_1, b_3]) \right], 2, 3, 3, 4] + \\
 & \delta aa \left[\frac{1}{b_1} f[b_1, b_2, b_3, b_4] b_3 (e^{b_2} (-1 + e^{b_1}) ff_{14}[b_2, b_3] + e^{b_1} (-1 + e^{b_2}) b_1 ff_{20}[b_1, b_3]) \right], \\
 & 1, 3, 3, 4] + \delta aa \left[\frac{1}{2 b_1 b_2^2} f[b_1, b_2, b_3, b_4] b_3 \right. \\
 & \quad \left. \left(2 (-1 + e^{b_1}) (-1 + e^{b_2}) - (-1 + e^{b_1}) (1 + e^{b_2}) b_2 + 2 b_2^2 (e^{b_2} (-1 + e^{b_1}) ff_{11}[b_2, b_3] - \right. \right. \\
 & \quad \left. \left. e^{b_1} (-1 + e^{b_2}) b_1 ff_{20}[b_1, b_2] + (-1 + e^{b_1}) gg_2[b_2]) \right) \right], 1, 2, 2, 4] == 0
 \end{aligned}$$

ff₁₆[b_j, b_k] := gg₄[b_j]; ff₁₁[b_j, b_k] := gg₅[b_j];

VerifyR3[a[f[b₁, b₂, b₃, b₄], 3, 4]]

a[f[b₁, b₂, b₃, b₄], 3, 4] →

$$\begin{aligned}
& c[f[b_1, b_2, b_3, b_4] b_3 (-e^{b_1} (-1 + e^{b_2}) ff_2[b_1, b_2] + e^{b_1} (-1 + e^{b_2}) ff_2[b_1, b_3] + \\
& \quad e^{b_2} ff_2[b_2, b_3] - e^{b_1+b_2} ff_2[b_2, b_3] - e^{b_1} b_3 ff_{14}[b_1, b_3] + \\
& \quad e^{b_1+b_2} b_3 ff_{14}[b_1, b_3] + e^{b_2} b_3 ff_{14}[b_2, b_3] - e^{b_1+b_2} b_3 ff_{14}[b_2, b_3] - \\
& \quad e^{b_1} b_1 b_2 ff_{20}[b_1, b_2] + e^{b_1+b_2} b_1 b_2 ff_{20}[b_1, b_2] + b_2 gg_2[b_2] - e^{b_1} b_2 gg_2[b_2] + \\
& \quad gg_3[b_2] - e^{b_1} gg_3[b_2] + e^{b_2} b_2 gg_5[b_2] - e^{b_1+b_2} b_2 gg_5[b_2]), 4] + \\
& ca[f[b_1, b_2, b_3, b_4] b_3 (e^{b_1} (-1 + e^{b_2}) ff_{14}[b_1, b_3] - e^{b_2} (-1 + e^{b_1}) ff_{14}[b_2, b_3]), 3, 3, 4] + \\
& ca[-e^{b_1} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_3 (ff_{14}[b_1, b_2] + b_1 ff_{20}[b_1, b_2]), 4, 2, 2] + \\
& ca[-\frac{1}{b_2} e^{b_1} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_3^2 (ff_{14}[b_1, b_3] + b_1 ff_{20}[b_1, b_3]), 3, 2, 4] + \\
& ca[\frac{1}{b_2} e^{b_1} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_3^2 (ff_{14}[b_1, b_3] + b_1 ff_{20}[b_1, b_3]), 4, 2, 3] + \\
& ca[-\frac{1}{b_1} f[b_1, b_2, b_3, b_4] b_3^2 (e^{b_2} (-1 + e^{b_1}) ff_{14}[b_2, b_3] + e^{b_1} (-1 + e^{b_2}) b_1 ff_{20}[b_1, b_3]), \\
& \quad 4, 1, 3] + ca[\frac{1}{b_1} \\
& \quad f[b_1, b_2, b_3, b_4] b_3^2 (e^{b_2} (-1 + e^{b_1}) ff_{14}[b_2, b_3] + e^{b_1} (-1 + e^{b_2}) b_1 ff_{20}[b_1, b_3]), 3, 1, 4] + \\
& ca[\frac{1}{2 b_2^2} f[b_1, b_2, b_3, b_4] b_3 (-2 (-1 + e^{b_1}) (-1 + e^{b_2}) + (-1 + e^{b_1}) (1 + e^{b_2}) b_2 + \\
& \quad 2 b_2^2 (e^{b_1} (-1 + e^{b_2}) b_1 ff_{20}[b_1, b_2] - (-1 + e^{b_1}) (gg_2[b_2] + e^{b_2} gg_5[b_2])), 2, 2, 4] + \\
& ca[-\frac{1}{2 b_1 b_2} f[b_1, b_2, b_3, b_4] b_3 (-2 (-1 + e^{b_1}) (-1 + e^{b_2}) + (-1 + e^{b_1}) (1 + e^{b_2}) b_2 + \\
& \quad 2 b_2^2 (e^{b_1} (-1 + e^{b_2}) b_1 ff_{20}[b_1, b_2] - (-1 + e^{b_1}) (gg_2[b_2] + e^{b_2} gg_5[b_2])), 2, 1, 4] + \\
& ca[\frac{1}{2 b_1 b_2} f[b_1, b_2, b_3, b_4] b_3 (-2 (-1 + e^{b_1}) (-1 + e^{b_2}) + (-1 + e^{b_1}) (1 + e^{b_2}) b_2 + \\
& \quad 2 b_2^2 (e^{b_1} (-1 + e^{b_2}) b_1 ff_{20}[b_1, b_2] - (-1 + e^{b_1}) (gg_2[b_2] + e^{b_2} gg_5[b_2])), 4, 1, 2] + \\
& \delta a[-\frac{1}{b_2} e^{b_1} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_3 (-ff_2[b_1, b_2] + ff_2[b_1, b_3] - \\
& \quad b_1 ff_4[b_1, b_2] + b_1 ff_4[b_1, b_3] + b_3 ff_{14}[b_1, b_3] + b_1 b_3 ff_{20}[b_1, b_3]), 2, 4] + \\
& \delta a[\frac{1}{b_1} f[b_1, b_2, b_3, b_4] b_3 (e^{b_2} (-1 + e^{b_1}) ff_2[b_2, b_3] - e^{b_1} (-1 + e^{b_2}) b_1 \\
& \quad (ff_4[b_1, b_2] - ff_4[b_1, b_3] + b_2 ff_{20}[b_1, b_2] - b_3 ff_{20}[b_1, b_3]) + \\
& \quad (-1 + e^{b_1}) (e^{b_2} b_3 ff_{14}[b_2, b_3] + gg_3[b_2] + b_2 (gg_2[b_2] + e^{b_2} gg_5[b_2])), 1, 4] + \\
& \delta aa[\frac{1}{b_2} e^{b_1} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_3 (ff_{14}[b_1, b_2] + b_1 ff_{20}[b_1, b_2]), 2, 2, 2, 4] + \\
& \delta aa[-\frac{1}{b_2} e^{b_1} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_3 (ff_{14}[b_1, b_3] + b_1 ff_{20}[b_1, b_3]), 2, 3, 3, 4] + \\
& \delta aa[\frac{1}{b_1} f[b_1, b_2, b_3, b_4] b_3 (e^{b_2} (-1 + e^{b_1}) ff_{14}[b_2, b_3] + e^{b_1} (-1 + e^{b_2}) b_1 ff_{20}[b_1, b_3]), \\
& \quad 1, 3, 3, 4] + \\
& \delta aa[-\frac{1}{2 b_1 b_2^2} f[b_1, b_2, b_3, b_4] b_3 (-2 (-1 + e^{b_1}) (-1 + e^{b_2}) + (-1 + e^{b_1}) (1 + e^{b_2}) b_2 + 2 b_2^2 \\
& \quad (e^{b_1} (-1 + e^{b_2}) b_1 ff_{20}[b_1, b_2] - (-1 + e^{b_1}) (gg_2[b_2] + e^{b_2} gg_5[b_2])), 1, 2, 2, 4] = 0
\end{aligned}$$

ff₁₄[b_{j_}, b_{k_}] := -b_j ff₂₀[b_j, b_k];

VerifyR3[a[f[b₁, b₂, b₃, b₄], 3, 4]]

a[f[b₁, b₂, b₃, b₄], 3, 4] →

$$\begin{aligned}
& c[f[b_1, b_2, b_3, b_4] b_3 (-e^{b_1} ff_2[b_1, b_3] + e^{b_1+b_2} ff_2[b_1, b_3] + e^{b_2} ff_2[b_2, b_3] - e^{b_1+b_2} \\
& \quad ff_2[b_2, b_3] - e^{b_1} (-1 + e^{b_2}) (ff_2[b_1, b_2] - b_1 b_2 ff_{20}[b_1, b_2]) + e^{b_1} b_1 b_3 ff_{20}[b_1, b_3] - \\
& \quad e^{b_1+b_2} b_1 b_3 ff_{20}[b_1, b_3] - e^{b_2} b_2 b_3 ff_{20}[b_2, b_3] + e^{b_1+b_2} b_2 b_3 ff_{20}[b_2, b_3] - \\
& \quad (-1 + e^{b_1}) b_2 gg_2[b_2] - (-1 + e^{b_1}) gg_3[b_2] + e^{b_2} b_2 gg_5[b_2] - e^{b_1+b_2} b_2 gg_5[b_2]), 4] + \\
& ca[-f[b_1, b_2, b_3, b_4] b_3 (e^{b_1} (-1 + e^{b_2}) b_1 ff_{20}[b_1, b_3] - e^{b_2} (-1 + e^{b_1}) b_2 ff_{20}[b_2, b_3]), \\
& \quad 3, 3, 4] + \\
& ca[-\frac{1}{b_1} f[b_1, b_2, b_3, b_4] b_3^2 (e^{b_1} (-1 + e^{b_2}) b_1 ff_{20}[b_1, b_3] - e^{b_2} (-1 + e^{b_1}) b_2 ff_{20}[b_2, b_3]), \\
& \quad 4, 1, 3] + ca[\frac{1}{b_1} f[b_1, b_2, b_3, b_4] b_3^2 \\
& \quad (e^{b_1} (-1 + e^{b_2}) b_1 ff_{20}[b_1, b_3] - e^{b_2} (-1 + e^{b_1}) b_2 ff_{20}[b_2, b_3]), 3, 1, 4] + \\
& ca[\frac{1}{2 b_2^2} f[b_1, b_2, b_3, b_4] b_3 (-2 (-1 + e^{b_1}) (-1 + e^{b_2}) + (-1 + e^{b_1}) (1 + e^{b_2}) b_2 + \\
& \quad 2 b_2^2 (e^{b_1} (-1 + e^{b_2}) b_1 ff_{20}[b_1, b_2] - (-1 + e^{b_1}) (gg_2[b_2] + e^{b_2} gg_5[b_2])), 2, 2, 4] + \\
& ca[-\frac{1}{2 b_1 b_2} f[b_1, b_2, b_3, b_4] b_3 (-2 (-1 + e^{b_1}) (-1 + e^{b_2}) + (-1 + e^{b_1}) (1 + e^{b_2}) b_2 + \\
& \quad 2 b_2^2 (e^{b_1} (-1 + e^{b_2}) b_1 ff_{20}[b_1, b_2] - (-1 + e^{b_1}) (gg_2[b_2] + e^{b_2} gg_5[b_2])), 2, 1, 4] + \\
& ca[\frac{1}{2 b_1 b_2} f[b_1, b_2, b_3, b_4] b_3 (-2 (-1 + e^{b_1}) (-1 + e^{b_2}) + (-1 + e^{b_1}) (1 + e^{b_2}) b_2 + \\
& \quad 2 b_2^2 (e^{b_1} (-1 + e^{b_2}) b_1 ff_{20}[b_1, b_2] - (-1 + e^{b_1}) (gg_2[b_2] + e^{b_2} gg_5[b_2])), \\
& \quad 4, 1, 2] + \delta a[\frac{1}{b_2} e^{b_1} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_3 \\
& \quad (ff_2[b_1, b_2] - ff_2[b_1, b_3] + b_1 (ff_4[b_1, b_2] - ff_4[b_1, b_3])), 2, 4] + \\
& \delta a[\frac{1}{b_1} f[b_1, b_2, b_3, b_4] b_3 (e^{b_2} (-1 + e^{b_1}) ff_2[b_2, b_3] - \\
& \quad e^{b_1} (-1 + e^{b_2}) b_1 (ff_4[b_1, b_2] - ff_4[b_1, b_3] + b_2 ff_{20}[b_1, b_2] - b_3 ff_{20}[b_1, b_3]) + \\
& \quad (-1 + e^{b_1}) (gg_3[b_2] + b_2 (-e^{b_2} b_3 ff_{20}[b_2, b_3] + gg_2[b_2] + e^{b_2} gg_5[b_2])), 1, 4] + \\
& \delta aa[\frac{1}{b_1} f[b_1, b_2, b_3, b_4] b_3 (e^{b_1} (-1 + e^{b_2}) b_1 ff_{20}[b_1, b_3] - e^{b_2} (-1 + e^{b_1}) b_2 ff_{20}[b_2, b_3]), \\
& \quad 1, 3, 3, 4] + \\
& \delta aa[-\frac{1}{2 b_1 b_2^2} f[b_1, b_2, b_3, b_4] b_3 (-2 (-1 + e^{b_1}) (-1 + e^{b_2}) + (-1 + e^{b_1}) (1 + e^{b_2}) b_2 + 2 b_2^2 \\
& \quad (e^{b_1} (-1 + e^{b_2}) b_1 ff_{20}[b_1, b_2] - (-1 + e^{b_1}) (gg_2[b_2] + e^{b_2} gg_5[b_2])), 1, 2, 2, 4] = 0
\end{aligned}$$

Simplify[(e^{b₁} (-1 + e^{b₂}) b₁ ff₂₀[b₁, b₃] - e^{b₂} (-1 + e^{b₁}) b₂ ff₂₀[b₂, b₃]) = 0]

$$e^{b_1} (-1 + e^{b_2}) b_1 ff_{20}[b_1, b_3] = e^{b_2} (-1 + e^{b_1}) b_2 ff_{20}[b_2, b_3]$$

$$\left(\frac{e^{b_1} b_1 ff_{20}[b_1, b_3]}{(-1 + e^{b_1})} = \frac{e^{b_2} b_2 ff_{20}[b_2, b_3]}{(-1 + e^{b_2})} \right) /. ff_{20}[x_, y_] \Rightarrow \frac{-1 + e^x}{e^x x} gg_6[y]$$

True

$$\mathbf{ff}_{20}[\mathbf{x}_-, \mathbf{y}_-] := \frac{-1 + e^{\mathbf{x}}}{e^{\mathbf{x}} \mathbf{x}} \mathbf{gg}_6[\mathbf{y}];$$

VerifyR3[a[f[b₁, b₂, b₃, b₄], 3, 4]]

a[f[b₁, b₂, b₃, b₄], 3, 4] →

$$\begin{aligned} & c \left[f[b_1, b_2, b_3, b_4] b_3 \left(-e^{b_1} (-1 + e^{b_2}) \mathbf{ff}_2[b_1, b_2] + e^{b_1} (-1 + e^{b_2}) \mathbf{ff}_2[b_1, b_3] - \right. \right. \\ & \quad \left. \left. (-1 + e^{b_1}) \left(e^{b_2} \mathbf{ff}_2[b_2, b_3] + \mathbf{gg}_3[b_2] + b_2 \left(\mathbf{gg}_2[b_2] + e^{b_2} \mathbf{gg}_5[b_2] - (-1 + e^{b_2}) \mathbf{gg}_6[b_2] \right) \right) \right), \right. \\ & \quad \left. 4 \right] + \text{ca} \left[-\frac{1}{2 b_2^2} (-1 + e^{b_1}) f[b_1, b_2, b_3, b_4] b_3 \right. \\ & \quad \left. \left(2 (-1 + e^{b_2}) - (1 + e^{b_2}) b_2 + 2 b_2^2 \left(\mathbf{gg}_2[b_2] + e^{b_2} \mathbf{gg}_5[b_2] - (-1 + e^{b_2}) \mathbf{gg}_6[b_2] \right) \right) \right), \right. \\ & \quad \left. 2, 2, 4 \right] + \text{ca} \left[-\frac{1}{2 b_1 b_2} (-1 + e^{b_1}) f[b_1, b_2, b_3, b_4] b_3 \right. \\ & \quad \left. \left(2 (-1 + e^{b_2}) - (1 + e^{b_2}) b_2 + 2 b_2^2 \left(\mathbf{gg}_2[b_2] + e^{b_2} \mathbf{gg}_5[b_2] - (-1 + e^{b_2}) \mathbf{gg}_6[b_2] \right) \right) \right), \right. \\ & \quad \left. 4, 1, 2 \right] + \text{ca} \left[\frac{1}{2 b_1 b_2} (-1 + e^{b_1}) f[b_1, b_2, b_3, b_4] b_3 \right. \\ & \quad \left. \left(2 (-1 + e^{b_2}) - (1 + e^{b_2}) b_2 + 2 b_2^2 \left(\mathbf{gg}_2[b_2] + e^{b_2} \mathbf{gg}_5[b_2] - (-1 + e^{b_2}) \mathbf{gg}_6[b_2] \right) \right) \right), 2, 1, 4 \right] + \\ & \delta a \left[\frac{1}{b_2} e^{b_1} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_3 \left(\mathbf{ff}_2[b_1, b_2] - \mathbf{ff}_2[b_1, b_3] + \right. \right. \\ & \quad \left. \left. b_1 \left(\mathbf{ff}_4[b_1, b_2] - \mathbf{ff}_4[b_1, b_3] \right) \right) \right), 2, 4 \right] + \delta a \left[\frac{1}{b_1} f[b_1, b_2, b_3, b_4] b_3 \right. \\ & \quad \left. \left(e^{b_2} (-1 + e^{b_1}) \mathbf{ff}_2[b_2, b_3] - e^{b_1} (-1 + e^{b_2}) b_1 \left(\mathbf{ff}_4[b_1, b_2] - \mathbf{ff}_4[b_1, b_3] \right) + \right. \right. \\ & \quad \left. \left. (-1 + e^{b_1}) \left(\mathbf{gg}_3[b_2] + b_2 \left(\mathbf{gg}_2[b_2] + e^{b_2} \mathbf{gg}_5[b_2] - (-1 + e^{b_2}) \mathbf{gg}_6[b_2] \right) \right) \right) \right), 1, 4 \right] + \\ & \delta a a \left[\frac{1}{2 b_1 b_2^2} (-1 + e^{b_1}) f[b_1, b_2, b_3, b_4] b_3 \left(2 (-1 + e^{b_2}) - (1 + e^{b_2}) b_2 + \right. \right. \\ & \quad \left. \left. 2 b_2^2 \left(\mathbf{gg}_2[b_2] + e^{b_2} \mathbf{gg}_5[b_2] - (-1 + e^{b_2}) \mathbf{gg}_6[b_2] \right) \right) \right), 1, 2, 2, 4 \right] = 0 \end{aligned}$$

Solve [(2 (-1 + e^{b₂}) - (1 + e^{b₂}) b₂ + 2 b₂² (gg₂[b₂] + e^{b₂} gg₅[b₂] - (-1 + e^{b₂}) gg₆[b₂])) = 0, gg₆[b₂]] /. b₂ → x

$$\left\{ \left\{ \mathbf{gg}_6[\mathbf{x}] \rightarrow \frac{-2 + 2 e^{\mathbf{x}} - \mathbf{x} - e^{\mathbf{x}} \mathbf{x} + 2 \mathbf{x}^2 \mathbf{gg}_2[\mathbf{x}] + 2 e^{\mathbf{x}} \mathbf{x}^2 \mathbf{gg}_5[\mathbf{x}]}{2 (-1 + e^{\mathbf{x}}) \mathbf{x}^2} \right\} \right\}$$

$$\mathbf{gg}_6[\mathbf{x}_-] := \frac{-2 + 2 e^{\mathbf{x}} - \mathbf{x} - e^{\mathbf{x}} \mathbf{x} + 2 \mathbf{x}^2 \mathbf{gg}_2[\mathbf{x}] + 2 e^{\mathbf{x}} \mathbf{x}^2 \mathbf{gg}_5[\mathbf{x}]}{2 (-1 + e^{\mathbf{x}}) \mathbf{x}^2};$$

VerifyR3[a[f[b₁, b₂, b₃, b₄], 3, 4]]

a[f[b₁, b₂, b₃, b₄], 3, 4] →

$$\begin{aligned} & c \left[-\frac{1}{2 b_2} f[b_1, b_2, b_3, b_4] b_3 \left(-2 (-1 + e^{b_1}) (-1 + e^{b_2}) + b_2 \left(2 e^{b_1} (-1 + e^{b_2}) \mathbf{ff}_2[b_1, b_2] - \right. \right. \right. \\ & \quad \left. \left. 2 e^{b_1} (-1 + e^{b_2}) \mathbf{ff}_2[b_1, b_3] + (-1 + e^{b_1}) \left(1 + e^{b_2} + 2 e^{b_2} \mathbf{ff}_2[b_2, b_3] + 2 \mathbf{gg}_3[b_2] \right) \right) \right), \right. \\ & \quad \left. 4 \right] + \delta a \left[\frac{1}{b_2} e^{b_1} (-1 + e^{b_2}) f[b_1, b_2, b_3, b_4] b_3 \left(\mathbf{ff}_2[b_1, b_2] - \mathbf{ff}_2[b_1, b_3] + \right. \right. \\ & \quad \left. \left. b_1 \left(\mathbf{ff}_4[b_1, b_2] - \mathbf{ff}_4[b_1, b_3] \right) \right) \right), 2, 4 \right] + \delta a \left[\frac{1}{2 b_1 b_2} f[b_1, b_2, b_3, b_4] b_3 \right. \\ & \quad \left. \left(-2 (-1 + e^{b_1}) (-1 + e^{b_2}) + b_2 \left(2 e^{b_2} (-1 + e^{b_1}) \mathbf{ff}_2[b_2, b_3] - 2 e^{b_1} (-1 + e^{b_2}) b_1 \right. \right. \right. \\ & \quad \left. \left. \left(\mathbf{ff}_4[b_1, b_2] - \mathbf{ff}_4[b_1, b_3] \right) + (-1 + e^{b_1}) \left(1 + e^{b_2} + 2 \mathbf{gg}_3[b_2] \right) \right) \right), 1, 4 \right] = 0 \end{aligned}$$

Expand[(ff₂[b₁, b₂] - ff₂[b₁, b₃] + b₁ (ff₄[b₁, b₂] - ff₄[b₁, b₃])) == 0]

ff₂[b₁, b₂] - ff₂[b₁, b₃] + b₁ ff₄[b₁, b₂] - b₁ ff₄[b₁, b₃] == 0

ff₂[b₁, b₂] + b₁ ff₄[b₁, b₂] == b₁ ff₄[b₁, b₃] + ff₂[b₁, b₃]

ff₂[b₁, b₂] + b₁ ff₄[b₁, b₂] == ff₂[b₁, b₃] + b₁ ff₄[b₁, b₃] /.

ff₂[x_, y_] => gg₇[x] - x ff₄[x, y]

True

ff₂[x_, y_] := gg₇[x] - x ff₄[x, y];

VerifyR3[a[f[b₁, b₂, b₃, b₄], 3, 4]]

a[f[b₁, b₂, b₃, b₄], 3, 4] →

$$\begin{aligned} & c \left[\frac{1}{2 b_2} f[b_1, b_2, b_3, b_4] b_3 \left(2 (-1 + e^{b_1}) (-1 + e^{b_2}) + 2 e^{b_2} (-1 + e^{b_1}) b_2^2 ff_4[b_2, b_3] + \right. \right. \\ & \quad \left. \left. b_2 \left(2 e^{b_1} (-1 + e^{b_2}) b_1 (ff_4[b_1, b_2] - ff_4[b_1, b_3]) - \right. \right. \right. \\ & \quad \left. \left. \left. (-1 + e^{b_1}) (1 + e^{b_2} + 2 gg_3[b_2] + 2 e^{b_2} gg_7[b_2]) \right) \right) \right], 4] + \\ & \delta a \left[-\frac{1}{2 b_1 b_2} f[b_1, b_2, b_3, b_4] b_3 \left(2 (-1 + e^{b_1}) (-1 + e^{b_2}) + 2 e^{b_2} (-1 + e^{b_1}) b_2^2 ff_4[b_2, b_3] + \right. \right. \\ & \quad \left. \left. b_2 \left(2 e^{b_1} (-1 + e^{b_2}) b_1 (ff_4[b_1, b_2] - ff_4[b_1, b_3]) - \right. \right. \right. \\ & \quad \left. \left. \left. (-1 + e^{b_1}) (1 + e^{b_2} + 2 gg_3[b_2] + 2 e^{b_2} gg_7[b_2]) \right) \right) \right], 1, 4] == 0 \end{aligned}$$

t1 = Collect[(2 (-1 + e^{b₁}) (-1 + e^{b₂}) + 2 e^{b₂} (-1 + e^{b₁}) b₂² ff₄[b₂, b₃] + b₂ (2 e^{b₁} (-1 + e^{b₂}) b₁ (ff₄[b₁, b₂] - ff₄[b₁, b₃]) - (-1 + e^{b₁}) (1 + e^{b₂} + 2 gg₃[b₂] + 2 e^{b₂} gg₇[b₂]))), ff[_], Simplify]

2 e^{b₁} (-1 + e^{b₂}) b₁ b₂ ff₄[b₁, b₂] - 2 e^{b₁} (-1 + e^{b₂}) b₁ b₂ ff₄[b₁, b₃] + 2 e^{b₂} (-1 + e^{b₁}) b₂² ff₄[b₂, b₃] - (-1 + e^{b₁}) (2 - 2 e^{b₂} + b₂ (1 + e^{b₂} + 2 gg₃[b₂] + 2 e^{b₂} gg₇[b₂]))

D[t1, b₃] / (2 b₂) // Simplify

-e^{b₁} (-1 + e^{b₂}) b₁ ff₄^(0,1)[b₁, b₃] + e^{b₂} (-1 + e^{b₁}) b₂ ff₄^(0,1)[b₂, b₃]

Simplify $\left[\frac{e^{b_1} b_1 ff_4^{(0,1)}[b_1, b_3]}{(-1 + e^{b_1})} == \frac{e^{b_2} b_2 ff_4^{(0,1)}[b_2, b_3]}{(-1 + e^{b_2})} \right]$

$\frac{e^{b_1} b_1 ff_4^{(0,1)}[b_1, b_3]}{-1 + e^{b_1}} == \frac{e^{b_2} b_2 ff_4^{(0,1)}[b_2, b_3]}{-1 + e^{b_2}}$

ff₄^(0,1)[b₂, b₃] == $\frac{(-1 + e^{b_2}) g_8[b_3]}{e^{b_2} b_2}$

ff₄^(0,1)[b₂, b₃] == $\frac{e^{-b_2} (-1 + e^{b_2}) g_8[b_3]}{b_2}$

ff₄[x_, y_] := gg₉[x] + $\frac{e^{-x} (-1 + e^x) gg_8[y]}{x}$;

```
VerifyR3[a[f[b1, b2, b3, b4], 3, 4]]
```

```
a[f[b1, b2, b3, b4], 3, 4] →
```

$$c \left[\frac{1}{2 b_2} (-1 + e^{b_1}) f[b_1, b_2, b_3, b_4] b_3 \left(2 (-1 + e^{b_2}) - b_2 (1 + e^{b_2} + 2 gg_3[b_2] + 2 e^{b_2} gg_7[b_2] + 2 gg_8[b_2] - 2 e^{b_2} gg_8[b_2]) + 2 e^{b_2} b_2^2 gg_9[b_2] \right), 4 \right] +$$

$$\delta a \left[-\frac{1}{2 b_1 b_2} (-1 + e^{b_1}) f[b_1, b_2, b_3, b_4] b_3 \left(2 (-1 + e^{b_2}) - b_2 (1 + e^{b_2} + 2 gg_3[b_2] + 2 e^{b_2} gg_7[b_2] + 2 gg_8[b_2] - 2 e^{b_2} gg_8[b_2]) + 2 e^{b_2} b_2^2 gg_9[b_2] \right), 1, 4 \right] == 0$$

```
Solve[(2 (-1 + e^b2) - b2 (1 + e^b2 + 2 gg3[b2] + 2 e^b2 gg7[b2] + 2 gg8[b2] - 2 e^b2 gg8[b2]) + 2 e^b2 b2^2 gg9[b2]) == 0, #] & /@ {gg3[b2], gg7[b2], gg8[b2], gg9[b2]} /. b2 → x
```

$$\left\{ \left\{ \left\{ gg_3[x] \rightarrow \frac{1}{2 x} \right. \right. \right.$$

$$\left. \left. \left. (-2 + 2 e^x - x - e^x x - 2 e^x x gg_7[x] - 2 x gg_8[x] + 2 e^x x gg_8[x] + 2 e^x x^2 gg_9[x]) \right\} \right\}, \left\{ \left\{ gg_7[x] \rightarrow \frac{1}{2 x} e^{-x} (-2 + 2 e^x - x - e^x x - 2 x gg_3[x] - 2 x gg_8[x] + 2 e^x x gg_8[x] + 2 e^x x^2 gg_9[x]) \right\} \right\},$$

$$\left\{ \left\{ gg_8[x] \rightarrow \frac{1}{2 (-1 + e^x) x} (2 - 2 e^x + x + e^x x + 2 x gg_3[x] + 2 e^x x gg_7[x] - 2 e^x x^2 gg_9[x]) \right\} \right\},$$

$$\left\{ \left\{ gg_9[x] \rightarrow \frac{1}{2 x^2} e^{-x} (2 - 2 e^x + x + e^x x + 2 x gg_3[x] + 2 e^x x gg_7[x] + 2 x gg_8[x] - 2 e^x x gg_8[x]) \right\} \right\}$$

$$gg_9[x_] := \frac{1}{2 x^2} e^{-x} (2 - 2 e^x + x + e^x x + 2 x gg_3[x] + 2 e^x x gg_7[x] + 2 x gg_8[x] - 2 e^x x gg_8[x]);$$

```
VerifyR3[a[f[b1, b2, b3, b4], 3, 4]]
```

```
a[f[b1, b2, b3, b4], 3, 4] → True
```

Verifying R

```
Print[VerifyR3[#]] & /@
```

```
{a[f[b1, b2, b3, b4], 1, 4], a[f[b1, b2, b3, b4], 2, 4], a[f[b1, b2, b3, b4], 3, 4], a[f[b1, b2, b3, b4], 4, 1], a[f[b1, b2, b3, b4], 4, 2], a[f[b1, b2, b3, b4], 4, 3]};
```

```
a[f[b1, b2, b3, b4], 1, 4] → True
```

```
a[f[b1, b2, b3, b4], 2, 4] → True
```

```
a[f[b1, b2, b3, b4], 3, 4] → True
```

```
a[f[b1, b2, b3, b4], 4, 1] → True
```

```
a[f[b1, b2, b3, b4], 4, 2] → True
```

```
a[f[b1, b2, b3, b4], 4, 3] → True
```

```
Print[VerifyR3[#]] & /@
```

```
{a[f[b1, b2, b3, b4], 1, 2], a[f[b1, b2, b3, b4], 1, 3], a[f[b1, b2, b3, b4], 2, 3], a[f[b1, b2, b3, b4], 2, 1], a[f[b1, b2, b3, b4], 3, 1], a[f[b1, b2, b3, b4], 3, 2]};
```

```

a[f[b1, b2, b3, b4], 1, 2] → True
a[f[b1, b2, b3, b4], 1, 3] → True
a[f[b1, b2, b3, b4], 2, 3] → True
a[f[b1, b2, b3, b4], 2, 1] → True
a[f[b1, b2, b3, b4], 3, 1] → True
a[f[b1, b2, b3, b4], 3, 2] → True

```

Playing with R

S[ρ0[j, k]]

$$\begin{aligned}
& c[gg_3[b_k], j] + c\left[\frac{1}{2 b_j}\right. \\
& \quad \left. e^{-b_j} (2 (-1 + e^{b_j}) + b_j (-1 - e^{b_j} - 2 gg_3[b_j] + 2 (-1 + e^{b_j}) gg_8[b_j] + 2 gg_8[b_k] - 2 e^{b_j} gg_8[b_k]))\right], \\
& k] + ca\left[\frac{e^{-b_j} (2 - 2 e^{b_j} + (1 + e^{b_j}) b_j)}{2 b_j^2}, k, j, k\right] + ca[gg_2[b_k], j, k, k] + ca[gg_5[b_j], k, j, j] + \\
& ca\left[-\frac{1}{2 (-1 + e^{b_k}) b_k^2} e^{-b_j} (-1 + e^{b_j}) (2 (-1 + e^{b_k}) - (1 + e^{b_k}) b_k + 2 b_k^2 (gg_2[b_k] + e^{b_k} gg_5[b_k]))\right], \\
& k, k, k] + \delta a[ff_3[b_j, b_k], j, j] + \delta a[ff_6[b_j, b_k], k, k] + \delta a\left[\frac{1}{2 b_j^2} e^{-b_j} (2 - 2 e^{b_j} + b_j\right. \\
& \quad \left. (1 + e^{b_j} + 2 gg_3[b_j] + 2 e^{b_j} gg_7[b_j] + 2 gg_8[b_j] - 2 e^{b_j} gg_8[b_j] - 2 gg_8[b_k] + 2 e^{b_j} gg_8[b_k]))\right], \\
& j, k] + \delta aa\left[-\frac{e^{-b_j} (2 - 2 e^{b_j} + (1 + e^{b_j}) b_j)}{2 b_j^3}, j, k, j, k\right] + \delta aa[ff_{15}[b_j, b_k], j, j, j, j] + \\
& \delta aa[ff_{18}[b_j, b_k], j, j, k, k] + \delta aa[ff_{23}[b_j, b_k], k, k, k, k] + \\
& \delta aa[gg_4[b_j], j, j, j, k] + \\
& \delta aa\left[(e^{-b_j} (-1 + e^{b_j}) (2 (-1 + e^{b_k}) - (1 + e^{b_k}) b_k + 2 b_k^2 (gg_2[b_k] + e^{b_k} gg_5[b_k]))\right) / \\
& \quad (2 (-1 + e^{b_k}) b_j b_k^2), j, k, k, k]
\end{aligned}$$

S[ρ0[j, k] /. {gg2|3|4|5[_] → 0, ff3|6|15|18|23[___] → 0,

$$\begin{aligned}
& \left. gg_8[x_] \Rightarrow (-1 + e^x)^{-1}, gg_7[y_] \Rightarrow -\frac{e^{-y} (-1 + e^y) (2 - 2 e^{b_k} + y + e^{b_k} y)}{2 (-1 + e^{b_k}) y}\right\}] \\
& c\left[-\frac{e^{-b_j} (-1 + e^{b_j}) (2 - 2 e^{b_k} + (1 + e^{b_k}) b_j)}{2 (-1 + e^{b_k}) b_j}, k\right] + ca\left[\frac{e^{-b_j} (2 - 2 e^{b_j} + (1 + e^{b_j}) b_j)}{2 b_j^2}, k, j, k\right] + \\
& ca\left[\frac{e^{-b_j} (-1 + e^{b_j}) (2 - 2 e^{b_k} + (1 + e^{b_k}) b_k)}{2 (-1 + e^{b_k}) b_k^2}, k, k, k\right] + \\
& \delta aa\left[-\frac{e^{-b_j} (2 - 2 e^{b_j} + (1 + e^{b_j}) b_j)}{2 b_j^3}, j, k, j, k\right] + \\
& \delta aa\left[-\frac{e^{-b_j} (-1 + e^{b_j}) (2 - 2 e^{b_k} + (1 + e^{b_k}) b_k)}{2 (-1 + e^{b_k}) b_j b_k^2}, j, k, k, k\right]
\end{aligned}$$

S[$\rho_0[j, k]$ /. {**gg**_{2|3|4|5|7|8}[_] → 0, **ff**_{3|6|15|18|23}[_] → 0}]

$$c\left[\frac{1}{2 b_j} e^{-b_j} \left(2(-1 + e^{b_j}) + b_j(-1 - e^{b_j} + 2(-1 + e^{b_j}) \text{gg}_8[b_j] - 2(-1 + e^{b_j}) \text{gg}_8[b_k])\right), k\right] +$$

$$ca\left[\frac{e^{-b_j} (2 - 2 e^{b_j} + (1 + e^{b_j}) b_j)}{2 b_j^2}, k, j, k\right] +$$

$$ca\left[\frac{e^{-b_j} (-1 + e^{b_j}) (2 - 2 e^{b_k} + (1 + e^{b_k}) b_k)}{2 (-1 + e^{b_k}) b_k^2}, k, k, k\right] +$$

$$\delta a\left[\frac{1}{2 b_j^2} e^{-b_j} \left(2 - 2 e^{b_j} + b_j (1 + e^{b_j} - 2(-1 + e^{b_j}) \text{gg}_8[b_j] + 2(-1 + e^{b_j}) \text{gg}_8[b_k])\right), j, k\right] +$$

$$\delta aa\left[-\frac{e^{-b_j} (2 - 2 e^{b_j} + (1 + e^{b_j}) b_j)}{2 b_j^3}, j, k, j, k\right] +$$

$$\delta aa\left[-\frac{e^{-b_j} (-1 + e^{b_j}) (2 - 2 e^{b_k} + (1 + e^{b_k}) b_k)}{2 (-1 + e^{b_k}) b_j b_k^2}, j, k, k, k\right]$$

$$\frac{1}{2 b_j} e^{-b_j} \left(2(-1 + e^{b_j}) + b_j(-1 - e^{b_j} + 2(-1 + e^{b_j}) \text{gg}_8[b_j] - 2(-1 + e^{b_j}) \text{gg}_8[b_k])\right) +$$

$$b_j \frac{1}{2 b_j^2} e^{-b_j} \left(2 - 2 e^{b_j} + b_j (1 + e^{b_j} - 2(-1 + e^{b_j}) \text{gg}_8[b_j] + 2(-1 + e^{b_j}) \text{gg}_8[b_k])\right) // \text{Simplify}$$

0

$$\frac{e^{-b_j} (2 - 2 e^{b_j} + (1 + e^{b_j}) b_j)}{2 b_j^2} + b_j \left(-\frac{e^{-b_j} (2 - 2 e^{b_j} + (1 + e^{b_j}) b_j)}{2 b_j^3}\right) // \text{Simplify}$$

0

$$\frac{e^{-b_j} (-1 + e^{b_j}) (2 - 2 e^{b_k} + (1 + e^{b_k}) b_k)}{2 (-1 + e^{b_k}) b_k^2} +$$

$$b_j \left(-\frac{e^{-b_j} (-1 + e^{b_j}) (2 - 2 e^{b_k} + (1 + e^{b_k}) b_k)}{2 (-1 + e^{b_k}) b_j b_k^2}\right) // \text{Simplify}$$

0

S[$\rho_0[j, k]$ /. {**gg**_{2|3|4|5|7|8}[_] → 0, **ff**_{3|6|15|18|23}[_] → 0}]

$$c\left[-\frac{e^{-b_j} (2 - 2 e^{b_j} + (1 + e^{b_j}) b_j)}{2 b_j}, k\right] + ca\left[\frac{e^{-b_j} (2 - 2 e^{b_j} + (1 + e^{b_j}) b_j)}{2 b_j^2}, k, j, k\right] +$$

$$ca\left[\frac{e^{-b_j} (-1 + e^{b_j}) (2 - 2 e^{b_k} + (1 + e^{b_k}) b_k)}{2 (-1 + e^{b_k}) b_k^2}, k, k, k\right] +$$

$$\delta a\left[\frac{e^{-b_j} (2 - 2 e^{b_j} + (1 + e^{b_j}) b_j)}{2 b_j^2}, j, k\right] + \delta aa\left[-\frac{e^{-b_j} (2 - 2 e^{b_j} + (1 + e^{b_j}) b_j)}{2 b_j^3}, j, k, j, k\right] +$$

$$\delta aa\left[-\frac{e^{-b_j} (-1 + e^{b_j}) (2 - 2 e^{b_k} + (1 + e^{b_k}) b_k)}{2 (-1 + e^{b_k}) b_j b_k^2}, j, k, k, k\right]$$

R[1, 2]@a[1, 1, 3] /. {gg2|3|4|5|7[_] → 0, ff3|6|15|18|23[_] → 0} // S

$$a[1, 1, 3] + ca[1, 3, 1, 2] + ca\left[\frac{-1 + e^{-b_1}}{b_1}, 2, 1, 3\right] + \delta aa\left[-\frac{-1 + e^{-b_1} + b_1}{b_1^2}, 1, 2, 1, 3\right]$$

R[1, 2]@a[1, 2, 3] /. {gg2|3|4|5|7[_] → 0, ff3|6|15|18|23[_] → 0} // S

$$\begin{aligned} & a[e^{b_1}, 2, 3] + a\left[-\frac{(-1 + e^{b_1}) b_2}{b_1}, 1, 3\right] + c\left[\frac{1}{2} \left(-\frac{(-1 + e^{b_1}) (2 - 2 e^{b_2} + (1 + e^{b_2}) b_2)}{-1 + e^{b_2}} - \frac{1}{b_1}\right.\right. \\ & \quad \left.\left. b_2 (2 (-1 + e^{b_1}) + b_1 (-1 - e^{b_1} + 2 (-1 + e^{b_1}) gg_8[b_1] - 2 (-1 + e^{b_1}) gg_8[b_2]))\right), 3\right] + \\ & ca\left[-\frac{e^{b_2} (-1 + e^{b_1}) b_2}{(-1 + e^{b_2}) b_1}, 3, 1, 2\right] + ca\left[\frac{(-1 + e^{b_1}) (-1 + e^{b_2} - e^{b_2} b_2)}{(-1 + e^{b_2}) b_2}, 2, 2, 3\right] + \\ & ca\left[\frac{e^{-b_1} (-1 + e^{b_1}) (-(-1 + e^{b_1}) (-1 + e^{b_2}) + e^{b_1+b_2} b_2)}{(-1 + e^{b_2}) b_1}, 2, 1, 3\right] + \\ & \delta a\left[-\frac{1}{2 b_1^2} \left(-\frac{(-1 + e^{b_1}) b_1 (2 - 2 e^{b_2} + (1 + e^{b_2}) b_2)}{-1 + e^{b_2}} + \right.\right. \\ & \quad \left.\left. b_2 (2 - 2 e^{b_1} + b_1 (1 + e^{b_1} - 2 (-1 + e^{b_1}) gg_8[b_1] + 2 (-1 + e^{b_1}) gg_8[b_2]))\right), 1, 3\right] + \\ & \delta aa\left[\frac{e^{-b_1} (-1 + e^{b_1})^2}{b_1^2}, 1, 2, 1, 3\right] + \delta aa\left[\frac{(-1 + e^{b_1}) (1 - e^{b_2} + e^{b_2} b_2)}{(-1 + e^{b_2}) b_1 b_2}, 1, 2, 2, 3\right] \end{aligned}$$

R[1, 2]@a[1, 2, 3] /. {gg2|3|4|5|7|8[_] → 0, ff3|6|15|18|23[_] → 0} // S

$$\begin{aligned} & a[e^{b_1}, 2, 3] + a\left[-\frac{(-1 + e^{b_1}) b_2}{b_1}, 1, 3\right] + \\ & c\left[\frac{1}{(-1 + e^{b_2}) b_1} (-(-1 + e^{b_1}) (-1 + e^{b_2}) b_2 + b_1 ((-1 + e^{b_1}) (-1 + e^{b_2}) + (-e^{b_1} + e^{b_2}) b_2)), 3\right] + \\ & ca\left[-\frac{e^{b_2} (-1 + e^{b_1}) b_2}{(-1 + e^{b_2}) b_1}, 3, 1, 2\right] + ca\left[\frac{(-1 + e^{b_1}) (-1 + e^{b_2} - e^{b_2} b_2)}{(-1 + e^{b_2}) b_2}, 2, 2, 3\right] + \\ & ca\left[\frac{e^{-b_1} (-1 + e^{b_1}) (-(-1 + e^{b_1}) (-1 + e^{b_2}) + e^{b_1+b_2} b_2)}{(-1 + e^{b_2}) b_1}, 2, 1, 3\right] + \\ & \delta a\left[\frac{1}{(-1 + e^{b_2}) b_1^2} ((-1 + e^{b_1}) (-1 + e^{b_2}) b_2 + b_1 (-(-1 + e^{b_1}) (-1 + e^{b_2}) + (e^{b_1} - e^{b_2}) b_2)), 1, 3\right] + \\ & \delta aa\left[\frac{e^{-b_1} (-1 + e^{b_1})^2}{b_1^2}, 1, 2, 1, 3\right] + \delta aa\left[\frac{(-1 + e^{b_1}) (1 - e^{b_2} + e^{b_2} b_2)}{(-1 + e^{b_2}) b_1 b_2}, 1, 2, 2, 3\right] \end{aligned}$$

$(\mathbf{a}[1, 3, 4] // \mathbf{R}[1, 2] // \mathbf{R}[1, 3] // \mathbf{R}[2, 3]) / .$

$\{\mathbf{gg}_2|3|4|5|7|8[_] \rightarrow 0, \mathbf{ff}_3|6|15|18|23[_] \rightarrow 0\} // \mathbf{S}$

$$\begin{aligned}
& a \left[e^{b_1+b_2}, 3, 4 \right] + a \left[-\frac{(-1+e^{b_1}) b_3}{b_1}, 1, 4 \right] + a \left[-\frac{e^{b_1} (-1+e^{b_2}) b_3}{b_2}, 2, 4 \right] + \\
& c \left[\left(-(-1+e^{b_1}) (-1+e^{b_3}) b_2 b_3 + b_1 (-e^{b_1} (-1+e^{b_2}) (-1+e^{b_3}) b_3 + \right. \right. \\
& \quad \left. \left. b_2 \left((-1+e^{b_1+b_2}) (-1+e^{b_3}) + (-e^{b_1} - e^{b_1+b_2} + e^{b_3} + e^{b_1+b_3}) b_3 \right) \right) \right] / \\
& \left((-1+e^{b_3}) b_1 b_2 \right), 4 \left] + ca \left[\frac{(-1+e^{b_1+b_2}) (-1+e^{b_3} - e^{b_3} b_3)}{(-1+e^{b_3}) b_3}, 3, 3, 4 \right] + \\
& ca \left[-\frac{e^{-b_2} (-1+e^{b_1}) \left((-1+e^{b_2}) (-1+e^{b_3}) + e^{b_2+b_3} b_3 \right)}{(-1+e^{b_3}) b_1}, 4, 1, 3 \right] + \\
& ca \left[\frac{e^{-b_1-b_2} (-1+e^{b_1}) \left(-(-1+e^{b_1}) (-1+e^{b_3}) + e^{b_1+b_2+b_3} b_3 \right)}{(-1+e^{b_3}) b_1}, 3, 1, 4 \right] + \\
& ca \left[-\frac{e^{-b_2} (-1+e^{b_2}) \left(-(-1+e^{b_1}) (-1+e^{b_3}) + e^{b_1+b_2+b_3} b_3 \right)}{(-1+e^{b_3}) b_2}, 4, 2, 3 \right] + \\
& ca \left[\frac{e^{-b_2} (-1+e^{b_2}) \left(-(-2+e^{b_1} + e^{b_1+b_2}) (-1+e^{b_3}) + e^{b_1+b_2+b_3} b_3 \right)}{(-1+e^{b_3}) b_2}, 3, 2, 4 \right] + \\
& \delta a \left[\frac{1}{(-1+e^{b_3}) b_1^2} \left((-1+e^{b_1}) (-1+e^{b_3}) b_3 + b_1 \left(-(-1+e^{b_1}) (-1+e^{b_3}) + (e^{b_1} - e^{b_3}) b_3 \right) \right), 1, 4 \right] + \\
& \delta a \left[\frac{1}{(-1+e^{b_3}) b_2^2} e^{b_1} \left((-1+e^{b_2}) (-1+e^{b_3}) b_3 + b_2 \left(-(-1+e^{b_2}) (-1+e^{b_3}) + (e^{b_2} - e^{b_3}) b_3 \right) \right), \right. \\
& \quad \left. 2, 4 \right] + \delta aa \left[\frac{e^{-b_1-b_2} (-1+e^{b_1})^2}{b_1^2}, 1, 3, 1, 4 \right] + \\
& \delta aa \left[\frac{e^{b_1-b_2} (-1+e^{b_2})^2}{b_2^2}, 2, 3, 2, 4 \right] + \delta aa \left[\frac{2 e^{-b_2} (-1+e^{b_1}) (-1+e^{b_2})}{b_1 b_2}, 1, 3, 2, 4 \right] + \\
& \delta aa \left[\frac{(-1+e^{b_1}) (1 - e^{b_3} + e^{b_3} b_3)}{(-1+e^{b_3}) b_1 b_3}, 1, 3, 3, 4 \right] + \\
& \delta aa \left[\frac{e^{b_1} (-1+e^{b_2}) (1 - e^{b_3} + e^{b_3} b_3)}{(-1+e^{b_3}) b_2 b_3}, 2, 3, 3, 4 \right]
\end{aligned}$$

$(\mathbf{a}[1, 3, 4] // \mathbf{R}[1, 2] // \mathbf{R}[1, 3] // \mathbf{R}[2, 3] // \mathbf{R}[3, 1] // \mathbf{R}[2, 1]) / .$

$\{\mathbf{gg}_2|3|4|5|7|8[_] \rightarrow 0, \mathbf{ff}_3|6|15|18|23[_] \rightarrow 0\} // \mathbf{S}$

$$\begin{aligned}
& a \left[e^{b_1+b_2} + (-1+e^{b_1}) (-1+e^{b_3}), 3, 4 \right] + \\
& a \left[-\frac{e^{b_2+b_3} (-1+e^{b_1}) b_3}{b_1}, 1, 4 \right] + a \left[\frac{(-1+e^{b_2}) (-e^{b_1} - e^{b_3} + e^{b_1+b_3}) b_3}{b_2}, 2, 4 \right] + \\
& c \left[\frac{1}{(-1+e^{b_3}) b_1 b_2} e^{-b_1-b_2} \left(-e^{b_1+2 b_2+b_3} (-1+e^{b_1}) (-1+e^{b_3}) b_2 b_3 + \right. \right. \\
& \quad \left. \left. b_1 \left(e^{b_1+b_2} (-1+e^{b_2}) (-1+e^{b_3}) \left(-e^{b_1} - e^{b_3} + e^{b_1+b_3} \right) b_3 + b_2 \left((-1+e^{b_3}) \right. \right. \right. \right. \\
& \quad \left. \left. \left. \left(-1+e^2 b_1 + e^{b_1+b_2} + e^2 (b_1+b_2) - 2 e^2 b_1+b_2 + e^2 b_3 + 2 e^{b_1+b_3} + e^2 (b_1+b_3) - 2 e^2 b_1+b_3 + \right. \right. \right. \right. \\
& \quad \left. \left. \left. e^{b_2+b_3} - 4 e^{b_1+b_2+b_3} + 3 e^2 b_1+b_2+b_3 - 2 e^{b_1+2 b_3} - e^{b_2+2 b_3} + 2 e^{b_1+b_2+2 b_3} - e^2 b_1+b_2+2 b_3 \right) \right) \right) - \\
& \quad \left. \left. \left. e^{b_1+b_2} \left(e^{b_1+b_2} + e^{b_3} - 2 e^2 b_3 - 2 e^{b_1+b_3} + e^{b_2+b_3} + 2 e^{b_1+2 b_3} - e^{b_2+2 b_3} \right) b_3 \right) \right) \right),
\end{aligned}$$

$$\begin{aligned}
 & 4] + \text{ca} \left[- \frac{e^{-b_1+b_3} (-1 + e^{b_1})^2 (-1 + e^{b_3})}{b_1}, \right. \\
 & 1, \\
 & 1, \\
 & 4] + \\
 & \text{ca} \left[\frac{e^{-b_1-b_2} (-1 + e^{b_1}) (-1 + e^{b_3}) (1 - e^{b_2+b_3} + e^{b_1+b_2+b_3})}{b_1}, \right. \\
 & 4, \\
 & 1, \\
 & 1] + \\
 & \text{ca} \left[- \frac{e^{-b_1+b_2+2b_3} (-1 + e^{b_1}) ((-1 + e^{b_1}) (-1 + e^{b_3}) - e^{b_1} b_3)}{(-1 + e^{b_3}) b_1}, \right. \\
 & 3, \\
 & 1, \\
 & 4] + \\
 & \text{ca} \left[- \frac{1}{(-1 + e^{b_3}) b_2} e^{-b_1-b_2} (-1 + e^{b_2}) (-e^{b_1} - e^{b_3} + e^{b_1+b_3}) \right. \\
 & \left. ((-1 + e^{b_1}) (-1 + e^{b_3}) (-1 + e^{b_3} + e^{b_2+b_3}) - e^{b_1+b_2+b_3} b_3), 4, 2, 3 \right] + \\
 & \text{ca} \left[\frac{1}{(-1 + e^{b_3}) b_1} e^{-b_1+b_3} (-1 + e^{b_1}) ((-1 + e^{b_3}) (1 - e^{b_1+b_2} - e^{b_2+b_3} + e^{b_1+b_2+b_3}) - e^{b_1+b_2+b_3} b_3), \right. \\
 & 4, \\
 & 1, \\
 & 3] + \text{ca} \left[\frac{1}{(-1 + e^{b_3}) b_2} \right. \\
 & e^{-b_1-b_2} (-1 + e^{b_2}) (-e^{b_1} - e^{b_3} + e^{b_1+b_3}) \\
 & \left. ((-1 + e^{b_3}) (-e^{b_1} + e^{b_1+b_2} - e^{b_3} + e^{b_1+b_3} - e^{b_2+b_3} + e^{b_1+b_2+b_3}) - e^{b_1+b_2+b_3} b_3), 3, 2, 4 \right] + \\
 & \text{ca} \left[\frac{1}{b_3} e^{-b_1-b_2} (-1 + e^{b_1}) (-(-1 + e^{b_3}) (-e^{b_1} + e^{b_1+b_2} - e^{b_3} + e^{b_1+b_3}) + e^{b_1+b_2+b_3} b_3), \right. \\
 & 4, 3, 3] + \\
 & \text{ca} \left[\frac{1}{b_2} e^{-b_1-2b_2-b_3} (-1 + e^{b_2}) (-1 + e^{b_3}) \right. \\
 & \left. ((-e^{b_1} - e^{b_3} + e^{b_1+b_3}) (-e^{b_1} + e^{b_1+b_2} - e^{b_3} + e^{b_1+b_3} - e^{b_2+b_3} + e^{b_1+b_2+b_3}) - e^{2b_1+b_2+b_3} b_3), 1, 2, 4 \right] + \\
 & \text{ca} \left[\frac{1}{b_3} e^{-b_1-2b_2-b_3} (-1 + e^{b_3}) (-1 + e^{2b_1} + e^{2(b_1+b_2)} - 2e^{2b_1+b_2} + e^{2b_3} + 2e^{b_1+b_3} + e^{2(b_1+b_3)} - 2e^{2b_1+b_3} - \right. \\
 & \left. 2e^{b_1+b_2+b_3} + 2e^{2b_1+b_2+b_3} - 2e^{b_1+2b_3} + e^{2b_1+b_2+b_3} (-1 + e^{b_2}) b_3), 1, 3, 4 \right] + \text{ca} \left[\frac{1}{(-1 + e^{b_3}) b_3} \right. \\
 & e^{-b_1-b_2} ((-1 + e^{b_3}) (-1 + e^{2b_1} + e^{b_1+b_2} + e^{2(b_1+b_2)} - 2e^{2b_1+b_2} + e^{2b_3} + 2e^{b_1+b_3} + e^{2(b_1+b_3)} - 2e^{2b_1+b_3} - \\
 & \left. 2e^{b_1+b_2+b_3} + 2e^{2b_1+b_2+b_3} - 2e^{b_1+2b_3}) - e^{b_1+b_2+b_3} (-e^{b_1} + e^{b_1+b_2} - e^{b_3} + e^{b_1+b_3}) b_3), 3, 3, 4 \right] + \\
 & \text{ca} \left[\frac{1}{b_3} e^{-b_1-2b_2-b_3} (-(-1 + e^{b_3}) (-e^{b_1} + e^{2b_1} + e^{b_1+b_2} + e^{2(b_1+b_2)} - 2e^{2b_1+b_2} - e^{b_3} + e^{2b_3} + \right. \\
 & \left. 3e^{b_1+b_3} + e^{2(b_1+b_3)} - 2e^{2b_1+b_3} - 2e^{b_1+b_2+b_3} + 2e^{2b_1+b_2+b_3} - 2e^{b_1+2b_3}) + \right.
 \end{aligned}$$

$$\begin{aligned}
& e^{2b_1+b_2} (1 - e^{b_2} - 2e^{b_3} + e^{2b_3} + 2e^{b_2+b_3}) b_3), 4, 3, 1] + ca \left[\frac{1}{b_2} e^{-b_1-2b_2-b_3} (-1 + e^{b_2}) \right. \\
& \left. (-(-1 + e^{b_3}) (-e^{b_1} - e^{b_3} + e^{b_1+b_3}) (1 - e^{b_1} + e^{b_1+b_2} - e^{b_3} + e^{b_1+b_3}) - e^{b_2+b_3} + e^{b_1+b_2+b_3}) + \right. \\
& \left. e^{2b_1+b_2} (1 - 2e^{b_3} + e^{2b_3} + e^{b_2+2b_3}) b_3), 4, 2, 1] + \right. \\
\delta a \left[\frac{1}{b_3} e^{-b_1-b_2} (-1 + e^{b_1}) \left((-1 + e^{b_3}) (1 - e^{b_1} + e^{b_1+b_2} - e^{b_3} + e^{b_1+b_3}) + e^{b_1+b_2+b_3} b_3 \right), \right. \\
& \left. 3, 4] + \right. \\
\delta a \left[\frac{1}{(-1 + e^{b_3}) b_2^2} e^{-b_1-b_2} (-e^{b_1} - e^{b_3} + e^{b_1+b_3}) (-e^{b_1+b_2} (-1 + e^{b_2}) (-1 + e^{b_3}) b_3 + \right. \\
& \left. b_2 \left((-1 + e^{b_2}) (-1 + e^{b_3}) (2 - 2e^{b_1} + e^{b_1+b_2} - 2e^{b_3} + 2e^{b_1+b_3}) - e^{b_1+b_2} (e^{b_2} - e^{b_3}) b_3 \right), 2, 4] + \right. \\
\delta a \left[\frac{1}{(-1 + e^{b_3}) b_1^2} e^{b_3} (-e^{-b_1} (-1 + e^{b_1})^2 (-1 + e^{b_3})^2 b_1 + \right. \\
& \left. e^{b_2} \left((-1 + e^{b_1}) (-1 + e^{b_3}) b_3 + b_1 \left(-(-1 + e^{b_1}) (-1 + e^{b_3}) + (e^{b_1} - e^{b_3}) b_3 \right) \right), 1, 4] + \right. \\
\delta aa \left[\frac{e^{-b_1+b_2+2b_3} (-1 + e^{b_1})^2}{b_1^2}, 1, 3, 1, 4] + \delta aa \left[\frac{e^{-b_1+b_3} (-1 + e^{b_1})^2 (-1 + e^{b_3})}{b_1^2}, \right. \\
& \left. 1, 1, 1, 4] + \right. \\
\delta aa \left[\frac{e^{-b_1-b_2} (-1 + e^{b_2})^2 (e^{b_1} + e^{b_3} - e^{b_1+b_3})^2}{b_2^2}, 2, 3, 2, 4] + \right. \\
\delta aa \left[\frac{2e^{-b_1+b_3} (-1 + e^{b_1}) (-1 + e^{b_2}) (-e^{b_1} - e^{b_3} + e^{b_1+b_3})}{b_1 b_2}, \right. \\
& \left. 1, 3, 2, 4] + \right. \\
\delta aa \left[-\frac{1}{b_1 b_2} 2e^{-b_1-b_2} (-1 + e^{b_1}) (-1 + e^{b_2}) (-1 + e^{b_3}) (-e^{b_1} - e^{b_3} + e^{b_1+b_3}), \right. \\
& \left. 1, 1, 2, 4] + \right. \\
\delta aa \left[-\frac{2e^{-b_1-b_2} (-1 + e^{b_1}) (-1 + e^{b_3}) (1 - e^{b_1} + e^{b_1+b_2} - e^{b_3} + e^{b_1+b_3})}{b_1 b_3}, \right. \\
& \left. 1, 1, 3, 4] + \right. \\
\delta aa \left[\frac{(-1 + e^{b_1}) (-1 + e^{b_3} + e^{-b_1-b_2} (-1 + e^{b_1}) (-1 + e^{b_3})^2 - e^{b_3} b_3)}{b_3^2}, \right. \\
& \left. 3, 3, 3, 4] + \right. \\
\delta aa \left[\left(e^{-b_1+b_3} (-1 + e^{b_1}) \left(-(-1 + e^{b_3}) (2 - 2e^{b_1} + e^{b_1+b_2} - 2e^{b_3} + 2e^{b_1+b_3}) + e^{b_1+b_2+b_3} b_3 \right) \right) / \right. \\
& \left. \left((-1 + e^{b_3}) b_1 b_3 \right), 1, 3, 3, 4] + \right. \\
\delta aa \left[\left(\left(e^{-b_1-b_2} (-1 + e^{b_2}) (-e^{b_1} - e^{b_3} + e^{b_1+b_3}) \right. \right. \right. \\
& \left. \left. \left(-(-1 + e^{b_3}) (2 - 2e^{b_1} + e^{b_1+b_2} - 2e^{b_3} + 2e^{b_1+b_3}) + e^{b_1+b_2+b_3} b_3 \right) \right) / \left((-1 + e^{b_3}) b_2 b_3 \right) \right), 2, \right. \\
& \left. 3, 3, 4] + \delta aa \left[\frac{1}{b_2^2} e^{-b_1-2b_2-b_3} (-1 + e^{b_2})^2 (-1 + e^{b_3}) \left((e^{b_1} + e^{b_3} - e^{b_1+b_3})^2 - e^{2b_1+b_2+b_3} b_3 \right), \right. \\
& \left. 2, 1, 2, 4] + \right. \\
\delta aa \left[-\frac{1}{b_2 b_3} e^{-b_1-2b_2-b_3} (-1 + e^{b_2}) (-2(-1 + e^{b_3}) (-e^{b_1} - e^{b_3} + e^{b_1+b_3}) (1 - e^{b_1} + e^{b_1+b_2} - e^{b_3} + e^{b_1+b_3}) + \right. \\
& \left. e^{2b_1+b_2} (1 - 3e^{b_3} + 2e^{2b_3} + e^{b_2+b_3}) b_3), 2, 1, 3, 4] + \delta aa \left[\frac{1}{b_3^2} \right.
\end{aligned}$$

$$e^{-b_1-2b_2-b_3} \left((-1 + e^{b_3}) (1 - e^{b_1} + e^{b_1+b_2} - e^{b_3} + e^{b_1+b_3})^2 - e^{2b_1+b_2} (1 - e^{b_2} - 2e^{b_3} + e^{2b_3} + 2e^{b_2+b_3}) b_3 \right),$$

$$3, 1, 3, 4]$$

rule2 = {**gg**_{3|4|5|7|8}[_] → 0, **gg**₂[**x**_] → **e^x / x**, **ff**[_] → 0};

ρ0[**j**, **k**] /. **rule2** // **S**

$$c \left[-\frac{e^{-b_j} (2 - 2e^{b_j} + (1 + e^{b_j}) b_j)}{2 b_j}, k \right] +$$

$$ca \left[\frac{e^{-b_j} (2 - 2e^{b_j} + (1 + e^{b_j}) b_j)}{2 b_j^2}, k, j, k \right] + ca \left[\frac{e^{b_k}}{b_k}, j, k, k \right] +$$

$$ca \left[-\frac{e^{-b_j} (-1 + e^{b_j}) (2 + b_k)}{2 b_k^2}, k, k, k \right] + \delta a \left[\frac{e^{-b_j} (2 - 2e^{b_j} + (1 + e^{b_j}) b_j)}{2 b_j^2}, j, k \right] +$$

$$\delta aa \left[-\frac{e^{-b_j} (2 - 2e^{b_j} + (1 + e^{b_j}) b_j)}{2 b_j^3}, j, k, j, k \right] + \delta aa \left[\frac{e^{-b_j} (-1 + e^{b_j}) (2 + b_k)}{2 b_j b_k^2}, j, k, k, k \right]$$

R[1, 2]@**a**[1, 1, ∞] /. **rule2** // **S**

$$a[1, 1, \infty] + c[-e^{b_2} b_1, \infty] + ca[1 - e^{b_2}, \infty, 1, 2] + ca \left[e^{b_2} + \frac{-1 + e^{-b_1}}{b_1}, 2, 1, \infty \right] +$$

$$ca \left[-\frac{e^{b_2} b_1}{b_2}, 2, 2, \infty \right] + \delta a[e^{b_2}, 1, \infty] + \delta aa \left[-\frac{-1 + e^{-b_1} + b_1}{b_1^2}, 1, 2, 1, \infty \right] + \delta aa \left[\frac{e^{b_2}}{b_2}, 1, 2, 2, \infty \right]$$

R[1, 2]@**a**[1, 2, ∞] /. **rule2** // **S**

$$a[e^{b_1}, 2, \infty] + a \left[-\frac{(-1 + e^{b_1}) b_2}{b_1}, 1, \infty \right] +$$

$$c \left[-\frac{(-1 + e^{b_1}) b_2 + b_1 (-1 + e^{b_1} + (e^{b_1} - e^{b_2} + e^{b_1+b_2}) b_2)}{b_1}, \infty \right] + ca \left[\frac{e^{b_2} (-1 + e^{b_1}) b_2}{b_1}, \infty, 1, 2 \right] +$$

$$ca \left[\frac{(-1 + e^{b_1}) (1 + e^{b_2} b_2)}{b_2}, 2, 2, \infty \right] + ca \left[-\frac{e^{-b_1} (-1 + e^{b_1}) (-1 + e^{b_1} + e^{b_1+b_2} b_2)}{b_1}, 2, 1, \infty \right] +$$

$$\delta a \left[\frac{(-1 + e^{b_1}) b_2 - b_1 (-1 + e^{b_1} + (e^{b_1} - e^{b_2} + e^{b_1+b_2}) b_2)}{b_1^2}, 1, \infty \right] +$$

$$\delta aa \left[\frac{e^{-b_1} (-1 + e^{b_1})^2}{b_1^2}, 1, 2, 1, \infty \right] + \delta aa \left[-\frac{(-1 + e^{b_1}) (1 + e^{b_2} b_2)}{b_1 b_2}, 1, 2, 2, \infty \right]$$