

Pensieve header: Finding the most general R in UU notation.

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SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\OneCo-1604";
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<< MostGeneralR-UU.m;
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ExportButton
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In the $U(T) \otimes U(H)$ conventions. Internal use symbols: {rr, pp}

Export

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gg2|4|5|7[_] = 0; gg6[x_] :=  $\frac{2-x}{2x^2}$ ; gg8[x_] := 1/x; cc1 = 0;
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Column@Table[x → CF[AForm[(δaForm@UU@x) // R[1, 2]]],
  {x, {β[f[b1, b2]], a[1, 1, h∞], a[1, 2, h∞], a[f[b1, b2], 1, h∞], a[f[b1, b2], 2, h∞],
    c[1, h∞], ao[1, 1, h∞], ao[1, 2, h∞], ca[1, 1, 1, h∞], ca[1, 2, 1, h∞],
    ca[1, h∞, 1, 1], ca[1, h∞, 1, 2], ca[1, h∞, 2, 1], ca[1, h∞, 2, 2],
    ca[1, 1, 2, h∞], ca[1, 2, 2, h∞], aao[1, 1, 1, 1, h∞], aao[1, 1, 2, 1, h∞],
    aao[1, 1, 1, 2, h∞], aao[1, 1, 2, 2, h∞], aao[1, 2, 1, 2, h∞], aao[1, 2, 2, 2, h∞]}}
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$$\begin{aligned}
 & \beta[f[b_1, b_2]] \rightarrow UU\left[ao\left[-\frac{e^{-b_1}(-1+e^{b_1})(f^{(0,1)}[b_1, b_2]-f^{(1,0)}[b_1, b_2])}{b_1}, 1, 2\right] + \beta[f[b_1, b_2]]\right] \\
 & a[1, 1, h\infty] \rightarrow \\
 & \quad UU\left[a[1, 1, h\infty] + aao\left[-\frac{-1+e^{-b_1}+b_1}{b_1^2}, 1, 2, 1, h\infty\right] + aao\left[\frac{1}{b_2}, 1, 2, 2, h\infty\right] + ca\left[\frac{1-e^{-b_1}}{b_1}, h\infty, 1, 2\right]\right] \\
 & a[1, 2, h\infty] \rightarrow UU\left[a\left[e^{b_1}, 2, h\infty\right] + a\left[-\frac{(-1+e^{b_1})b_2}{b_1}, 1, h\infty\right] + aao\left[\frac{e^{-b_1}(-1+e^{b_1})^2}{b_1^2}, 1, 2, 1, h\infty\right] + \right. \\
 & \quad \left. aao\left[\frac{1-e^{-b_1}}{b_1 b_2}, 1, 2, 2, h\infty\right] + ao\left[\frac{-1+e^{b_1}-e^{b_1}b_2}{b_1}, 1, h\infty\right] + ca\left[-\frac{1-e^{-b_1}}{b_1}, h\infty, 1, 2\right]\right] \\
 & a[f[b_1, b_2], 1, h\infty] \rightarrow UU\left[a[f[b_1, b_2], 1, h\infty] + aao\left[\frac{f[b_1, b_2]}{b_2}, 1, 2, 2, h\infty\right] + \right. \\
 & \quad \left. aao\left[\frac{e^{-b_1}(f[b_1, b_2](-1+e^{b_1}-e^{b_1}b_1)-(-1+e^{b_1})b_1(f^{(0,1)}[b_1, b_2]-f^{(1,0)}[b_1, b_2]))}{b_1^2}, 1, 2, 1, h\infty\right] + \right. \\
 & \quad \left. ca\left[\frac{e^{-b_1}(-1+e^{b_1})(f[b_1, b_2]+b_1(-f^{(0,1)}[b_1, b_2]+f^{(1,0)}[b_1, b_2]))}{b_1}, h\infty, 1, 2\right]\right] \\
 & a[f[b_1, b_2], 2, h\infty] \rightarrow UU\left[a\left[e^{b_1}f[b_1, b_2], 2, h\infty\right] + a\left[-\frac{(-1+e^{b_1})f[b_1, b_2]b_2}{b_1}, 1, h\infty\right] + \right. \\
 & \quad \left. aao\left[\frac{e^{-b_1}(-1+e^{b_1})^2(f[b_1, b_2]+b_2(f^{(0,1)}[b_1, b_2]-f^{(1,0)}[b_1, b_2]))}{b_1^2}, 1, 2, 1, h\infty\right] + \right. \\
 & \quad \left. aao\left[-\frac{(-1+e^{b_1})(f[b_1, b_2]+b_2(f^{(0,1)}[b_1, b_2]-f^{(1,0)}[b_1, b_2]))}{b_1 b_2}, 1, 2, 2, h\infty\right] + \right. \\
 & \quad \left. ao\left[\frac{f[b_1, b_2](-1+e^{b_1}-e^{b_1}b_2)+(-1+e^{b_1})b_2(f^{(0,1)}[b_1, b_2]-f^{(1,0)}[b_1, b_2])}{b_1}, 1, h\infty\right] + \right. \\
 & \quad \left. ca\left[-\frac{e^{-b_1}(-1+e^{b_1})(f[b_1, b_2]+b_2(f^{(0,1)}[b_1, b_2]-f^{(1,0)}[b_1, b_2]))}{b_1}, h\infty, 1, 2\right]\right] \\
 & c[1, h\infty] \rightarrow UU[c[1, h\infty]] \\
 & ao[1, 1, h\infty] \rightarrow UU[ao[1, 1, h\infty]] \\
 & ao[1, 2, h\infty] \rightarrow UU\left[ao\left[e^{b_1}, 2, h\infty\right] + ao\left[-\frac{(-1+e^{b_1})b_2}{b_1}, 1, h\infty\right]\right] \\
 & ca[1, 1, 1, h\infty] \rightarrow UU\left[aa\left[-\frac{1-e^{-b_1}}{b_1}, 1, 2, 1, h\infty\right] + ca[1, 1, 1, h\infty] + ca\left[-1+e^{-b_1}, h\infty, 1, 2\right]\right] \\
 & ca[1, 2, 1, h\infty] \rightarrow UU\left[aa\left[\frac{1-e^{-b_1}}{b_1}, 1, 2, 1, h\infty\right] + ca[1, 2, 1, h\infty] + ca\left[1-e^{-b_1}, h\infty, 1, 2\right]\right] \\
 & ca[1, h\infty, 1, 1] \rightarrow UU[ca[1, h\infty, 1, 1]] \\
 & ca[1, h\infty, 1, 2] \rightarrow UU[ca[1, h\infty, 1, 2]] \\
 & ca[1, h\infty, 2, 1] \rightarrow UU\left[ca\left[e^{b_1}, h\infty, 2, 1\right] + \right. \\
 & \quad \left. ca\left[-1+e^{b_1}, h\infty, 2, 2\right] + ca\left[-\frac{(-1+e^{b_1})b_2}{b_1}, h\infty, 1, 1\right] + ca\left[-\frac{(-1+e^{b_1})b_2}{b_1}, h\infty, 1, 2\right]\right] \\
 & ca[1, h\infty, 2, 2] \rightarrow UU[ca[1, h\infty, 2, 2]] \\
 & ca[1, 1, 2, h\infty] \rightarrow UU\left[aa\left[\frac{1-e^{-b_1}}{b_1}, 1, 2, 2, h\infty\right] + aao\left[\frac{e^{-b_1}(-1+e^{b_1})^2 b_2}{b_1^2}, 1, 2, 1, h\infty\right] + \right. \\
 & \quad \left. ca\left[e^{b_1}, 1, 2, h\infty\right] + ca\left[-\frac{(1-e^{-b_1})b_2}{b_1}, h\infty, 1, 2\right] + ca\left[-\frac{(-1+e^{b_1})b_2}{b_1}, 1, 1, h\infty\right]\right] \\
 & ca[1, 2, 2, h\infty] \rightarrow UU\left[aa\left[\frac{-1+e^{b_1}}{b_1}, 1, 2, 2, h\infty\right] + aao\left[-\frac{e^{-b_1}(-1+e^{b_1})^2 b_2}{b_1^2}, 1, 2, 1, h\infty\right] + \right. \\
 & \quad \left. ca\left[e^{b_1}, 2, 2, h\infty\right] + ca\left[\frac{(1-e^{-b_1})b_2}{b_1}, h\infty, 1, 2\right] + ca\left[-\frac{(-1+e^{b_1})b_2}{b_1}, 2, 1, h\infty\right]\right] \\
 & aao[1, 1, 1, 1, h\infty] \rightarrow \\
 & \quad UU\left[aa\left[1, 1, 1, 1, h\infty\right] + aao\left[1-e^{-b_1}, 1, 2, 1, h\infty\right] + ca\left[(1-e^{-b_1})b_1, h\infty, 1, 2\right]\right] \\
 & aao[1, 1, 2, 1, h\infty] \rightarrow UU\left[aa\left[e^{-b_1}, 1, 2, 1, h\infty\right] + ca\left[(-1+e^{-b_1})b_1, h\infty, 1, 2\right]\right] \\
 & aao[1, 1, 1, 2, h\infty] \rightarrow \\
 & \quad UU\left[aa\left[e^{b_1}, 1, 1, 2, h\infty\right] + aao\left[-1+e^{b_1}, 1, 2, 2, h\infty\right] + aao\left[-\frac{(-1+e^{b_1})b_2}{b_1}, 1, 1, 1, h\infty\right] + \right. \\
 & \quad \left. aao\left[-\frac{e^{-b_1}(-1+e^{b_1})^2 b_2}{b_1}, 1, 2, 1, h\infty\right] + ca\left[(1-e^{-b_1})b_2, h\infty, 1, 2\right]\right] \\
 & aao[1, 1, 2, 2, h\infty] \rightarrow \\
 & \quad UU\left[aa\left[1, 1, 2, 2, h\infty\right] + aao\left[-\frac{(1-e^{-b_1})b_2}{b_1}, 1, 2, 1, h\infty\right] + ca\left[-(1-e^{-b_1})b_2, h\infty, 1, 2\right]\right] \\
 & aao[1, 2, 1, 2, h\infty] \rightarrow UU\left[aa\left[e^{2b_1}, 2, 1, 2, h\infty\right] + aao\left[e^{b_1}(-1+e^{b_1}), 2, 2, 2, h\infty\right] + \right. \\
 & \quad \left. aao\left[-\frac{2e^{b_1}(-1+e^{b_1})b_2}{b_1}, 1, 1, 2, h\infty\right] + aao\left[-\frac{2(-1+e^{b_1})^2 b_2}{b_1}, 1, 2, 2, h\infty\right] + \right. \\
 & \quad \left. aao\left[\frac{(-1+e^{b_1})^2 b_2^2}{b_1^2}, 1, 1, 1, h\infty\right] + aao\left[\frac{e^{-b_1}(-1+e^{b_1})^3 b_2^2}{b_1^2}, 1, 2, 1, h\infty\right] + ca\left[\frac{e^{-b_1}(-1+e^{b_1})b_2^2}{b_1}, h\infty, 1, 2\right]\right] \\
 & aao[1, 2, 2, 2, h\infty] \rightarrow UU\left[aa\left[e^{b_1}, 2, 2, 2, h\infty\right] + aao\left[-\frac{2(-1+e^{b_1})b_2}{b_1}, 1, 2, 2, h\infty\right] + \right. \\
 & \quad \left. aao\left[\frac{e^{-b_1}(-1+e^{b_1})^2 b_2^2}{b_1^2}, 1, 2, 1, h\infty\right] + ca\left[-\frac{e^{-b_1}(-1+e^{b_1})b_2^2}{b_1}, h\infty, 1, 2\right]\right]
 \end{aligned}$$

Turbo Gassner is spanned by v_i (for a), w_i (for ao), z_{ij} (for $c[i,j,h\infty]$), and $u_{ij,k}$ (for $aao[i,k,j,h\infty]$). Also, $t_i := e^{b_i}$.

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τ[expr_] := Simplify[expr /. b_i_ :=> Log[t_i] /. {
  t_j f^(1,0)[t_j, t_k] - t_k f^(0,1)[t_j, t_k] -> ε_{j,k}[f],
  t_k f^(0,1)[t_j, t_k] - t_j f^(1,0)[t_j, t_k] -> -ε_{j,k}[f],
  f[t_j, t_k] -> f
}];
TGData = Table[
  y = x /. f -> f[t_j, t_k] /. t_i_ :=> e^{b_i} /. {
    f_ . v_i_ :=> a[f/b_i, i, h∞],
    w_i_ :=> ao[1/b_i, i, h∞],
    u_{i_,j_,k_} :=> aao[1/(b_i b_j), i, k, j, h∞]
  };
  x -> ÅForm[(δaForm@UU@y) // R[j, k]] [[1]] /. {
    ca[_ , h∞, __] -> 0,
    a[f_, i_, h∞] :=> τ[b_i f] v_i,
    ao[f_, i_, h∞] :=> τ[b_i f] w_i,
    aao[f_, i_, j_, k_, h∞] | aao[f_, k_, h∞, i_, j_] :=> τ[b_i b_k f] u_{i,k,j}
  },
  {x, {f v_j, f v_k, f v_i, w_j, w_k, w_i, u_{j,j,j}, u_{j,j,k}, u_{j,k,j}, u_{j,k,k}, u_{k,j,j}, u_{k,j,k},
    u_{k,k,j}, u_{k,k,k}, u_{j,j,1}, u_{j,k,1}, u_{k,j,1}, u_{k,k,1}, u_{i,j,j}, u_{i,j,k}, u_{i,k,j}, u_{i,k,k}, u_{j,i,j},
    u_{j,i,k}, u_{k,i,j}, u_{k,i,k}, u_{j,i,1}, u_{k,i,1}, u_{i,j,1}, u_{i,k,1}, u_{i1,i2,j}, u_{i1,i2,k}, u_{i1,i2,1}}}
];
Column@TGData

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$$\begin{aligned}
f v_j &\rightarrow f v_j + f u_{k,j,k} + u_{j,j,k} \left(-f + \left(1 - \frac{1}{t_j} \right) \varepsilon_{j,k}[f] \right) \\
f v_k &\rightarrow (f - f t_j) v_j + f t_j v_k - \frac{(-1+t_j)^2 u_{j,i,k} \varepsilon_{j,k}[f]}{t_j} + \\
&\quad (-1 + t_j) u_{k,j,k} \varepsilon_{j,k}[f] + w_j (\varepsilon_{j,k}[f] - t_j (f + \varepsilon_{j,k}[f])) \\
f v_i &\rightarrow f v_i + \frac{(-1+t_j) u_{j,i,k} \varepsilon_{j,k}[f]}{t_j} \\
w_j &\rightarrow w_j \\
w_k &\rightarrow (1 - t_j) w_j + t_j w_k \\
w_i &\rightarrow w_i \\
u_{j,j,j} &\rightarrow u_{j,j,j} + \left(1 - \frac{1}{t_j} \right) u_{j,j,k} \\
u_{j,j,k} &\rightarrow \frac{u_{j,i,k}}{t_j} \\
u_{j,k,j} &\rightarrow (1 - t_j) u_{j,j,j} - \frac{(-1+t_j)^2 u_{j,i,k}}{t_j} + t_j u_{k,j,j} + (-1 + t_j) u_{k,j,k} \\
u_{j,k,k} &\rightarrow \left(-1 + \frac{1}{t_j} \right) u_{j,j,k} + u_{k,j,k} \\
u_{k,j,j} &\rightarrow (1 - t_j) u_{j,j,j} - \frac{(-1+t_j)^2 u_{j,i,k}}{t_j} + t_j u_{k,j,j} + (-1 + t_j) u_{k,j,k} \\
u_{k,j,k} &\rightarrow \left(-1 + \frac{1}{t_j} \right) u_{j,j,k} + u_{k,j,k} \\
u_{k,k,j} &\rightarrow (-1 + t_j)^2 u_{j,j,j} + \frac{(-1+t_j)^3 u_{j,i,k}}{t_j} - \\
&\quad 2(-1 + t_j) t_j u_{k,j,j} - 2(-1 + t_j)^2 u_{k,j,k} + t_j^2 u_{k,k,j} + (-1 + t_j) t_j u_{k,k,k} \\
u_{k,k,k} &\rightarrow \left(-2 + \frac{1}{t_j} + t_j \right) u_{j,j,k} + (2 - 2 t_j) u_{k,j,k} + t_j u_{k,k,k} \\
u_{j,j,1} &\rightarrow u_{j,j,1} \\
u_{j,k,1} &\rightarrow (1 - t_j) u_{j,j,1} + t_j u_{k,j,1} \\
u_{k,j,1} &\rightarrow (1 - t_j) u_{j,j,1} + t_j u_{k,j,1} \\
u_{k,k,1} &\rightarrow (-1 + t_j)^2 u_{j,j,1} - 2(-1 + t_j) t_j u_{k,j,1} + t_j^2 u_{k,k,1} \\
u_{i,j,j} &\rightarrow u_{j,i,j} + \left(1 - \frac{1}{t_j} \right) u_{j,i,k} \\
u_{i,j,k} &\rightarrow \frac{u_{j,i,k}}{t_j} \\
u_{i,k,j} &\rightarrow (1 - t_j) u_{j,i,j} - \frac{(-1+t_j)^2 u_{j,i,k}}{t_j} + t_j u_{k,i,j} + (-1 + t_j) u_{k,i,k} \\
u_{i,k,k} &\rightarrow \left(-1 + \frac{1}{t_j} \right) u_{j,i,k} + u_{k,i,k} \\
u_{j,i,j} &\rightarrow u_{j,i,j} + \left(1 - \frac{1}{t_j} \right) u_{j,i,k} \\
u_{j,i,k} &\rightarrow \frac{u_{j,i,k}}{t_j} \\
u_{k,i,j} &\rightarrow (1 - t_j) u_{j,i,j} - \frac{(-1+t_j)^2 u_{j,i,k}}{t_j} + t_j u_{k,i,j} + (-1 + t_j) u_{k,i,k} \\
u_{k,i,k} &\rightarrow \left(-1 + \frac{1}{t_j} \right) u_{j,i,k} + u_{k,i,k} \\
u_{j,i,1} &\rightarrow u_{j,i,1} \\
u_{k,i,1} &\rightarrow (1 - t_j) u_{j,i,1} + t_j u_{k,i,1} \\
u_{i,j,1} &\rightarrow u_{j,i,1} \\
u_{i,k,1} &\rightarrow (1 - t_j) u_{j,i,1} + t_j u_{k,i,1} \\
u_{i1,i2,j} &\rightarrow u_{i2,i1,j} + \left(1 - \frac{1}{t_j} \right) u_{i2,i1,k} \\
u_{i1,i2,k} &\rightarrow \frac{u_{i2,i1,k}}{t_j} \\
u_{i1,i2,1} &\rightarrow u_{i2,i1,1}
\end{aligned}$$

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Fs = First /@ TGData; Ts = Last /@ TGData;
Fs = Fs /. {f -> f_., i -> i_., i1 -> i1_., i2 -> i2_., l -> l_};
TGRule = Thread[Fs -> Ts] /. Rule -> RuleDelayed /.  $\delta_{j_-,k_-}[f_-] \Rightarrow t_j \partial_{t_j} f - t_k \partial_{t_k} f$ ;
AppendTo[TGRule, u_-,_- -> Error]

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$$\{f_- . v_j \Rightarrow f v_j + f u_{k,j,k} + u_{j,j,k} \left(-f + \left(1 - \frac{1}{t_j}\right) (t_j \partial_{t_j} f - t_k \partial_{t_k} f) \right),$$

$$f_- . v_k \Rightarrow (f - f t_j) v_j + f t_j v_k - \frac{(-1 + t_j)^2 u_{j,j,k} (t_j \partial_{t_j} f - t_k \partial_{t_k} f)}{t_j} +$$

$$(-1 + t_j) u_{k,j,k} (t_j \partial_{t_j} f - t_k \partial_{t_k} f) + w_j ((t_j \partial_{t_j} f - t_k \partial_{t_k} f) - t_j (f + (t_j \partial_{t_j} f - t_k \partial_{t_k} f))),$$

$$f_- . v_{i_-} \Rightarrow f v_i + \frac{(-1 + t_j) u_{j,i,k} (t_j \partial_{t_j} f - t_k \partial_{t_k} f)}{t_j}, w_j \Rightarrow w_j, w_k \Rightarrow (1 - t_j) w_j + t_j w_k,$$

$$w_{i_-} \Rightarrow w_i, u_{j,j,j} \Rightarrow u_{j,j,j} + \left(1 - \frac{1}{t_j}\right) u_{j,j,k}, u_{j,j,k} \Rightarrow \frac{u_{j,j,k}}{t_j},$$

$$u_{j,k,j} \Rightarrow (1 - t_j) u_{j,j,j} - \frac{(-1 + t_j)^2 u_{j,j,k}}{t_j} + t_j u_{k,j,j} + (-1 + t_j) u_{k,j,k},$$

$$u_{j,k,k} \Rightarrow \left(-1 + \frac{1}{t_j}\right) u_{j,j,k} + u_{k,j,k},$$

$$u_{k,j,j} \Rightarrow (1 - t_j) u_{j,j,j} - \frac{(-1 + t_j)^2 u_{j,j,k}}{t_j} + t_j u_{k,j,j} + (-1 + t_j) u_{k,j,k},$$

$$u_{k,j,k} \Rightarrow \left(-1 + \frac{1}{t_j}\right) u_{j,j,k} + u_{k,j,k}, u_{k,k,j} \Rightarrow (-1 + t_j)^2 u_{j,j,j} + \frac{(-1 + t_j)^3 u_{j,j,k}}{t_j} -$$

$$2(-1 + t_j) t_j u_{k,j,j} - 2(-1 + t_j)^2 u_{k,j,k} + t_j^2 u_{k,k,j} + (-1 + t_j) t_j u_{k,k,k},$$

$$u_{k,k,k} \Rightarrow \left(-2 + \frac{1}{t_j} + t_j\right) u_{j,j,k} + (2 - 2 t_j) u_{k,j,k} + t_j u_{k,k,k}, u_{j,j,l_-} \Rightarrow u_{j,j,l},$$

$$u_{j,k,l_-} \Rightarrow (1 - t_j) u_{j,j,l} + t_j u_{k,j,l}, u_{k,j,l_-} \Rightarrow (1 - t_j) u_{j,j,l} + t_j u_{k,j,l},$$

$$u_{k,k,l_-} \Rightarrow (-1 + t_j)^2 u_{j,j,l} - 2(-1 + t_j) t_j u_{k,j,l} + t_j^2 u_{k,k,l}, u_{i_-,j,j} \Rightarrow u_{j,i,j} + \left(1 - \frac{1}{t_j}\right) u_{j,i,k},$$

$$u_{i_-,j,k} \Rightarrow \frac{u_{j,i,k}}{t_j}, u_{i_-,k,j} \Rightarrow (1 - t_j) u_{j,i,j} - \frac{(-1 + t_j)^2 u_{j,i,k}}{t_j} + t_j u_{k,i,j} + (-1 + t_j) u_{k,i,k},$$

$$u_{i_-,k,k} \Rightarrow \left(-1 + \frac{1}{t_j}\right) u_{j,i,k} + u_{k,i,k}, u_{j,i_-,j} \Rightarrow u_{j,i,j} + \left(1 - \frac{1}{t_j}\right) u_{j,i,k}, u_{j,i_-,k} \Rightarrow \frac{u_{j,i,k}}{t_j},$$

$$u_{k,i_-,j} \Rightarrow (1 - t_j) u_{j,i,j} - \frac{(-1 + t_j)^2 u_{j,i,k}}{t_j} + t_j u_{k,i,j} + (-1 + t_j) u_{k,i,k},$$

$$u_{k,i_-,k} \Rightarrow \left(-1 + \frac{1}{t_j}\right) u_{j,i,k} + u_{k,i,k}, u_{j,i_-,l_-} \Rightarrow u_{j,i,l}, u_{k,i_-,l_-} \Rightarrow (1 - t_j) u_{j,i,l} + t_j u_{k,i,l},$$

$$u_{i_-,j,l_-} \Rightarrow u_{j,i,l}, u_{i_-,k,l_-} \Rightarrow (1 - t_j) u_{j,i,l} + t_j u_{k,i,l}, u_{i1_-,i2_-,j} \Rightarrow u_{i2,i1,j} + \left(1 - \frac{1}{t_j}\right) u_{i2,i1,k},$$

$$u_{i1_-,i2_-,k} \Rightarrow \frac{u_{i2,i1,k}}{t_j}, u_{i1_-,i2_-,l_-} \Rightarrow u_{i2,i1,l}, u_-,_- - \Rightarrow \text{Error}\}$$

```

σj,k = . ;
SetDelayed@@ {σj,k, Expand@*
  ReplaceAll[ujj,ii,kk /; ! OrderedQ[{jj, ii}] => uii,jj,kk]@*ReplaceAll[TGRule]}
checks = {f[t1, t2, t3] v1, f[t1, t2, t3] v2, f[t1, t2, t3] v3, w1, w2, w3, u1,1,1, u1,1,2, u1,2,1,
  u1,2,2, u1,2,1, u1,2,2, u2,2,1, u2,2,2, u1,1,3, u1,2,3, u1,2,3, u2,2,3, u0,1,1, u0,1,2, u0,2,1,
  u0,2,2, u0,1,1, u0,1,2, u0,2,1, u0,2,2, u0,1,3, u0,2,3, u0,1,3, u0,2,3, u0,1,1, u0,1,2, u0,1,3};
Column[R31 = (# // σ1,2 // σ1,3 // σ2,3 // Simplify) & /@checks]

```

$$\frac{1}{t_1 t_2} \left(f[t_1, t_2, t_3] t_1 t_2 (v_1 - u_{1,1,2} - u_{1,1,3} + u_{1,2,2} + u_{1,3,3}) + \right.$$

$$t_3 \left((-1 + t_1) u_{1,1,3} - t_1 (-1 + t_2) u_{1,2,3} \right) f^{(0,0,1)}[t_1, t_2, t_3] +$$

$$t_2 \left(t_2 \left((-1 + t_1) u_{1,1,2} - (-1 + t_1) u_{1,1,3} + t_1 u_{1,2,3} \right) f^{(0,1,0)}[t_1, t_2, t_3] + \right.$$

$$\left. (-1 + t_1) u_{1,1,3} \left(f^{(0,1,0)}[t_1, t_2, t_3] + t_1 f^{(1,0,0)}[t_1, t_2, t_3] \right) + \right.$$

$$\left. t_1 \left(-u_{1,2,3} f^{(0,1,0)}[t_1, t_2, t_3] + (-1 + t_1) u_{1,1,2} f^{(1,0,0)}[t_1, t_2, t_3] \right) \right)$$

$$\frac{1}{t_1 t_2} \left(f[t_1, t_2, t_3] t_1 t_2 \left((-1 + t_1) v_1 - u_{1,2,3} + u_{1,3,3} + t_1 (v_2 - w_1 + u_{1,2,3} - u_{1,3,3} - u_{2,2,3} + u_{2,3,3}) \right) + \right.$$

$$t_3 \left((-1 + t_1)^2 u_{1,1,3} + t_1 \left((-1 + t_1) (-2 + t_2) u_{1,2,3} - t_1 (-1 + t_2) u_{2,2,3} \right) \right)$$

$$f^{(0,0,1)}[t_1, t_2, t_3] +$$

$$t_2 \left((-u_{1,1,3} + t_2 (u_{1,1,2} + u_{1,1,3})) f^{(0,1,0)}[t_1, t_2, t_3] - t_1^3 (w_1 + u_{1,1,2} + u_{1,1,3} - u_{1,2,2} - u_{1,2,3}) \right.$$

$$f^{(1,0,0)}[t_1, t_2, t_3] + t_1^2 \left(2 u_{1,2,3} f^{(0,1,0)}[t_1, t_2, t_3] - u_{2,2,3} f^{(0,1,0)}[t_1, t_2, t_3] + \right.$$

$$t_2 (w_1 + u_{1,1,2} + u_{1,1,3} - u_{1,2,2} - 2 u_{1,2,3} + u_{2,2,3}) f^{(0,1,0)}[t_1, t_2, t_3] -$$

$$u_{1,1,3} \left(f^{(0,1,0)}[t_1, t_2, t_3] - 2 f^{(1,0,0)}[t_1, t_2, t_3] \right) + w_1 f^{(1,0,0)}[t_1, t_2, t_3] +$$

$$2 u_{1,1,2} f^{(1,0,0)}[t_1, t_2, t_3] - u_{1,2,2} f^{(1,0,0)}[t_1, t_2, t_3] - u_{1,2,3} f^{(1,0,0)}[t_1, t_2, t_3] \left. \right) -$$

$$t_1 \left(t_2 (w_1 + 2 u_{1,1,2} + 2 u_{1,1,3} - u_{1,2,2} - 2 u_{1,2,3}) f^{(0,1,0)}[t_1, t_2, t_3] + \right.$$

$$2 u_{1,2,3} f^{(0,1,0)}[t_1, t_2, t_3] + u_{1,1,2} f^{(1,0,0)}[t_1, t_2, t_3] +$$

$$\left. u_{1,1,3} \left(-2 f^{(0,1,0)}[t_1, t_2, t_3] + f^{(1,0,0)}[t_1, t_2, t_3] \right) \right)$$

$$\frac{1}{t_1 t_2} \left(-f[t_1, t_2, t_3] t_1 t_2 ((-1 + t_1) v_1 + t_1 ((-1 + t_2) v_2 + w_1 + t_2 (-v_3 + w_2))) + \right. \\
t_3 u_{1,1,3} f^{(0,0,1)} [t_1, t_2, t_3] + t_2^2 u_{1,1,2} f^{(0,1,0)} [t_1, t_2, t_3] - \\
t_2 u_{1,1,3} f^{(0,1,0)} [t_1, t_2, t_3] + t_2^2 u_{1,1,3} f^{(0,1,0)} [t_1, t_2, t_3] - \\
t_1^3 t_2 (w_1 + u_{1,1,2} + u_{1,1,3} - u_{1,2,2} + t_2 u_{1,2,2} - u_{1,2,3} + t_2 u_{1,2,3} - t_2 u_{1,3,2} - t_2 u_{1,3,3}) \\
f^{(1,0,0)} [t_1, t_2, t_3] + t_1^2 (t_3 (u_{1,1,3} - 2 u_{1,2,3} + u_{2,2,3}) f^{(0,0,1)} [t_1, t_2, t_3] - \\
t_2^2 (w_2 - u_{1,2,2} - u_{1,2,3} + u_{1,3,2} + u_{1,3,3} + u_{2,2,3} - u_{2,3,3}) f^{(0,1,0)} [t_1, t_2, t_3] + \\
t_2 (t_3 (w_1 - w_2 + 2 u_{1,2,3} - u_{1,3,3} - 2 u_{2,2,3} + u_{2,3,3}) f^{(0,0,1)} [t_1, t_2, t_3] + \\
2 u_{1,2,3} f^{(0,1,0)} [t_1, t_2, t_3] - u_{2,2,3} f^{(0,1,0)} [t_1, t_2, t_3] - \\
u_{1,1,3} (f^{(0,1,0)} [t_1, t_2, t_3] - 2 f^{(1,0,0)} [t_1, t_2, t_3]) + w_1 f^{(1,0,0)} [t_1, t_2, t_3] + \\
2 u_{1,1,2} f^{(1,0,0)} [t_1, t_2, t_3] - u_{1,2,2} f^{(1,0,0)} [t_1, t_2, t_3] - u_{1,2,3} f^{(1,0,0)} [t_1, t_2, t_3]) + \\
t_2^2 (t_3 (w_2 + u_{2,2,3} - u_{2,3,3}) f^{(0,0,1)} [t_1, t_2, t_3] + w_2 f^{(0,1,0)} [t_1, t_2, t_3] + \\
u_{1,1,2} f^{(0,1,0)} [t_1, t_2, t_3] + u_{1,1,3} f^{(0,1,0)} [t_1, t_2, t_3] - u_{1,2,2} f^{(0,1,0)} [t_1, t_2, t_3] - \\
3 u_{1,2,3} f^{(0,1,0)} [t_1, t_2, t_3] + u_{1,3,3} f^{(0,1,0)} [t_1, t_2, t_3] + 2 u_{2,2,3} f^{(0,1,0)} [t_1, t_2, t_3] - \\
u_{2,3,3} f^{(0,1,0)} [t_1, t_2, t_3] + u_{1,2,2} f^{(1,0,0)} [t_1, t_2, t_3] + u_{1,2,3} f^{(1,0,0)} [t_1, t_2, t_3] - \\
u_{1,3,2} f^{(1,0,0)} [t_1, t_2, t_3] - u_{1,3,3} f^{(1,0,0)} [t_1, t_2, t_3]) \left. \right) - \\
t_1 (2 t_3 (u_{1,1,3} - u_{1,2,3}) f^{(0,0,1)} [t_1, t_2, t_3] + t_2^2 (u_{1,2,2} + u_{1,2,3} - u_{1,3,2} - u_{1,3,3}) f^{(0,1,0)} [\\
t_1, t_2, t_3] + t_2^2 (2 u_{1,1,2} + 2 u_{1,1,3} - u_{1,2,2} - 3 u_{1,2,3} + u_{1,3,3}) f^{(0,1,0)} [t_1, t_2, t_3] + \\
t_2 (t_3 (w_1 + 2 u_{1,2,3} - u_{1,3,3}) f^{(0,0,1)} [t_1, t_2, t_3] + 2 u_{1,2,3} f^{(0,1,0)} [t_1, t_2, t_3] + \\
u_{1,1,2} f^{(1,0,0)} [t_1, t_2, t_3] + u_{1,1,3} (-2 f^{(0,1,0)} [t_1, t_2, t_3] + f^{(1,0,0)} [t_1, t_2, t_3])))$$

w_1

$$-(-1 + t_1) w_1 + t_1 w_2 \\
-(-1 + t_1) w_1 + t_1 (-(-1 + t_2) w_2 + t_2 w_3)$$

$$u_{1,1,1} + \frac{(-1+t_1)(u_{1,1,2}+u_{1,1,3})}{t_1} \\
\frac{-u_{1,1,3}+t_2(u_{1,1,2}+u_{1,1,3})}{t_1 t_2}$$

$$\frac{1}{t_1} (-u_{1,1,2} - u_{1,1,3} + t_1 (u_{1,1,1} + 2 u_{1,1,2} + 2 u_{1,1,3} - u_{1,2,2} - u_{1,2,3}) + \\
t_1^2 (-u_{1,1,1} - u_{1,1,2} - u_{1,1,3} + u_{1,2,1} + u_{1,2,2} + u_{1,2,3})) \\
\frac{(-1+t_1) u_{1,1,3} - t_1 u_{1,2,3} + t_2 (-(-1+t_1) u_{1,1,2} - (-1+t_1) u_{1,1,3} + t_1 (u_{1,2,2} + u_{1,2,3}))}{t_1 t_2}$$

$$\frac{1}{t_1} (-u_{1,1,2} - u_{1,1,3} + t_1 (u_{1,1,1} + 2 u_{1,1,2} + 2 u_{1,1,3} - u_{1,2,2} - u_{1,2,3}) + \\
t_1^2 (-u_{1,1,1} - u_{1,1,2} - u_{1,1,3} + u_{1,2,1} + u_{1,2,2} + u_{1,2,3})) \\
\frac{(-1+t_1) u_{1,1,3} - t_1 u_{1,2,3} + t_2 (-(-1+t_1) u_{1,1,2} - (-1+t_1) u_{1,1,3} + t_1 (u_{1,2,2} + u_{1,2,3}))}{t_1 t_2}$$

$$-\frac{1}{t_1} (u_{1,1,2} + u_{1,1,3} - t_1 (u_{1,1,1} + 3 u_{1,1,2} + 3 u_{1,1,3} - 2 u_{1,2,2} - 2 u_{1,2,3}) + \\
t_1^2 (2 u_{1,1,1} + 3 u_{1,1,2} + 3 u_{1,1,3} - 2 u_{1,2,1} - 4 u_{1,2,2} - 4 u_{1,2,3} + u_{2,2,2} + u_{2,2,3}) - \\
t_1^3 (u_{1,1,1} + u_{1,1,2} + u_{1,1,3} - 2 u_{1,2,1} - 2 u_{1,2,2} - 2 u_{1,2,3} + u_{2,2,1} + u_{2,2,2} + u_{2,2,3}))$$

$$\frac{1}{t_1 t_2}$$

$$(-(-1 + t_1)^2 u_{1,1,3} + t_1 (2 (-1 + t_1) u_{1,2,3} - t_1 u_{2,2,3}) + t_2 ((-1 + t_1)^2 u_{1,1,2} + (-1 + t_1)^2 u_{1,1,3} + \\
t_1 (-2 (-1 + t_1) u_{1,2,2} - 2 (-1 + t_1) u_{1,2,3} + t_1 (u_{2,2,2} + u_{2,2,3})))$$

$$\frac{u_{1,1,3}}{t_1 t_2}$$

$$\frac{-(-1+t_1) u_{1,1,3} + t_1 u_{1,2,3}}{t_1 t_2}$$

$$\frac{-(-1+t_1) u_{1,1,3} + t_1 u_{1,2,3}}{t_1 t_2}$$

$$\frac{(-1+t_1)^2 u_{1,1,3} + t_1 (-2 (-1+t_1) u_{1,2,3} + t_1 u_{2,2,3})}{t_1 t_2}$$

$$u_{0,1,1} + \frac{(-1+t_1)(u_{0,1,2}+u_{0,1,3})}{t_1}$$

$$\frac{-u_{0,1,3}+t_2(u_{0,1,2}+u_{0,1,3})}{t_1 t_2}$$

$$\frac{1}{t_1} (-u_{1,1,2} - u_{1,1,3} + t_1 (u_{1,1,1} + 2 u_{1,1,2} + 2 u_{1,1,3} - u_{1,2,2} - u_{1,2,3}) + t_1^2 (-u_{1,1,1} - u_{1,1,2} - u_{1,1,3} + u_{1,2,1} + u_{1,2,2} + u_{1,2,3}))$$

$$\left(-1 + \frac{1}{t_1}\right) u_{1,1,2} + u_{1,2,2}$$

$$- \frac{1}{t_1} (u_{1,1,2} + u_{1,1,3} - t_1 (u_{1,1,1} + 3 u_{1,1,2} + 3 u_{1,1,3} - 2 u_{1,2,2} - 2 u_{1,2,3}) + t_1^2 (2 u_{1,1,1} + 3 u_{1,1,2} + 3 u_{1,1,3} - 2 u_{1,2,1} - 4 u_{1,2,2} - 4 u_{1,2,3} + u_{2,2,2} + u_{2,2,3}) - t_1^3 (u_{1,1,1} + u_{1,1,2} + u_{1,1,3} - 2 u_{1,2,1} - 2 u_{1,2,2} - 2 u_{1,2,3} + u_{2,2,1} + u_{2,2,2} + u_{2,2,3}))$$

$$\frac{(-1+t_1)^2 u_{1,1,2} + t_1 (-2 (-1+t_1) u_{1,2,2} + t_1 u_{2,2,2})}{t_1}$$

$$\frac{u_{1,1,3}}{t_1}$$

$$\left(-1 + \frac{1}{t_1}\right) u_{1,1,3} + u_{1,2,3}$$

$$\left(-1 + \frac{1}{t_1}\right) u_{1,1,3} + u_{1,2,3}$$

$$\frac{(-1+t_1)^2 u_{1,1,3} + t_1 (-2 (-1+t_1) u_{1,2,3} + t_1 u_{2,2,3})}{t_1}$$

$$u_{0,1,1} + \frac{(-1+t_1) (u_{0,1,2} + u_{0,1,3})}{t_1}$$

$$\frac{u_{0,1,2}}{t_1}$$

$$\frac{1}{t_1} (-u_{0,1,2} - u_{0,1,3} + t_1 (u_{0,1,1} + 2 u_{0,1,2} + 2 u_{0,1,3} - u_{0,2,2} - u_{0,2,3}) + t_1^2 (-u_{0,1,1} - u_{0,1,2} - u_{0,1,3} + u_{0,2,1} + u_{0,2,2} + u_{0,2,3}))$$

$$\left(-1 + \frac{1}{t_1}\right) u_{0,1,2} + u_{0,2,2}$$

$$u_{0,1,1} + \frac{(-1+t_1) (u_{0,1,2} + u_{0,1,3})}{t_1}$$

$$\frac{u_{0,1,2}}{t_1}$$

$$\frac{1}{t_1} (-u_{0,1,2} - u_{0,1,3} + t_1 (u_{0,1,1} + 2 u_{0,1,2} + 2 u_{0,1,3} - u_{0,2,2} - u_{0,2,3}) + t_1^2 (-u_{0,1,1} - u_{0,1,2} - u_{0,1,3} + u_{0,2,1} + u_{0,2,2} + u_{0,2,3}))$$

$$\left(-1 + \frac{1}{t_1}\right) u_{0,1,2} + u_{0,2,2}$$

$$\frac{u_{0,1,3}}{t_1}$$

$$\left(-1 + \frac{1}{t_1}\right) u_{0,1,3} + u_{0,2,3}$$

$$\frac{u_{0,1,3}}{t_1}$$

$$\left(-1 + \frac{1}{t_1}\right) u_{0,1,3} + u_{0,2,3}$$

$$u_{0,1,1} + \frac{(-1+t_1) (u_{0,1,2} + u_{0,1,3})}{t_1}$$

$$\frac{u_{0,1,2}}{t_1}$$

$$\frac{u_{0,1,3}}{t_1}$$

OCr = (# // $\sigma_{1,3}$ // $\sigma_{1,2}$ // **Simplify**) & /@ checks;

OC1 - OCr // **Simplify**

$$\{0, f[t_1, t_2, t_3] (-1+t_1) (u_{1,2,3} - u_{1,3,3}), f[t_1, t_2, t_3] (-1+t_1) (u_{1,2,2} - u_{1,3,2}), 0\}$$

Column[**UC1** = (# // $\sigma_{1,3}$ // $\sigma_{2,3}$ // **Simplify**) & /@ checks]

$$\frac{1}{t_1 t_2} \left(f[t_1, t_2, t_3] t_1 (-u_{1,1,3} + u_{1,2,3} + t_2 (v_1 - u_{1,2,3} + u_{1,3,3})) + t_3 (-(-1+t_1) u_{1,1,3} - t_1 (-1+t_2) u_{1,2,3}) f^{(0,0,1)}[t_1, t_2, t_3] + t_1 (-t_2 u_{1,2,3} f^{(0,1,0)}[t_1, t_2, t_3] + t_2^2 u_{1,2,3} f^{(0,1,0)}[t_1, t_2, t_3] + (-1+t_1) u_{1,1,3} f^{(1,0,0)}[t_1, t_2, t_3]) \right)$$

$$\begin{aligned} & \frac{1}{t_1 t_2} \left(f[t_1, t_2, t_3] t_1 t_2 (v_2 - u_{2,2,3} + u_{2,3,3}) + \right. \\ & t_3 (-(-1 + t_1) u_{1,2,3} - t_1 (-1 + t_2) u_{2,2,3}) f^{(0,0,1)}[t_1, t_2, t_3] + \\ & t_1 (-t_2 u_{2,2,3} f^{(0,1,0)}[t_1, t_2, t_3] + \\ & \left. t_2^2 u_{2,2,3} f^{(0,1,0)}[t_1, t_2, t_3] + (-1 + t_1) u_{1,2,3} f^{(1,0,0)}[t_1, t_2, t_3] \right) \\ & \frac{1}{t_1 t_2} \left(-f[t_1, t_2, t_3] t_1 t_2 ((-1 + t_1) v_1 + t_1 ((-1 + t_2) v_2 + w_1 + t_2 (-v_3 + w_2))) + t_3 u_{1,1,3} \right. \\ & f^{(0,0,1)}[t_1, t_2, t_3] - t_1^3 (u_{1,1,3} - u_{1,2,3} + t_2 (w_1 + u_{1,2,3} - u_{1,3,3})) f^{(1,0,0)}[t_1, t_2, t_3] + \\ & t_1 (2 t_3 (-u_{1,1,3} + u_{1,2,3}) f^{(0,0,1)}[t_1, t_2, t_3] + t_2^2 u_{1,2,3} f^{(0,1,0)}[t_1, t_2, t_3] - \\ & t_2 (t_3 (w_1 + 2 u_{1,2,3} - u_{1,3,3}) f^{(0,0,1)}[t_1, t_2, t_3] + u_{1,2,3} f^{(0,1,0)}[t_1, t_2, t_3]) - \\ & u_{1,1,3} f^{(1,0,0)}[t_1, t_2, t_3]) + \\ & t_1^2 (t_3 (u_{1,1,3} - 2 u_{1,2,3} + u_{2,2,3}) f^{(0,0,1)}[t_1, t_2, t_3] - t_2^3 (w_2 + u_{2,2,3} - u_{2,3,3}) \\ & f^{(0,1,0)}[t_1, t_2, t_3] + t_2^2 (t_3 (w_2 + u_{2,2,3} - u_{2,3,3}) f^{(0,0,1)}[t_1, t_2, t_3] + \\ & (w_2 - u_{1,2,3} + 2 u_{2,2,3} - u_{2,3,3}) f^{(0,1,0)}[t_1, t_2, t_3]) + \\ & (2 u_{1,1,3} - u_{1,2,3}) f^{(1,0,0)}[t_1, t_2, t_3] + t_2 (t_3 (w_1 - w_2 + 2 u_{1,2,3} - u_{1,3,3} - 2 u_{2,2,3} + u_{2,3,3}) \\ & f^{(0,0,1)}[t_1, t_2, t_3] - u_{2,2,3} f^{(0,1,0)}[t_1, t_2, t_3] + w_1 f^{(1,0,0)}[t_1, t_2, t_3] - \\ & \left. u_{1,3,3} f^{(1,0,0)}[t_1, t_2, t_3] + u_{1,2,3} (f^{(0,1,0)}[t_1, t_2, t_3] + f^{(1,0,0)}[t_1, t_2, t_3])) \right) \end{aligned}$$

w₁

w₂

$$-(-1 + t_1) w_1 + t_1 (-(-1 + t_2) w_2 + t_2 w_3)$$

$$u_{1,1,1} + \frac{(-1+t_1) u_{1,1,3}}{t_1 t_2}$$

$$u_{1,1,2} + \frac{(-1+t_2) u_{1,1,3}}{t_2}$$

$$u_{1,2,1} + \frac{(-1+t_1) u_{1,2,3}}{t_1 t_2}$$

$$u_{1,2,2} + \frac{(-1+t_2) u_{1,2,3}}{t_2}$$

$$u_{1,2,1} + \frac{(-1+t_1) u_{1,2,3}}{t_1 t_2}$$

$$u_{1,2,2} + \frac{(-1+t_2) u_{1,2,3}}{t_2}$$

$$u_{2,2,1} + \frac{(-1+t_1) u_{2,2,3}}{t_1 t_2}$$

$$u_{2,2,2} + \frac{(-1+t_2) u_{2,2,3}}{t_2}$$

$\frac{u_{1,1,3}}{t_1 t_2}$

$\frac{u_{1,2,3}}{t_1 t_2}$

$\frac{u_{1,2,3}}{t_1 t_2}$

$\frac{u_{1,2,3}}{t_1 t_2}$

$\frac{u_{2,2,3}}{t_1 t_2}$

$\frac{u_{2,2,3}}{t_1 t_2}$

$$u_{0,1,1} + \frac{(-1+t_1) u_{0,1,3}}{t_1 t_2}$$

$$u_{0,1,2} + \frac{(-1+t_2) u_{0,1,3}}{t_2}$$

$$u_{0,2,1} + \frac{(-1+t_1) u_{0,2,3}}{t_1 t_2}$$

$$u_{0,2,2} + \frac{(-1+t_2) u_{0,2,3}}{t_2}$$

$$u_{0,1,1} + \frac{(-1+t_1) u_{0,1,3}}{t_1 t_2}$$

$$u_{0,1,2} + \frac{(-1+t_2) u_{0,1,3}}{t_2}$$

$$u_{0,2,1} + \frac{(-1+t_1) u_{0,2,3}}{t_1 t_2}$$

$$u_{0,2,2} + \frac{(-1+t_2) u_{0,2,3}}{t_2}$$

$\frac{u_{0,1,3}}{t_1 t_2}$

$\frac{u_{0,2,3}}{t_1 t_2}$

$\frac{u_{0,2,3}}{t_1 t_2}$

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$\frac{u_{0,1,3}}{t_1 t_2}$

$$\frac{u_{0,2,3}}{t_1 t_2}$$

$$u_{0,1,1} + \frac{(-1+t_1) u_{0,1,3}}{t_1 t_2}$$

$$u_{0,1,2} + \frac{(-1+t_2) u_{0,1,3}}{t_2}$$

$$\frac{u_{0,1,3}}{t_1 t_2}$$

UCr = (# // $\sigma_{2,3}$ // $\sigma_{1,3}$ // Simplify) & /@ checks;

Uc1 - UCr // Simplify

$$\left\{ \frac{1}{t_1 t_2} \right.$$

$$\left. (-1+t_2) \left(f[t_1, t_2, t_3] t_1 (u_{1,1,3} - u_{1,2,3}) + (-1+t_1) \left(t_3 (u_{1,1,3} - u_{1,2,3}) f^{(0,0,1)}[t_1, t_2, t_3] + \right. \right. \right.$$

$$\left. \left. t_2 u_{1,2,3} f^{(0,1,0)}[t_1, t_2, t_3] - t_1 u_{1,1,3} f^{(1,0,0)}[t_1, t_2, t_3] \right) \right), \frac{1}{t_1 t_2}$$

$$\left. (-1+t_1) \left(f[t_1, t_2, t_3] t_2 (u_{1,2,3} - u_{2,2,3}) + (-1+t_2) \left(t_3 (u_{1,2,3} - u_{2,2,3}) f^{(0,0,1)}[t_1, t_2, t_3] + \right. \right. \right.$$

$$\left. \left. t_2 u_{2,2,3} f^{(0,1,0)}[t_1, t_2, t_3] - t_1 u_{1,2,3} f^{(1,0,0)}[t_1, t_2, t_3] \right) \right), \frac{1}{t_1 t_2} \left(f[t_1, t_2, t_3] t_1 \right.$$

$$t_2 \left((-1+t_1) (-1+t_2) v_1 - (-1+t_1) (-1+t_2) v_2 - t_1 w_1 + t_1 t_2 w_1 + t_2 w_2 - t_1 t_2 w_2 \right) +$$

$$\left. (-1+t_1) (-1+t_2) \left(t_3 \left((1+t_2) u_{1,1,3} - u_{2,2,3} + t_2 (-2 u_{1,2,3} + u_{2,2,3}) \right) f^{(0,0,1)}[t_1, t_2, t_3] + \right. \right.$$

$$t_2 \left(t_2 (u_{1,2,3} - u_{2,2,3}) + u_{2,2,3} \right) f^{(0,1,0)}[t_1, t_2, t_3] +$$

$$t_1^2 (u_{1,1,3} - u_{1,2,3} + t_2 (w_1 + u_{1,1,3} - u_{1,3,3})) f^{(1,0,0)}[t_1, t_2, t_3] -$$

$$t_1 \left(t_3 (u_{1,1,3} - 2 u_{1,2,3} + u_{2,2,3}) f^{(0,0,1)}[t_1, t_2, t_3] + t_2^2 (w_2 + u_{2,2,3} - u_{2,3,3}) \right.$$

$$f^{(0,1,0)}[t_1, t_2, t_3] + u_{1,1,3} f^{(1,0,0)}[t_1, t_2, t_3] + t_2 \left(t_3 (w_1 - w_2 + u_{1,1,3} - u_{1,3,3} - \right.$$

$$u_{2,2,3} + u_{2,3,3}) f^{(0,0,1)}[t_1, t_2, t_3] - u_{2,2,3} f^{(0,1,0)}[t_1, t_2, t_3] + u_{1,2,3}$$

$$\left. \left. \left(f^{(0,1,0)}[t_1, t_2, t_3] - f^{(1,0,0)}[t_1, t_2, t_3] \right) + u_{1,1,3} f^{(1,0,0)}[t_1, t_2, t_3] \right) \right) \right),$$

$$0, 0, (-1+t_1) (-1+t_2) (w_1 - w_2), - \frac{(-1+t_1) (-1+t_2) u_{1,1,3}}{t_1 t_2},$$

$$\frac{(-1+t_1) (-1+t_2) u_{1,1,3}}{t_1 t_2},$$

$$- \frac{(-1+t_1) (-1+t_2) u_{1,2,3}}{t_1 t_2},$$

$$\frac{(-1+t_1) (-1+t_2) u_{1,2,3}}{t_1 t_2},$$

$$- \frac{(-1+t_1) (-1+t_2) u_{1,2,3}}{t_1 t_2},$$

$$\frac{(-1+t_1) (-1+t_2) u_{1,2,3}}{t_1 t_2},$$

$$- \frac{(-1+t_1) (-1+t_2) u_{2,2,3}}{t_1 t_2},$$

$$\frac{(-1+t_1) (-1+t_2) u_{2,2,3}}{t_1 t_2},$$

$$0,$$

$$0,$$

$$0,$$

$$0,$$

$$- \frac{(-1+t_1) (-1+t_2) u_{0,1,3}}{t_1 t_2},$$

$$\begin{aligned}
 & \frac{(-1 + t_1) (-1 + t_2) u_{0,1,3}}{t_1 t_2}, \\
 - & \frac{(-1 + t_1) (-1 + t_2) u_{0,2,3}}{t_1 t_2}, \\
 & \frac{(-1 + t_1) (-1 + t_2) u_{0,2,3}}{t_1 t_2}, \\
 - & \frac{(-1 + t_1) (-1 + t_2) u_{0,1,3}}{t_1 t_2}, \\
 & \frac{(-1 + t_1) (-1 + t_2) u_{0,1,3}}{t_1 t_2}, \\
 - & \frac{(-1 + t_1) (-1 + t_2) u_{0,2,3}}{t_1 t_2}, \\
 & \frac{(-1 + t_1) (-1 + t_2) u_{0,2,3}}{t_1 t_2}, \\
 & 0, \\
 & 0, \\
 & 0, \\
 & 0, \\
 - & \frac{(-1 + t_1) (-1 + t_2) u_{0,1,3}}{t_1 t_2}, \\
 & \frac{(-1 + t_1) (-1 + t_2) u_{0,1,3}}{t_1 t_2}, \\
 & 0 \}
 \end{aligned}$$