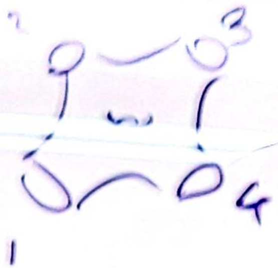
$$\Gamma_{g,n} = \pi_0 \,\mathrm{Diff}^+(\Sigma_{g,n}, \partial)$$



$$\Gamma_{0,n} \cong RB_{n-1}$$ (ribbon braid group)

↑ framed

General idea: construct a presentation of the mapping class groups (Hatcher–Thurston '80's)

Build a CW complex

Dim 2: Faces between

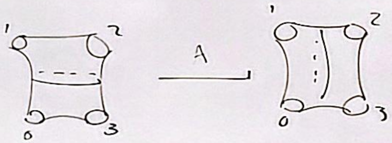
Build a CW-complex:

Dim 0: vertices $(\Sigma_{g,n}; P)$ pants decomposition.

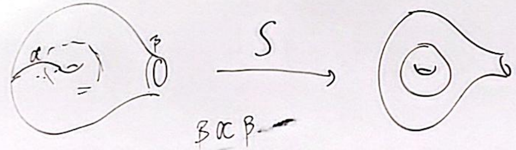
- In genus 0: "vertices are trivalent (unrooted) trees."

Dim 1: Edges $(\Sigma_{g,n}; P) \longrightarrow (\Sigma_{g,n}; P')$

are generated by elementary diffeomorphisms

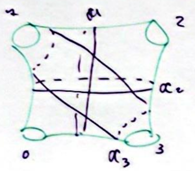
(1) 

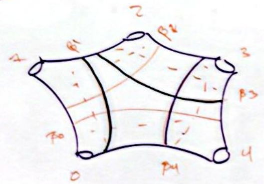
(2) Dehn twists around any curve in the decomposition.

(3) 

Dim 2: Faces between edges are of the form:

3A)
$$\begin{tikzcd} \alpha_1 \arrow[r, "A"] \arrow[rrr, bend right, "id_\alpha"'] & \alpha_2 \arrow[r, "A"] & \alpha_3 \arrow[r, "A"] & \alpha_1 \end{tikzcd}$$





(5A) $\{\beta_2, \beta_0\}$ ← is a pants decomposition.

$$\begin{tikzcd} \{\beta_1, \beta_3\} \arrow[r] & \{\beta_1, \beta_4\} \arrow[r] & \{\beta_2, \beta_4\} \arrow[r] & \{\beta_2, \beta_0\} \arrow[d] \\ & \{\beta_3, \beta_5\} \arrow[r] & \{\beta_3, \beta_1\} & \end{tikzcd}$$

Build a CW-complex:

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- In genus 0: "vertices are trivalent (unrooted) trees."

Dim 1: Edges $(\Sigma_{g,n}; P) \longrightarrow (\Sigma_{g,n}; P')$

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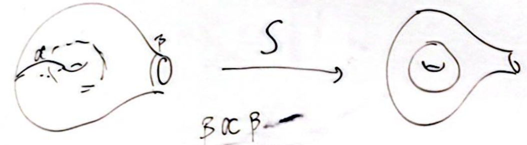
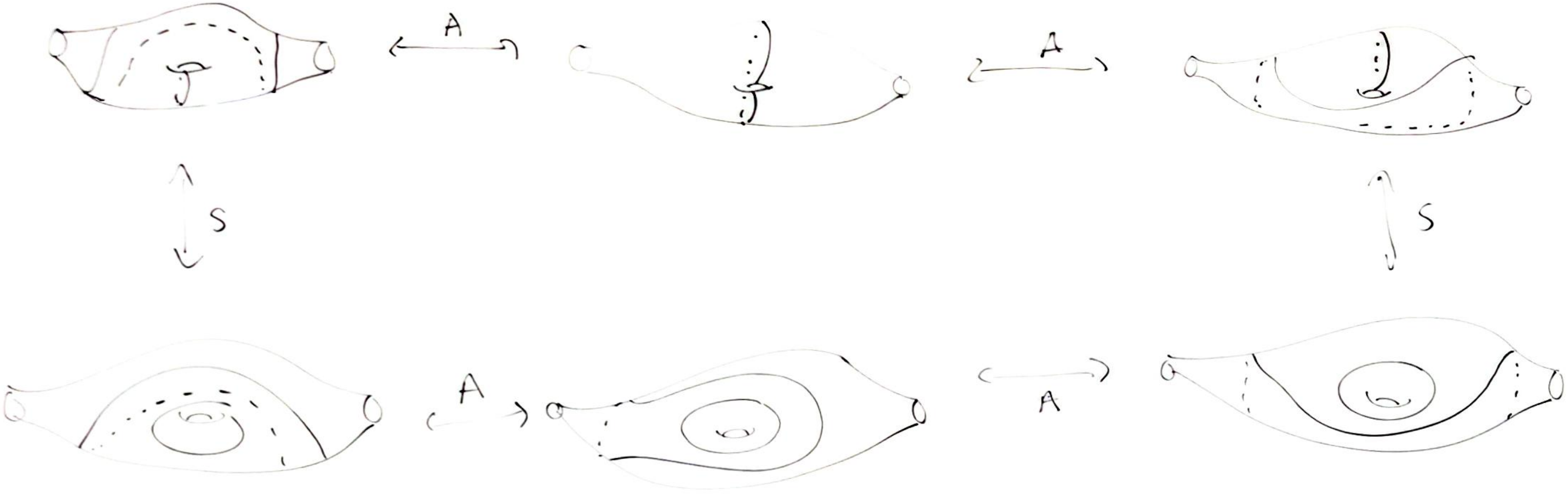
(1)

(2) Dehn twists around any curve in the decomposition.

(3)

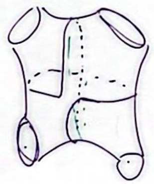




(6AS)

Dim 2: Faces between edges are all of the form:

Thm (HT 80): This (maximal) curve complex on $\Sigma_{g,n}$ is connected and simply connected.



Idea: (Hatcher - Lochak - Schneps)

If we want to know how

$\mathrm{Gal}\,\bar{\mathbb{Q}} \hookrightarrow \widehat{GT} \hookrightarrow \mathrm{Aut}\left(\coprod \hat{\Gamma}_{g,n}\right)$ we can

(1) Build groupoids $S(g,n)$ which are approximations of the curve complex

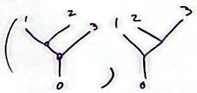
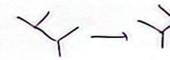
$B(S(g,n)) \simeq MCC(\Sigma_{g,n})$ ← maximal curve complex

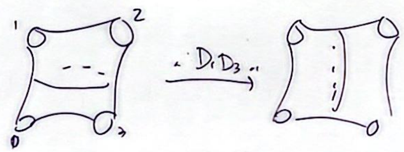
Build a groupoid $S(g,n)$

obj: trivalent, connected, graphs w/ $\beta_1(G) = g$ and $|\partial(G)| = n$ ← and a labelling

$\mathrm{ob}(S(1,2)) = \{$  ,  $\}$

morphisms: $S(g,n)(G_1, G_2) =$

eg: $S(0,4)$() $= \{$ A-moves  in the $MCC(\Sigma_{0,4})$; Dehn twists around boundaries and separating curves;



- $S(g,n)$

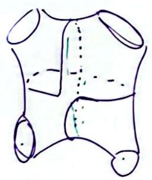
$$|B(S(g,n))| \simeq MCC(\Sigma_{g,n})$$

Idea: Morphisms in $S(g,n)$ are composites of Dehn twists A-moves and S-moves.

$$BS(g,n)_k = \left\{ G_1 \xrightarrow{\phi_1} \cdots \xrightarrow{\phi_{k-1}} G_k \right\}$$

where ϕ_i is a Dehn twist, A-move or a S-move, or relabelling.

In particular, we can see all of the two-dim faces of the $MCC(\Sigma_{g,n})$ via some triangulation



HLS: Let's look at the action (construct)

$$\widehat{GT} \xrightarrow{\eta} \mathrm{Aut}(S(g,n))$$

By the study of $\widehat{GT}$-actions on Braid groups, we know how this should look in genus 0.

$$\widehat{GT} \xrightarrow{\eta} \mathrm{Aut}(S(0,n))$$

$F = (f,\lambda) \in \widehat{GT}$

$\eta_F : S(0,n) \longrightarrow S(0,n)$

$\eta_F = id$ on objects

$\eta_F(A) = A \cdot f(x_{12}, x_{23})$

$\eta_F(D_\alpha) = D_\alpha^{\lambda}$

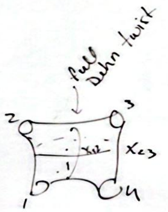
$\eta_F(D_\alpha^{1/2}) = D_\alpha^{\mu}$ $\quad \mu = \frac{\lambda - 1}{2}$

$$\mathrm{Gal}\,\overline{\mathbb{Q}} \hookrightarrow \pi_1^{et}(M_{g,n,\overline{\mathbb{Q}}})^{\wedge} \hat{\Gamma}_{g,n}$$

$$\widehat{GT} \hookrightarrow \mathrm{Aut}(\hat{B}_n) + \text{strand deletion} + \text{strand doubling}$$

$n \geq 5$

$$PB_3 \longrightarrow \Gamma_{0,4}$$



$\eta_F(\mathrm{id}_G) = \mathrm{id}_G$

$\eta_F(\alpha_1 \xrightarrow{A} \alpha_2 \xrightarrow{A} \alpha_3 \xrightarrow{A} \alpha_1)$

$= A \cdot f(x_{12}, x_{23}) \cdot A f(-,-) \cdot A f(\quad)$

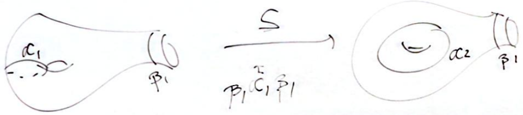
$=$ the hexagon in GT_1

HLS: $\exists\, \Lambda \subseteq GT_1$
which acts on $S(g,n)$ for all $g \geq 0$, $n \geq 4$

Boavida-Horel-R:
$\widehat{GT} \cong \mathrm{Aut}_{op}(\widehat{S}(0,*))$

Bonatto-R:
$NS \subseteq \widehat{GT}$
$NS \cong \mathrm{Aut}_{modop}(S(g,*))$

what happens to the S-move?



$(f,1)(S) = f(\alpha_1^2, \beta_1^2)\, \alpha_2\, f(\alpha_1^2, \beta_1^2)$

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$\mathrm{Gal}\,\overline{\mathbb{Q}} \hookrightarrow \pi_1^{et}(M_{g,n,x_0}) \cong \widehat{\Gamma}_{g,n}$

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$n \geq 5$

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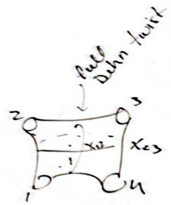

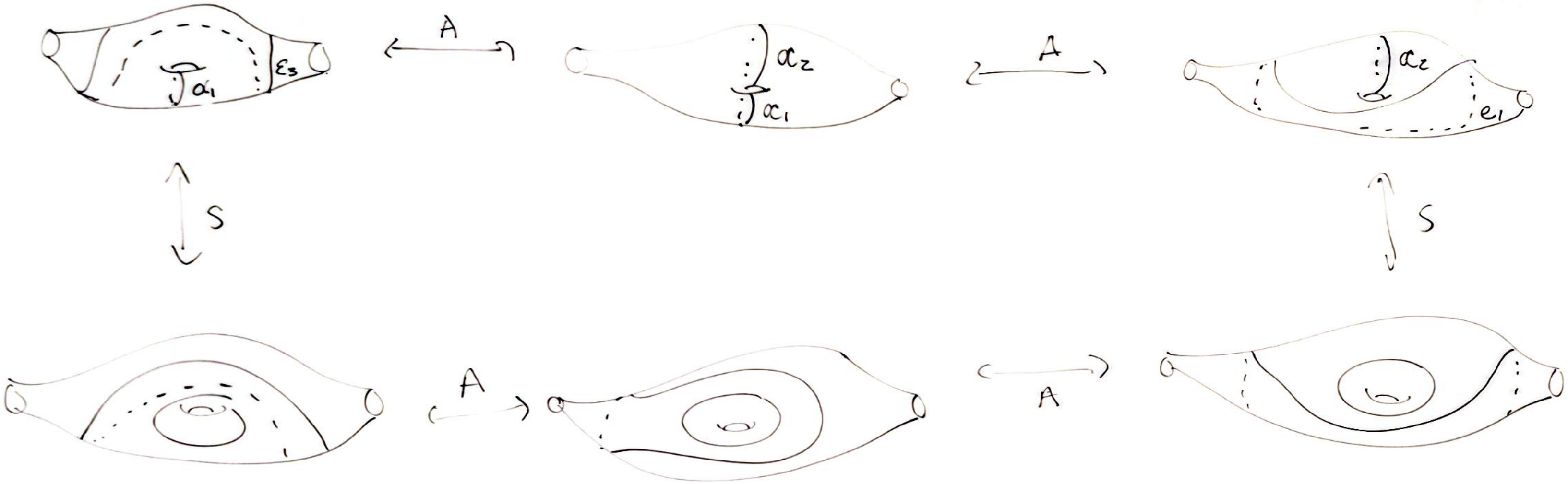

(6AS)

$$(R)\ f(e_3,a_1)\,\underbrace{f(a_1^2,a_2^2)}_{\text{S moves}}\,\underbrace{f(e_2,e_3)}_{\text{A-moves}}\,f(e_1,e_2)\,f(a_1^2,a_2^2)\,f(a_3,e_1)=1$$

$\eta_F(id_G) = id_G$

$\eta_F(\alpha_1 \xrightarrow{A} \alpha_2 \xrightarrow{A} \alpha_3 \xrightarrow{A} \alpha_1)$

$= A \cdot f(x_{12}, x_{23}) \cdot A f(-,-) \cdot A f(\quad)$

= the hexagon in GT_1

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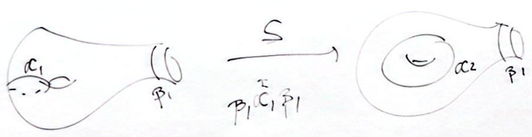
$\{S(g,n)\}_{g \geq 0, n \geq 1}$

what happens to the S-move?



$(f,1)(S) = f(\alpha_1^2, \beta_1^2)\,\alpha_2\, f(\alpha_1^2, \beta_1^2)$

$Aut(S_{\leq 1})$

$S_{\leq 1}$ = truncated modular operad

$S(g,n) = *$ if $g > 1$

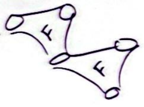
$\Lambda \xrightarrow{\eta} Aut(S(g,n))$

$GT_1 \longrightarrow Aut(Sol\,KV(g,n))$

$GT_1 \xrightarrow{\Delta} GT_1^{2g+3n+1} \longrightarrow KV(0,3)^{2g+3n+1} \longrightarrow KV(g,n) \longrightarrow Aut(Sol\,KV$

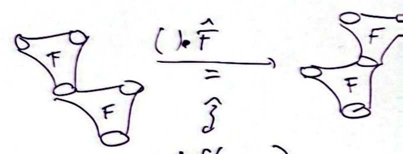
$GT_1 \longrightarrow KV(0,3)$

$\psi \longmapsto v(\psi) = F$



$\psi \longmapsto v(\psi) = F$

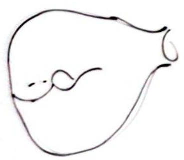
$\hat{F} = (F^{12} \cdot F^{12,3})^{-1} (F^{23} \cdot F^{1,23}) \in KRV(0,3$



$$
\begin{tikzcd}
\mathfrak{grt}_1 \arrow[d, bend right] \\
\mathfrak{grt}^{ell} \arrow[u, "\text{section}"'] & & \mathrm{Lie}(\Lambda) \arrow[ll, "?"']
\end{tikzcd}
$$

ker ?

SolKV(1,1)



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$\{S(g,n)\}_{g \geq 0, n \geq 1}$

$\Lambda \leq NS = \{(f,\lambda) \in \widehat{GT}, \text{ satisfying } \mathrm{IV} \}$

R

$(f,1)$

Pentagon

$f \in \widehat{GT}$

$f(x,y) = g(x,y) g^{-1}(y,x)$

$\exists! g \in F_2^{ab}$ $\quad F_2 / aba^{-1}b^{-1}$

$GT = \{(f,\lambda) \in \hat{F}_2' \times \hat{\mathbb{Z}} \mid f(x,y) f(y,x) = 1, \text{ hexagon}, \text{ pentagon} \}$

$(f,\lambda) \in \hat{F}_2' \times \hat{\mathbb{Z}}$ is a word $f(x,y) \in F_2$

which gives an automorphism of F_2.

$$
\begin{aligned}
F_2 &\longrightarrow F_2 \\
x &\longmapsto x^{\lambda} \\
y &\longmapsto f(x,y)^{-1} y f(x,y)
\end{aligned}
$$

Harbater-Schneps: Prove $\widehat{GT} \hookrightarrow \mathrm{Out}(\hat{B}_4)$

for any $(f,\lambda) \in GT$ $\exists! g \in F_2^{ab}$

such that $f(x,y) = \tilde{g}^{-1}(y,x) \tilde{g}(x,y)$

f = word in x, y $\quad \pi(g)$ is unique in

$$GT = \{ (f,\lambda) \in \hat{F}_2' \times \hat{\mathbb{Z}} \mid f(x,y)f(y,x)=1,\ \text{hexagon},\ \text{pentagon} \}$$

$$\Lambda \leq GT$$

which acts on $S(g,n)$ for all $g \geq 0$, $n \geq 4$

$$\begin{array}{c} grt_1 \\ \uparrow\downarrow \\ grt^{ell} \\ \uparrow \\ \text{ker ?} \end{array} \quad \text{section.} \qquad \xleftarrow{?} \ Lie(\Lambda)$$

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$$\hat{GT} \cong Aut_{op}(S(0,*))$$

Bonatto-R:

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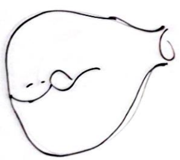
$$NS \cong Aut_{modop}(S(*,*)) \quad \{S(g,n)\}_{g\geq 0, n\geq 1}$$

SolKV(1,1)



$$\Lambda \leq NS = \{ (f,\lambda) \in GT, \text{ satisfying } : , \underset{R}{IV} \}$$

(f, 1)

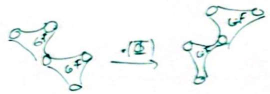
$$f \in \hat{GT} \quad \text{pentagon}$$

$$f(x,y) = g(x,y)g^{-1}(y,x)$$

$$\exists g \in F_2^{ab} \qquad F_2 / aba^{-1}b^{-1}$$

$$GT_1 \longrightarrow Aut(SolKV(0,n+1)) \}$$

$$\begin{array}{c} GT_1 \xrightarrow{v} KV(0,3) \xrightarrow{\Delta} KV(0,3)^{\times n-1} \longrightarrow KV(0,n+1) \xrightarrow{\cong} Aut(SolKV(\\ \cong \\ Aut^{op}(S(0,n+1)) \end{array}$$



$$F = (f,1) \in GT_1$$

$$F : F_{r_2} \longrightarrow F_{r_2}$$

$$x \longmapsto x$$

$$y \longmapsto f^{-1}yf$$

$$F \rightsquigarrow \underbrace{(F^{12}F^{12,3})^{-1}(F^{23}F^{1,23})}_{\text{associator}} = \Phi \quad \text{element of } KRV(0,3)$$

is an automorphism of F_3

Q: Are we using the pentagon here?

Ingredient 1:

$KRV(4)$ acts freely & trans. on $SolKV(4)$

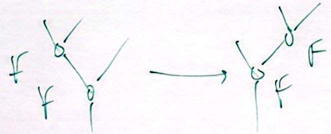
KRV, SolKV form operads

$$GT = \{(f,\lambda) \in \hat{F}_2' \times \hat{\mathbb{Z}} \mid f(x,y)f(y,x)=1, \text{ hexagon, pentagon}\}$$

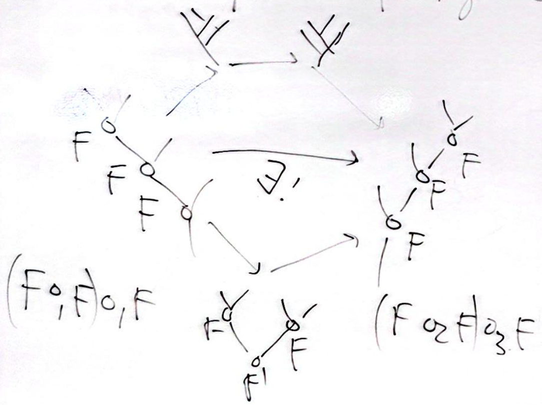
$\to Aut(SolKV(0,n+1))\}$

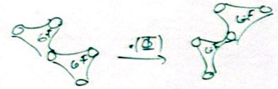
$\to KV(0,3) \xrightarrow{\Delta} KV(0,3)^{\times n-1} \longrightarrow KV(0,n+1) \to Aut(SolKV(0,n))$

$\Longrightarrow \forall F \in SolKV(2),\ \Phi_F := (F \circ_1 F)^{-1}(F \circ_2 F)$



satisfies the pentagon equation.





n1)

1) $\in GT_1$

element of $KRV(0,3)$

$$F \rightsquigarrow \underbrace{(F^{12}F^{12,3})^{-1}(F^{23}F^{1,23})}_{\text{association}} = \Phi$$

is an automorphism of F_3