$\mathrm{SolKV}(n) := \{ \text{KV solutions of type } (g, n+1) \}$.

$\mathrm{SolKV}(2) = \mathrm{SolKV}$ classical

Thm: The sets $\mathrm{SolKV} = \{\mathrm{SolKV}(n)\}_{n}$ form a colored operad.

Def: $F \in \mathrm{SolKV}(n)$

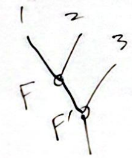
$F \in \mathrm{TAut}_n$ s.t.

$F(x_1 + \dots + x_n) = \mathrm{bch}(x_1, \dots, x_n)$

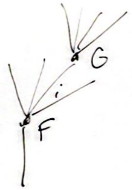
$+$ Jacobian condition.

Idea: We restrict to the suboperad generated by $\mathrm{SolKV}(2)$

$$\mathrm{SolKV}_{\langle 2 \rangle} \subseteq \mathrm{SolKV}$$



For any $1 \leq i \leq m$, there are



$$\circ_i : \mathrm{SolKV}(m) \times \mathrm{SolKV}(n) \longrightarrow \mathrm{SolKV}(m+n-1)$$

$$(F, F') \longmapsto F \circ_i F' = (F \circ_i 1) \cdot (1 \circ_i F')$$

operations which are associative and Σ_n-equivariant.

(*) whenever the bch functions agree

We form an operad of groupoids:

$\mathrm{SolKV}_{\langle 2 \rangle}(n)$ a groupoid with

$\rightarrow$ objects:

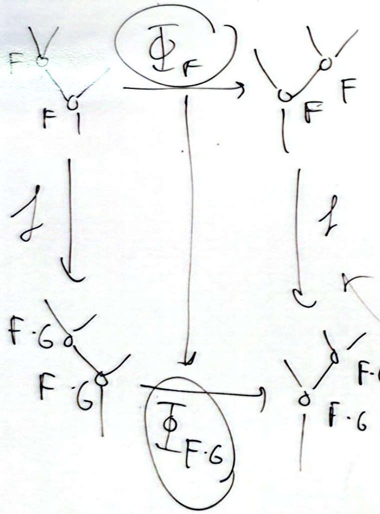
$$\psi = \mathrm{krv}(2) \qquad \frac{\psi^{1,2} + \psi^{12,3} - \psi^{2,3} + \psi^{1,23}}{\psi(t_{12}, t_{23})}$$

We want a morphism

$$GRT_1 \hookrightarrow KRV(2) \xrightarrow{\eta} Aut_{Op}(SolKV_{\langle 2\rangle})$$

$$G \longmapsto \left(SolKV_{\langle 2\rangle}(2) \longrightarrow SolKV_{\langle 2\rangle}(2) \right)$$

$$F \longmapsto F\cdot G$$



Extend on objects via operadic composition.

$$F\circ_1 F \longmapsto (F\circ_1 F)\cdot(G\circ_1 G) = (F\cdot G \circ_1 F\cdot G)$$

Extend to morphisms in the only possible way.

$$\Phi_{F\cdot G} = (G\circ_1 G)^{-1}\,\Phi_F\,(G\circ_2 G)$$

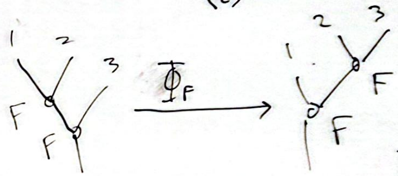
Drinfeld twist

Question: Is η an isomorphism?

Idea: We restrict to the suboperad generated by $SolKV(2)$

$$SolKV_{\langle 2\rangle} \subseteq SolKV$$

(2) morphisms which change the underlying tree



$$\Phi_F = (F\circ_1 F)^{-1}(F\circ_2 F) = (F^{12,3}\cdot F^{1,2})^{-1}(F^{1,23}\cdot F^{2,3})$$

We form an operad of groupoids:

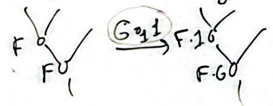
$SolKV_{\langle 2\rangle}(n)$ a groupoid with

→ objects: elements of $SolKV_{\langle 2\rangle}(n)$

→ morphisms between F, F' is the unique element $G\in KRV(n)$ s.t. $F' = F\cdot G$.

these are generated by 2 types of morphisms

(1)

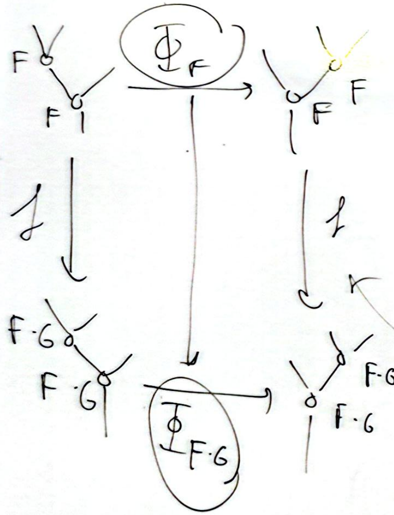


with same underlying tree

We want a morphism

$$GRT_1 \hookrightarrow KRV(2) \xrightarrow{\eta} \mathrm{Aut}_{op}\left(SolKV_{\langle 2\rangle}\right)$$

$$G \longmapsto \left(\begin{array}{c} SolKV_{\langle 2\rangle}(2) \xrightarrow{f} SolKV_{\langle 2\rangle}(2) \\ F \longmapsto F\cdot G \end{array} \right)$$



Extend on objects via operadic composition.

$$F\circ_1 F \longmapsto (F\circ_1 F)\cdot(G\circ_1 G)$$
$$\parallel$$
$$(F\cdot G\circ_1 F\cdot G)$$

Extend to morphisms in <u>the only possible way</u>.

$$\Phi_{F\cdot G} = (G\circ_1 G)^{-1}\,\Phi_F\,(G\circ_2 G)$$

<u>Drinfeld twist</u>

<u>Question</u>: Is η an isomorphism?

$SolKV(n)$ = groupoid with

objects: pairs $(F, \mathbb{F})$ $\quad F\in SolKV(n)$, $\mathbb{F}\in PT_n$

morphisms: ? $\quad G\in KRV(n)$

Define $KRV(2) \longrightarrow \mathrm{Aut}_{op}(SolKV)$

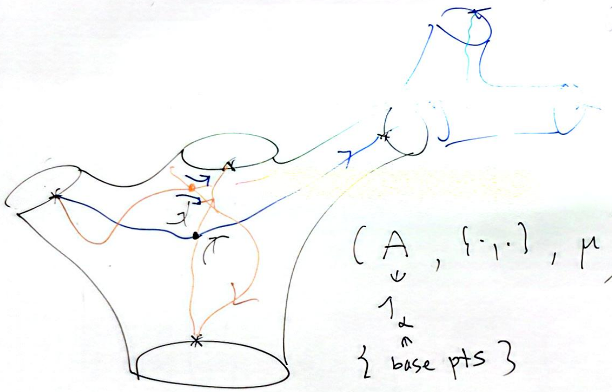
$$G \longmapsto \begin{array}{c} SolKV(3) \longrightarrow SolKV(3) \\ (F, \mathbb{F}) \longmapsto (F\cdot G\,\mathbb{F}, \mathbb{F}) \end{array}$$

Assume $krv(n) \cong grt_1^{(n)} + t_n$

$krv(n)/t_n = X(n)$. $\quad X(n) \overset{?}{\cong} grt_1$?

$\circ_i : \Sigma_m \sqcup_i \Sigma_n \longrightarrow \Sigma_{m+n-1}$

$\mathfrak{g}(\Sigma_m \sqcup \Sigma_n) \longrightarrow \mathfrak{g}(\Sigma_{m+n-1})$



$\mathfrak{g}(\Sigma_m \sqcup \Sigma_n)$

$\uparrow$

$\mathfrak{g}(\Sigma_m) \oplus \mathfrak{g}(\Sigma_n)$

$(A, \{\cdot,\cdot\}, \mu)$

$\uparrow_\alpha$

{ base pts }

$\theta : \gamma_1 \mapsto g_1(x_1, x_2).e^{x_1} g_1^{-1}$

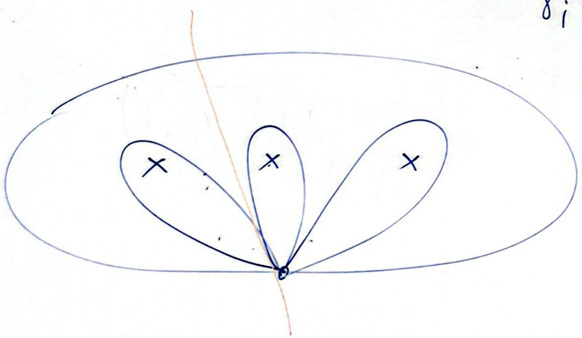
$\gamma_2 \mapsto g_2 e^{x_2} g_2^{-1}$

$\tilde{g}_1(x_1,x_2)$

$\boxed{(\theta \circ_1 \tilde{\theta})} : \gamma_1 \mapsto g_1(x_1+x_2, x_3) e^{x_1} g_1^{-1}$

$\gamma_2 \mapsto g_1(x_1+x_2, x_3) e^{x_2} g_1^{-1}$

$\gamma_2 \mapsto g_2(x_1+x_2, x_3) e^{x_3} g_3^{-1}$



$\gamma_i = e^{x_i}$

$\gamma_1\gamma_2\gamma_3^{-1} \longrightarrow \gamma_1 e^{x_2} e^{-x_3}$

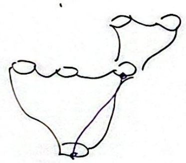
QEG * FA

PA(n) = based paths on
and loops

$PA = \sqcup PA(n)$

$\Sigma_n \hookrightarrow \Sigma_m \quad m > n$

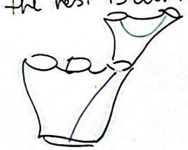
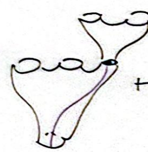
$PA(n) \overset{?}{\hookrightarrow} PA(m)$



paths w/ one end at the ith basepoint & the other on a different ∂

$\circ_i : PA^i(n) \times PA^i(m) \longrightarrow PA(m+n-1)$ non-trivial

the rest is defined to be zero:

 $\longmapsto 0$  $\longmapsto 0$

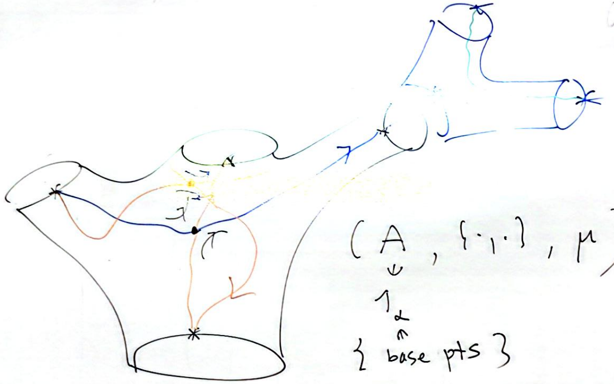
$$o_i : \Sigma_m \coprod_i \Sigma_n \longrightarrow \Sigma_{m+n-1}$$

$$\mathfrak{g}(\Sigma_m \coprod \Sigma_n) \longrightarrow \mathfrak{g}(\Sigma_{m+n-1})$$

$$\mathfrak{g}(\Sigma_m \coprod \Sigma_n)$$

$$\mathfrak{g}(\Sigma_m) \oplus \mathfrak{g}(\Sigma_n)$$

$U(\mathrm{tder} \ltimes \mathrm{cyc})$



$(A, \{\cdot,\cdot\}, \mu)$

1_α

{ base pts }

$$\theta : \gamma_1 \mapsto g_1(x_1, x_2) \cdot e^{x_1} g_1^{-1}$$

$$\gamma_2 \mapsto g_2 e^{x_2} g_2^{-1}$$

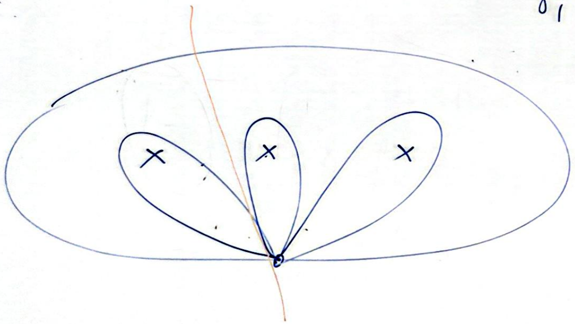
a^{12} $b^{12,3}$

$$\begin{array}{ll} \mathrm{TAut}_n \ni G & G(a) = b \\ \downarrow\ \downarrow \quad s \swarrow\ \downarrow t & G(c) + J(G) = d \\ \mathrm{Lie}_n \ni a, b & \\ + & \\ \mathrm{Cyclic}_n \quad c, d & \end{array}$$

$\gamma_i = e^{x_i}$

$\gamma_1 \gamma_2 \gamma_3^{-1} \rightsquigarrow \gamma_1 e^{x_2} e^{-x_3}$

QEG * FA

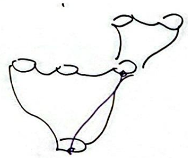


PA(n) = based paths and loops on

PA = $\sqcup$ PA(n)

$\Sigma_n \hookrightarrow \Sigma_m$ $m > n$

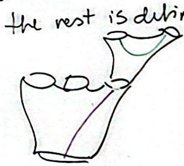
PA(n) $\hookrightarrow$? PA(m)



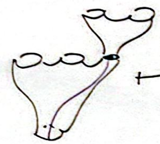
$o_i : PA^i(n) \times PA^0(m) \longrightarrow PA(m+n-1)$ non-trivial

paths w/ one end at the ith basepoint & the other on a different ∂

the rest is defined to be zero:



$\longmapsto 0$



$\longmapsto 0$