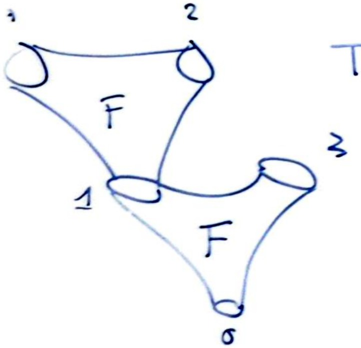
I) GT_1-actions on (higher) genus KV-solutions.

<u>Genus 0</u>: A KV-solution of type $(0, n+1)$, $n \geq 2$

is an element $F \in TAut_n$ s.t.

(KVI) $F(x_1 + \dots + x_n) = \log(e^{x_1} \cdots e^{x_n})$

(KVII) $\exists\, h(z) \in K[[z]] \ni J(F^{-1}) = tr\left(\sum_{i=1}^{n} h(x_i) - h\left(\sum_{i=1}^{n} x_i\right)\right)$



Thm (AKKN '18) Such KV solutions exist and can be obtained via gluing surfaces along boundaries.

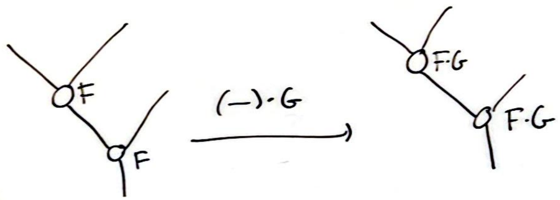
$$F \circ_1 F = F^{12} \cdot F^{12,3} \qquad F \circ_2 F = F^{23} \cdot F^{1,23}$$

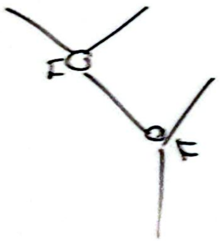
$F \in SolKV(0,3)$

$$\begin{tikzcd} KV(0,3) \arrow[r, phantom, "\curvearrowright"] & \{SolKV(0,3)\} & KRV(0,3) \arrow[l, phantom, "\circlearrowleft"] \\ GT_1 \arrow[u, "r"] \arrow[r, phantom, "\curvearrowright"] & \{Ass\} \arrow[u, dotted, "F"] & GRT_1 \arrow[u, "v"] \arrow[l, phantom, "\circlearrowleft"] \end{tikzcd}$$

Let $G \in KRV(0,3)$, $G \circ_1 G = G^{12} \cdot G^{12,3}$ and $G \circ_2 G = G^{23} \cdot G^{1,23}$

Q: $\exists$ a solution of type $(0,n+1)$ which is not obtained via gluing?





$F \cdot G = F'$ in $SolKV(0,3)$

$(F \circ_1 F)(G \circ_1 G) = FG \circ_1 FG = F' \circ_1 F'$ in $SolKV(0,4)$

$$\mathfrak{grt}_1 \hookrightarrow \mathfrak{krv}$$

$$\psi \longmapsto (\psi(-x-y,x), \psi(-x-y,y)).$$

$$t_n \longrightarrow tder_n$$

$$t_{12} \longmapsto \quad + $$

Define an action: $GRT_1 \xrightarrow{\eta} Aut(SolKV(0,4))$

$$\psi \longmapsto \eta_{v(\psi)} : F \circ_i F \longrightarrow F \cdot v(\psi) \circ_i F \cdot v(\psi) \quad i = 1,2.$$

$v(\psi) = g$

$$\mathrm{KV}(0,3) \circlearrowright \{\mathrm{SolKV}(0,3)\} \circlearrowleft \mathrm{KRV}(0,3) \qquad \hat{g} = (g^{12,3})^{-1}(g^{12})^{-1} g^{23} \cdot g^{1,23} = h(t^{1,2}, t^{2,3}) =$$ 

$$\begin{array}{ccccc} \mathrm{KV}(0,3) & \circlearrowright & \{\mathrm{SolKV}(0,3)\} & \circlearrowleft & \mathrm{KRV}(0,3) \\ \uparrow v & & \hat{\vdots}\ F & & \uparrow v \\ GT_1 & \circlearrowright & \{\mathrm{Ass}\} & \circlearrowleft & \mathrm{GRT}_1 \ni \psi \end{array}$$

if $F \in \mathrm{SolKV}(0,3)$ is constructed from an associator $F \cdot g$ = GRT_1-action on associators

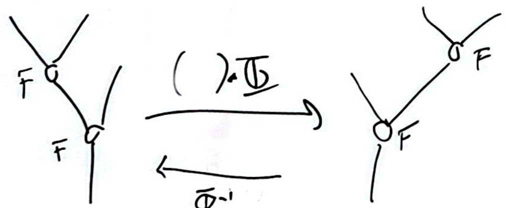
$$F \circ_1 F \xrightarrow{(\ )\cdot \hat{g}} F \circ_2 F$$

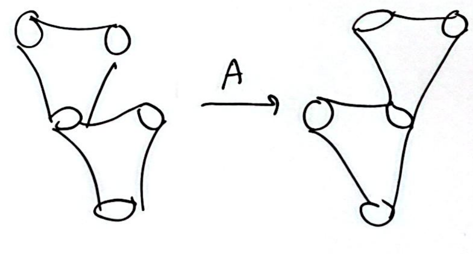
Let's define a groupoid $\mathrm{SolKV}(0,4)$

where F and F' have same Duflo function.

objects: binary rooted trees with 3-leaves decorated by composible KV-solutions.

morphisms: $\Phi = (F^{12,3})^{-1}(F^{1,2})^{-1} F^{2,3} F^{1,23} = (F \circ_1 F)^{-1} \cdot (F \circ_2 F) \in \mathrm{KRV}(0,4)$





$\upsilon(\psi) = g$

$$\mathrm{KV}(0,3) \circlearrowright \{\mathrm{SolKV}(0,3)\} \circlearrowleft \mathrm{KRV}^{\mathcal{I}}(0,3) \qquad \hat{g} = (g^{12,3})^{-1}(g^{12})^{-1} g^{23} \cdot g^{1,23} = h(t^{1,2}, t^{2,3}) =$$

$\uparrow r \qquad \hat{\vdots}\ F \qquad \uparrow v$

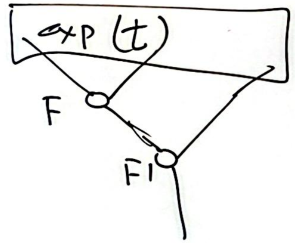
$GT_1 \circlearrowright \{\mathrm{Ass}\} \circlearrowleft GRT_1 \ni \psi$

if $F \in \mathrm{SolKV}(0,3)$ is constructed from an associator $F \cdot g$ = GRT_1-action on associators

$$F \circ_1 F \xrightarrow{(\ )\cdot \hat{g}} F \circ_2 F$$

$$GRT_1 \oplus t = \mathrm{KRV}(0,3)$$

Let's define a groupoid $\mathrm{SolKV}(0,4)$

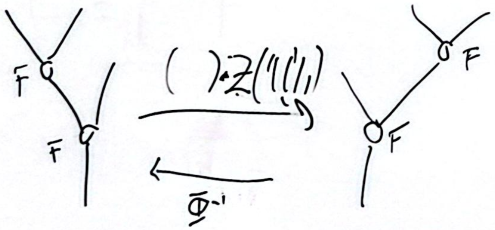


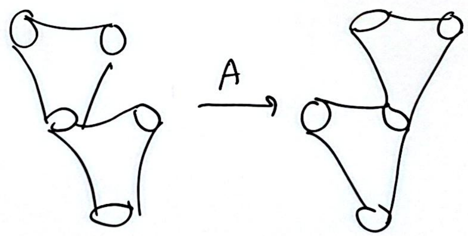
$\overset{?}{=}$ expansions

$$\mathrm{PaB} \xrightarrow{Z} \widehat{\mathrm{PaCD}}$$

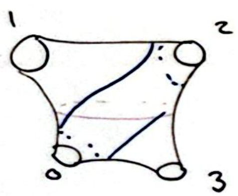
objects: binary rooted trees with 3-leaves decorated by composible KV-solutions.

morphisms: $\Phi = (F^{12,3})^{-1}(F^{1,2})^{-1} F^{2,3} F^{1,23} = (F\circ_1 F)^{-1} \cdot (F \circ_2 F) \in \mathrm{KRV}(0,4)$



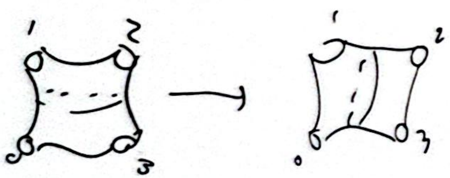


Def: Let $S(0,3+1)$ be a groupoid whose objects are (special) parts decompositions of $\Sigma_{0,3+1}$ (up to isotopy).

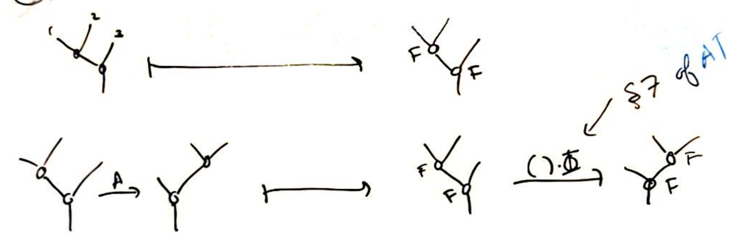


$ob(S(0,4))$ are rooted binary trees T with leaves labelled $\{1,2,3\} \to T$

$$S(0,4)(T_1, T_2) \subseteq \Gamma_{0,4} \cong RB_3 \cong B_3 \ltimes \mathbb{Z}^3$$



$$S(0,3+1) \xrightarrow{Z} SolKV(0,3+1)$$



∂T

$$qT \cong \widehat{PaRB}^{cyc} \xrightarrow{Z} \widehat{PaCD}^{cyc} = \coprod_{n\geq 1} PaCD(n) \cong A^{sh}$$

chord diagrams cups and caps

$\widehat{PaCD}^{cyc} \xrightarrow{W_g} Cat_{k[[\hbar]]}$

$\widehat{PaRB}^{cyc} \overset{ZA}{\dashrightarrow} Cat_{k[[\hbar]]}$

$\hbar$

parenthesized category of tangles of Tureav

$A(\uparrow_n \uparrow_k)$

$\downarrow T_g$

$U(g \oplus g^*)^{\otimes n} \otimes U(\ldots$

$\longmapsto A(\uparrow) = \mathcal{C}$

$\mathcal{C}$ = infin braided monoidal cat