

BALLOONS AND HOOPS AND THEIR UNIVERSAL FINITE TYPE INVARIANT, BF THEORY, AND AN ULTIMATE ALEXANDER INVARIANT

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ABSTRACT. Balloons are two-dimensional spheres. Hoops are one dimensional loops. Knotted Balloons and Hoops (KBH) in 4-space behave much like the first and second homotopy groups of a topological space — hoops can be composed as in π_1 , balloons as in π_2 , and hoops “act” on balloons as π_1 acts on π_2 . We observe that ordinary knots and tangles in 3-space map into KBH in 4-space and become amalgams of both balloons and hoops.

We give an ansatz for a tree and wheel (that is, free-Lie and cyclic word) -valued invariant ζ of (ribbon) KBHs in terms of the said compositions and action and we explain its relationship with finite type invariants. We speculate that ζ is a complete evaluation of the BF topological quantum field theory in 4D. We show that a certain “reduction and repackaging” of ζ is an “ultimate Alexander invariant” that contains the Alexander polynomial (multivariable, if you wish), has extremely good composition properties, is evaluated in a topologically meaningful way, and is least-wasteful in a computational sense. If you believe in categorification, that should be a wonderful playground.



Web resources for this paper are available at

[Web/]:=<http://www.math.toronto.edu/~drorbn/papers/KBH/>, including an electronic version, source files, computer programs, lecture handouts and lecture videos; one of the handouts is attached at the end of this paper. *Throughout this paper we follow the notational conventions and notations outlined in Section 10.5.*

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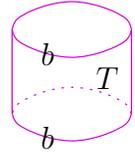
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1. INTRODUCTION

Riddle 1.1. The set of homotopy classes of maps of a tube $T = S^1 \times [0, 1]$ into a based topological space (X, b) which map the rim $\partial T = S^1 \times \{0, 1\}$ of T to the basepoint b is a group with an obvious “stacking” composition; we call that group $\pi_T(X)$. Homotopy theorists often study $\pi_1(X) = [S^1, X]$ and $\pi_2(X) = [S^2, X]$ but seldom if ever do they study $\pi_T(X) = [T, X]$. Why?



The solution of this riddle is on Page 14. Whatever it may be, the moral is that it is better to study the homotopy classes of circles and spheres in X rather than the homotopy classes of tubes in X , and by morphological transfer, it is better to study isotopy classes of embeddings of circles and spheres into some ambient space than isotopy classes of embeddings of tubes into the same space.

In [BND1, BND2], Zsuzsanna Dancso and I studied the finite-type knot theory of ribbon tubes in \mathbb{R}^4 and found it to be closely related to deep results by Alekseev and Torossian [AT] on the Kashiwara-Vergne conjecture and Drinfel’d’s associators. At some point we needed a computational tool with which to make and to verify conjectures.

This paper started in being that computational tool. After a lengthy search I found a language in which all the operations and equations needed for [BND1, BND2] could be expressed and computed. Upon reflection, it turned out that the key to that language was to work with knotted balloons and hoops, meaning spheres and circles, rather than with knotted tubes.

Then I realized that there may be independent interest in that computational tool. For (ribbon) knotted balloons and hoops in \mathbb{R}^4 (\mathcal{K}^{rbh} , Section 2) in themselves form a lovely algebraic structure (an MMA, Section 3), and the “tool” is really a well-behaved invariant ζ . More precisely, ζ is a “homomorphism ζ of the MMA \mathcal{K}_0^{rbh} to the MMA M of trees and wheels” (trees in Section 4 and wheels in Section 5). Here \mathcal{K}_0^{rbh} is a variant of \mathcal{K}^{rbh} defined using generators and relations (Definition 3.5). Assuming a sorely missing Reidemeister theory for ribbon-knotted tubes in \mathbb{R}^4 (Conjecture 3.7), \mathcal{K}_0^{rbh} is actually equal to \mathcal{K}^{rbh} .

The invariant ζ has a rather concise definition that uses only basic operations written in the language of free Lie algebras. In fact, a nearly complete definition appears within Figure 4, with lesser extras in Figures 5 and 1. These definitions are relatively easy to implement on a computer, and as that was my original goal, the implementation along with some computational examples is described in Section 6. Further computations, more closely related to [AT] and to [BND1, BND2], will be described in [BN4].

In Section 7 we sketch a conceptual interpretation of ζ . Namely, we sketch the statement and the proof of the following theorem:

Theorem 1.2. *The invariant ζ is (the logarithm of) a universal finite type invariant of the objects in \mathcal{K}_0^{rbh} (assuming Conjecture 3.7, of ribbon-knotted balloons and hoops in \mathbb{R}^4).*

While the formulae defining ζ are reasonably simple, the proof that they work using only notions from the language of free Lie algebras involves some painful computations — the more reasonable parts of the proof are embedded within Sections 4 and 5, and the less reasonable parts are postponed to Section 10.4. An added benefit of the results of Section 7 is that they constitute an alternative construction of ζ and an alternative proof of its invariance — the

construction requires more words than the free-Lie construction, yet the proof of invariance becomes simpler and more conceptual.

In Section 8 we discuss the relationship of ζ with the BF topological quantum field theory, and in Section 9 we explain how a certain reduction of ζ becomes a system of formulae for the (multivariable) Alexander polynomial which, in some senses, is better than any previously available formula.

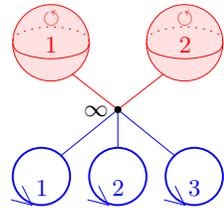
Section 10 is for “odds and ends” — things worth saying, yet those that are better postponed to the end. This includes the details of some definitions and proofs, some words about our conventions, and an attempt at explaining how I think about “meta” structures.

Remark 1.3. Nothing of substance places this paper in \mathbb{R}^4 . Everything works just as well in \mathbb{R}^d for any $d \geq 4$, with “balloons” meaning $(d - 2)$ -dimensional spheres and “hoops” always meaning 1-dimensional circles. We have only specialized to $d = 4$ only for reasons of concreteness.

2. THE OBJECTS

2.1. Ribbon Knotted Balloons and Hoops. This paper is about ribbon-knotted balloons (S^2 's) and hoops (or loops, or S^1 's) in \mathbb{R}^4 or, equivalently, in S^4 . Throughout this paper T and H will denote finite¹ (not necessarily disjoint) sets of “labels”, where the labels in T label the balloons (though for reasons that will become clear later, they are also called “tail labels” and the things they label are sometimes called “tails”), and the labels in H label the hoops (though they are sometimes called “head labels” and they sometimes label “heads”).

Definition 2.1. A $(T; H)$ -labelled rKBH (ribbon-Knotted Balloons and Hoops) is a ribbon² up-to-isotopy embedding into \mathbb{R}^4 or into S^4 of $|T|$ oriented 2-spheres labelled by the elements of T (the “balloons”), of $|H|$ oriented circles labelled by the elements of H (the “hoops”), and of $|T| + |H|$ strings (namely, intervals) connecting the $|T|$ balloons and the $|H|$ hoops to some fixed base point, often denoted ∞ . Thus a $(\underline{2}; \underline{3})$ -labelled³ rKBH, for example, is a ribbon up-to-isotopy embedding into \mathbb{R}^4 or into S^4 of the space drawn on the right. Let $\mathcal{K}^{r_{bh}}(T; H)$ denote the set of all $(T; H)$ -labelled rKBHs.



Recall that 1-dimensional objects cannot be knotted in 4-dimensional space. Hence the hoops in an rKBH are not in themselves knotted, and hence an rKBH may be viewed as a knotting of the $|T|$ balloons in it, along with a choice of $|H|$ elements of the fundamental group of the complement of the balloons. Likewise, the $|T| + |H|$ strings in an rKBH only matter as homotopy classes of paths in the complement of the balloons. In particular, they can be modified arbitrarily in the vicinity of ∞ , and hence they don't even need to be specified near ∞ — it is enough that we know that they emerge from a small neighbourhood

¹The bulk of the paper easily generalizes to the case where H (not T !) is infinite, though nothing is gained by allowing H to be infinite.

²The adjective “ribbon” will be explained in Definition 2.4 below.

³See “notational conventions”, Section 10.5.

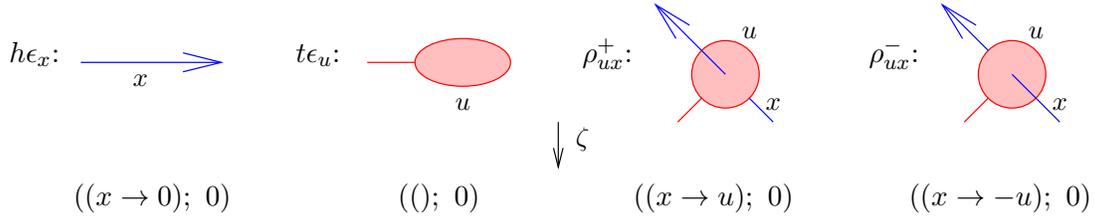


Figure 1. The four generators $h\epsilon_x$, $t\epsilon_u$, and ρ_{ux}^\pm , drawn in \mathbb{R}_{xyz}^3 (ρ_{ux}^\pm differ in the direction in which x pierces u — from below at ρ_{ux}^+ and from above at ρ_{ux}^-). The lower part of the figure previews the values of the main invariant ζ discussed in this paper on these generators. More later, in Section 5.

of ∞ (small enough so as to not intersect the balloons) and that each reaches its balloon or its hoop.

Conveniently we often pick our base point to be the point ∞ of the formula $S^4 = \mathbb{R}^4 \cup \{\infty\}$ and hence we can draw rKBHs in \mathbb{R}^4 (meaning, of course, that we draw in \mathbb{R}^2 and adopt conventions on how to lift these drawings to \mathbb{R}^4).

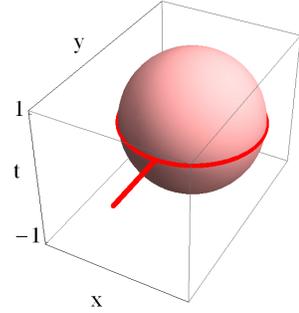
We will usually reserve the labels x , y , and z for hoops, the labels u , v , and w for balloons, and the labels a , b , and c for things that could be either balloons or hoops. With almost no risk of ambiguity, we also use x , y , z , along also with t , to denote the coordinates of \mathbb{R}^4 . Thus \mathbb{R}_{xy}^2 is the xy plane within \mathbb{R}^4 , \mathbb{R}_{txy}^3 is the hyperplane perpendicular to the z axis, and \mathbb{R}_{txyz}^4 is just another name for \mathbb{R}^4 .

Examples 2.2 and 2.3 below are more than just examples, for they introduce much notation that we use later on.

Example 2.2. The first four examples of rKBHs are the “four generators” shown in Figure 1:

- $h\epsilon_x$ is an element of $\mathcal{K}^{rbh} (; x)$ (more precisely, $\mathcal{K}^{rbh}(\emptyset; \{x\})$). It has a single hoop extending from near ∞ and back to near ∞ , and as indicated above, we didn’t bother to indicate how it closes near ∞ and how it is connected to ∞ with an extra piece of string. Clearly, $h\epsilon_x$ is the “unknotted hoop”.

- $t\epsilon_u$ is an element of $\mathcal{K}^{rbh}(u; \cdot)$. As a picture in \mathbb{R}_{xyz}^3 , it looks like a simplified tennis racket, consisting of a handle, a rim, and a net. To interpret a tennis racket in \mathbb{R}^4 , we embed \mathbb{R}_{xyz}^3 into \mathbb{R}_{txyz}^4 as the hyperplane $[t = 0]$, and inside it we place the handle and the rim as they were placed in \mathbb{R}_{xyz}^3 . We also make two copies of the net, the “upper” copy and the “lower” copy. We place the upper copy so that its boundary is the rim and so that its interior is pushed into the $[t > 0]$ half-space (relative to the pictured $[t = 0]$ placement) by an amount proportional to the distance from the boundary. Similarly we place the lower copy, except we push it into the $[t < 0]$ half space. Thus the two nets along with the rim make a 2-sphere in \mathbb{R}^4 , which is connected to ∞ using the handle. Clearly, $t\epsilon_u$ is the “unknotted balloon”. We orient $t\epsilon_u$ by adopting the conventions that surfaces drawn in the plane are oriented counterclockwise (unless otherwise noted) and that when pushed to $4D$, the upper copy retains the original orientation while the lower copy reverses it.

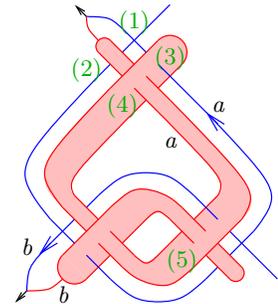


Warning: the vertical direction here is the “time” coordinate t , so this is an \mathbb{R}_{txy}^3 picture.

- ρ_{ux}^+ is an element of $\mathcal{K}^{rbh}(u; x)$. It is the 4D analog of the “positive Hopf link”. Its picture in Figure 1 should be interpreted in much the same way as the previous two — what is displayed should be interpreted as a 3D picture using standard conventions (what’s hidden is “below”), and then it should be placed within the $[t = 0]$ copy of \mathbb{R}_{xyz}^3 in \mathbb{R}^4 . This done, the racket’s net should be split into two copies, one to be pushed to $[t > 0]$ and the other to $[t < 0]$. In \mathbb{R}_{xyz}^3 it appears as if the hoop x intersects the balloon u right in the middle. Yet in \mathbb{R}^4 our picture represents a legitimate knot as the hoop is embedded in $[t = 0]$, the nets are pushed to $[t \neq 0]$, and the apparent intersection is eliminated.
- ρ_{ux}^- is the “negative Hopf link”. It is constructed out of its picture in exactly the same way as ρ_{ux}^+ . We postpone to Section 10.1 the explanation of why ρ_{ux}^+ is “positive” and ρ_{ux}^- is “negative”.

Example 2.3. Below on the right is a somewhat more sophisticated example of an rKBH with two balloons labelled a and b and two hoops labelled with the same labels (hence it is an element of $\mathcal{K}^{rbh}(a, b; a, b)$). It should be interpreted using the same conventions as in the previous example, though some further comments are in order:

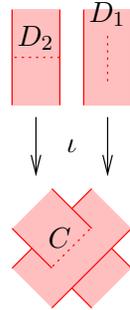
- The “crossing” marked (1) on the right is between two hoops and in 4D it matters not if it is an overcrossing or an undercrossing. Hence we did not bother to indicate which of the two it is. A similar comment applies in two other places.
- Likewise, crossing (2) is between a 1D strand and a thin tube, and its sense is immaterial. For no real reason we’ve drawn the strand “under” the tube, but had we drawn it “over”, it would be the same rKBH. A similar comment applies in two other places.
- Crossing (3) is “real” and is similar to ρ^- in the previous example. Two other crossings in the picture are similar to ρ^+ .



- Crossing (4) was not seen before, though its 4D meaning should be clear from our interpretation rules: nets are pushed up (or down) along the t coordinate by an amount proportional to the distance from the boundary. Hence the wider net in (4) gets pushed more than the narrower one, and hence in 4D they do not intersect even though their projections to 3D do intersect, as the figure indicates. A similar comment applies in two other places.
- Our example can be simplified a bit using isotopies. Most notably, crossing (5) can be eliminated by pulling the narrow “\” finger up and out of the wider “/” membrane. Yet note that a similar feat cannot be achieved near (3) and (4). Over there the wider “/” finger cannot be pulled down and away from the narrower “\” membrane and strand without a singularity along the way.

We can now complete Definition 2.1 by providing the the definition of “ribbon embedding”.

Definition 2.4. We say that an embedding of a collection of 2-spheres S_i into \mathbb{R}^4 (or into S^4) is “ribbon” if it can be extended to an immersion ι of a collection of 3-balls B_i whose boundaries are the S_i ’s, so that the singular set $\Sigma \subset \mathbb{R}^4$ of ι consists of transverse self-intersections, and so that each connected component C of Σ is a “ribbon singularity”: $\iota^{-1}(C)$ consists of two closed disks D_1 and D_2 , with D_1 embedded in the interior of one of the B_i and with D_2 embedded with its interior in the interior of some B_j and with its boundary in $\partial B_j = S_j$. A dimensionally-reduced illustration is on the right. The ribbon condition does not place any restriction on the hoops of an rKBH.



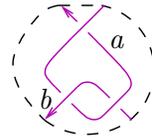
It is easy to verify that all the examples above are ribbon, and that all the operations we define below preserve the ribbon condition.

There is much literature about ribbon knots in \mathbb{R}^4 . See e.g. [HKS, HS, KS, Sa, Wa1, BND1, BND2].

2.2. Usual tangles and the map δ . For the purposes of this paper, a “usual tangle”⁴, or a “u-tangle”, is a “framed pure labelled tangle in a disk”. In detail, it is a piece of an oriented knot diagram drawn in a disk, having no closed components and with its components labelled by the elements of some set S , with all regarded modulo the Reidemeister moves R1’, R2, and R3:

$$\text{R1': } \bigcirc = \bigcirc \quad \text{R2: } \left(\begin{array}{c} \diagup \\ \diagdown \end{array} \right) = \left(\begin{array}{c} \diagdown \\ \diagup \end{array} \right) \quad \text{R3: } \left(\begin{array}{c} \diagup \\ \diagdown \end{array} \right) = \left(\begin{array}{c} \diagdown \\ \diagup \end{array} \right)$$

The set of all tangles with components labelled by S is denoted $u\mathcal{T}(S)$. An example of a member of $u\mathcal{T}(a, b)$ is on the right. Note that our u-tangles do not have a specific “up” direction so they do not form a category, and that the condition “no closed components” prevents them from being a planar algebra. In fact, $u\mathcal{T}$ carries almost no interesting algebraic structure. Yet it contains knots (as 1-component tangles) and more generally, by restricting to a subset, it contains “pure tangles”



⁴Better English would be “ordinary tangle”, but I want the short form to be “u-tangle”, which fits better with the “v-tangles” and “w-tangles” that arise later in this paper.

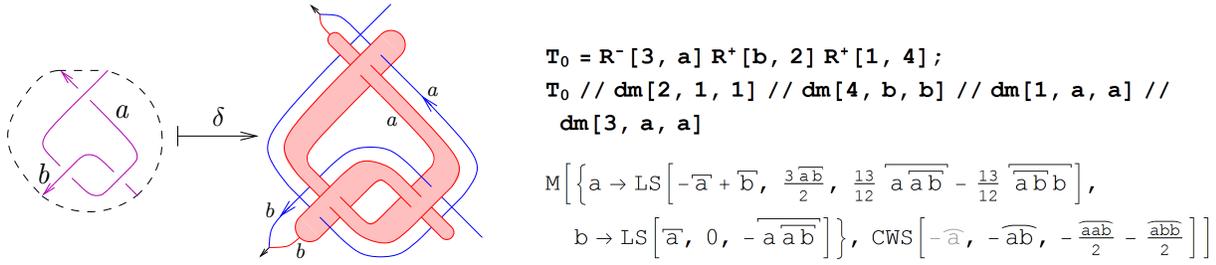


Figure 2. A $T_0 \mapsto \delta(T_0)$ example, and its invariant ζ of Section 5 (computed to degree 3).

or “string links” [HL]. And in the next section $u\mathcal{T}$ will be generalized to $v\mathcal{T}$ and to $w\mathcal{T}$, which do carry much interesting structure.

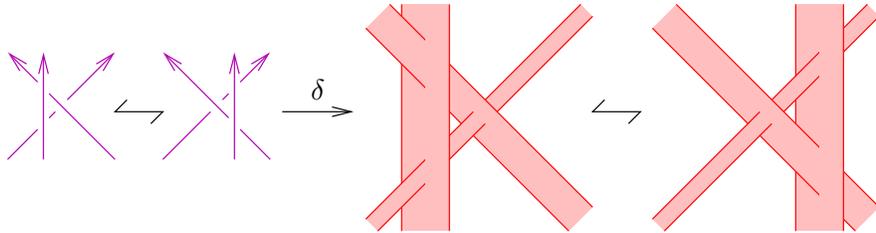
There is a map $\delta: u\mathcal{T}(S) \rightarrow \mathcal{K}^{rbh}(S; S)$. The picture should precede the words, and it appears as the left half of Figure 2.

In words, if $T \in u\mathcal{T}(S)$, to make $\delta(T)$ we convert each strand $s \in S$ of T into a pair of parallel entities: a copy of s on the right and a band on the left (T is a planar diagram and s is oriented, so “left” and “right” make sense). We cap the resulting band near its beginning and near its end, connecting the cap at its end to ∞ (namely, to outside the picture) with an extra piece of string — so that when the bands are pushed to 4D in the usual way, they become balloons with strings. Finally, near the crossings of T we apply the following (sign-preserving) local rules:



Proposition 2.5. *The map δ is well defined.*

Proof. We need to check that the Reidemeister moves in $u\mathcal{T}$ are carried to isotopies in \mathcal{K}^{rbh} . We’ll only display the “band part” of the third Reidemeister move, as everything else is similar or easier:



The fact that the two “band diagrams” above are isotopic before “inflation” to \mathbb{R}^4 , and hence also after, is visually obvious. \square

2.3. The Fundamental Invariant and the Near-Injectivity of δ . The “Fundamental Invariant” $\pi(K)$ of $K \in \mathcal{K}^{rbh}(u_i; x_j)$ is the triple $(\pi_1(K^c); m; l)$, where within this triple:

- The first entry is the fundamental group of the complement of the balloons of K , with basepoint taken to be at ∞ .
- The second entry m is the function $m: T \rightarrow \pi_1(K^c)$ which assigns to a balloon $u \in T$ its “base meridian” m_u — the path obtained by travelling along the string of u from ∞ to near the balloon, then Hopf-linking with the balloon u once in the positive direction much like in the generator ρ^+ of Figure 1, and then travelling back to the basepoint again along the string of u .
- The third entry l is the function $l: H \rightarrow \pi_1(K^c)$ which assigns to hoop $x \in H$ its longitude l_x — it is simply the hoop x itself regarded as an element of $\pi_1(K^c)$.

Thus for example, with $\langle \alpha \rangle$ denoting the group generated by a single element α and following the “notational conventions” of Section 10.5 for “inline functions”,

$$\begin{aligned} \pi(h\epsilon_x) &= (1; (); (x \rightarrow 1)), & \pi(t\epsilon_u) &= (\langle \alpha \rangle; (u \rightarrow \alpha); ()), \\ \text{and } \pi(\rho_{ux}^\pm) &= (\langle \alpha \rangle; (u \rightarrow \alpha); (x \rightarrow \alpha^{\pm 1})). \end{aligned}$$

We leave the following proposition as an exercise for the reader:

Proposition 2.6. *If T is an \underline{n} -labelled u -tangle, then $\pi(\delta(T))$ is the fundamental group of the complement of T (within a 3-dimensional space!), followed by the list of meridians of T (placed near the outgoing ends of the components of T), followed by the list of longitudes of T . \square*

It is well known (e.g. [Kaw, Theorem 6.1.7]) that knots are determined by the fundamental group of their complements, along with their “peripheral systems”, namely their meridians and longitudes regarded as elements of the fundamental groups of their complements. Thus we have:

Theorem 2.7. *When restricted to long knots (which are the same as knots), δ is injective. \square*

Remark 2.8. A similar map studied by Winter [Win1] is (sometimes) 2 to 1, as it retains less orientation information.

I expect that δ is also injective on arbitrary tangles and that experts in geometric topology would consider this trivial, but this result would be outside of my tiny puddle.

2.4. The Extension to v/w-Tangles and the Near-Surjectivity of δ . The map δ can be extended to “virtual crossings” [Kau] using the local assignment

$$\begin{array}{c} \text{X} \\ \delta \rightarrow \text{X} = \text{X} \end{array} \tag{1}$$

In a few more words, u -tangles can be extended to “ v -tangles” by allowing “virtual crossings” as on the left hand side of (1), and then modding out by the “virtual Reidemeister moves” and the “mixed move” / “detour move” of [Kau]⁵. One may then observe, as in Figure 3, that δ respects those moves as well as the “overcrossings commute” relation (yet not the “undercrossings commute” relation). Hence δ descends to the space $w\mathcal{T}$ of w -tangles, which are the quotient of v -tangles by the overcrossings commute relation.

⁵ In [Kau] the mixed / detour move was yet unnamed, and was simply “move (c) of Figure 2”.



Figure 3. The “Overcrossing Commute” (OC) relation and the gist of the proof that it is respected by δ , and the “Undercrossing Commute” (UC) relation and the gist of the reason why it is not respected by δ .

A topological-flavoured construction of δ appears in Section 10.2.

The newly extended $\delta: w\mathcal{T} \rightarrow \mathcal{K}^{rbh}$ cannot possibly be surjective, for the rKBHs in its image always have an equal number of balloons as hoops, with the same labels. Yet if we allow the deletion of components, δ becomes surjective:

Theorem 2.9. *For any KTG K there is some w -tangle T so that K is obtained from $\delta(T)$ by the deletion of some of its components.*

Proof. (Sketch) This is a variant of Theorem 3.1 of Satoh’s [Sa]. Clearly every knotting of 2-spheres in \mathbb{R}^4 can be obtained from a knotting of tubes by capping those tubes. Satoh shows that any knotting of tubes is in the image of a map he calls “Tube”, which is identical to our δ except our δ also includes the capping (good) and an extra hoop component for each balloon (harmless as they can be deleted). Finally to get the hoops of K simply put them in as extra strands in T , and then delete the spurious balloons that δ would produce next to each hoop. \square

3. THE OPERATIONS

3.1. The Meta-Monoid-Action. Loosely speaking, an rKBH K is a map of several S^1 ’s and several S^2 ’s into some ambient space. The former (the hoops of K) resemble elements of π_1 , and the latter (the balloons of K) resemble elements of π_2 . In general in homotopy theory, π_1 and π_2 are groups, and further, there is an action of π_1 on π_2 . Thus we find that on \mathcal{K}^{rbh} there are operations that resemble the group multiplication of π_1 , and the group multiplication of π_2 , and the action of π_1 on π_2 .

Let us describe these operations more carefully. Let $K \in \mathcal{K}^{rbh}(T; H)$.

- Analogously to the product in π_1 , there is the operation of “concatenating two hoops”. Specifically, if x and y are two distinct labels in H and z is a label not in H (except possibly equal to x or to y), we let⁶ $K // hm_z^{xy}$ be K with the x and y hoops removed and replaced with a single hoop labelled z that traces the path of them both. See Figure 4.

⁶ See “notational conventions”, Section 10.5.

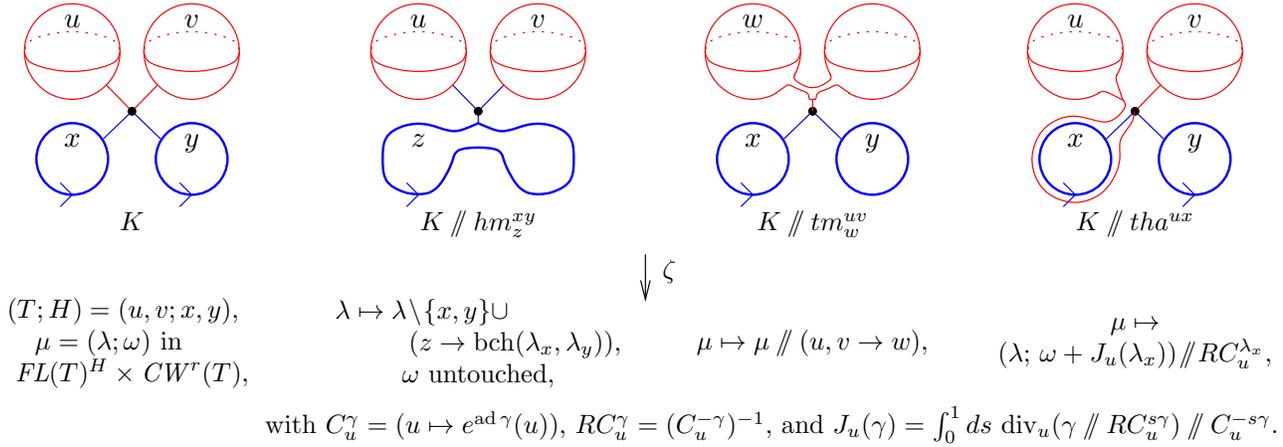
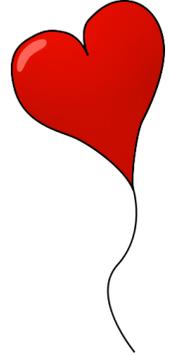


Figure 4. An rKBH K and the three basic unary operators applied to it. We use schematic notation; K may have plenty more components, and it may actually be knotted. The lower part of the figure is a summary of the main invariant ζ defined in this paper. See Section 5.

- Analogously to the homotopy-theoretic product of π_2 , there is the operation of “merging two balloons”. Specifically, if u and v are two distinct labels in T and w is a label not in T (except possibly equal to u or to v), we let $K // tm_w^{uv}$ be K with the u and v balloons removed and replaced by a single two-lobed balloon (topologically, still a sphere!) labelled w which spans them both. See Figure 4, or the even nicer two-lobed balloon displayed on the right.
- Analogously to the homotopy-theoretic action of π_1 on π_2 , there is the operation tha^{ux} (“tail by head action on u by x ”) of re-routing the string of the balloon u to go along the hoop x , as illustrated in Figure 4. In balloon-theoretic language, after the isotopy which pulls the neck of u along its string, this is the operation of “tying the balloon”, commonly performed to prevent the leakage of air (though admittedly, this will fail in 4D).



In addition, \mathcal{K}^{rbh} affords the further unary operations $t\eta^u$ (when $u \in T$) of “puncturing” the balloon u (implying, deleting it) and $h\eta^x$ (when $x \in H$) of “cutting” the hoop x (implying, deleting it). These two operations were already used in the statement and proof of Theorem 2.9.

In addition, \mathcal{K}^{rbh} affords the binary operation $*$ of “connected sum”, sketched on the right (along with its ζ formulae of Section 5). Whenever we have disjoint label sets $T_1 \cap T_2 = \emptyset = H_1 \cap H_2$, it is an operation $\mathcal{K}^{rbh}(T_1; H_1) \times \mathcal{K}^{rbh}(T_2; H_2) \rightarrow \mathcal{K}^{rbh}(T_1 \cup T_2; H_1 \cup H_2)$. We often suppress the $*$ symbol and write $K_1 K_2$ for $K_1 * K_2$.

Finally, there are re-labelling operations $h\sigma_b^a$ and $t\sigma_b^a$ on \mathcal{K}^{rbh} , which take a label a (either a head or a tail) and rename it b (provided b is “new”).

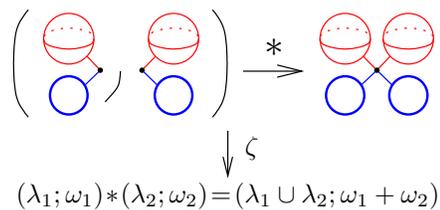


Figure 5. Connected sums.

Proposition 3.1. *The operations $*$, $t\sigma_v^u$, $h\sigma_y^x$, $t\eta^u$, $h\eta^x$, hm_z^{xy} , tm_w^{uv} and tha^{ux} and the special elements $t\epsilon_u$ and $h\epsilon_x$ have the following properties:*

- If the labels involved are distinct, the unary operations all commute with each other.
- The re-labelling operations have some obvious properties and interactions: $\sigma_b^a \parallel \sigma_c^b = \sigma_c^a$, $hm_x^{xy} \parallel h\sigma_z^x = hm_z^{xy}$, etc., and similarly for the deletion operations η^a .
- $*$ is commutative and associative; where it makes sense, it bi-commutes with the unary operations ($(K_1 \parallel hm_z^{xy}) * K_2 = (K_1 * K_2) \parallel hm_z^{xy}$, etc.).
- $t\epsilon_u$ and $h\epsilon_x$ are “units”:

$$\begin{aligned} (K * t\epsilon_u) \parallel tm_w^{uv} &= K \parallel t\sigma_w^v, & (K * t\epsilon_u) \parallel tm_w^{vu} &= K \parallel t\sigma_w^v, \\ (K * h\epsilon_x) \parallel hm_z^{xy} &= K \parallel h\sigma_z^y, & (K * h\epsilon_x) \parallel hm_z^{yx} &= K \parallel h\sigma_z^y. \end{aligned}$$

- Meta-associativity of hm , similar to the associativity in π_1 :

$$hm_x^{xy} \parallel hm_x^{xz} = hm_y^{yz} \parallel hm_x^{xy}. \quad (2)$$

- Meta-associativity of tm , similar to the associativity in π_2 :

$$tm_u^{uv} \parallel tm_u^{uw} = tm_v^{vw} \parallel tm_u^{uv}. \quad (3)$$

- Meta-actions commute. The following is a special case of the first property above, yet it deserves special mention because later in this paper it will be the only such commutativity that is non-obvious to verify:

$$tha^{ux} \parallel tha^{vy} = tha^{vy} \parallel tha^{ux}. \quad (4)$$

- Meta-action axiom t , similar to $(uv)^x = u^x v^x$:

$$tm_w^{uv} \parallel tha^{wx} = tha^{ux} \parallel tha^{vx} \parallel tm_w^{uv}. \quad (5)$$

- Meta-action axiom h , similar to $u^{xy} = (u^x)^y$:

$$hm_z^{xy} \parallel tha^{uz} = tha^{ux} \parallel tha^{uy} \parallel hm_z^{xy}. \quad (6)$$

Proof. The first four properties say almost nothing and we did not even specify them in full⁷. The remaining four deserve attention, especially in the light of the fact that the verification of their analogs later in this paper will be non-trivial. Yet in the current context, their verification is straightforward. \square

⁷ We feel that the clarity of this paper is enhanced by this omission.

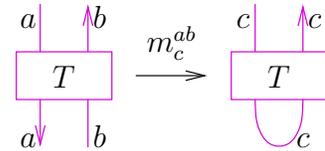
Later we will seek to construct invariants of rKBH's by specifying their values on some generators and by specifying their behaviour under our list of operations. Thus it is convenient to introduce a name for the algebraic structure of which \mathcal{K}^{rbh} is an instance:

Definition 3.2. A meta-monoid-action (MMA) M is a collections of sets $M(T; H)$, one for each pair of finite sets of labels T and H , along with partially-defined operations⁸ $*$, $t\sigma_v^u$, $h\sigma_y^x$, $t\eta^u$, $h\eta^x$, hm_z^{xy} , tm_w^{uv} and tha^{ux} , and with special elements $t\epsilon_u \in M(\{u\}; \emptyset)$ and $h\epsilon_x \in M(\emptyset; \{x\})$, which together satisfy the properties in Proposition 3.1.

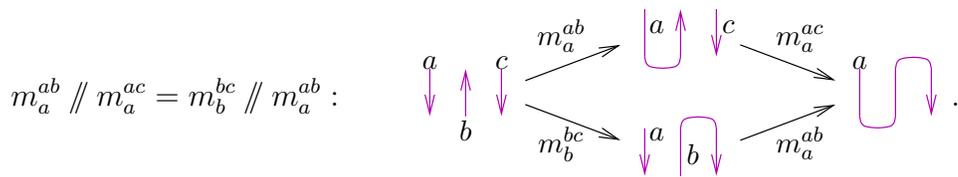
For the rationale behind the name “meta-monoid-action” see Section 10.3. In Section 10.3.5 we note that \mathcal{K}^{rbh} in fact has the further structure making it a meta-group-action (or more precisely, a meta-Hopf-algebra-action).

3.2. The Meta-Monoid of Tangles and the Homomorphism δ . Our aim in this section is to show that the map $\delta: w\mathcal{T} \rightarrow \mathcal{K}^{rbh}$ of Sections 2.2 and 2.4, which maps w-tangles to knotted balloons and hoops, is a “homomorphism”. But first we have to discuss the relevant algebraic structures on $w\mathcal{T}$ and on \mathcal{K}^{rbh} .

$w\mathcal{T}$ is a “meta-monoid” (see Section 10.3.2). Namely, for any finite set S of “strand labels” $w\mathcal{T}(S)$ is a set, and whenever we have a set S of labels and three labels $a \neq b$ and c not in it, we have the operation $m_c^{ab}: w\mathcal{T}(S \cup \{a, b\}) \rightarrow w\mathcal{T}(S \cup \{c\})$ of “concatenating strand a with strand b and calling the resulting strand c ”. See the picture on the right, and note that while on $u\mathcal{T}$ the operation m_c^{ab} would be defined only if the head of a happens to be adjacent to the tail of b , on $v\mathcal{T}$ and on $w\mathcal{T}$ this operation is always defined, as the head of a can always be brought near the tail of b by adding some virtual crossings, if necessary. $w\mathcal{T}$ trivially also carries the rest of the necessary structure to form a meta-monoid — namely, strand relabelling operations σ_b^a , strand deletion operations η^a , and a disjoint union operation $*$, and “units” ϵ_a (tangles with a single unknotted strand labelled a).



It is easy to verify the associativity property (compare with Equation (32) of Section 10.3.1):



It is also easy to verify that if a tangle $T \in w\mathcal{T}(a, b)$ is non-split, then $T \neq (T \parallel \eta^b) * (T \parallel \eta^a)$, so in the sense of Section 10.3.2, $w\mathcal{T}$ is non-classical.

⁸ tm_w^{uv} , for example, is defined on $M(T; H)$ exactly when $u, v \in T$ yet $w \notin T \setminus \{u, v\}$. All other operations behave similarly.

\mathcal{K}^{rbh} is an analog of both π_1 and π_2 . In homotopy theory, the group π_1 acts on π_2 so one may form the semi-direct product $\pi_1 \ltimes \pi_2$. In a similar manner, one may put a “combined” multiplication on that part of \mathcal{K}^{rbh} in which the balloons and the hoops are matched together. More precisely, given a finite set of labels S , let $\mathcal{K}^{b=h}(S) := \mathcal{K}^{rbh}(S; S)$ be the set of rKBHs whose balloons and whose hoops are both labelled with labels in S . Then define $dm_c^{ab} : \mathcal{K}^{b=h}(S \cup \{a, b\}) \rightarrow \mathcal{K}^{b=h}(S \cup \{c\})$ (the prefix d is for “diagonal”, or “double”) by

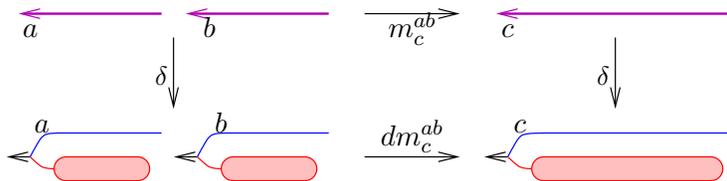
$$dm_c^{ab} = tha^{ab} \parallel tm_c^{ab} \parallel hm_c^{ab}. \quad (7)$$

It is a routine exercise to verify that the properties (2)–(6) of hm , tm , and tha imply that dm is meta-associative:

$$dm_a^{ab} \parallel dm_a^{ac} = dm_b^{bc} \parallel dm_a^{ab}.$$

Thus dm (along with “diagonal” η ’s and σ ’s and an unmodified $*$) puts a meta-monoid structure on $\mathcal{K}^{b=h}$.

Proposition 3.3. $\delta : w\mathcal{T} \rightarrow \mathcal{K}^{b=h}$ is a meta-monoid homomorphism. (A rough picture is on the right: in the picture a and b are strands within the same tangle, and they may be knotted with each other and with possible further components of that tangle). \square



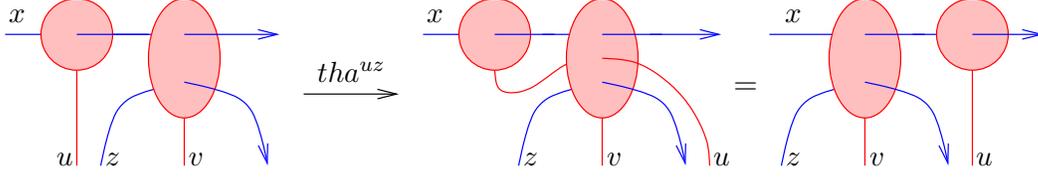
3.3. Generators and Relations for \mathcal{K}^{rbh} . It is always good to know that a certain algebraic structure is finitely presented. If we had a complete set of generators and relations for \mathcal{K}^{rbh} , for example, we could define a “homomorphic invariant” of rKBHs by picking some target MMA \mathcal{M} (Definition 3.2), declaring the values of the invariant on the generators, and verifying that the relations are satisfied. Hence it’s good to know the following:

Theorem 3.4. *The MMA \mathcal{K}^{rbh} is generated (as an MMA) by the four rKBHs $h\epsilon_x$, $t\epsilon_u$, and ρ_{ux}^\pm of Figure 1.*

Proof. By Theorem 2.9 and the fact that the MMA operations include component deletions $t\eta^u$ and $h\eta^x$ it follows that \mathcal{K}^{rbh} is generated by the image of δ . By the previous proposition and the fact (7) that dm can be written in terms of the MMA operations of \mathcal{K}^{rbh} , it follows that \mathcal{K}^{rbh} is generated by the δ -images of the generators of $w\mathcal{T}$. But the generators of $w\mathcal{T}$ are the virtual crossing $\underset{a}{\overset{b}{\curvearrowright}}$ and the right-handed and left-handed crossings $\underset{a}{\overset{b}{\curvearrowright}}$ and $\underset{a}{\overset{b}{\curvearrowleft}}$, and

so the theorem follows from the following easily verified assertions: $\delta \left(\underset{a}{\overset{b}{\curvearrowright}} \right) = t\epsilon_a h\epsilon_a t\epsilon_b h\epsilon_b$, $\delta \left(\underset{a}{\overset{b}{\curvearrowleft}} \right) = \rho_{ab}^+ t\epsilon_b h\epsilon_a$, and $\delta \left(\underset{a}{\overset{b}{\curvearrowright}} \right) = \rho_{ba}^- t\epsilon_a h\epsilon_b$. \square

We now turn to the study of relations. Our first is the hardest and most significant, the “Conjugation Relation”, whose name is inspired by the group theoretic relation $vu^v = uv$ (here u^v denotes group conjugation, $u^v = v^{-1}uv$). Consider the following equality:



Easily, the rKBH on the very left is $\rho_{ux}^+(\rho_{vy}^+\rho_{wz}^+ \parallel tm_v^{vw}) \parallel hm_x^{xy}$ and the one on the very right is $(\rho_{vx}^+\rho_{wz}^+ \parallel tm_v^{vw})\rho_{uy}^+ \parallel hm_x^{xy}$, and so

$$\rho_{ux}^+\rho_{vy}^+\rho_{wz}^+ \parallel tm_v^{vw} \parallel hm_x^{xy} \parallel tha^{uz} = \rho_{vx}^+\rho_{wz}^+\rho_{uy}^+ \parallel tm_v^{vw} \parallel hm_x^{xy}. \quad (8)$$

Definition 3.5. Let \mathcal{K}_0^{rbh} be the MMA freely generated by symbols $\rho_{ux}^\pm \in \mathcal{K}_0^{rbh}(u; x)$, modulo the following relations:

- Relabelling: $\rho_{ux}^\pm \parallel h\sigma_y^x \parallel t\sigma_v^u = \rho_{vy}^\pm$.
- Cutting and puncturing: $\rho_{ux}^\pm \parallel h\eta^x = t\epsilon_u$ and $\rho_{ux}^\pm \parallel t\eta^u = h\epsilon_x$.
- Inverses: $\rho_{ux}^+\rho_{vy}^- \parallel tm_w^{uv} \parallel hm_z^{xy} = t\epsilon_w h\epsilon_z$.
- Conjugation relations: for any $s_{1,2} \in \{\pm\}$,

$$\rho_{ux}^{s_1}\rho_{vy}^{s_2}\rho_{wz}^{s_2} \parallel tm_v^{vw} \parallel hm_x^{xy} \parallel tha^{uz} = \rho_{vx}^{s_2}\rho_{wz}^{s_2}\rho_{uy}^{s_1} \parallel tm_v^{vw} \parallel hm_x^{xy}.$$

- Tail-commutativity: on any inputs, $tm_w^{uv} = tm_w^{vu}$.
- Framing independence: $\rho_{ux}^\pm \parallel tha^{ux} = \rho_{ux}^\pm$.

The following proposition, whose proof we leave as an exercise, says that \mathcal{K}_0^{rbh} is a pretty good approximation to \mathcal{K}^{rbh} :

Proposition 3.6. *The obvious maps $\pi: \mathcal{K}_0^{rbh} \rightarrow \mathcal{K}^{rbh}$ and $\delta: w\mathcal{T} \rightarrow \mathcal{K}_0^{rbh}$ are well defined. \square*

Conjecture 3.7. *The projection $\pi: \mathcal{K}_0^{rbh} \rightarrow \mathcal{K}^{rbh}$ is an isomorphism.*

We expect that there should be a Reidemeister-style combinatorial calculus of ribbon knots in \mathbb{R}^4 . The above conjecture is that the definition of \mathcal{K}_0^{rbh} is such a calculus. We expect that given any such calculus, the proof of the conjecture should be easy. In particular, the above conjecture is equivalent to the statement that the stated relations in the definition of $w\mathcal{T}$ generate the relations in the kernel of Satoh's Tube map δ_0 (see Section 10.2), and this is equivalent to the conjecture whose proof was attempted at [Win2]. Though I understood by private communication with B. Winter that [Win2] is presently flawed.

In the absence of a combinatorial description of \mathcal{K}^{rbh} , we replace it by \mathcal{K}_0^{rbh} throughout the rest of this paper. Hence we construct invariants of elements of \mathcal{K}_0^{rbh} instead of invariants of genuine rKBHs. Yet note that the map $\delta: w\mathcal{T} \rightarrow \mathcal{K}_0^{rbh}$ is well-defined, so our invariants are always good enough to yield invariants of tangles and virtual tangles.

3.4. Example: The Fundamental Invariant. The ‘‘Fundamental Invariant’’ π of Section 2.3 is defined in a direct manner on \mathcal{K}^{rbh} and does not need to suffer from the difficulties of the previous section. Yet it can also serve as an example for our approach for defining invariants on \mathcal{K}_0^{rbh} using generators and relations.

Definition 3.8. Let $\Pi(T; H)$ denote the set of all triples $(G; m; l)$ of a group G along with functions $m \in G^T$ and $l \in G^H$, regarded modulo group isomorphisms with their obvious action on m and l ⁹. Define MMA operations $(*, t\sigma_v^u, h\sigma_y^x, t\eta^u, h\eta^x, tm_w^{uv}, hm_z^{xy}, tha^{ux})$ on $\Pi = \{\Pi(T; H)\}$ and units $t\epsilon_u$ and $h\epsilon_x$ as follows:

- $*$ is the operation of taking the free product $G_1 * G_2$ of groups and concatenating the lists of heads and tails:

$$(G_1; m_1; l_1) * (G_2; m_2; l_2) := (G_1 * G_2; m_1 \cup m_2; l_1 \cup l_2).$$

- $t\sigma_b^a / h\sigma_b^a$ relabels an element labelled a to be labelled b .
- $t\eta^u / h\eta^x$ removes the element labelled u / x .
- tm_w^{uv} “combines” u and v to make w . Precisely, it replaces the input group G with $G' = G / \langle m_u = m_v \rangle$, removes the tail labels u and v , and introduces a new tail, the element $m_u = m_v$ of G' and labels it w :

$$tm_w^{uv}(G; m; l) := (G / \langle m_u = m_v \rangle; (m \setminus \{u, v\}) \cup (w \rightarrow m_u); l).$$

- hm_z^{xy} replaces two elements in l by their product:

$$hm_z^{xy}(G; m; l) := (G, m, (l \setminus \{x, y\}) \cup (z \rightarrow l_x l_y)).$$

- The best way to understand the action of tha^{ux} is as “the thing that makes the fundamental invariant π a homomorphism, given the geometric interpretation of tha^{ux} on \mathcal{K}^{rbh} in Section 3.1”. In formulae, this becomes

$$tha^{ux}(G; m; l) := (G * \langle \alpha \rangle / \langle m_u = l_x \alpha l_x^{-1} \rangle; (m \setminus u) \cup (u \rightarrow \alpha), l),$$

where α is some new element that is added to G .

- $t\epsilon_u = (\langle \alpha \rangle; (u \rightarrow \alpha); ())$ and $h\epsilon_x = (1; (); (x \rightarrow 1))$.

We state the following without its easy topological proof:

Proposition 3.9. $\pi: \mathcal{K}^{rbh} \rightarrow \Pi$ is a homomorphism of MMAs. □

A consequence is that π can be computed on any rKBH starting from its values on the generators of \mathcal{K}^{rbh} as listed in Section 2.3 and then using the operations of Definition 3.8.

Comment 3.10. The fundamental groups of ribbon 2-knots are “labelled oriented tree” (LOT) groups in the sense of Howie [Ho1, Ho2]. Howie’s definition has an obvious extension to labelled oriented forests, yielding a class of groups that may be called “LOF groups”. One may show that the the fundamental groups of complements of rKBHs are always LOF groups. One may also show that the subset Π^{LOF} of Π in which the group component G is an LOF group is a sub-MMA of Π . Therefore $\pi: \mathcal{K}^{rbh} \rightarrow \Pi^{\text{LOF}}$ is also a homomorphism of MMAs; I expect it to be an isomorphism or very close to an isomorphism. Thus much of the rest of this paper can be read as a “theory of homomorphic (in the MMA sense) invariants of LOF groups”. I don’t know how much it may extend to a similar theory of homomorphic invariants of bigger classes of groups.

⁹ I ignore set-theoretic difficulties. If you insist, you may restrict to countable groups or to finitely presented groups.

4. THE FREE LIE INVARIANT

In this section we construct ζ_0 , the “tree” part to our main tree-and-wheel valued invariant ζ , by following the scheme of Section 3.3. Yet before we succeed, it is useful to aim a bit higher and fail, and thus appreciate that even ζ_0 is not entirely trivial.

4.1. A Free Group Failure. If the balloon part of an rKBH K is unknotted, the fundamental group $\pi_1(K^c)$ of its complement is the free group generated by the meridians $(m_u)_{u \in T}$. The hoops of K are then elements in that group and hence they can be written as words $(w_x)_{x \in H}$ in the m_u ’s and their inverses. Perhaps we can make an MMA \mathcal{W} out of lists (w_x) of free words in letters $m_u^{\pm 1}$ and use it to define a homomorphic invariant $W: \mathcal{K}^{rbh} \rightarrow \mathcal{W}$? All we need, it seems, is to trace how MMA operations on K affect the corresponding list (w_x) of words.

The beginning is promising. $*$ acts on pairs of lists of words by taking the union of those lists. hm_z^{xy} acts on a list of words by replacing w_x and w_y by their concatenation, now labelled z . tm_r^{pq} acts on $\bar{w} = (w_x)$ by replacing every occurrence of the letter m_p and every occurrence of the letter m_q in \bar{w} by a single new letter, m_r .

The problem is with tha^{ux} . Imitating the topology, tha^{ux} should act on $\bar{w} = (w_y)$ by replacing every occurrence of m_u in \bar{w} with $w_x \alpha w_x^{-1}$, where α is a new letter, destined to replace m_u . But w_x may also contain instances of m_u , so after the replacement $m_u \mapsto \alpha^{w_x}$ is performed, it should be performed again to get rid of the m_u ’s that appear in the “conjugator” w_x . But new m_u ’s are then created, and the replacement should be carried out yet again... The process clearly doesn’t stop, and our attempt failed.

Yet not all is lost. The later and later replacements occur within conjugators of conjugators, deeper and deeper into the lower central series of the free groups involved. Thus if we replace free groups by some completion thereof in which deep members of the lower central series are “small”, the process becomes convergent. This is essentially what will be done in the next section.

4.2. A Free Lie Algebra Success. Given a set T , let $FL(T)$ denote the graded completion of the free Lie algebra on the generators in T (sometimes we will write “ FL ” for “ $FL(T)$ for some set T ”). We define a meta-monoid-action M_0 as follows. For any finite set T of “tail labels” and any finite set H or “head labels”, we let

$$M_0(T; H) := FL(T)^H$$

be the set of H -labelled arrays of elements of $FL(T)$. On $M_0 := \{M_0(T; H)\}$ we define operations as follows, starting from the trivial and culminating with the most interesting, tha^{ax} . All of our definitions are directly motivated by the “failure” of the previous section; in establishing the correspondence between the definitions below and the ones above, one should interpret $\lambda = (\lambda_x) \in M_0(T; H)$ as “a list of logarithms of a list of words (w_x) ”.

- $h\sigma_y^x$ is simply σ_y^x as explained in the conventions section, Section 10.5.
- $t\sigma_v^u$ is induced by the map $FL(T) \rightarrow FL((T \setminus u) \cup \{v\})$ in which the generator u is mapped to the generator v .

- $t\eta$ acts by setting one of the tail variables to 0, and $h\eta$ acts by dropping an array element. Thus for $\lambda \in M_0(T; H)$,

$$\lambda \parallel t\eta^u = \lambda \parallel (u \mapsto 0) \quad \text{and} \quad \lambda \parallel h\eta^x = \eta \setminus x.$$

- If $\lambda_1 \in M_0(T_1; H_1)$ and $\lambda_2 \in M_0(T_2; H_2)$ (and, of course, $T_1 \cap T_2 = \emptyset = H_1 \cap H_2$), then

$$\lambda_1 * \lambda_2 := (\lambda_1 \parallel \iota_1) \cup (\lambda_2 \parallel \iota_2)$$

where ι_i are the natural embeddings $\iota_i: FL(T_i) \hookrightarrow FL(T_1 \cup T_2)$, for $i = 1, 2$.

- If $\lambda \in M_0(T; H)$ then

$$\lambda \parallel tm_w^{uv} := \lambda \parallel (u, v \mapsto w),$$

where $(u, v \mapsto w)$ denotes the morphism $FL(T) \rightarrow FL(T \setminus \{u, v\} \cup \{w\})$ defined by mapping the generators u and v to the generator w .

- If $\lambda \in M_0(T; H)$ then

$$\lambda \parallel hm_z^{xy} := \lambda \setminus \{x, y\} \cup (z \rightarrow \text{bch}(\lambda_x, \lambda_y)),$$

where bch stands for the Baker-Campbell-Hausdorff formula:

$$\text{bch}(a, b) := \log(e^a e^b) = a + b + \frac{1}{2}[a, b] + \dots$$

- If $\lambda \in M_0(T; H)$ then

$$\lambda \parallel tha^{ux} := \lambda \parallel (C_u^{-\lambda_x})^{-1} = \lambda \parallel RC_u^{\lambda_x} \tag{10}$$

In the above formula $C_u^{-\lambda_x}$ denotes the automorphism of $FL(T)$ defined by mapping the generator u to its “conjugate” $e^{-\lambda_x} u e^{\lambda_x}$. More precisely, u is mapped to $e^{-\text{ad } \lambda_x}(u)$, where ad denotes the adjoint action, and e^{ad} is taken in the formal sense. Thus

$$C_u^{-\lambda_x} : u \mapsto e^{-\text{ad } \lambda_x}(u) = u - [\lambda_x, u] + \frac{1}{2}[\lambda_x, [\lambda_x, u]] - \dots \tag{11}$$

Also in Equation (10), $RC_u^{\lambda_x} := (C_u^{-\lambda_x})^{-1}$ denotes the inverse of the automorphism $C_u^{-\lambda_x}$.

- $t\epsilon_u = ()$ and $h\epsilon_x = (x \rightarrow 0)$.

Warning 4.1. When $\gamma \in FL$, the inverse of $C_u^{-\gamma}$ may *not* be C_u^γ . If γ does not contain the generator u , then indeed $C_u^{-\gamma} \parallel C_u^\gamma = I$. But in general applying $C_u^{-\gamma}$ creates many “new” u ’s, within the γ ’s that appear in the right hand side of (11), and the “new” u ’s are then conjugated by C_u^γ instead of being left in place. Yet $C_u^{-\gamma}$ is invertible, so we simply name its inverse RC_u^γ .

The name “ RC ” stands either for “Reverse Conjugation”, or for “Repeated Conjugation”. The rationale for the latter naming is that if $\alpha \in FL(T)$ and \bar{u} is a name for a new “temporary” free-Lie generator, then $RC_u^\gamma(\alpha)$ is the result of applying the transformation $u \mapsto e^{\text{ad } \gamma}(\bar{u})$ repeatedly to α until it stabilizes (at any fixed degree this will happen after a finite number of iterations), followed by the eventual renaming $\bar{u} \mapsto u$.

Comment 4.2. Some further insight into RC_u^γ can be obtained by studying the triangle on the right. The space at the bottom of the triangle is the quotient of the free Lie algebra on $T \cup \{\bar{u}\}$ (where \bar{u} is a new “temporary” generator) by either of the two relations shown there; these two relations are of course equivalent. The map ϕ is induced from the obvious inclusion of $FL(T)$ into $FL(T \cup \{\bar{u}\})$, and in the presence of the relation $\bar{u} = e^{-\text{ad}^\gamma u}$, it is clearly an isomorphism. The map $\bar{\phi}$ is likewise induced from the renaming $u \mapsto \bar{u}$. It too is an isomorphism, but slightly less trivially — indeed, using the relation $u = e^{\text{ad}^\gamma \bar{u}}$ repeatedly, any element in $FL(T \cup \{\bar{u}\})$ can be written in form that does not include u , and hence is in the image of $\bar{\phi}$. It is clear that $C_u^{-\gamma} = \bar{\phi} \parallel \phi^{-1}$. Hence $RC_u^\gamma = \phi \parallel \bar{\phi}^{-1}$, and as $\bar{\phi}^{-1}$ is described in terms of repeated applications of the relation $u = e^{\text{ad}^\gamma \bar{u}}$, it is clear that RC_u^γ indeed involves “repeated conjugation” as asserted in the previous paragraph.

$$\begin{array}{ccc}
 FL(T) & \begin{array}{c} \xleftarrow{C_u^{-\gamma}} \\ \xrightarrow{RC_u^\gamma} \end{array} & FL(T) \\
 \searrow \phi & & \swarrow \bar{\phi} \\
 & & u \rightarrow \bar{u} \\
 FL(T \cup \{\bar{u}\}) & / & \left(\begin{array}{l} \bar{u} = e^{-\text{ad}^\gamma u} \\ \text{and / or} \\ u = e^{\text{ad}^\gamma \bar{u}} \end{array} \right)
 \end{array}$$

Warning 4.3. Equation (10) does *not* say that $tha^{ux} = RC_u^{\lambda_x}$ as abstract operations, only that they are equal when evaluated on λ . In general it is not the case that $\mu \parallel tha^{ux} = \mu \parallel RC_u^{\lambda_x}$ for arbitrary μ — the latter equality is only guaranteed if $\mu_x = \lambda_x$.

As another example of the difference, the operations hm_z^{xy} and tha^{ux} do not commute — in fact, the composition $hm_z^{xy} \parallel tha^{ux}$ does not even make sense, for by the time tha^{ux} is evaluated its input does not have an entry labelled x . Yet the commutativity

$$\lambda \parallel hm_z^{xy} \parallel RC_u^{\lambda_x} = \lambda \parallel RC_u^{\lambda_x} \parallel hm_z^{xy} \quad (12)$$

makes perfect sense and holds true, for the operation hm_z^{xy} only involves the heads / roots of trees, while $RC_u^{\lambda_x}$ only involves their tails / leafs.

Theorem 4.4. M_0 , with the operations defined above, is a meta-monoid-action (MMA).

Proof. Most MMA axioms are trivial to verify. The most important ones are the ones in Equations (2) through (6). Of these, the meta-associativity of hm follows from the associativity of the bch formula, $\text{bch}(\text{bch}(\lambda_x, \lambda_y), \lambda_z) = \text{bch}(\lambda_x, \text{bch}(\lambda_y, \lambda_z))$, the meta-associativity of tm and is trivial, and it remains to prove that meta-actions commute (Equation (4); all other required commutativities are easy) and the the meta-action axiom t (Equation (5)) and h (Equation (6)).

Meta-actions commute. Expanding (4) using the above definitions and denoting $\alpha := \lambda_x$, $\beta = \lambda_y$, $\alpha' := \alpha \parallel RC_v^\beta$, and $\beta' := \beta \parallel RC_u^\alpha$, we see that we need to prove the identity

$$RC_u^\alpha \parallel RC_v^{\beta'} = RC_v^\beta \parallel RC_u^{\alpha'}. \quad (13)$$

Consider the commutative diagram on the right. In it $FL(u, v)$ means “the (completed) free Lie algebra with generators u and v , and some additional fixed collection of generators”, and likewise for $FL(u, \bar{u}, v, \bar{v})$. The diagonal arrows are all substitution homomorphisms as indicated, and they are all isomorphisms. We put the elements α and β in the upper-left space, and by comparing with the diagram in Comment 4.2, we see that the upper horizontal map is RC_u^α and the left vertical map is RC_v^β . Therefore β' is the image of β in the top left space, and α' is the image of α in the bottom left space. Therefore again using the diagram in Comment 4.2, the right vertical map is $RC_v^{\beta'}$ and the lower horizontal map is $RC_u^{\alpha'}$, and (13) follows from the commutativity of the external square in the above diagram.

$$\begin{array}{ccc}
 FL(u, v) & \xrightarrow[RC_u^\alpha]{\beta} & FL(u, v) \\
 \downarrow \alpha & \searrow^{(u,v) \rightarrow (u,v)} & \swarrow_{(u,v) \rightarrow (\bar{u}, v)} \\
 RC_v^\beta & FL(u, \bar{u}, v, \bar{v}) / \left(\begin{array}{l} u = e^{\text{ad } \alpha} \bar{u} \\ v = e^{\text{ad } \beta} \bar{v} \end{array} \right) & \downarrow RC_v^{\beta'} \\
 \downarrow \alpha' & \swarrow_{(u,v) \rightarrow (u, \bar{v})} & \searrow_{(u,v) \rightarrow (\bar{u}, \bar{v})} \\
 FL(u, v) & \xrightarrow[RC_u^{\alpha'}]{} & FL(u, v)
 \end{array}$$

For use later, we record the fact that by reading all the horizontal and vertical arrows backwards, the above argument also proves the identity

$$C_u^{-\alpha} // RC_v^\beta // C_v^{-\beta} = C_v^{-\beta} // RC_u^\alpha // C_u^{-\alpha}. \quad (14)$$

Meta-action axiom t . Expanding (5) and denoting $\gamma := \lambda_x$, we need to prove the identity

$$t_w^{uv} // RC_w^\gamma // t_w^{uv} = RC_u^\gamma // RC_v^\gamma // RC_u^\gamma // t_w^{uv}. \quad (15)$$

Consider the diagram on the right. In it, the vertical and diagonal arrows are all substitution homomorphisms as indicated. The horizontal arrows are RC maps as indicated. The element γ lives in the upper left corner of the diagram, but equally makes sense in the upper of the central spaces. We denote its image via RC_u^γ by γ_2 , and think of it as an element of the middle space in the top row. Likewise $\gamma_4 := \gamma // t_w^{uv}$ lives in both the bottom left space and the bottom of the two middle spaces.

$$\begin{array}{ccccc}
 \gamma \in FL(u, v) & \xrightarrow[RC_u^\gamma]{} & \gamma_2 \in FL(u, v) & \xrightarrow[RC_v^{\gamma_2}]{} & FL(u, v) \\
 \downarrow \phi_1 & \searrow^{u,v \downarrow u,v} & \downarrow \phi_2 & \swarrow_{u,v \downarrow \bar{u}, v} & \downarrow \phi_3 \\
 \gamma \in FL(u, \bar{u}, v, \bar{v}) & / \left(\begin{array}{l} u = e^{\text{ad } \gamma} \bar{u} \\ v = e^{\text{ad } \gamma} \bar{v} \end{array} \right) & & & \\
 \downarrow u,v \downarrow w & & \downarrow u,v \downarrow w & & \downarrow u,v \downarrow w \\
 & & \gamma_4 \in FL(w, \bar{w}) / w = e^{\text{ad } \gamma_4} \bar{w} & & \\
 \downarrow \phi_4 & \swarrow_{w \rightarrow w} & & \searrow_{w \rightarrow \bar{w}} & \downarrow \phi_5 \\
 \gamma_4 \in FL(w) & \xrightarrow[RC_w^{\gamma_4}]{} & & & FL(w)
 \end{array}$$

It requires a minimal effort to show that the map at the very centre of the diagram is well defined. The commutativity of the triangles in the diagram follows from Comment 4.2, and the commutativity of the trapezoids is obvious. Hence the diagram is overall commutative. Reading it from the top left to the bottom right along the left and the bottom edges gives the left hand side of Equation (15), and along the top and the right edges gives the right hand side.

Meta-action axiom h . Expanding (6), we need to prove

$$\lambda \parallel hm_z^{xy} \parallel RC_u^{\text{bch}(\lambda_x, \lambda_y)} = \lambda \parallel RC_u^{\lambda_x} \parallel RC_u^{\lambda_y} \parallel RC_u^{\lambda_x} \parallel hm_z^{xy}.$$

Using commutativities as in Equation (12) and denoting $\alpha = \lambda_x$ and $\beta = \lambda_y$ we can cancel the hm_z^{xy} 's, and we are left with

$$RC_u^{\text{bch}(\alpha, \beta)} \stackrel{?}{=} RC_u^\alpha \parallel RC_u^{\beta'}, \quad \text{where} \quad \beta' := \beta \parallel RC_u^\alpha. \quad (16)$$

This last equality follows from a careful inspection of the following commutative diagram:

$$\begin{array}{ccccc} FL(u) & \xrightarrow{RC_u^\alpha} & FL(u) & \xrightarrow{RC_u^{\beta'}} & FL(u) \\ & \searrow & \swarrow^{u \rightarrow \bar{u}} & \searrow^{u \rightarrow \bar{u}} & \swarrow^{u \rightarrow \bar{u}} \\ & & FL(u, \bar{u}) / (u = e^{\text{ad } \alpha} \bar{u}) & & FL(\bar{u}, \bar{\bar{u}}) / (\bar{u} = e^{\text{ad } \beta'} \bar{\bar{u}}) \\ & & \searrow & & \swarrow \\ & & & & FL(u, \bar{u}, \bar{\bar{u}}) / \left(\begin{array}{l} u = e^{\text{ad } \alpha} \bar{u}, \\ \bar{u} = e^{\text{ad } \beta'} \bar{\bar{u}} \end{array} \right) \end{array} \quad (17)$$

Indeed, by the definition of RC_u^α we have that $\beta' = \beta$ modulo the relation $u = e^{\text{ad } \alpha} \bar{u}$. So in the bottom space, $u = e^{\text{ad } \alpha} \bar{u} = e^{\text{ad } \alpha} e^{\text{ad } \beta'} \bar{\bar{u}} = e^{\text{ad } \alpha} e^{\text{ad } \beta} \bar{\bar{u}} = e^{\text{bch}(\text{ad } \alpha, \text{ad } \beta)} \bar{\bar{u}} = e^{\text{ad } \text{bch}(\alpha, \beta)} \bar{\bar{u}}$. Hence if we concentrate on the three corners of (17), we see the diagram on the right, whose top row is both $RC_u^\alpha \parallel RC_u^{\beta'}$ and the definition of $RC_u^{\text{bch}(\alpha, \beta)}$. \square

$$\begin{array}{ccc} FL(u) & \text{-----} & FL(u) \\ & \searrow & \swarrow^{u \rightarrow \bar{u}} \\ & & FL(u, \bar{u}) / (u = e^{\text{ad } \text{bch}(\alpha, \beta)} \bar{u}) \end{array}$$

It remains to construct $\zeta_0: \mathcal{K}_0^{rhh} \rightarrow M_0$ by proclaiming its values on the generators:

$$\zeta_0(t\epsilon_u) := (), \quad \zeta_0(h\epsilon_x) := (x \rightarrow 0), \quad \text{and} \quad \zeta_0(\rho_{ux}^\pm) := (x \rightarrow \pm u).$$

Proposition 4.5. ζ_0 is well defined; namely, the values above satisfy the relations in Definition 3.5.

Proof. We only verify the Conjugation Relation (8), as all other relations are easy. On the left we have

$$\begin{aligned} \rho_{ux}^+ \rho_{vy}^+ \rho_{wz}^+ &\xrightarrow{\zeta_0} (x \rightarrow u, y \rightarrow v, z \rightarrow w) \xrightarrow{tm_v^{vw}} (x \rightarrow u, y \rightarrow v, z \rightarrow v) \\ &\xrightarrow{hm_x^{xy}} (x \rightarrow \text{bch}(u, v), z \rightarrow v) \xrightarrow{tha^{uz}} (x \rightarrow \text{bch}(e^{\text{ad } v}(u), v), z \rightarrow v), \end{aligned}$$

while on the right it is

$$\rho_{vx}^+ \rho_{wz}^+ \rho_{uy}^+ \xrightarrow{\zeta_0} (x \rightarrow v, y \rightarrow u, z \rightarrow w) \xrightarrow{tm_v^{vw} \parallel hm_x^{xy}} (x \rightarrow \text{bch}(v, u), z \rightarrow v),$$

and the equality follows because $\text{bch}(e^{\text{ad } v}(u), v) = \log(e^v e^u e^{-v} \cdot e^v) = \text{bch}(v, u)$. \square

As we shall see in Section 7, ζ_0 is related to the tree part of the Kontsevitch integral. Thus by finite-type folklore [BN1, HM], when evaluated on string links (i.e., pure tangles) ζ_0 should be equivalent to the collection of all Milnor μ invariants [Mi]. No proof of this fact will be provided here.

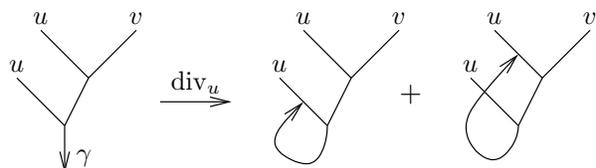
5. THE WHEEL-VALUED SPICE AND THE INVARIANT ζ

This is perhaps the most important section of this paper. In it we construct the wheels part of the full trees-and-wheels MMA M and the full tree-and-wheels invariant $\zeta: \mathcal{K}^{rbh} \rightarrow M$.

5.1. Cyclic words, div_u , and J_u . The target MMA, M , of the extended invariant ζ is an extension of M_0 by “wheels”, or equally well, by “cyclic words”, and the main difference between M and M_0 is the addition of a wheel-valued “spice” term $J_u(\lambda_x)$ to the meta-action tha^{ux} . We first need the “infinitesimal version” div_u of J_u .

Recall that if T is a set (normally, of tail labels), we denote by $FL(T)$ the graded completion of the free Lie algebra on the generators in T . Similarly we denote by $FA(T)$ the the graded completion of the free associative algebra on the generators in T , and by $CW(T)$ the graded completion of the vector space of cyclic words on T , namely, $CW(T) := FA(T)/\{uw = wu: u \in T, w \in FA(T)\}$. Note that the last is a vector space quotient — we mod out by the vector-space span of $\{uw = wu\}$, and not by the ideal generated by that set. Hence CW is not an algebra and not “commutative”; merely, the words in it are invariant under cyclic permutations of their letters. We often call the elements of CW “wheels”. Denote by tr the projection $\text{tr}: FA \rightarrow CW$ and by ι the standard inclusion $\iota: FL(T) \rightarrow FA(T)$ (ι is defined to be the identity on letters in T , and is then extended to the rest of FL using $\iota([\lambda_1, \lambda_2]) := \iota(\lambda_1)\iota(\lambda_2) - \iota(\lambda_2)\iota(\lambda_1)$). Note that operations defined by “letter substitutions” make sense on FA and on CW . In particular, the operation RC_u^γ of Section 4.2 makes sense on FA and on CW .

The inclusion ι can be extended from “trees” (elements of FL) to “wheels of trees” (elements of $CW(FL)$). Given a letter $u \in T$ and an element $\gamma \in FL(T)$, we let $\text{div}_u \gamma$ be the sum of all ways of gluing the root of γ to near any one



of the u -labelled leaves of γ ; each such gluing is a wheel of trees, and hence can be interpreted as an element of $CW(T)$. An example is on the right, and a formula-level definition follows: we first define $\sigma_u: FL(T) \rightarrow FA(T)$ by setting $\sigma_u(v) := \delta_{uv}$ for letters $v \in T$ and then setting $\sigma_u([\lambda_1, \lambda_2]) := \iota(\lambda_1)\sigma_u(\lambda_2) - \iota(\lambda_2)\sigma_u(\lambda_1)$, and then we set $\text{div}_u(\gamma) := \text{tr}(u\sigma_u(\gamma))$. An alternative definition of a similar functional div is in [AT, Proposition 3.20], and some further discussion is in [BND2, Section 3.2].

Now given $u \in T$ and $\gamma \in FL(T)$ define

$$J_u(\gamma) := \int_0^1 ds \text{div}_u(\gamma // RC_u^{s\gamma}) // C_u^{-s\gamma}. \quad (18)$$

Note that at degree d , the integrand in the above formula is a degree d element of $CW(T)$ with coefficients that are polynomials of degree at most $d - 1$ in s . Hence the above formula

is entirely algebraic. The following (difficult!) proposition contains all that we will need to know about J_u .

Proposition 5.1. *If $\alpha, \beta, \gamma \in FL$ then the following three equations hold:*

$$J_u(\text{bch}(\alpha, \beta)) = J_u(\alpha) + J_u(\beta \parallel RC_u^\alpha) \parallel C_u^{-\alpha}, \quad (19)$$

$$J_u(\alpha) - J_u(\alpha \parallel RC_v^\beta) \parallel C_v^{-\beta} = J_v(\beta) - J_v(\beta \parallel RC_u^\alpha) \parallel C_u^{-\alpha} \quad (20)$$

$$J_w(\gamma \parallel tm_w^{uv}) = (J_u(\gamma) + J_v(\gamma \parallel RC_u^\gamma) \parallel C_u^{-\gamma}) \parallel tm_w^{uv} \quad (21)$$

We postpone the proof of this proposition to Section 10.4.

Remark 5.2. J_u can be characterized as the unique functional $J_u: FL(T) \rightarrow CW(T)$ which satisfies Equation (19) as well as the conditions $J_u(0) = 0$ and

$$\left. \frac{d}{d\epsilon} J_u(\epsilon\gamma) \right|_{\epsilon=0} = \text{div}_u(\gamma), \quad (22)$$

which in themselves are easy consequences of the definition of J_u , Equation (18). Indeed, taking $\alpha = s\gamma$ and $\beta = \epsilon\gamma$ in Equation (19), where s and ϵ are scalars, we find that

$$J_u((s + \epsilon)\gamma) = J_u(s\gamma) + J_u(\epsilon\gamma \parallel RC_u^{s\gamma}) \parallel C_u^{-s\gamma}.$$

Differentiating the above equation with respect to ϵ at $\epsilon = 0$ and using Equation (22), we find that

$$\frac{d}{ds} J_u(s\gamma) = \text{div}_u(\gamma \parallel RC_u^{s\gamma}) \parallel C_u^{-s\gamma},$$

and integrating from 0 to 1 we get Equation (18).

Finally for this section, one may easily verify that the degree 1 piece of CW is preserved by the actions of C_u^γ and RC_u^γ , and hence it is possible to reduce modulo degree 1. Namely, set $CW^r(T) := CW(T)/\text{deg } 1 = CW^{>1}(T)$, and all operations remain well defined and satisfy the same identities.

5.2. The MMA M . Let M be the collection $\{M(T; H)\}$, where

$$M(T; H) := FL(T)^H \times CW^r(T) = M_0(T; H) \times CW^r(T)$$

(I really mean \times , not \otimes). The collection M has MMA operations as follows:

- $t\sigma_v^u$, $t\eta^u$, and tm_w^{uv} are defined by the same formulae as in Section 4.2. Note that these formulae make sense on CW and on CW^r just as they do on FL .
- $h\sigma_y^x$, $h\eta^x$, and hm_z^{xy} are extended to act as the identity on the $CW^r(T)$ factor of $M(T; H)$.
- If $\mu_i = (\lambda_i; \omega_i) \in M(T_i; H_i)$ for $i = 1, 2$ (and, of course, $T_1 \cap T_2 = \emptyset = H_1 \cap H_2$), set

$$\mu_1 * \mu_2 := (\lambda_1 * \lambda_2; \iota_1(\omega_1) + \iota_2(\omega_2)),$$

where ι_i are the obvious inclusions $\iota_i: CW^r(T_i) \rightarrow CW^r(T_1 \cup T_2)$.

- The only truly new definition is that of tha^{ux} :

$$(\lambda; \omega) \parallel tha^{ux} := (\lambda; \omega + J_u(\lambda_x)) \parallel RC_u^{\lambda_x}.$$

Thus the “new” tha^{ux} is just the “old” tha^{ux} , with an added term of $J_u(\lambda_x)$.

- $t\epsilon_u := ((); 0)$ and $h\epsilon_x := ((x \rightarrow 0); 0)$.

Theorem 5.3. *M, with the operations defined above, is a meta-monoid-action (MMA). Furthermore, if $\zeta: \mathcal{K}_0^{rbh} \rightarrow M$ is defined on the generators in the same way as ζ_0 , except extended by 0 to the CW^r factor, $\zeta(\rho_{ux}^\pm) := ((x \rightarrow \pm u); 0)$, then it is well-defined; namely, the values above satisfy the relations in Definition 3.5.*

Proof. Given Theorem 4.4 and Proposition 4.5, the only non-obvious checks remaining are the “wheel parts” of the main equations defining and MMA, Equations (2)–(6), the Conjugation Relation (8), and the FI relation (9). As the only interesting wheels-creation occurs with the operation *tha*, (2) and (3) are easy. As easily $J_u(v) = 0$ if $u \neq v$, no wheels are created by the *tha* action within the proof of Proposition 4.5, so that proof still holds. We are left with (4)–(6) and (8)–(9).

Let us start with the wheels part of Equation (4). If $\mu = ((x \rightarrow \alpha, y \rightarrow \beta, \dots); \omega) \in M$, then

$$\mu \parallel tha^{ux} = ((x \rightarrow \alpha \parallel RC_u^\alpha, y \rightarrow \beta \parallel RC_u^\alpha, \dots); (\omega + J_u(\alpha)) \parallel RC_u^\alpha)$$

and hence the wheels-only part of $\mu \parallel tha^{ux} \parallel tha^{vy}$ is

$$\begin{aligned} \omega \parallel RC_u^\alpha \parallel RC_v^{\beta \parallel RC_u^\alpha} + J_u(\alpha) \parallel RC_u^\alpha \parallel RC_v^{\beta \parallel RC_u^\alpha} + J_v(\beta \parallel RC_u^\alpha) \parallel RC_v^{\beta \parallel RC_u^\alpha} \\ = [\omega + J_u(\alpha) + J_v(\beta \parallel RC_u^\alpha) \parallel C_u^{-\alpha}] \parallel RC_u^\alpha \parallel RC_v^{\beta \parallel RC_u^\alpha}. \end{aligned}$$

In a similar manner, the wheels-only part of $\mu \parallel tha^{vy} \parallel tha^{ux}$ is

$$[\omega + J_v(\beta) + J_u(\alpha \parallel RC_v^\beta) \parallel C_v^{-\beta}] \parallel RC_v^\beta \parallel RC_u^{\alpha \parallel RC_v^\beta}.$$

Using Equation (13) the operators outside the square brackets in the above two formulae are the same, and so we only need to verify that

$$\omega + J_u(\alpha) + J_v(\beta \parallel RC_u^\alpha) \parallel C_u^{-\alpha} = \omega + J_v(\beta) + J_u(\alpha \parallel RC_v^\beta) \parallel C_v^{-\beta}.$$

But this is Equation (20). In a similar manner, the wheels parts of Equations (5) and (6) reduce to Equations (21) and (19), respectively. One may also verify that no wheels appear within Equation (8), and that wheels appear in Equation (9) only in degree 1, which is eliminated in CW^r . \square

Thus we have a tree-and-wheel valued invariant ζ defined on \mathcal{K}_0^{rbh} , and thus $\delta \parallel \zeta$ is a tree-and-wheel valued invariant of tangles and w-tangles.

As we shall see in Section 7, the wheels part ω of ζ is related to the wheels part of the Kontsevitch integral. Thus by finite-type folklore (e.g. [Kr]), the Abelianization of ω (obtained by declaring all the letters in $CW(T)$ to be commuting) should be closely related to the multi-variable Alexander polynomial. More on that in Section 9. I don’t know what the bigger (non-commutative) part of ω measures.

6. SOME COMPUTATIONAL EXAMPLES

Part of the reason I am happy about the invariant ζ is that it is relatively easily computable. Cyclic words are easy to implement, and using the Lyndon basis (e.g. [Re, Chapter 5]), free Lie algebras are easy too. Hence I include here a demo-run of a rough implementation, written in *Mathematica*. The full source files are available at [Web/].

6.1. **The Program.** First we load the package `FreeLie.m`, which contains a collection of programs to manipulate series in completed free Lie algebras and series of cyclic words. We tell `FreeLie.m` to show series by default only up to degree 3, and that if two (infinite) series are compared, they are to be compared by default only up to degree 5:

```

<< FreeLie.m
$SeriesShowDegree = 3; $SeriesCompareDegree = 5;
    
```

Merely as a test of `FreeLie.m`, we tell it to set `t1` to be `bch(u, v)`. The computer’s response is to print that series to degree 3:

```

t1 = BCH[⟨u⟩, ⟨v⟩]
    
```

```

LS [u + v, u v / 2, 1/12 u u v + 1/12 u v v]
    
```

Note that by default Lie series are printed in “top bracket form”, which means that brackets are printed above their arguments, rather than around them. Hence $\overline{u \overline{uv}}$ means $[u, [u, v]]$. This practise is especially advantageous when it is used on highly-nested expressions, when it becomes difficult for the eye to match left brackets with the their corresponding right brackets.

Note also that that `FreeLie.m` utilizes *lazy evaluation*, meaning that when a Lie series (or a series of cyclic words) is defined, its definition is stored but no computations take place until it is printed or until its value (at a certain degree) is explicitly requested. Hence `t1` is a reference to the entire Lie series `bch(u, v)`, and not merely to the degrees 1–3 parts of that series, which are printed above. Hence when we request the value of `t1` to degree 6, the computer complies:

```

t1@{6}
    
```

```

LS [u + v, u v / 2, 1/12 u u v + 1/12 u v v, 1/24 u u v v, -1/720 u u u u v +
1/180 u u u v v + 1/180 u u v v v + 1/120 u v u v v + 1/360 u u v u v - 1/720 u v v v v,
-1/1440 u u u u v v + 1/360 u u u v v v + 1/240 u u v u v v + 1/720 u u u v u v - 1/1440 u u v v v v]
    
```

(It is surprisingly easy to compute `bch` to a high degree and some amazing patterns emerge. See [\[Web/mo\]](#) and [\[Web/bch\]](#).)

The package `FreeLie.m` know about various free Lie algebra operations, but not about our specific circumstances. Hence we have to make some further definitions. The first few are set-theoretic in nature. We define the “domain” of a function stored as a list of *key*→*value* pairs to be the set of “first elements” of these pairs; meaning, the set of keys. We define

what it means to remove a key (and its corresponding value), and likewise for a list of keys. We define what it means for two functions to be equal (their domains must be equal, and for every key $\#$, we are to have $\# // f_1 = \# // f_2$). We also define how to apply a Lie morphism mor to a function (apply it to each value), and how to compare (λ, ω) pairs (in $FL(T)^H \times CW^r(T)$):

```

(oo) Domain[f_List] := First /@ f;
(heart) f \ key_ := DeleteCases[f, key -> _];
(heart) f \ keys_List := Fold[#1 \ #2 &, f, keys];
(heart) f1_List == f2_List := Domain[f1] === Domain[f2] && (And @@ (
  ((# /. f1) == (# /. f2)) & /@ Domain[f1]));
LieMorphism[mor_] [f_List] := MapAt[LieMorphism[mor], f, {All, 2}];
M[\lambda_1, \omega_1] == M[\lambda_2, \omega_2] := (\lambda1 == \lambda2) && (\omega1 == \omega2);

```

Next we enter some free-Lie definitions that are not a part of `FreeLie.m`. Namely we define $RC_{u, \bar{u}}^\gamma(s)$ to be the result of “stable application” of the morphism $u \rightarrow e^{\text{ad}(\gamma)}(\bar{u})$ to s (namely, apply the morphism repeatedly until things stop changing; at any fixed degree this happens after a finite number of iterations). We define RC_u^γ to be $RC_{u, \bar{u}}^\gamma // (\bar{u} \rightarrow u)$. Finally, we define J as in Equation (18):

```

(oo) RC_u[\gamma_LieSeries, ub_] [s_] := StableApply[LieMorphism[<u> -> Ad[\gamma][<ub>]], s];
(heart) RC_u[\gamma_LieSeries] [s_] := s // RC_u[\gamma, <u>] // LieMorphism[<u> -> <u>];
(heart) J_u[\gamma_] :=
  Module[{s}, \int_0^1 (\gamma // RC_u[s \gamma] // div_u // LieMorphism[u -> Ad[-s \gamma][u]]) ds];

```

Mostly to introduce our notation for cyclic words, let us compute $J_v(\text{bch}(u, v))$ to degree 4. Note that when a series of wheels is printed out here, its degree 1 piece is greyed out to honour the fact that it “does not count” within ζ :

```

(oo) J_v[t1] @ {4}
(heart)

```

```

(heart) CWS[\nabla, uv, \frac{uuv}{2} - \frac{uvv}{2}, \frac{uuuv}{6} + \frac{3uuvv}{4} - \frac{3uvuv}{2} + \frac{uvvv}{6}]

```

Next is a series of definitions that implement the definitions of $*$, tm , hm , and tha following Sections 4.2 and 5.2:

```

(◯◯) M /: M[λ1_, ω1_] * M[λ2_, ω2_] := M[λ1 ∪ λ2, ω1 + ω2];
(♥) tm[u_, v_, w_][λ_List] := λ // LieMorphism[⟨u⟩ → ⟨w⟩, ⟨v⟩ → ⟨w⟩];
tm[u_, v_, w_][M[λ_, ω_]] := LieMorphism[⟨u⟩ → ⟨w⟩, ⟨v⟩ → ⟨w⟩] /@ M[λ, ω];
hm[x_, y_, z_][λ_List] := Union[λ \ {x, y}, {z → BCH[x /. λ, y /. λ]}];
hm[x_, y_, z_][M[λ_, ω_]] := M[λ // hm[x, y, z], ω];
tha[u_, x_][λ_List] := MapAt[RC_u[x /. λ], λ, {All, 2}];
tha[u_, x_][M[λ_, ω_]] :=
  M[λ // tha[u, x], (ω + J_u[x /. λ]) // RC_u[x /. λ]];

```

Next we set the values of $\zeta(t\epsilon_x)$ and $\zeta(\rho_{ux}^\pm)$, which we simply denote $t\epsilon_x$ and ρ_{ux}^\pm :

```

(◯◯) he[x_] := M[{x → MakeLieSeries[0]}, MakeCWSeries[0]]
(♥) ρ+[u_, x_] := M[{x → MakeLieSeries[⟨u⟩]}, MakeCWSeries[0]];
ρ-[u_, x_] := M[{x → MakeLieSeries[-⟨u⟩]}, MakeCWSeries[0]];

```

The final bit of definitions have to do with 3-dimensional tangles. We set R^+ to be the value of $\zeta(\delta(\))$ as in the proof of Theorem 3.4, likewise for R^- , and we define dm following Equation (7):

```

(◯◯) R+[a_, b_] := ρ+[a, b] * he[a]; R-[a_, b_] := ρ-[a, b] * he[a];
(♥) dm[a_, b_, c_][μ_] := μ // tha[⟨a⟩, b] // tm[⟨a⟩, ⟨b⟩, ⟨c⟩] // hm[a, b, c];

```

6.2. Testing Properties and Relations. It is always good to test both the program and the math by verifying that the operations we have implemented satisfy the relations predicted by the mathematics. As a first example, we verify the meta-associativity of tm . Hence in line 1 below we set $\mathbf{t1}$ to be the element $t_1 = ((x \rightarrow u + v + w, y \rightarrow [u, v] + [v, w]); uvw)$ of $M(u, v, w; x, y)$. In line 2 we compute $t_1 // tm_u^{uv}$, in line 3 we compute $t_2 := t_1 // tm_u^{uv} // tm_u^{uw}$ and store its value in $\mathbf{t2}$, in line 4 we compute $t_1 // tm_v^{vw}$, in line 5 we compute $t_3 := t_1 // tm_v^{vw} // tm_u^{uv}$ and store its value in $\mathbf{t3}$, and then in line 6 we test if t_2 is equal to t_3 . The computer thinks the answer is “True”, at least to the degree tested:

```

(◯◯) Print /@ {{u = ⟨"u"⟩, v = ⟨"v"⟩, w = ⟨"w"⟩}};
(♥) 1 → (t1 = M[{
  x → MakeLieSeries[u + v + w], y → MakeLieSeries[b[u, v] + b[v, w]]
}, MakeCWSeries[CW["uvw"]]]),
2 → (t1 // tm[u, v, u]),
3 → (t2 = t1 // tm[u, v, u] // tm[u, w, u]),
4 → (t1 // tm[v, w, v]),
5 → (t3 = t1 // tm[v, w, v] // tm[u, v, u]),
6 → (t2 ≡ t3)};

```

```

 1 → M[{x → LS[ $\overline{u} + \overline{v} + \overline{w}$ , 0, 0], y → LS[0,  $\overline{u\overline{v}} + \overline{v\overline{w}}$ , 0]}, CWS[0, 0,  $\overline{u\overline{v\overline{w}}}$ ]]
2 → M[{x → LS[2  $\overline{u} + \overline{w}$ , 0, 0], y → LS[0,  $\overline{u\overline{w}}$ , 0]}, CWS[0, 0,  $\overline{u\overline{u\overline{w}}}$ ]]
3 → M[{x → LS[3  $\overline{u}$ , 0, 0], y → LS[0, 0, 0]}, CWS[0, 0,  $\overline{u\overline{u\overline{u}}}$ ]]
4 → M[{x → LS[ $\overline{u} + 2 \overline{v}$ , 0, 0], y → LS[0,  $\overline{u\overline{v}}$ , 0]}, CWS[0, 0,  $\overline{u\overline{v\overline{v}}}$ ]]
5 → M[{x → LS[3  $\overline{u}$ , 0, 0], y → LS[0, 0, 0]}, CWS[0, 0,  $\overline{u\overline{u\overline{u}}}$ ]]
6 → True

```

The corresponding test for the meta-associativity of hm is a bit harder, yet produces the same result. Note that we have declared `SeriesCompareDegree` to be higher than `SeriesShowDegree`, so the “True” output below means a bit more than the visual comparison of lines 3 and 5:

```

 Print /@ {
 1 → (t1 =  $\rho^+[u, x] \rho^+[v, y] \rho^+[w, z]$ ),
2 → (t1 // hm[x, y, x]),
3 → (t2 = t1 // hm[x, y, x] // hm[x, z, x]),
4 → (t1 // hm[y, z, y]),
5 → (t3 = t1 // hm[y, z, y] // hm[x, y, x]),
6 → (t2 ≡ t3);

```

```

 1 → M[{x → LS[ $\overline{u}$ , 0, 0], y → LS[ $\overline{v}$ , 0, 0], z → LS[ $\overline{w}$ , 0, 0]}, CWS[0, 0, 0]]
2 → M[{x → LS[ $\overline{u} + \overline{v}$ ,  $\frac{\overline{u\overline{v}}}{2}$ ,  $\frac{1}{12} \overline{u\overline{u\overline{v}}} + \frac{1}{12} \overline{u\overline{v\overline{v}}}$ ], z → LS[ $\overline{w}$ , 0, 0]}, CWS[0, 0, 0]]
3 →
M[{x → LS[ $\overline{u} + \overline{v} + \overline{w}$ ,  $\frac{\overline{u\overline{v}}}{2} + \frac{\overline{u\overline{w}}}{2} + \frac{\overline{v\overline{w}}}{2}$ ,  $\frac{1}{12} \overline{u\overline{u\overline{v}}} + \frac{1}{12} \overline{u\overline{u\overline{w}}} + \frac{1}{3} \overline{u\overline{v\overline{w}}} + \frac{1}{12} \overline{v\overline{v\overline{w}}} + \frac{1}{12} \overline{u\overline{v\overline{v}}} + \frac{1}{6} \overline{u\overline{w\overline{v}}} + \frac{1}{12} \overline{u\overline{w\overline{w}}} + \frac{1}{12} \overline{v\overline{w\overline{w}}}$ ], CWS[0, 0, 0]]
4 → M[{x → LS[ $\overline{u}$ , 0, 0], y → LS[ $\overline{v} + \overline{w}$ ,  $\frac{\overline{v\overline{w}}}{2}$ ,  $\frac{1}{12} \overline{v\overline{v\overline{w}}} + \frac{1}{12} \overline{v\overline{w\overline{w}}}$ ], CWS[0, 0, 0]]
5 →
M[{x → LS[ $\overline{u} + \overline{v} + \overline{w}$ ,  $\frac{\overline{u\overline{v}}}{2} + \frac{\overline{u\overline{w}}}{2} + \frac{\overline{v\overline{w}}}{2}$ ,  $\frac{1}{12} \overline{u\overline{u\overline{v}}} + \frac{1}{12} \overline{u\overline{u\overline{w}}} + \frac{1}{3} \overline{u\overline{v\overline{w}}} + \frac{1}{12} \overline{v\overline{v\overline{w}}} + \frac{1}{12} \overline{u\overline{v\overline{v}}} + \frac{1}{6} \overline{u\overline{w\overline{v}}} + \frac{1}{12} \overline{u\overline{w\overline{w}}} + \frac{1}{12} \overline{v\overline{w\overline{w}}}$ ], CWS[0, 0, 0]]
6 → True

```

We next test the meta-action axiom t on $((x \rightarrow u + [u, t], y \rightarrow u + [u, t]); uu + tuv)$ and the meta-action axiom h on $((x \rightarrow u + [u, v], y \rightarrow v + [u, v]); uu + uvv)$:

```

 Print /@ {u = <"u">, v = <"v">, w = <"w">, t = <"t">};
 1 → (t1 = M[{
x → MakeLieSeries[u + b[u, t]], y → MakeLieSeries[u + b[u, t]]
}, MakeCWSeries[CW["uu"] + CW["tuv"]]]),
2 → (t2 = t1 // tm[u, v, w] // tha[w, x]),
3 → (t3 = t1 // tha[u, x] // tha[v, x] // tm[u, v, w]),
4 → (t2 ≡ t3);

```

```

 1 → M[{x → LS[ $\overline{u}$ ,  $-\overline{t}u$ , 0], y → LS[ $\overline{u}$ ,  $-\overline{t}u$ , 0]}, CWS[0,  $\overline{uu}$ ,  $\overline{tuv}$ ]]
2 → M[{x → LS[ $\overline{w}$ ,  $-\overline{t}w$ ,  $-\overline{t}ww$ ], y → LS[ $\overline{w}$ ,  $-\overline{t}w$ ,  $-\overline{t}ww$ ]}, CWS[ $\overline{w}$ ,  $-\overline{t}w + \overline{ww}$ ,  $\frac{3}{2}\overline{tw}$ ]]
3 → M[{x → LS[ $\overline{w}$ ,  $-\overline{t}w$ ,  $-\overline{t}ww$ ], y → LS[ $\overline{w}$ ,  $-\overline{t}w$ ,  $-\overline{t}ww$ ]}, CWS[ $\overline{w}$ ,  $-\overline{t}w + \overline{ww}$ ,  $\frac{3}{2}\overline{tw}$ ]]
4 → True

```

```

 Print /@ {{u = <"u">, v = <"v">}};
 1 → (t1 = M[{
    x → MakeLieSeries[u + b[u, v]], y → MakeLieSeries[v + b[u, v]]
  }, MakeCWSeries[CW["uu"] + CW["uvv"]]]),
2 → (t2 = t1 // hm[x, y, z] // tha[u, z]),
3 → (t3 = t1 // tha[u, x] // tha[u, y] // hm[x, y, z]),
4 → (t2 ≡ t3);

```

```

 1 → M[{x → LS[ $\overline{u}$ ,  $\overline{u}v$ , 0], y → LS[ $\overline{v}$ ,  $\overline{u}v$ , 0]}, CWS[0,  $\overline{uu}$ ,  $\overline{uvv}$ ]]
2 → M[{z → LS[ $\overline{u} + \overline{v}$ ,  $\frac{3}{2}\overline{uv}$ ,  $-\frac{17}{12}\overline{u}u\overline{v} - \frac{17}{12}\overline{u}v\overline{v}$ ]}, CWS[ $\overline{u}$ ,  $\overline{uu} - 2\overline{uv}$ ,  $\frac{\overline{uv}}{2} + \frac{\overline{uvv}}{2}$ ]]
3 → M[{z → LS[ $\overline{u} + \overline{v}$ ,  $\frac{3}{2}\overline{uv}$ ,  $-\frac{17}{12}\overline{u}u\overline{v} - \frac{17}{12}\overline{u}v\overline{v}$ ]}, CWS[ $\overline{u}$ ,  $\overline{uu} - 2\overline{uv}$ ,  $\frac{\overline{uv}}{2} + \frac{\overline{uvv}}{2}$ ]]
4 → True

```

And finally for this testing section, we test the Conjugation Relation of Equation (8):

```

 Print /@ {
 1 → (t1 =  $\rho^+[u, x] \rho^+[v, y] \rho^+[w, z]$ ),
2 → (t2 = t1 // tm[v, w, v] // hm[x, y, x] // tha[u, z]),
3 → (t3 =  $\rho^+[v, x] \rho^+[w, z] \rho^+[u, y]$ ),
4 → (t4 = t3 // tm[v, w, v] // hm[x, y, x]),
5 → (t2 ≡ t4);

```

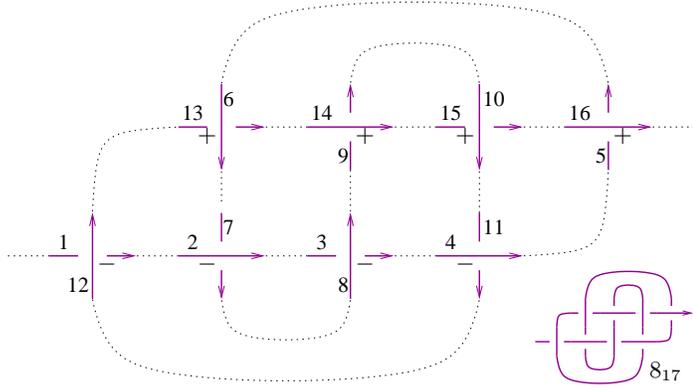
```

 1 → M[{x → LS[ $\overline{u}$ , 0, 0], y → LS[ $\overline{v}$ , 0, 0], z → LS[ $\overline{w}$ , 0, 0]}, CWS[0, 0, 0]]
2 → M[{x → LS[ $\overline{u} + \overline{v}$ ,  $-\frac{\overline{uv}}{2}$ ,  $\frac{1}{12}\overline{u}u\overline{v} + \frac{1}{12}\overline{u}v\overline{v}$ ], z → LS[ $\overline{v}$ , 0, 0]}, CWS[0, 0, 0]]
3 → M[{x → LS[ $\overline{v}$ , 0, 0], y → LS[ $\overline{u}$ , 0, 0], z → LS[ $\overline{w}$ , 0, 0]}, CWS[0, 0, 0]]
4 → M[{x → LS[ $\overline{u} + \overline{v}$ ,  $-\frac{\overline{uv}}{2}$ ,  $\frac{1}{12}\overline{u}u\overline{v} + \frac{1}{12}\overline{u}v\overline{v}$ ], z → LS[ $\overline{v}$ , 0, 0]}, CWS[0, 0, 0]]
5 → True

```

6.3. Demo Run 1 — the Knot 8_{17} . We are ready for a more substantial computation — the invariant of the knot 8_{17} . We draw 8_{17} in the plane, with all but the neighbourhoods of the crossings dashed-out. We thus get a tangle T_1 which is the disjoint union of 8 individual crossings (4 positive and 4 negative). We number the 16 strands that appear in these 8 crossings in the order of their eventual appearance within 8_{17} , as seen below.

The 8-crossing tangle T_1 we just got has a rather boring ζ invariant, a disjoint merge of 8 ρ^\pm 's. We store it in $\mu 1$. Note that we used numerals as labels, and hence in the expression below top-bracketed numerals should be interpreted as symbols and not as integers. Note also that the program automatically converts two-digit numerical labels into alphabetical symbols, when these appear within Lie elements.



Hence in the output below, “a” is “10”, “c” is “12”, “e” is “14”, and “g” is “16”:

```

☹️  $\mu 1 = R^- [12, 1] R^- [2, 7] R^- [8, 3] R^- [4, 11] R^+ [16, 5] R^+ [6, 13] R^+ [14, 9] R^+ [10, 15]$ 
☹️

```

```

🖥️ M[ { 1 → LS[-c̄, 0, 0], 2 → LS[0, 0, 0],
        3 → LS[-8̄, 0, 0], 4 → LS[0, 0, 0], 5 → LS[ā, 0, 0], 6 → LS[0, 0, 0],
        7 → LS[-2̄, 0, 0], 8 → LS[0, 0, 0], 9 → LS[ē, 0, 0], 10 → LS[0, 0, 0],
        11 → LS[-4̄, 0, 0], 12 → LS[0, 0, 0], 13 → LS[b̄, 0, 0],
        14 → LS[0, 0, 0], 15 → LS[ā, 0, 0], 16 → LS[0, 0, 0] }, CWS[0, 0, 0] ]

```

Next is the key part of the computation. We “sew” together the strands of T_1 in order by first sewing 1 and 2 and naming the result 1, then sewing 1 and 3 and naming the result 1 once more, and on until everything is sewn together to a single strand named 1. This is done by applying dm_1^k repeatedly to $\mu 1$, for $k = 2, \dots, 16$, each time storing the result back again in $\mu 1$. Finally, we only wish to print the wheels part of the output, and this we do to degree 6:

```

☹️ Do[ $\mu 1 = \mu 1 // dm[1, k, 1], \{k, 2, 16\}$ ];
☹️ Last[ $\mu 1$ ]@{6}

```

```

🖥️ CWS[0, -11̄, 0, - $\frac{31 \overline{1111}}{12}$ , 0, - $\frac{1351 \overline{111111}}{360}$ ]

```

Let $A(X)$ be the Alexander polynomial of 8_{17} . Namely, $A(X) = -X^{-3} + 4X^{-2} - 8X^{-1} + 11 - 8X + 4X^2 - X^3$. For comparison with the above computation, we print the series expansion of $\log A(e^x)$, also to degree 6:

```

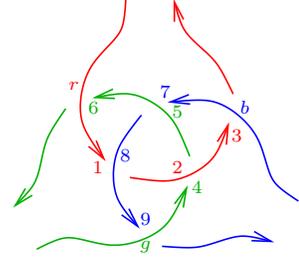
☹️ Series[Log[- $\frac{1}{x^3} + \frac{4}{x^2} - \frac{8}{x} + 11 - 8x + 4x^2 - x^3$  /. x → ex], {x, 0, 6}]
☹️

```

$$\text{computer} -x^2 - \frac{31 x^4}{12} - \frac{1351 x^6}{360} + o[x]^7$$

6.4. Demo Run 2 — the Borromean Tangle. In a similar manner we compute the invariant of the *rgb*-coloured Borromean tangle, shown below.

We label the edges near the crossings as shown, using the labels $\{r, 1, 2, 3\}$ for the *r* component, $\{g, 4, 5, 6\}$ for the *g* component, and $\{b, 7, 8, 9\}$ for the *b* component. We let $\mu 2$ store the invariant of the disjoint union of 6 independent crossings labelled as in the Borromean tangle, we concatenate the numerically-labelled strands into their corresponding letter-labelled strands, and we then print $\mu 2$, which now contains the invariant we seek:



```

(oo) μ2 = R-[r, 6] R+[2, 4] R-[g, 9] R+[5, 7] R-[b, 3] R+[8, 1];
(heart) (Do[μ2 = μ2 // dm[r, k, r], {k, 1, 3}]; Do[μ2 = μ2 // dm[g, k, g], {k, 4, 6}];
Do[μ2 = μ2 // dm[b, k, b], {k, 7, 9}]; μ2)

```

```

(oo) M[ { b → LS[0, gr̄, 1/2 gr̄gr̄ + br̄rḡ + 1/2 gr̄rr̄],
g → LS[0, -br̄, 1/2 bbr̄ - bgr̄ - br̄rḡ + 1/2 brr̄],
r → LS[0, bḡ, 1/2 bbḡ + bgr̄ + 1/2 bgḡ] }, CWS[0, 0, 2 bgr̄] ]

```

We then print the *r*-head part of the tree part of the invariant to degree 5 (the *g*-head and *b*-head parts can be computed in a similar way, or deduced from the cyclic symmetry of *r*, *g*, and *b*), and the wheels part to the same degree:

```

(oo) (r /. First[μ2]) @ {5}
(heart)

```

```

(oo) LS[0, bḡ, 1/2 bbḡ + bgr̄ + 1/2 bgḡ,
1/6 b bbḡ + 1/2 b bgr̄ + 1/2 b ggr̄ + 1/4 b bbḡ + 1/2 b gr̄r̄ + 1/6 b ggḡ,
1/24 b b bbḡ + 1/6 b b bgr̄ + 1/4 b b ggr̄ + 1/12 b b bbḡ + 1/4 b b gr̄r̄ +
1/6 b g ggr̄ + 1/4 b g gr̄r̄ - b bgr̄ḡ + 1/12 b bgḡḡ - 2 b br̄gḡ + 1/6 b gr̄r̄r̄ +
1/2 b g bgr̄ - b g br̄ḡ - 1/12 b bḡbḡ - 1/2 b gr̄gr̄ + 1/24 b ggḡḡ]

```

```

(oo) Last[μ2] @ {5}
(heart)

```

$$\begin{aligned}
 \text{CWS} & \left[0, 0, 2 \overline{\text{bgr}}, \overline{\text{bbgr}} - \overline{\text{bgbr}} + \overline{\text{bggr}} - \overline{\text{bgrg}} + \overline{\text{bgrr}} - \overline{\text{brgr}}, \right. \\
 & \frac{\overline{\text{bbbgr}}}{3} - \frac{\overline{\text{bbgbr}}}{2} + \frac{\overline{\text{bbggr}}}{2} + \frac{\overline{\text{bbgrg}}}{2} + \frac{\overline{\text{bbgrr}}}{2} + \frac{\overline{\text{bbrbg}}}{2} - \frac{3 \overline{\text{bbrgr}}}{2} + \frac{\overline{\text{bgbrr}}}{2} - \frac{3 \overline{\text{bggbr}}}{2} + \\
 & \left. \frac{\overline{\text{bgggr}}}{3} - \frac{\overline{\text{bggrg}}}{2} + \frac{\overline{\text{bggrr}}}{2} + \frac{\overline{\text{bgrgg}}}{2} - \frac{3 \overline{\text{bgrrg}}}{2} + \frac{\overline{\text{bgrrr}}}{3} + \frac{\overline{\text{brggr}}}{2} - \frac{\overline{\text{brgrr}}}{2} + \frac{\overline{\text{brrgr}}}{2} \right]
 \end{aligned}$$

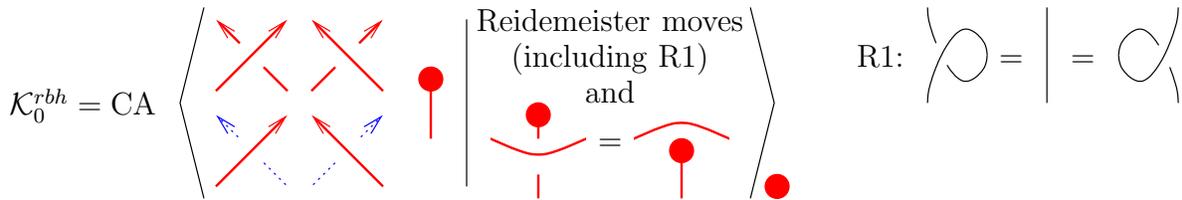
A more graphically-pleasing presentation of the same values, with the degree raised to 6, appears in Figure 6.

7. SKETCH OF THE RELATION WITH FINITE TYPE INVARIANTS

One way to view the invariant ζ of Section 5 is as a mysterious extension of the reasonably natural invariant ζ_0 of Section 4. Another is as a solution to a universal problem — as we shall see in this section, ζ is a universal finite type invariant of objects in \mathcal{K}_0^{rbh} . Given that \mathcal{K}_0^{rbh} is closely related to $w\mathcal{T}$ (w-tangles), and given that much was already said on finite type invariants of w-tangles in [BND2], this section will be merely a sketch, difficult to understand without reading much of [BND1] and sections 1–3 of [BND2], as well as the parts of section 4 that concern with caps.

Over all, defining ζ using the language of Sections 4 and 5 is about as difficult as using finite type invariants. Yet computing it using the language of Sections 4 and 5 is much easier while proving invariance is significantly harder.

7.1. A circuit algebra description of \mathcal{K}_0^{rbh} . A w-tangle represents a collection of ribbon-knotted tubes in \mathbb{R}^4 . It follows from Theorem 2.9 that every rKBH can be obtained from a w-tangle by capping some of its tubes and “puncturing” the rest, where “puncturing” a tube means “replacing it with its spine, a strand that runs along it”. Using thick red lines to denote tubes, red bullets to denote caps, and dotted blue lines to denote punctured tubes, we find that



Note that punctured tubes (meaning strands or “hoops”) can only go under capped tubes (balloons), and that while it is allowed to slide tubes “over” caps, it is not allowed to slide them “under” caps. Further explanations and the meaning of “CA” are in [BND2]. The “red bullet” subscript on the right hand side indicates that we restrict our attention to the subspace in which all red strands are eventually capped. We leave it to the reader to interpret the operations hm , tha , and tm in this language (tm is non-obvious!).

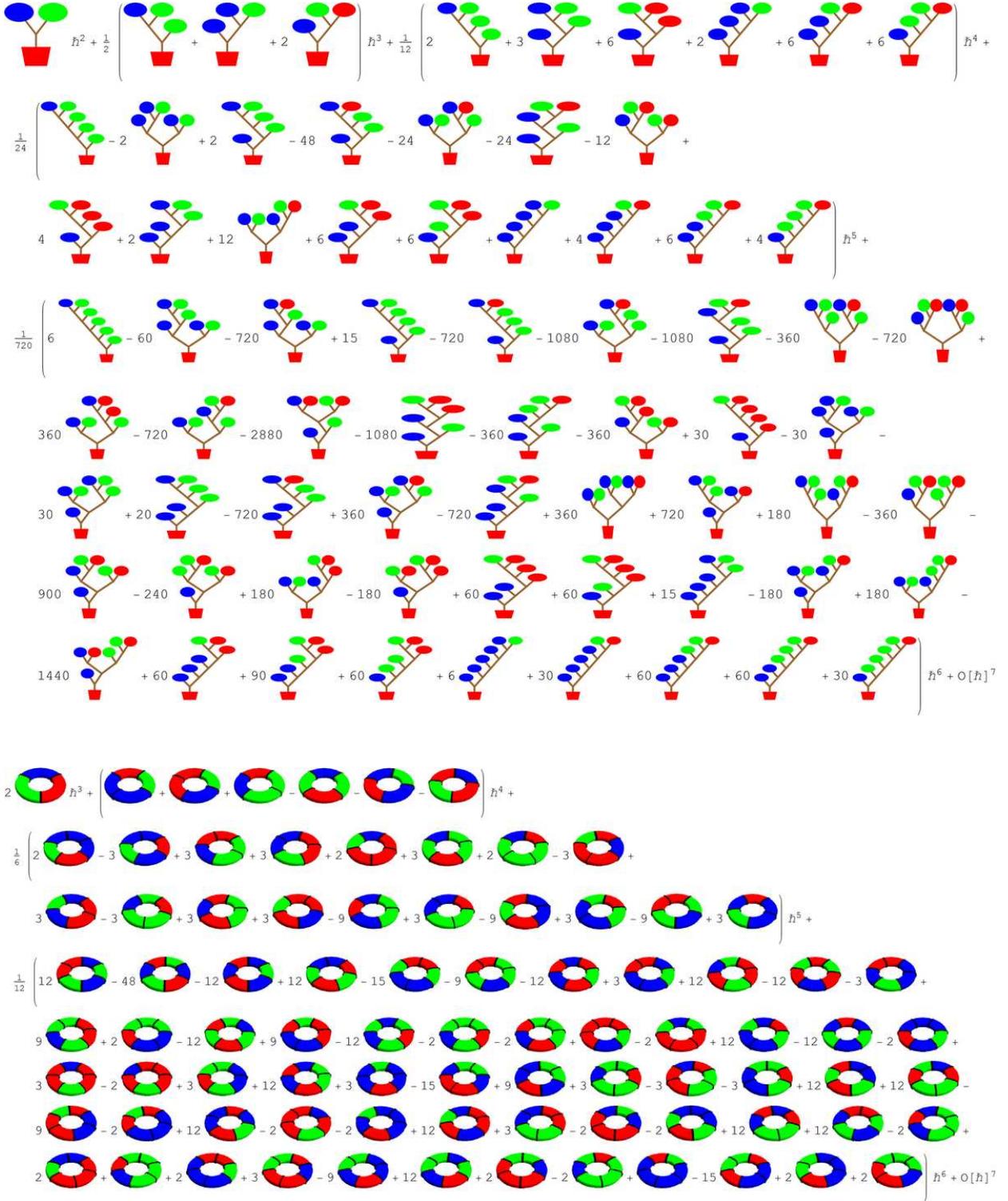
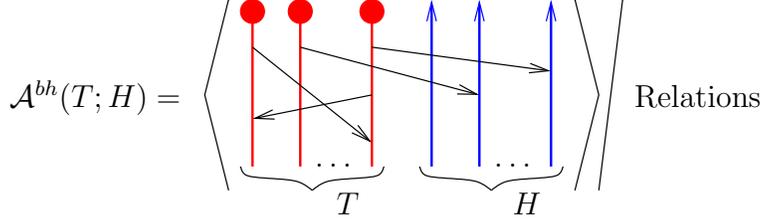


Figure 6. The redhead part of the tree part, and the wheels part, of the invariant of the Borromean tangle, to degree 6.

7.2. **Arrow diagrams for \mathcal{K}_0^{rhh} .** As in [BND1, BND2], one we finite type invariants of elements on \mathcal{K}_0^{rhh} bi considering iterated differences of crossings and non-crossings (“virtual crossings”), and then again as in [BND1, BND2], we find that the arrow-diagram space $\mathcal{A}^{bh}(T; H)$ corresponding to these invariants may be described schematically as follows:



In the above, arrow tails may land only on the red “tail” strands, but arrow heads may land on either kind of strand. The “Relations” are the TC and $\overrightarrow{4T}$ relations of [BND1, Section 2.3], the CP relation of [BND2, Section 4.2], and the relation $D_L = D_R = 0$, which corresponds to the R1 relation (D_L and D_R are defined in [BND1, Section 3]).

The operation hm acts on \mathcal{A}^{bh} by concatenating two head stands. The operation tha acts by duplicating a head strand (with the usual summation over all possible ways of reconnecting arrow-heads as in [BND1, Section 2.5.1.6]), changing the colour of one of the duplicates to red, and then concatenating it to the beginning of some tail strand.

We note that modulo the relations, one may eliminate all arrow-heads from all tail strands. For diagrams in which there are no arrow-heads on tail strands, the operation tm is defined by merging together two tail strands. The TC relation implies that arrow-tails on the resulting tail-strand can be order in any desired way.

As in [BND1, Section 3.5], \mathcal{A}^{bh} has an alternative model in which internal “2-in 1-out” trivalent vertices are allowed, and in which we also impose the \overrightarrow{AS} , \overrightarrow{STU} and $\overrightarrow{IH\bar{X}}$ relations (ibid.).

7.3. **The algebra structure on \mathcal{A}^{bh} and its primitives.** For any fixed finite sets T and H , the space $\mathcal{A}^{bh}(T; H)$ is a co-commutative bi-algebra. Its product defined using the disjoint union followed by the tm operation on all tail strands and the hm operation on all head strands, and its co-product is the “sum of all splittings” as in [BND1, Section 3.2]. Thus by Milnor-Moore, $\mathcal{A}^{bh}(T; H)$ is the universal enveloping algebra of its set of primitives \mathcal{P}^{bh} . The latter is the set of connected diagrams in \mathcal{A}^{bh} (modulo relations), and those, as in [BND2, Section 3.2], are the trees and the degree > 1 wheels. (Though note that even if $T = H = \{1, \dots, n\}$, the algebra structure on $\mathcal{A}^{bh}(T; H)$ is different from the algebra structure on the space $\mathcal{A}^w(\uparrow_n)$ of ibid.). Identifying trees with $FL(T)$ and wheels with $CW^r(T)$, we find that

$$\mathcal{P}^{bh}(T; H) \cong FL(T)^H \times CW^r(T) = M(T; H).$$

Theorem 7.1. *By taking logarithms (using formal power series and the algebra structure of \mathcal{A}^{bh}), $\mathcal{P}^{bh}(T; H)$ inherits the structure of an MMA from the group-like elements of \mathcal{A}^{bh} . Furthermore, $\mathcal{P}^{bh}(T; H)$ and $M(T; H)$ are isomorphic as MMAs.*

Sketch of the proof. Once it is established that $\mathcal{P}^{bh}(T; H)$ is an MMA, that tm and hm act in the same way as on M and that the tree part of the action of tha is given using the RC operation, it follows that the wheels part of the action of tha is given by some functional J' which necessarily satisfies Equation (19). But according to Remark 5.2, Equation (19) and a few auxiliary conditions determine J uniquely. These conditions are easily verified for J' , and hence $J' = J$. This concludes the proof.

Note that the above theorem and the fact that $\mathcal{P}^{bh}(T; H)$ is an MMA provide an alternative proof of Proposition 5.1 which bypasses the hard computations of Section 10.4. In fact, personally I first knew that J exists and satisfies Proposition 5.1 using the reasoning of this section, and only then I observed using the reasoning of Remark 5.2 that J must be given by the formula in Equation (18).

7.4. The homomorphic expansion Z^{bh} . As in [BND1, Section 3.4] and [BND2, Section 3.1], there is a homomorphic expansion (a universal finite type invariant with good composition properties) $Z^{bh}: \mathcal{K}_0^{rbh} \rightarrow \mathcal{A}^{bh}$ defined by mapping crossings to exponentials of arrows. It is easily verified that Z^{bh} is a morphism of MMAs, and therefore it is determined by its values on the generators ρ^\pm of \mathcal{K}_0^{rbh} , which are single crossings in the language of Section 7.1. Taking logarithms we find that $\log Z^{bh} = \zeta$ on the generators and hence always, and hence ζ is the logarithm of a universal finite type invariant of elements of \mathcal{K}_0^{rbh} .

8. THE RELATION WITH THE BF TOPOLOGICAL QUANTUM FIELD THEORY

8.1. Tensorial Interpretation. Given a Lie algebra \mathfrak{g} , any element of $FL(T)$ can be interpreted as a function taking $|T|$ inputs in \mathfrak{g} and producing a single output in \mathfrak{g} . Hence, putting aside issues of completion and convergence, there is a map $\tau_1: FL(T) \rightarrow \text{Fun}(\mathfrak{g}^T \rightarrow \mathfrak{g})$, where in general, $\text{Fun}(X \rightarrow Y)$ denotes the space of functions from X to Y . To deal with completions more precisely, we pick a formal parameter \hbar , multiply the degree k part of τ_1 by \hbar^k , and get a perfectly good $\tau = \tau_{\mathfrak{g}}: FL(T) \rightarrow \text{Fun}(\mathfrak{g}^T \rightarrow \mathfrak{g}[[\hbar]])$, where in general, $V[[\hbar]] := \mathbb{Q}[[\hbar]] \otimes V$ for any vector space V . The map τ obviously extends to $\tau: FL(T)^H \rightarrow \text{Fun}(\mathfrak{g}^T \rightarrow \mathfrak{g}^H[[\hbar]])$.

Similarly, if also \mathfrak{g} is finite dimensional, then by taking traces in the adjoint representation we get a map $\tau = \tau_{\mathfrak{g}}: CW(T) \rightarrow \text{Fun}(\mathfrak{g}^T \rightarrow \mathbb{Q}[[\hbar]])$. Multiplying this τ with the τ from the previous paragraph we get $\tau = \tau_{\mathfrak{g}}: M(T; H) \rightarrow \text{Fun}(\mathfrak{g}^T \rightarrow \mathfrak{g}^H[[\hbar]])$. Exponentiating, we get

$$e^\tau: M(T; H) \rightarrow \text{Fun}(\mathfrak{g}^T \rightarrow \mathcal{U}(\mathfrak{g})^{\otimes H}[[\hbar]]).$$

8.2. ζ and BF Theory. Fix a finite dimensional Lie algebra \mathfrak{g} . In [CR] (see especially section 4), Cattaneo and Rossi discuss the BF quantum field theory with fields $A \in \Omega^1(\mathbb{R}^4, \mathfrak{g})$ and $B \in \Omega^2(\mathbb{R}^4, \mathfrak{g}^*)$ and construct an observable “ $U(A, B, \Xi)$ ” for each “long” \mathbb{R}^2 in \mathbb{R}^4 ; meaning, for each 2-sphere in S^4 with a prescribed behaviour at ∞ . We interpret these as observables defined on our “balloons”. The Cattaneo-Rossi observables are functions of a variable $\Xi \in \mathfrak{g}$, and they can be interpreted as power series in a formal parameter \hbar . Further, given the connection-field A , one may always consider its formal holonomy along a closed path (a “hoop”) and interpret it as an element in $\mathcal{U}(\mathfrak{g})[[\hbar]]$. Multiplying these hoop observables and also the Cattaneo-Rossi balloon observables, we get an observable \mathcal{O}_γ for any KBH γ , taking values in $\text{Fun}(\mathfrak{g}^T \rightarrow \mathcal{U}(\mathfrak{g})^{\otimes H}[[\hbar]])$.

Conjecture 8.1. *If γ is an rKBH, then $\langle \mathcal{O}_\gamma \rangle_{BF} = e^\tau(\zeta(\gamma))$.*

Of course, some interpretation work is required before Conjecture 8.1 even becomes a well-posed mathematical statement.

We note that the Cattaneo-Rossi observable does not depend on the ribbon property of the KBH γ . I hesitate to speculate whether this is an indication that the work presented in this paper can be extended to non-ribbon knots or an indication that somewhere within the rigorous mathematical analysis of BF theory an obstruction will arise that will force one to restrict to ribbon knots (yet I speculate that one of these possibilities holds true).

Most likely the work of Watanabe [Wa2] is a proof of Conjecture 8.1 for the case of a single balloon and no hoops, and very likely it contains all key ideas necessary for a complete proof of Conjecture 8.1.

9. THE SIMPLEST NON-COMMUTATIVE REDUCTION AND AN ULTIMATE ALEXANDER INVARIANT

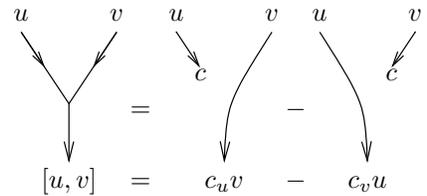
9.1. **Informal.** Let us start with some informal words. All the fundamental operations within the definition of M , namely $[\cdot, \cdot]$, C_u^γ , RC_u^γ and div_u , act by modifying trees and wheels near their extremities — their “tails” and their “heads” (for wheels, all extremities are “tails”). Thus all operations will remain well-defined and will continue to satisfy the MMA properties if we extend or reduce trees and wheels by objects or relations that are confined to their “inner” parts.

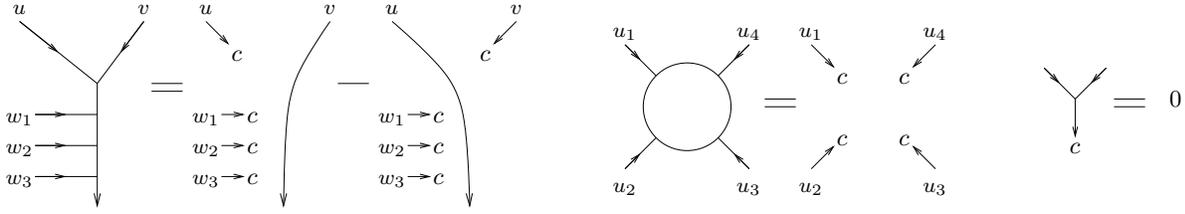
In this section we discuss the “ β -quotient of M ”, an extension/reduction of M as discussed above, which is even better-computable than M . As we have seen in Section 6, objects in M , and in particular the invariant ζ , are machine-computable. Yet the dimensions of FL and of CW grow exponentially in the degree, and so does the complexity of computations in M . Objects in the β -quotient are described in terms of commutative power series, their dimensions grow polynomially in the degree, and computations in the β -quotient are polynomial time. In fact, the power series appearing with the β -quotient can be “summed”, and *non-perturbative* formulae can be given to everything in sight.

Yet ζ^β , meaning ζ reduced to the β -quotient, remains strong enough to contain the (multi-variable) Alexander polynomial. I argue that in fact, the formulae obtained for the Alexander polynomial within this β -calculus are “better” than many standard formulae for the Alexander polynomial.

More on the relationship between the β -calculus and the Alexander polynomial (though nothing about its relationship with M and ζ), is in [BNS].

Still on the informal level, the β -quotient arises by allowing a new type of a “sink” vertex c and imposing the β -relation, shown on the right, on both trees and wheels. One easily sees that under this relation, trees can be shaved to single arcs union “ c -stubs”, wheels become unions of c -stubs, and c -stubs “commute with everything”:





Hence c -stubs can be taken as generators for a commutative power series ring R (with one generator c_u for each possible tail label u), $CW(T)$ becomes a copy of the ring R , elements of $FL(T)$ becomes column vectors whose entries are in R and whose entries correspond to the tail label in the remaining arc of a shaved tree, and elements of $FL(T)^H$ can be regarded as $T \times H$ matrices with entries in R . Hence in the β -quotient the MMA M reduces to an MMA $\{\beta_0(T; H)\}$ whose elements are $T \times H$ matrices of power series, with yet an additional power series to encode the wheels part. We will introduce β_0 more formally below, and then note that it can be simplified even further (with no further loss of information) to an MMA β whose entries and operations involve rational functions, rather than power series.

Remark 9.1. The β -relation arose from studying the (unique non-commutative) 2-dimensional Lie algebra $\mathfrak{g}_2 := FL(\xi_1, \xi_2)/([\xi_1, \xi_2] = \xi_2)$, as in Section 8.1. Loosely, within \mathfrak{g}_2 the β -relation is a “polynomial identity” in a sense similar to the “polynomial identities” of the theory of PI-rings [Ro]. For a more direct relationship between this Lie algebra and the Alexander polynomial, see [Web/chic1].

9.2. Less informal. For a finite set T let $R = R(T) := \mathbb{Q}[\{c_u\}_{u \in T}]$ denote the ring of power series with commuting generators c_u corresponding to the elements u of T , and let $L = L(T) := R \otimes \mathbb{Q}T$ be the the free R -module with generators T . Turn L into a Lie algebra over R by declaring that $[u, v] = c_u v - c_v u$ for any $u, v \in T$. Let $c: L \rightarrow R$ be the R -linear extension of $u \mapsto c_u$; namely,

$$\gamma = \sum_u \gamma_u u \in L \mapsto c_\gamma := \sum_u \gamma_u c_u \in R, \tag{23}$$

where the γ_u 's are coefficients in R . Note that with this definition we have $[\alpha, \beta] = c_\alpha \beta - c_\beta \alpha$ for any $\alpha, \beta \in L$. There are obvious surjections $\pi: FL \rightarrow L$ and $\pi: CW \rightarrow R$ (strictly speaking, the first of those maps has a small cokernel yet becomes a surjection once the ground ring of its domain space is extended to R).

The following Lemma-Definition may appear scary, yet its proof is nothing more than high school level algebra, and the messy formulae within it mostly get renormalized away by the end of this section. Hang on!

Lemma-Definition 9.2. The operations C_u , RC_u , bch , div_u , and J_u descend from FL/CW to L/R , and, for $\alpha, \beta, \gamma \in L$ (with $\gamma = \sum_v \gamma_v v$) they are given by

$$v \parallel C_u^{-\gamma} = v \parallel RC_u^\gamma = v \quad \text{for } u \neq v \in T, \quad (24)$$

$$\rho \parallel C_u^{-\gamma} = \rho \parallel RC_u^\gamma = \rho \quad \text{for } \rho \in R, \quad (25)$$

$$u \parallel C_u^{-\gamma} = e^{-c_\gamma} \left(u + c_u \frac{e^{c_\gamma} - 1}{c_\gamma} \gamma \right) \quad (26)$$

$$= e^{-c_\gamma} \left(\left(1 + c_u \gamma_u \frac{e^{c_\gamma} - 1}{c_\gamma} \right) u + c_u \frac{e^{c_\gamma} - 1}{c_\gamma} \sum_{v \neq u} \gamma_v v \right), \quad (27)$$

$$u \parallel RC_u^\gamma = \left(1 + c_u \gamma_u \frac{e^{c_\gamma} - 1}{c_\gamma} \right)^{-1} \left(e^{c_\gamma} u - c_u \frac{e^{c_\gamma} - 1}{c_\gamma} \sum_{v \neq u} \gamma_v v \right), \quad (28)$$

$$\text{bch}(\alpha, \beta) = \frac{c_\alpha + c_\beta}{e^{c_\alpha + c_\beta} - 1} \left(\frac{e^{c_\alpha} - 1}{c_\alpha} \alpha + e^{c_\alpha} \frac{e^{c_\beta} - 1}{c_\beta} \beta \right), \quad (29)$$

$$\text{div}_u \gamma = c_u \gamma_u, \quad (30)$$

$$J_u(\gamma) = \log \left(1 + \frac{e^{c_\gamma} - 1}{c_\gamma} c_u \gamma_u \right). \quad (31)$$

Proof. (Sketch) Equation (24) is obvious — C_u or RC_u conjugate or repeatedly conjugate u , but not v . Equation (25) is the statement that C_u and RC_u are R -linear, namely that they act on scalars as the identity. Informally this is the fact that 1-wheels commute with everything, and formally it follows from the fact that $\pi: FL \rightarrow L$ is a well defined morphism of Lie algebras.

To prove Equation (26), we need to compute $e^{-\text{ad} \gamma}(u)$, and it is enough to carry this computation out within the 2-dimensional subspace of L spanned by u and by γ . Hence the computation is an exercise in diagonalization — one needs to diagonalize the 2×2 matrix $\text{ad}(-\gamma)$ in order to exponentiate it. Here are some details: set $\delta = [-\gamma, u] = c_u \gamma - c_\gamma u$. Then clearly $\text{ad}(-\gamma)(\delta) = -c_\gamma \delta$, and hence $e^{-\text{ad} \gamma}(\delta) = e^{-c_\gamma} \delta$. Also note that $\text{ad}(-\gamma)(\gamma) = 0$, and hence $e^{-\text{ad} \gamma}(\gamma) = \gamma$. Thus

$$u \parallel C_u^{-\gamma} = e^{-\text{ad} \gamma}(u) = e^{-\text{ad} \gamma} \left(-\frac{\delta}{c_\gamma} + \frac{c_u \gamma}{c_\gamma} \right) = -\frac{e^{-c_\gamma} \delta}{c_\gamma} + \frac{c_u \gamma}{c_\gamma} = e^{-c_\gamma} \left(u + c_u \frac{e^{c_\gamma} - 1}{c_\gamma} \gamma \right).$$

Equation (27) is simply (26) rewritten using $\gamma = \sum_v \gamma_v v$. To prove Equation (28), take its right hand side and use Equations (27) and (24) to get u back again, and hence our formula for RC_u^γ indeed inverts the formula already established for $C_u^{-\gamma}$.

Equation (29) amounts to writing the group law of a 2-dimensional Lie group in terms of its 2-dimensional Lie algebra, $L_0 := \text{span}(\alpha, \beta)$, and this is again an exercise in 2×2 matrix algebra, though a slightly harder one. We work in the adjoint representation of L_0 and aim to compare the exponential of the left hand side of Equation (29) with the exponential of its right hand side. If a and b are scalars, let $e(a, b)$ be the matrix representing $e^{\text{ad}(a\alpha + b\beta)}$ on L_0 relative

to the basis (α, β) . Then using $[\alpha, \beta] = c_\alpha\beta - c_\beta\alpha$ we find that $e(a, b) = \exp \begin{pmatrix} bc_\beta & -ac_\beta \\ -bc_\alpha & ac_\alpha \end{pmatrix}$, and we need to show that $e(1, 0) \cdot e(0, 1) = e \left(\frac{c_\alpha + c_\beta}{e^{c_\alpha + c_\beta} - 1} \frac{e^{c_\alpha} - 1}{c_\alpha}, \frac{c_\alpha + c_\beta}{e^{c_\alpha + c_\beta} - 1} e^{c_\alpha} \frac{e^{c_\beta} - 1}{c_\beta} \right)$. Lazy bums do it as follows:

```

 e[a_, b_] := MatrixExp[{{b c_β, -a c_β},
 e[1, 0] . e[0, 1] == e[ $\frac{c_\alpha + c_\beta}{e^{c_\alpha + c_\beta} - 1} \frac{e^{c_\alpha} - 1}{c_\alpha}$ ,  $\frac{c_\alpha + c_\beta}{e^{c_\alpha + c_\beta} - 1} e^{c_\alpha} \frac{e^{c_\beta} - 1}{c_\beta}$ ] // Simplify

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True

Equation (30) is the fact that $\text{div}_u u = c_u$, along with the R -linearity of div_u .

For Equation (31), note that using Equation (28), the coefficient of u in $\gamma \parallel RC_u^{s\gamma}$ is $\gamma_u e^{sc_\gamma} \left(1 + c_u \gamma_u \frac{e^{sc_\gamma} - 1}{c_\gamma}\right)^{-1}$. Thus using Equation (30) and the fact that C_u acts trivially on R ,

$$\begin{aligned} J_u(\gamma) &= \int_0^1 ds \text{div}_u(\gamma \parallel RC_u^{s\gamma}) \parallel C_u^{-s\gamma} = \int_0^1 ds \left(1 + c_u \gamma_u \frac{e^{sc_\gamma} - 1}{c_\gamma}\right)^{-1} c_u \gamma_u e^{sc_\gamma} \\ &= \log \left(1 + \frac{e^{sc_\gamma} - 1}{c_\gamma} c_u \gamma_u\right) \Big|_0^1 = \log \left(1 + \frac{e^{c_\gamma} - 1}{c_\gamma} c_u \gamma_u\right). \quad \square \end{aligned}$$

9.3. The reduced invariant ζ^{β_0} . We now let $\beta_0(T; H)$ be the β -reduced version of $M(T; H)$. Namely, in parallel with Section 5.2 we define

$$\beta_0(T; H) := L(T)^H \times R^r(T) = R(T)^{T \times H} \times R^r(T).$$

In other words, elements of $\beta_0(T; H)$ are $T \times H$ matrices $A = (A_{ux})$ of power series in the variables $\{c_u\}_{u \in T}$, along with a single additional power series $\omega \in R^r$ (R^r is R modded out by its degree 1 piece) corresponding to the last factor above, which we write at the top left of A :

$$\beta_0(u, v, \dots; x, y, \dots) = \left\{ \left(\begin{array}{c|ccc} \omega & x & y & \cdots \\ u & A_{ux} & A_{uy} & \cdot \\ v & A_{vx} & A_{vy} & \cdot \\ \vdots & \cdot & \cdot & \ddots \end{array} \right) : \omega \in R^r(T), A. \in R(T) \right\}$$

Continuing in parallel with Section 5.2 and using the formulae from Lemma-Definition 9.2, we turn $\{\beta_0(T; H)\}$ into an MMA with operations defined as follows (on a typical element of β_0 , which is a decorated matrix (A, ω) as above):

- $t\sigma_v^u$ acts by renaming row u to v and sending the variable c_u to c_v everywhere. $t\eta^u$ acts by removing row u and sending c_u to 0. tm_w^{uv} acts by adding row u to row v calling the result row w , and by sending c_u and c_v to c_w everywhere.

- $h\sigma_y^x$ and $h\eta^x$ are clear. To define hm_z^{xy} , let $\alpha = (A_{ux})_{u \in T}$ and $\beta = (A_{uy})_{u \in T}$ denote the columns of x and y in A , let $c_\alpha := \sum_{u \in T} A_{ux} c_u$ and $c_\beta := \sum_{u \in T} A_{uy} c_u$ in parallel with Equation (23), and let hm_z^{xy} act by removing the x - and y -columns α and β and introducing a new column, labelled z , and containing $\frac{c_\alpha + c_\beta}{e^{c_\alpha + c_\beta} - 1} \left(\frac{e^{c_\alpha} - 1}{c_\alpha} \alpha + e^{c_\alpha} \frac{e^{c_\beta} - 1}{c_\beta} \beta \right)$, as in Equation (29).
- We now describe the action of tha^{ux} on an input (A, ω) as depicted on the right. Let $\gamma = \begin{pmatrix} \gamma_u \\ \gamma_{\text{rest}} \end{pmatrix}$ be the column of x , split into the “row u ” part γ_u and the rest, γ_{rest} . Let c_γ be $\sum_{v \in T} \gamma_v c_v$ as in Equation (23). Then tha^{ux} acts as follows:
 - As dictated by Equation (31), ω is replaced by $\omega + \log \left(1 + \frac{e^{c_\gamma} - 1}{c_\gamma} c_u \gamma_u \right)$.
 - As dictated by Equations (24) and (28), every column $\alpha = \begin{pmatrix} \alpha_u \\ \alpha_{\text{rest}} \end{pmatrix}$ in A (including the column γ itself) is replaced by

$$\left(1 + c_u \gamma_u \frac{e^{c_\gamma} - 1}{c_\gamma} \right)^{-1} \begin{pmatrix} e^{c_\gamma} \alpha_u \\ \alpha_{\text{rest}} - c_u \frac{e^{c_\gamma} - 1}{c_\gamma} (c_\gamma)_{\text{rest}} \end{pmatrix},$$

where $(c_\gamma)_{\text{rest}}$ is the column whose row v entry is $c_v \gamma_v$, for any $v \neq u$.

- The “merge” operation $*$ is $\frac{\omega_1}{T_1} \left| \begin{array}{c} H_1 \\ A_1 \end{array} \right. * \frac{\omega_2}{T_2} \left| \begin{array}{c} H_2 \\ A_2 \end{array} \right. := \frac{\omega_1 + \omega_2}{T_1} \left| \begin{array}{cc} H_1 & H_2 \\ A_1 & 0 \\ 0 & A_2 \end{array} \right.$
- $t\epsilon_u = \frac{0}{u} \left| \begin{array}{c} \emptyset \\ \emptyset \end{array} \right.$ and $h\epsilon_x = \frac{0}{\emptyset} \left| \begin{array}{c} x \\ \emptyset \end{array} \right.$ (these values correspond to a matrix with an empty set of columns and a matrix with an empty set of rows, respectively).

We have concocted the definition of the MMA β_0 so that the projection $\pi: M \rightarrow \beta_0$ would be a morphism of MMAs. Hence to completely compute $\zeta^{\beta_0} := \pi \circ \zeta$ on any rKBH (to all orders!), it is enough to note its values on the generators. These are determined by the values in Theorem 5.3: $\zeta^{\beta_0}(\rho_{ux}^\pm) = \frac{0}{u} \left| \begin{array}{c} x \\ \pm 1 \end{array} \right.$

9.4. The ultimate Alexander invariant ζ^β . Some repackaging is in order. Noting the ubiquity of factors of the form $\frac{e^c - 1}{c}$ in the previous section, it makes sense to multiply any column α of the matrix A by $\frac{e^{c_\alpha} - 1}{c_\alpha}$. Noting that row- u entries (things like γ_u) often appear multiplied by c_u , we multiply every row by its corresponding variable c_u . Doing this and rewriting the formulae of the previous section in the new variables, we find that the variables c_u only appear within exponentials of the form e^{c_u} . So we set $t_u := e^{c_u}$ and rewrite everything in terms of the t_u 's. Finally, the only formula that touches ω is additive and has a log term. So we replace ω with e^ω . The result is “ β -calculus”, which was described in detail in [BNS]. A summary version follows. In these formulae, α , β , γ , and δ denote entries, rows, columns, or submatrices as appropriate, and whenever α is a column, $\langle \alpha \rangle$ is the sum of its entries:

$$\beta(T; H) = \left\{ \begin{array}{c|ccc} \omega & x & y & \cdots \\ \hline u & \alpha_{ux} & \alpha_{uy} & \cdot \\ v & \alpha_{vx} & \alpha_{vy} & \cdot \\ \vdots & \cdot & \cdot & \cdot \end{array} \middle| \begin{array}{l} \omega \text{ and the } \alpha_{ux} \text{'s are rational func-} \\ \text{tions in variables } t_u, \text{ one for each} \\ u \in T. \text{ When all } t_u \text{'s are set to 1,} \\ \omega \text{ is 1 and every } \alpha_{ux} \text{ is 0.} \end{array} \right\},$$

$$tm_w^{uv} : \begin{array}{c|c} \omega & H \\ \hline u & \alpha \\ v & \beta \\ T & \gamma \end{array} \mapsto \left(\begin{array}{c|c} \omega & H \\ \hline w & \alpha + \beta \\ T & \gamma \end{array} \right) // (t_u, t_v \rightarrow t_w),$$

$$hm_z^{xy} : \begin{array}{c|ccc} \omega & x & y & H \\ \hline T & \alpha & \beta & \gamma \end{array} \mapsto \begin{array}{c|c} \omega & z \\ \hline T & \alpha + \beta + \langle \alpha \rangle \beta \end{array} \begin{array}{c|c} H \\ \hline \gamma \end{array},$$

$$tha^{ux} : \begin{array}{c|ccc} \omega & x & H \\ \hline u & \alpha & \beta \\ T & \gamma & \delta \end{array} \mapsto \begin{array}{c|ccc} \omega(1 + \alpha) & x & H \\ \hline u & \alpha(1 + \langle \gamma \rangle / (1 + \alpha)) & \beta(1 + \langle \gamma \rangle / (1 + \alpha)) \\ T & \gamma / (1 + \alpha) & \delta - \gamma \beta / (1 + \alpha) \end{array},$$

$$\frac{\omega_1}{T_1} \begin{array}{c|c} H_1 \\ \hline A_1 \end{array} * \frac{\omega_2}{T_2} \begin{array}{c|c} H_2 \\ \hline A_2 \end{array} := \frac{\omega_1 \omega_2}{T_1 T_2} \begin{array}{c|cc} H_1 & H_2 \\ \hline A_1 & 0 \\ T_2 & 0 & A_2 \end{array},$$

$$\zeta^\beta(t\epsilon_u) = \frac{1}{u} \begin{array}{c|c} \emptyset \\ \hline \emptyset \end{array}, \quad \zeta^\beta(h\epsilon_x) = \frac{1}{\emptyset} \begin{array}{c|c} x \\ \hline \emptyset \end{array}, \quad \text{and} \quad \zeta^\beta(\rho_{ux}^\pm) = \frac{1}{u} \begin{array}{c|c} x \\ \hline t_u^{\pm 1} - 1 \end{array}.$$

Theorem 9.3. *If K is a u -knot regarded as a 1-component pure tangle by cutting it open, then the ω part of $\zeta^\beta(\delta(K))$ is the Alexander polynomial of K .*

I know of three winding paths that constitute a proof of the above theorem:

- Use the results of Section 7 here, of [BND1, Section 3.7], and of [Lee].
- Use the results of Section 7 here, of [BND1, Section 3.9], and the known relation of the Alexander polynomial with the wheels part of the Kontsevich integral (e.g. [Kr]).
- Use the results of [KLW], where formulae very similar to ours appear.

Yet to me, the strongest evidence that Theorem 9.3 is true is that it was verified explicitly on very many knots — see the single example in Section 6.3 here and many more in [BNS].

In several senses, ζ^β is an “ultimate” Alexander invariant:

- The formulae in this section may appear complicated, yet note that if an rKBH consists of about n balloons and hoops, its invariant is described in terms of only $O(n^2)$ polynomials and each of the operations tm , hm and tha involves only $O(n^2)$ operations on polynomials.
- It is defined for tangles and has a prescribed behaviour under tangle compositions (in fact, it is defined in terms of that prescribed behaviour). This means that when ζ^β is computed on some large knot with (say) n crossings, the computation can be broken up into n steps of complexity $O(n^2)$ at the end of each the quantity computed is the

invariant of some topological object (a tangle), or even into $3n$ steps at the end of each the quantity computed is the invariant of some rKBH¹⁰.

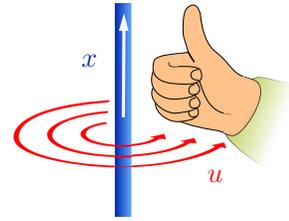
- ζ^β contains also the multivariable Alexander polynomial and the Burau representation (overwhelmingly verified by experiment, not written-up yet).
- ζ^β has an easily prescribed behaviour under hoop- and balloon-doubling, and $\zeta^\beta \circ \delta$ has an easily prescribed behaviour under strand-doubling (not shown here).

10. ODDS AND ENDS

10.1. Linking Numbers and Signs. If x is an oriented S^1 and u is an oriented S^2 in an oriented S^4 (or \mathbb{R}^4) and the two are disjoint, their linking number l_{ux} is defined as follows. Pick a ball B whose oriented boundary is u (using the “outward pointing normal” convention for orienting boundaries), and which intersects x in finitely many transversal intersection points p_i . At any of these intersection points p_i , the concatenation of the orientation of B at p_i (thought of a basis to the tangent space of B at p_i) with the tangent to x at p_i is a basis of the tangent space of S^4 at p_i , and as such it may either be positively oriented or negatively oriented. Define $\sigma(p_i) = +1$ in the former case and $\sigma(p_i) = -1$ in the latter case. Finally, let $l_{ux} := \sum_i \sigma(p_i)$. It is a standard fact that l_{ux} is an isotopy invariant of (u, x) .

Exercise 10.1. Verify that $l_{ux}(\rho_{ux}^\pm) = \pm 1$, where ρ_{ux}^+ and ρ_{ux}^- are the positive and negative Hopf links as in Example 2.2. For the purpose of this exercise the plane in which Figure 1 is drawn is oriented counterclockwise, the 3D space it represents has its third coordinate oriented “up” from the plane of the paper, and \mathbb{R}_{txyz}^4 is oriented so that the t coordinate is “first”.

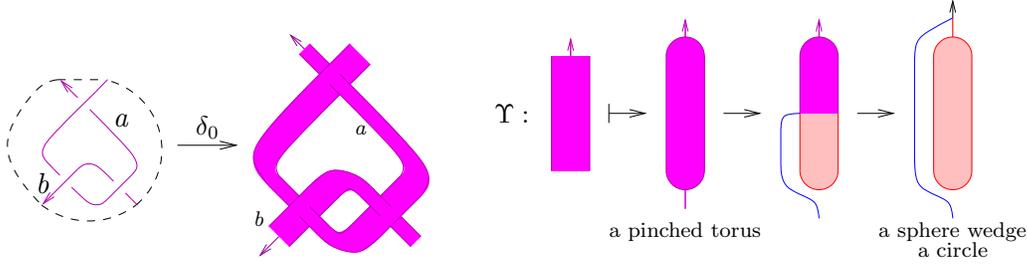
An efficient thumb rule for deciding the linking-number signs for a balloon u and a hoop x presented using our standard notation as in Section 2.1 is the “right-hand rule” of the figure on the right, shown here without further explanation. The lovely figure is adopted from [Wikipedia: Right-hand rule].



10.2. A topological construction of δ . The map δ is a composition $\delta_0 // \Upsilon$ (“ δ_0 followed by Υ ”, aka $\Upsilon \circ \delta_0$. See “notational conventions”, Section 10.5.). Here δ_0 is the standard “tubing” map δ_0 (called “Tube” in Satoh’s [Sa]), though with the tubes decorated by an additional arrowhead to retain orientation information. The map Υ caps and strings both ends of all tubes to ∞ and then uses, at the level of embeddings, the

¹⁰A similar statement can be made for Alexander formulae based on the Burau representation. Yet note that such formulae still end with a computation of a determinant which may take $O(n^3)$ steps. Note also that the presentation of knots as braid closures is typically inefficient — typically a braid with $O(n^2)$ crossings is necessary in order to present a knot with just n crossings.

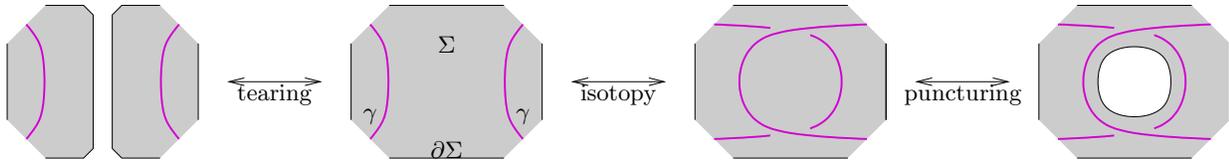
fact that a pinched torus is homotopy equivalent to a sphere wedge a circle:



It is worthwhile to give a completely “topological” definition of the tubing map δ_0 , thus giving $\delta = \delta_0 // \Upsilon$ a topological interpretation. We must start with a topological interpretation of v-tangles, and even before, with v-knots, also known as virtual knots.

The standard topological interpretation of v-knots (e.g. [Kup]) is that they are oriented knots drawn¹¹ on an oriented surface Σ , modulo “stabilization”, which is the addition and/or removal of empty handles (handles that do not intersect with the knot). We prefer an equivalent, yet even more bare-bones approach. For us, a virtual knot is an oriented knot γ drawn on a “virtual surface Σ for γ ”. More precisely, Σ is an oriented surface that may have a boundary, γ is drawn on Σ , and the pair (Σ, γ) is taken modulo the following relations:

- Isotopies of γ on Σ (meaning, in $\Sigma \times [-\epsilon, \epsilon]$).
- Tearing and puncturing parts of Σ away from γ :



(We call Σ a “virtual surface” because tearing and puncturing imply that we only care about it in the immediate vicinity of γ).

We can now define¹² a map δ_0 , defined on v-knots and taking values in ribbon tori in \mathbb{R}^4 : given (Σ, γ) , embed Σ arbitrarily in $\mathbb{R}^3_{xyz} \subset \mathbb{R}^4$. Note that the unit normal bundle of Σ in \mathbb{R}^4 is a trivial circle bundle and it has a distinguished trivialization, constructed using its positive- t -direction section and the orientation that gives each fibre a linking number $+1$ with the base Σ . We say that a normal vector to Σ in \mathbb{R}^4 is “near unit” if its norm is between $1 - \epsilon$ and $1 + \epsilon$. The near-unit normal bundle of Σ has as fibre an annulus that can be identified with $[-\epsilon, \epsilon] \times S^1$ (identifying the radial direction $[1 - \epsilon, 1 + \epsilon]$ with $[-\epsilon, \epsilon]$ in an orientation-preserving manner), and hence the near-unit normal bundle of Σ defines an embedding of $\Sigma \times [-\epsilon, \epsilon] \times S^1$ into \mathbb{R}^4 . On the other hand, γ is embedded in $\Sigma \times [-\epsilon, \epsilon]$ so $\gamma \times S^1$ is embedded in $\Sigma \times [-\epsilon, \epsilon] \times S^1$, and we can let $\delta_0(\Sigma, \gamma)$ be the composition

$$\gamma \times S^1 \hookrightarrow \Sigma \times [-\epsilon, \epsilon] \times S^1 \hookrightarrow \mathbb{R}^4,$$

¹¹Here and below, “drawn on Σ ” means “embedded in $\Sigma \times [-\epsilon, \epsilon]$ ”.

¹²Following a private discussion with Dylan Thurston.

which is a torus in \mathbb{R}^4 , oriented using the given orientation of γ and the standard orientation of S^1 .

We leave it to the reader to verify that $\delta_0(\Sigma, \gamma)$ is ribbon, that it is independent of the choices made within its construction, that it is invariant under isotopies of γ and under tearing and puncturing, that it is also invariant under the “overcrossing commute” relation of Figure 3, and that it is equivalent to Satoh’s tubing map.

The map δ_0 has straightforward generalizations to v-links, v-tangles, framed-v-links, v-knotted-graphs, etc.

10.3. Monoids, Meta-Monoids, Monoid-Actions and Meta-Monoid-Actions. How do we think about meta-monoid-actions? Why that name? Let us start with ordinary monoids.

10.3.1. *Monoids.* A monoid¹³ G gives rise to a slew of spaces and maps between them: the spaces would be the spaces of sequences $G^n = \{(g_1, \dots, g_n) : g_i \in G\}$, and the maps will be the maps “that can be written using the monoid structure” — they will include, for example, the map $m_i^{ij} : G^n \rightarrow G^{n-1}$ defined as “store the product $g_i g_j$ as entry number i in G^{n-1} while erasing the original entries number i and j and re-numbering all other entries as appropriate”. In addition, there is also an obvious binary “concatenation” map $*$: $G^n \times G^m \rightarrow G^{n+m}$ and a special element $\epsilon \in G^1$ (the monoid unit).

Equivalently but switching from “numbered registers” to “named registers”, a monoid G automatically gives rise to another slew of spaces and operations. The spaces are $G^X = \{f : X \rightarrow G\} = \{(x \rightarrow g_x)_{x \in X}\}$ of functions from a finite set X to G , or as we prefer to say it, of X -indexed sequences of elements in G , or how computer scientists may say it, of associative arrays of elements of G with keys in X . The maps between such spaces would now be the obvious “register multiplication maps” $m_z^{xy} : G^{X \cup \{x, y\}} \rightarrow G^{X \cup \{z\}}$ (defined whenever $x, y, z \notin X$ and $x \neq y$), and also the obvious “delete a register” map $\eta^x : G^X \rightarrow G^{X \setminus \{x\}}$, the obvious “rename a register” map $\sigma_y^x : G^{X \cup \{x\}} \rightarrow G^{X \cup \{y\}}$, and an obvious $*$: $G^X \times G^Y \rightarrow G^{X \cup Y}$, defined whenever X and Y are disjoint. Also, there are special elements, “units”, $\epsilon_x \in G^{\{x\}}$.

This collection of spaces and maps between them (and the units) satisfies some properties. Let us highlight and briefly discuss two of those:

- (1) The “associativity property”: For any $\Omega \in G^X$,

$$\Omega \parallel m_x^{xy} \parallel m_x^{xz} = \Omega \parallel m_y^{yz} \parallel m_x^{xy}. \quad (32)$$

This property is an immediate consequence of the associativity axiom of monoid theory. Note that it is a “linear property” — its subject, Ω , appears just once on each side of the equality. Similar linear properties include $\Omega \parallel \sigma_y^x \parallel \sigma_z^y = \Omega \parallel \sigma_z^x$, $\Omega \parallel m_z^{xy} \parallel \sigma_u^z = \Omega \parallel m_u^{xy}$, etc., and there are also “multi-linear” properties like $(\Omega_1 * \Omega_2) * \Omega_3 = \Omega_1 * (\Omega_2 * \Omega_3)$, which are “linear” in each of their inputs.

¹³A monoid is a group sans inverses. You lose nothing if you think “group” whenever the discussion below states “monoid”.

(2) If $\Omega \in G^{\{x,y\}}$, then

$$\Omega = (\Omega // \eta^y) * (\Omega // \eta^x) \quad (33)$$

(indeed, if $\Omega = (x \rightarrow g_x, y \rightarrow g_y)$, then $\Omega // \eta^y = (x \rightarrow g_x)$ and $\Omega // \eta^x = (y \rightarrow g_y)$ and so the right hand side is $(x \rightarrow g_x) * (y \rightarrow g_y)$, which is Ω back again), so an element of $G^{\{x,y\}}$ can be factored as an element of $G^{\{x\}}$ times an element of $G^{\{y\}}$. Note that Ω appears twice in the right hand side of this property, so this property is “quadratic”. In order to write this property one must be able to “make two copies of Ω ”.

10.3.2. Meta-Monoids.

Definition 10.2. A meta-monoid is a collection $(G_X, m_z^{xy}, \sigma_z^x, \eta^x, *)$ of sets G_X , one for each finite set X “of labels”, and maps between them $m_z^{xy}, \sigma_z^x, \eta^x, *$ with the same domains and ranges as above, and special elements $\epsilon_x \in G_{\{x\}}$, and with the same **linear and multi-linear** properties as above.

Very crucially, we do not insist on the non-linear property (33) of above, and so we may not have the factorization $G_{\{x,y\}} = G_{\{x\}} \times G_{\{y\}}$, and in general it need not be the case that $G_X = G^X$ for some monoid G . (Though of course, the case $G_X = G^X$ is an example of a meta-monoid, which perhaps may be called a “classical meta-monoid”).

Thus a meta-monoid is like a monoid in that it has sets G_X of “multi-elements” on which almost-ordinary monoid theoretic operations are defined. Yet the multi-elements in G_X need not simply be lists of elements as in G^X , and instead they may be somehow “entangled”. A relatively simple example of a meta-monoid which isn’t a monoid is $H^{\otimes X}$ where H is a Hopf algebra¹⁴. This simple example is similar to “quantum entanglement”. But a meta-monoid is not limited to the kind of entanglement that appears in tensor powers. Indeed many of the examples within the main text of this paper aren’t tensor powers and their “entanglement” is closer to that of the theory of tangles. This especially applied to the meta-monoid $w\mathcal{T}$ of Section 3.2.

10.3.3. *Monoid-Actions.* A monoid-action¹⁵ of a monoid G_1 on another monoid G_2 is a single algebraic structure MA consisting of two sets G_1 (“heads”) and G_2 (“tails”), a binary operation defined on G_1 , a binary operation defined on G_2 , and a mixed operation $G_1 \times G_2 \rightarrow G_2$ (denoted $(x, u) \mapsto u^x$) which satisfy some well known axioms, of which the most interesting are the associativities of the first two binary operations and the two action axioms $(uv)^x = u^x v^x$ and $u^{(xy)} = (u^x)^y$.

As in the case of individual monoids, a monoid-action MA gives rise to a slew of spaces and maps between them. The spaces are $\text{MA}(T; H) := G_2^T \times G_1^H$, defined whenever T and H are finite sets of “tail labels” and “head labels”. The main operations¹⁶ are $tm_w^{uv} : \text{MA}(T \cup \{u, v\}; H) \rightarrow \text{MA}(T \cup \{w\}; H)$ defined using the multiplication in G_2 (assuming $u, v, w \notin T$ and $u \neq v$), $hm_z^{xy} : \text{MA}(T; H \cup \{x, y\}) \rightarrow \text{MA}(T; H \cup \{z\})$ (assuming $x, y \notin H$ and $x \neq y$) defined using the multiplication in G_1 , and $tha^{ux} : \text{MA}(T; H) \rightarrow \text{MA}(T; H)$ (assuming $x \in H$

¹⁴Or merely an algebra.

¹⁵Think “group-action”.

¹⁶There are also $*$, $t\eta^u$, $h\eta^x$, $t\sigma_v^u$, and $h\sigma_y^x$ and units $t\epsilon_u$ and $h\epsilon_x$ as before.

and $u \in T$) defined using the action of G_1 on G_2 . These operations have the following properties, corresponding to the associativity of G_1 and G_2 and to the two action axioms of the previous paragraph:

$$\begin{aligned} hm_x^{xy} \parallel hm_x^{xz} &= hm_y^{yz} \parallel hm_x^{xy}, & tm_u^{uv} \parallel tm_u^{uw} &= tm_v^{vw} \parallel tm_u^{uv}, \\ tm_w^{uv} \parallel tha^{wx} &= tha^{ux} \parallel tha^{vx} \parallel tm_w^{uv}, & hm_z^{xy} \parallel tha^{uz} &= tha^{ux} \parallel tha^{uy} \parallel hm_z^{xy}. \end{aligned} \quad (34)$$

There are also routine properties involving also $*$, η 's and σ 's as before.

10.3.4. *Meta-Monoid-Actions.* Finally, a meta-monoid-action is to a monoid-action like a meta-monoid is to a monoid. Thus it is a collection

$$(M(T; H), tm_w^{uv}, hm_z^{xy}, tha^{ux}, t\sigma_w^u, h\sigma_y^x, t\eta^u, h\eta^x, *, t\epsilon_u, h\epsilon_x)$$

of sets $M(T; H)$, one for each pair of finite sets $(T; H)$ of “tail labels” and “head labels”, and maps between them tm_w^{uv} , hm_z^{xy} , tha^{ux} , $t\sigma_w^u$, $h\sigma_y^x$, $t\eta^u$, $h\eta^x$, $*$, and units $t\epsilon_u$ and $h\epsilon_x$, with the same domains and ranges as above and with the same **linear and multi-linear** properties as above; most importantly, the properties in (34).

Thus a meta-monoid-action is like a monoid-action in that it has sets $M(T; H)$ of “multi-elements” on which almost-ordinary monoid theoretic operations are defined. Yet the multi-elements in $M(T; H)$ need not simply be lists of elements as in $G_2^T \times G_1^H$, and instead they may be somehow “entangled”.

10.3.5. *Meta-Groups / Meta-Hopf-Algebras.* Clearly, the prefix “meta” can be added to many other types of algebraic structures, though sometimes a little care must be taken. To define a “meta-group”, for example, one may add to the definition of a meta-monoid in Section 10.3.2 a further collection of operations S^x , one for each $x \in X$, representing “invert the (meta-)element in register x ”. Except that the axiom for an inverse, $g \cdot g^{-1} = \epsilon$, is “quadratic” in g — one must have two copies of g in order to write the axiom, and hence it cannot be written using S^x and the operations in Section 10.3.2. Thus in order to define a meta-group, we need to also include “meta-co-product” operations $\Delta_{yz}^x : G_{X \cup \{x\}} \rightarrow G_{X \cup \{y, z\}}$. These operations should satisfy some further axioms, much like within the definition of a Hopf algebra. The major ones are: a meta-co-associativity, a meta-compatibility with the meta-multiplication, and a meta-inverse axiom $\Omega \parallel \Delta_{yz}^x \parallel S^y \parallel m_x^{yz} = (\Omega \parallel \eta^x) * \epsilon_x$.

A strict analogy with groups would suggest another axiom: a meta-co-commutativity of Δ , namely $\Delta_{yz}^x = \Delta_{zy}^x$. Yet experience shows that it is better to sometimes not insist on meta-co-commutativity. Perhaps the name “meta-group” should be used when meta-co-commutativity is assumed, and “meta-Hopf-algebra” when it isn't.

Similarly one may extend “meta-monoid-actions” to “meta-group-actions” and/or “meta-Hopf-actions”, in which new operations $t\Delta$ and $h\Delta$ are introduced, with appropriate axioms.

Note that $v\mathcal{T}$ and $w\mathcal{T}$ have a meta-co-product, defined using “strand doubling”. It is not meta-co-commutative.

Note also that \mathcal{K}^{rbh} and \mathcal{K}_0^{rbh} have operations $h\Delta$ and $t\Delta$, defined using “hoop doubling” and “balloon doubling”. The former is meta-co-commutative while the latter is not.

Note also that M and M_0 have an operation $h\Delta_{yz}^x$ defined by cloning one Lie-word, and an operation $t\Delta_{vw}^u$ defined using the substitution $u \rightarrow v + w$. Both of these operations are meta-co-commutative.

Thus ζ_0 and ζ cannot be homomorphic with respect to $t\Delta$. The discussion of trivalent vertices in [BND2, Section 4] can be interpreted as an analysis of the failure of ζ to be homomorphic with respect to $t\Delta$, but this will not be attempted in this paper.

10.4. Some Differentials and the Proof of Proposition 5.1. We prove Proposition 5.1, namely Equations (19) through (21), by verifying that each of these equations holds at one point, and then by differentiating each side of each equation and showing that the derivatives are equal. While routine, this argument appears complicated because the spaces involved are infinite dimensional and the operations involved are non-commutative. In fact, even the well-known derivative of the exponential function, which appears in the definition of C_u which appears in the definitions of RC_u and of J_u , may surprise readers who are used to the commutative case $de^x = e^x dx$.

Recall that FA denotes the graded completion of the free associative algebra on some alphabet T , and that the exponential map $\exp: FL \rightarrow FA$ defined by $\gamma \mapsto \exp(\gamma) = e^\gamma := \sum_{k=0}^{\infty} \frac{\gamma^k}{k!}$ makes sense in this completion.

Lemma 10.3. *If $\delta\gamma$ denotes an infinitesimal variation of γ , then the infinitesimal variation δe^γ of e^γ is given as follows:*

$$\delta e^\gamma = e^\gamma \cdot \left(\delta\gamma \parallel \frac{1 - e^{-\text{ad } \gamma}}{\text{ad } \gamma} \right) = \left(\delta\gamma \parallel \frac{e^{\text{ad } \gamma} - 1}{\text{ad } \gamma} \right) \cdot e^\gamma. \quad (35)$$

Above expressions such as $\frac{e^{\text{ad } \gamma} - 1}{\text{ad } \gamma}$ are interpreted via their power series expansions, $\frac{e^{\text{ad } \gamma} - 1}{\text{ad } \gamma} = 1 + \frac{1}{2} \text{ad } \gamma + \frac{1}{6} (\text{ad } \gamma)^2 + \dots$, and hence $\delta\gamma \parallel \frac{e^{\text{ad } \gamma} - 1}{\text{ad } \gamma} = \delta\gamma + \frac{1}{2} [\gamma, \delta\gamma] + \frac{1}{6} [\gamma, [\gamma, \delta\gamma]] + \dots$. Also, the precise meaning of (35) is that for any $\delta\gamma \in FL$, the derivative $\delta e^\gamma := \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (e^{\gamma + \epsilon \delta\gamma} - e^\gamma)$ is given by the right-hand-side of that equation. Equivalently, δe^γ is the term proportional to $\delta\gamma$ in $e^{\gamma + \delta\gamma}$, where during calculations we may assume that “ $\delta\gamma$ is an infinitesimal”, meaning that anything quadratic or higher in $\delta\gamma$ can be regarded as equal to 0.

Lemma 10.3 is rather standard (e.g. [DK, Section 1.5], [Me, Section 7]). Here’s a tweet:

Proof of Lemma 10.3. With an infinitesimal $\delta\gamma$, consider $F(s) := e^{-s\gamma} e^{s(\gamma + \delta\gamma)} - 1$. Then $F(0) = 0$ and $\frac{d}{ds} F(s) = e^{-s\gamma} (-\gamma) e^{s(\gamma + \delta\gamma)} + e^{-s\gamma} (\gamma + \delta\gamma) e^{s(\gamma + \delta\gamma)} = e^{-s\gamma} \delta\gamma e^{s(\gamma + \delta\gamma)} = e^{-s\gamma} \delta\gamma e^{s\gamma} = \delta\gamma \parallel e^{-s \text{ad } \gamma}$. So $e^{-\gamma} \delta\gamma = F(1) = \int_0^1 ds \frac{d}{ds} F(s) = \delta\gamma \parallel \int_0^1 ds e^{-s \text{ad } \gamma} = \delta\gamma \parallel \frac{1 - e^{-\text{ad } \gamma}}{\text{ad } \gamma}$. The second part of (35) is proven in a similar manner, starting with $G(s) := e^{s(\gamma + \delta\gamma)} e^{-s\gamma} - 1$. \square

Lemma 10.4. *If $\gamma = \text{bch}(\alpha, \beta)$ and $\delta\alpha$, $\delta\beta$, and $\delta\gamma$ are infinitesimals related by $\gamma + \delta\gamma = \text{bch}(\alpha + \delta\alpha, \beta + \delta\beta)$, then*

$$\delta\gamma \parallel \frac{1 - e^{-\text{ad } \gamma}}{\text{ad } \gamma} = \left(\delta\alpha \parallel \frac{1 - e^{-\text{ad } \alpha}}{\text{ad } \alpha} \parallel e^{-\text{ad } \beta} \right) + \left(\delta\beta \parallel \frac{1 - e^{-\text{ad } \beta}}{\text{ad } \beta} \right) \quad (36)$$

Proof. Use Leibniz' law on $e^\gamma = e^\alpha e^\beta$ to get $\delta e^\gamma = (\delta e^\alpha)e^\beta + e^\alpha(\delta e^\beta)$. Now use Lemma 10.3 three times to get

$$e^\gamma \left(\gamma \parallel \frac{1 - e^{-\text{ad } \gamma}}{\text{ad } \gamma} \right) = e^\alpha \left(\delta \alpha \parallel \frac{1 - e^{-\text{ad } \alpha}}{\text{ad } \alpha} \right) e^\beta + e^\alpha e^\beta \left(\delta \beta \parallel \frac{1 - e^{-\text{ad } \beta}}{\text{ad } \beta} \right),$$

conjugate the e^β in the first summand to the other side of the parenthesis, and cancel $e^\gamma = e^\alpha e^\beta$ from both sides of the resulting equation. \square

Recall that C_u^γ and RC_u^γ are automorphisms of FL . We wish to study their variations δC_u^γ and δRC_u^γ with respect to γ (these variations are ‘‘infinitesimal’’ automorphisms of FL). We need a definition and a property first.

Definition 10.5. For $u \in T$ and $\gamma \in FL(T)$ let $\text{ad}_u\{\gamma\} = \text{ad}_u^\gamma: FL(T) \rightarrow FL(T)$ denote the derivation of $FL(T)$ defined by its action of the generators as follows:

$$v \parallel \text{ad}_u\{\gamma\} = v \parallel \text{ad}_u^\gamma := \begin{cases} [\gamma, u] & v = u \\ 0 & \text{otherwise.} \end{cases}$$

Property 10.6. ad_u is the infinitesimal version of both C_u and RC_u . Namely, if $\delta\gamma$ is an infinitesimal, then $C_u^{\delta\gamma} = RC_u^{\delta\gamma} = 1 + \text{ad}_u\{\delta\gamma\}$.

We omit the easy proof of this property and move on to δC_u^γ and δRC_u^γ :

Lemma 10.7.
$$\delta C_u^\gamma = \text{ad}_u \left\{ \delta\gamma \parallel \frac{e^{\text{ad } \gamma} - 1}{\text{ad } \gamma} \parallel RC_u^{-\gamma} \right\} \parallel C_u^\gamma$$
 and
$$\delta RC_u^\gamma = RC_u^\gamma \parallel \text{ad}_u \left\{ \delta\gamma \parallel \frac{1 - e^{-\text{ad } \gamma}}{\text{ad } \gamma} \parallel RC_u^\gamma \right\}.$$

Proof. Substitute α and $\delta\beta$ into Equation (16) and get $RC_u^{\text{bch}(\alpha, \delta\beta)} = RC_u^\alpha \parallel RC_u^{\delta\beta \parallel RC_u^\alpha}$, and hence using Property 10.6 for the infinitesimal $\delta\beta \parallel RC_u^\alpha$ and Lemma 10.4 with $\delta\alpha = \beta = 0$ on $\text{bch}(\alpha, \delta\beta)$,

$$RC_u^{\alpha + (\delta\beta \parallel \frac{\text{ad } \alpha}{1 - e^{-\text{ad } \alpha}})} = RC_u^\alpha + RC_u^\alpha \parallel \text{ad}_u\{\delta\beta \parallel RC_u^\alpha\}.$$

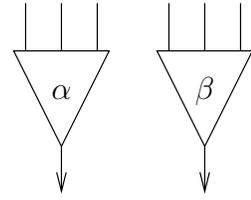
Now replacing $\alpha \rightarrow \gamma$ and $\delta\beta \rightarrow \delta\gamma \parallel \frac{1 - e^{-\text{ad } \gamma}}{\text{ad } \gamma}$, we get the equation for δRC_u^γ . The equation for δC_u^γ now follows by taking the variation of $C_u^\gamma \parallel RC_u^{-\gamma} = Id$. \square

Our next task is to compute $\delta J_u(\gamma)$. Yet before we can do that, we need to know one of the two properties of div_u that matter for us (besides its linearity):

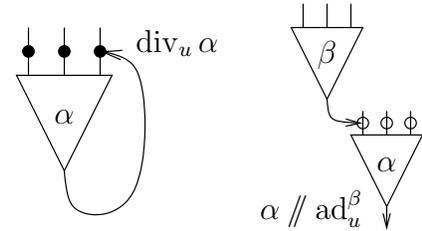
Proposition 10.8. For any $u, v \in T$ and any $\alpha, \beta \in FL$ and with δ_{uv} denoting the Kronecker delta function, the following ‘‘cocycle condition’’ holds: (compare with [AT, Proposition 3.20])

$$\underbrace{(\text{div}_u \alpha) \parallel \text{ad}_v^\beta}_A - \underbrace{(\text{div}_v \beta) \parallel \text{ad}_u^\alpha}_B = \underbrace{\delta_{uv} \text{div}_u[\alpha, \beta]}_C + \underbrace{\text{div}_u(\alpha \parallel \text{ad}_v^\beta)}_D - \underbrace{\text{div}_v(\beta \parallel \text{ad}_u^\alpha)}_E. \quad (37)$$

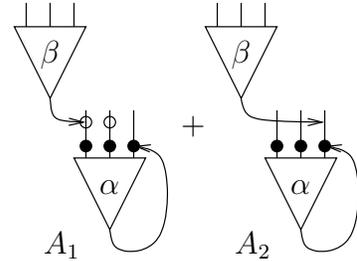
Proof. Start with the case where $u = v$. We draw each contribution to each of the terms above and note that all of these contributions cancel, but we must first explain our drawing conventions. We draw α and β as the “logic gates” appearing on the right. Each is really a linear combination, but (37) is bilinear so this doesn’t matter. Each is really a tree, but the proof does not use this so we don’t display this. Each may have many tail-legs labelled by other elements of T , but we care only about the legs labelled $u = v$ and so we display only those, and without real loss of generality, we draw it as if α and β each have exactly 3 such tails.



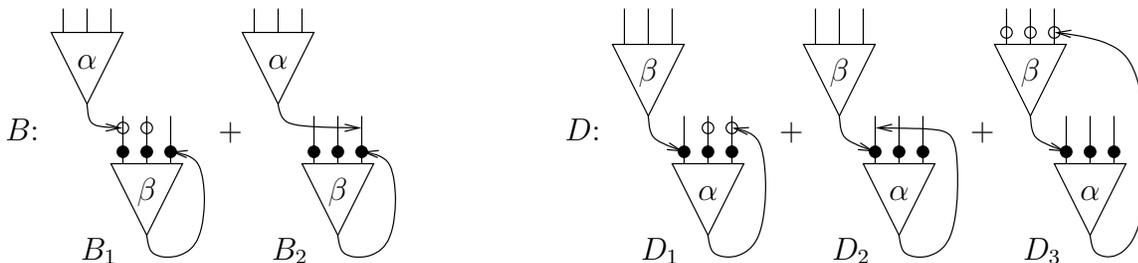
Objects such as $\text{div}_u \alpha$ and $\alpha // \text{ad}_u^\beta$ are obtained from α and β by connecting the head of one near its own tails, or near the other’s tails, in all possible ways. We draw just one summand from each sum, yet we indicate the other possible summands in each case by marking the other places where the relevant head could go with filled circles (\bullet) or empty circles (\circ) (the filling of the circles has no algebraic meaning; it is there only to separate summations in cases where two summations appear in the same formula). I hope the pictures on the right explain this better than the words.

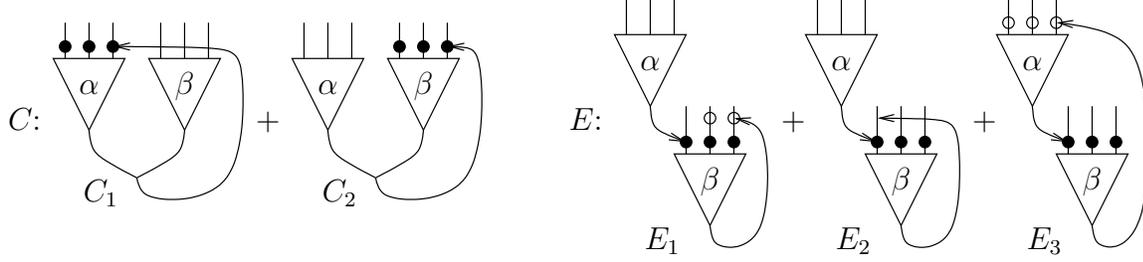


We illustrate our next convention with the pictorial representation of term A of Equation (37), $(\text{div}_u \alpha) // \text{ad}_u^\beta$, shown on the right. Namely, when the two relevant summations dictate that two heads may fall on the same arc, we split the sum into the generic part, A_1 on the right, in which the two heads do not fall on the same arc, and the exceptional part, A_2 on the right, in which the two heads do indeed fall on the same arc. The last convention is that \bullet indicates the first summation, and \circ the second. Hence in A_1 , the α head may fall in 3 places, and after that, the β head may only fall on one of the remaining relevant tails, whereas in A_2 the α is again free, but the β head must fall on the same arc.



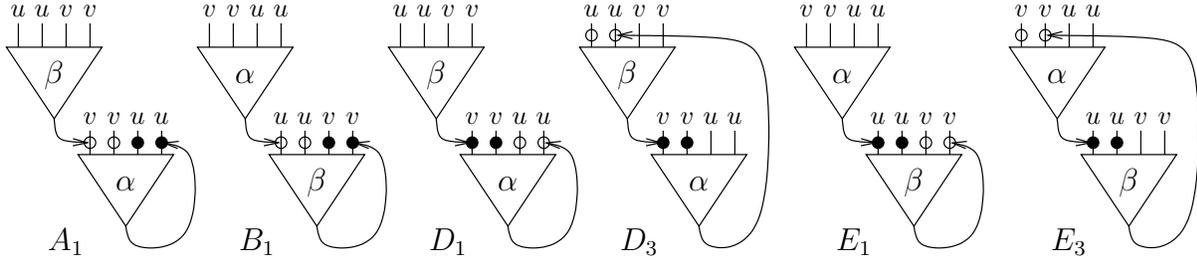
With all these conventions in place and with term A as above, we depict terms B - E :





Clearly, $A_1 = D_1$, $B_1 = E_1$, and $D_3 = E_3$ (the last equality is the only place in this paper that we need the cyclic property of cyclic words). Also, by the Jacobi identity, $A_2 - D_2 = C_1$ and $E_2 - B_2 = C_2$. So altogether, $A - B = C + D - E$.

The case where $u \neq v$ is similar, except we have to separate between u and v tails, the terms analogous to A_2 , B_2 , D_2 and E_2 cannot occur, and $C = 0$:



Clearly, $A - B = D - E$. □

For completeness and for use within the proof of Equation (21), here's the remaining property of div we need to know, presented without its easy proof:

Proposition 10.9. For any $\gamma \in FL$, $\gamma \parallel t_w^{uv} \parallel \text{div}_w = \gamma \parallel \text{div}_u \parallel t_w^{uv} + \gamma \parallel \text{div}_v \parallel t_w^{uv}$. □

Proposition 10.10. $\delta J_u(\gamma) = \delta\gamma \parallel \frac{1 - e^{-\text{ad}\gamma}}{\text{ad}\gamma} \parallel RC_u^\gamma \parallel \text{div}_u \parallel C_u^{-\gamma}$.

Proof. Let $I_s := \gamma \parallel RC_u^{s\gamma} \parallel \text{div}_u \parallel C_u^{-s\gamma}$ denote the integrand in the definition of J_u . Then under $\gamma \rightarrow \gamma + \delta\gamma$, using Leibniz, the linearity of div_u , and both parts of Lemma 10.7, we have

$$\begin{aligned} \delta I_s &= \delta\gamma \parallel RC_u^{s\gamma} \parallel \text{div}_u \parallel C_u^{-s\gamma} + \gamma \parallel RC_u^{s\gamma} \parallel \text{ad}_u \left\{ \delta\gamma \parallel \frac{1 - e^{-\text{ad}s\gamma}}{\text{ad}\gamma} \parallel RC_u^{s\gamma} \right\} \parallel \text{div}_u \parallel C_u^{-s\gamma} \\ &\quad - \gamma \parallel RC_u^{s\gamma} \parallel \text{div}_u \parallel \text{ad}_u \left\{ \delta\gamma \parallel \frac{1 - e^{-\text{ad}s\gamma}}{\text{ad}\gamma} \parallel RC_u^{s\gamma} \right\} \parallel C_u^{-s\gamma}. \end{aligned}$$

Taking the last two terms above as D and A of Equation (37), with $\alpha = \gamma \parallel RC_u^{s\gamma}$ and $\beta = \delta\gamma \parallel \frac{1 - e^{-\text{ad}s\gamma}}{\text{ad}\gamma} \parallel RC_u^{s\gamma}$, and using $[\alpha, \beta] = [\gamma, \delta\gamma \parallel \frac{1 - e^{-\text{ad}s\gamma}}{\text{ad}\gamma} \parallel RC_u^{s\gamma}] \parallel RC_u^{s\gamma} = \delta\gamma \parallel (1 - e^{-\text{ad}s\gamma}) \parallel RC_u^{s\gamma}$,

we get

$$\begin{aligned} \delta I_s &= \delta\gamma \parallel RC_u^{s\gamma} \parallel \operatorname{div}_u \parallel C_u^{-s\gamma} + \delta\gamma \parallel \frac{1 - e^{-\operatorname{ad} s\gamma}}{\operatorname{ad} \gamma} \parallel RC_u^{s\gamma} \parallel \operatorname{ad}_u \{ \gamma \parallel RC_u^{s\gamma} \} \parallel \operatorname{div}_u \parallel C_u^{-s\gamma} \\ &\quad - \delta\gamma \parallel \frac{1 - e^{-\operatorname{ad} s\gamma}}{\operatorname{ad} \gamma} \parallel RC_u^{s\gamma} \parallel \operatorname{div}_u \parallel \operatorname{ad}_u \{ \gamma \parallel RC_u^{s\gamma} \} \parallel C_u^{-s\gamma} \\ &\quad - \delta\gamma \parallel (1 - e^{-\operatorname{ad} s\gamma}) \parallel RC_u^{s\gamma} \parallel \operatorname{div}_u \parallel C_u^{-s\gamma}, \end{aligned}$$

and so, by combining the first and the last terms above,

$$\begin{aligned} \delta I_s &= \delta\gamma \parallel e^{-\operatorname{ad} s\gamma} \parallel RC_u^{s\gamma} \parallel \operatorname{div}_u \parallel C_u^{-s\gamma} \\ &\quad + \delta\gamma \parallel \frac{1 - e^{-\operatorname{ad} s\gamma}}{\operatorname{ad} \gamma} \parallel RC_u^{s\gamma} \parallel \operatorname{ad}_u \{ \gamma \parallel RC_u^{s\gamma} \} \parallel \operatorname{div}_u \parallel C_u^{-s\gamma} \\ &\quad - \delta\gamma \parallel \frac{1 - e^{-\operatorname{ad} s\gamma}}{\operatorname{ad} \gamma} \parallel RC_u^{s\gamma} \parallel \operatorname{div}_u \parallel \operatorname{ad}_u \{ \gamma \parallel RC_u^{s\gamma} \} \parallel C_u^{-s\gamma}, \end{aligned}$$

and hence, once again using Lemma 10.7 to differentiate $RC_u^{s\gamma}$ and $C_u^{-s\gamma}$ (except that things are now simpler because $s\gamma$ and $\delta(s\gamma) = \frac{d}{ds}(s\gamma) = \gamma$ commute), we get

$$\delta I_s = \frac{d}{ds} \left(\delta\gamma \parallel \frac{1 - e^{-\operatorname{ad} s\gamma}}{\operatorname{ad} \gamma} \parallel RC_u^{s\gamma} \parallel \operatorname{div}_u \parallel C_u^{-s\gamma} \right).$$

Integrating with respect to the variable s and using the fundamental theorem of calculus, we are done. \square

Proof of Equation (19). We fix α and show that Equation (19) holds for every β . For this it is enough to show that Equation (19) holds for $\beta = 0$ (it trivially does), and that the derivatives of both sides of Equation (19) in the radial direction are equal, for any given β . Namely, it is enough to verify that the variations of the two sides of Equation (19) under $\beta \rightarrow \beta + \delta\beta$ are equal, where $\delta\beta$ is proportional to β . Indeed, using the chain rule, Lemma 10.4, Proposition 10.10, the fact that β commutes with $\delta\beta$, and with $\gamma := \operatorname{bch}(\alpha, \beta)$,

$$\begin{aligned} \delta LHS &= \left(\delta\beta \parallel \frac{1 - e^{-\operatorname{ad} \beta}}{\operatorname{ad} \beta} \parallel \frac{\operatorname{ad} \gamma}{1 - e^{-\operatorname{ad} \gamma}} \right) \parallel \frac{1 - e^{-\operatorname{ad} \gamma}}{\operatorname{ad} \gamma} \parallel RC_u^\gamma \parallel \operatorname{div}_u \parallel C_u^{-\gamma} \\ &= \delta\beta \parallel RC_u^\gamma \parallel \operatorname{div}_u \parallel C_u^{-\gamma}. \end{aligned}$$

Similarly, using Proposition 10.10 and the fact that $\beta \parallel RC_u^\alpha$ commutes with $\delta\beta \parallel RC_u^\alpha$,

$$\delta RHS = \delta\beta \parallel RC_u^\alpha \parallel RC_u^{\beta \parallel RC_u^\alpha} \parallel \operatorname{div}_u \parallel C_u^{-\beta \parallel RC_u^\alpha} \parallel C_u^{-\alpha} = \delta\beta \parallel RC_u^\gamma \parallel \operatorname{div}_u \parallel C_u^{-\gamma},$$

where in the last equality we have used Equation (16) to combine the RC 's and its inverse to combine the C 's. \square

Proof of Equation (20). Equation (20) clearly holds when $\alpha = 0$, so as before, it is enough to prove it after taking the radial derivative with respect to α . So we need (ouch!)

$$\begin{aligned} & \alpha \parallel RC_u^\alpha \parallel \operatorname{div}_u \parallel C_u^{-\alpha} - \alpha \parallel RC_v^\beta \parallel RC_u^\alpha \parallel RC_v^\beta \parallel \operatorname{div}_u \parallel C_u^{-\alpha} \parallel RC_v^\beta \parallel C_v^{-\beta} \\ &= -\beta \parallel RC_u^\alpha \parallel \operatorname{ad}_u^\alpha \parallel RC_u^\alpha \parallel \frac{1 - e^{-\operatorname{ad}(\beta \parallel RC_u^\alpha)}}{\operatorname{ad}(\beta \parallel RC_u^\alpha)} \parallel RC_v^\beta \parallel RC_u^\alpha \parallel \operatorname{div}_v \parallel C_v^{-\beta} \parallel RC_u^\alpha \parallel C_u^{-\alpha} \\ & \qquad \qquad \qquad - \beta \parallel RC_u^\alpha \parallel J_v \parallel \operatorname{ad}_u^{-\alpha} \parallel RC_u^\alpha \parallel C_u^{-\alpha}. \end{aligned}$$

This we simplify using (13) and (14), cancel the $C_u^{-\alpha}$ on the right, and get

$$\begin{aligned} & \alpha \parallel RC_u^\alpha \parallel \operatorname{div}_u - \alpha \parallel RC_u^\alpha \parallel RC_v^\beta \parallel RC_u^\alpha \parallel \operatorname{div}_u \parallel C_v^{-\beta} \parallel RC_u^\alpha \\ & \stackrel{?}{=} -\beta \parallel RC_u^\alpha \parallel \operatorname{ad}_u^\alpha \parallel RC_u^\alpha \parallel \frac{1 - e^{-\operatorname{ad}(\beta \parallel RC_u^\alpha)}}{\operatorname{ad}(\beta \parallel RC_u^\alpha)} \parallel RC_v^\beta \parallel RC_u^\alpha \parallel \operatorname{div}_v \parallel C_v^{-\beta} \parallel RC_u^\alpha \\ & \qquad \qquad \qquad - \beta \parallel RC_u^\alpha \parallel J_v \parallel \operatorname{ad}_u^{-\alpha} \parallel RC_u^\alpha. \end{aligned}$$

We note that above α and β only appear within the combinations $\alpha \parallel RC_u^\alpha$ and $\beta \parallel RC_u^\alpha$, so we rename $\alpha \parallel RC_u^\alpha \rightarrow \alpha$ and $\beta \parallel RC_u^\alpha \rightarrow \beta$:

$$\begin{aligned} & \alpha \parallel \operatorname{div}_u - \alpha \parallel RC_v^\beta \parallel \operatorname{div}_u \parallel C_v^{-\beta} \\ & \stackrel{?}{=} -\beta \parallel \operatorname{ad}_u^\alpha \parallel \frac{1 - e^{-\operatorname{ad}(\beta)}}{\operatorname{ad}(\beta)} \parallel RC_v^\beta \parallel \operatorname{div}_v \parallel C_v^{-\beta} - \beta \parallel J_v \parallel \operatorname{ad}_u^{-\alpha}. \quad (38) \end{aligned}$$

Equation (38) still contains a J_v in it, so in order to prove it, we have to differentiate once again. So note that it holds at $\beta = 0$, multiply by -1 , and take the radial variation with respect to β (note that $\left. \frac{d}{ds} \frac{1 - e^{-\operatorname{ad}(s\beta)}}{\operatorname{ad}(s\beta)} \right|_{s=1} = \frac{e^{-\operatorname{ad}(\beta)}(1 + \operatorname{ad}(\beta) - e^{\operatorname{ad}(\beta)})}{\operatorname{ad}(\beta)}$):

$$\begin{aligned} & \alpha \parallel RC_v^\beta \parallel \operatorname{ad}_v^\beta \parallel RC_v^\beta \parallel \operatorname{div}_u \parallel C_v^{-\beta} - \alpha \parallel RC_v^\beta \parallel \operatorname{div}_u \parallel \operatorname{ad}_v^\beta \parallel RC_v^\beta \parallel C_v^{-\beta} \\ & \stackrel{?}{=} \beta \parallel \operatorname{ad}_u^\alpha \parallel \frac{1 - e^{-\operatorname{ad}(\beta)}}{\operatorname{ad}(\beta)} \parallel RC_v^\beta \parallel \operatorname{div}_v \parallel C_v^{-\beta} \\ & \quad + \beta \parallel \operatorname{ad}_u^\alpha \parallel \frac{e^{-\operatorname{ad}(\beta)}(1 + \operatorname{ad}(\beta) - e^{\operatorname{ad}(\beta)})}{\operatorname{ad}(\beta)} \parallel RC_v^\beta \parallel \operatorname{div}_v \parallel C_v^{-\beta} \\ & \quad + \beta \parallel \operatorname{ad}_u^\alpha \parallel \frac{1 - e^{-\operatorname{ad}(\beta)}}{\operatorname{ad}(\beta)} \parallel RC_v^\beta \parallel \operatorname{ad}_v^\beta \parallel RC_v^\beta \parallel \operatorname{div}_v \parallel C_v^{-\beta} \\ & \quad + \beta \parallel \operatorname{ad}_u^\alpha \parallel \frac{1 - e^{-\operatorname{ad}(\beta)}}{\operatorname{ad}(\beta)} \parallel RC_v^\beta \parallel \operatorname{div}_v \parallel \operatorname{ad}_v^{-\beta} \parallel RC_v^\beta \parallel C_v^{-\beta} \\ & \quad + \beta \parallel RC_v^\beta \parallel \operatorname{div}_v \parallel CC_v^{-\beta} \parallel \operatorname{ad}_u^{-\alpha} \end{aligned} \quad (39)$$

We massage three independent parts of the above desired equality at the same time:

- The div and the ad on the left hand side make terms D and A of Equation (37), with $\alpha \parallel RC_v^\beta \rightarrow \alpha$ and $\beta \parallel RC_v^\beta \rightarrow \beta$. We replace them by terms A and E .

- We combine the first two terms of the right hand side using $\frac{1-e^{-a}}{a} + \frac{e^{-a}(1+a-e^a)}{a} = e^{-a}$.
- In Equation (14), $C_u^{-\alpha} \parallel RC_v^\beta \parallel C_v^{-\beta} = C_v^{-\beta} \parallel RC_u^\alpha \parallel C_u^{-\alpha}$, take an infinitesimal α and use Property 10.6 and Lemma 10.7 to get

$$\text{ad}_u^{-\alpha} \parallel RC_v^\beta \parallel C_v^{-\beta} = \text{ad}_v^{-\beta} \parallel \text{ad}_u^\alpha \parallel \frac{1-e^{-\text{ad}(\beta)}}{\text{ad}(\beta)} \parallel RC_v^\beta \parallel C_v^{-\beta} + C_v^{-\beta} \parallel \text{ad}_u^{-\alpha}. \quad (40)$$

The last of that matches the last of (39), so we can replace the last of (39) with the start of (40).

All of this done, Equation (39) becomes the lowest point of this paper:

$$\begin{aligned} & \beta \parallel RC_v^\beta \parallel \text{ad}_u^\alpha \parallel RC_v^\beta \parallel \text{div}_v \parallel C_v^{-\beta} - \beta \parallel RC_v^\beta \parallel \text{div}_v \parallel \text{ad}_u^\alpha \parallel RC_v^\beta \parallel C_v^{-\beta} \\ & \stackrel{?}{=} \beta \parallel \text{ad}_u^\alpha \parallel e^{-\text{ad}(\beta)} \parallel RC_v^\beta \parallel \text{div}_v \parallel C_v^{-\beta} \\ & + \beta \parallel \text{ad}_u^\alpha \parallel \frac{1-e^{-\text{ad}(\beta)}}{\text{ad}(\beta)} \parallel RC_v^\beta \parallel \text{ad}_v^\beta \parallel RC_v^\beta \parallel \text{div}_v \parallel C_v^{-\beta} \\ & + \beta \parallel \text{ad}_u^\alpha \parallel \frac{1-e^{-\text{ad}(\beta)}}{\text{ad}(\beta)} \parallel RC_v^\beta \parallel \text{div}_v \parallel \text{ad}_v^{-\beta} \parallel RC_v^\beta \parallel C_v^{-\beta} \\ & + \beta \parallel RC_v^\beta \parallel \text{div}_v \parallel \text{ad}_u^{-\alpha} \parallel RC_v^\beta \parallel C_v^{-\beta} \\ & - \beta \parallel RC_v^\beta \parallel \text{div}_v \parallel \text{ad}_v^{-\beta} \parallel \text{ad}_u^\alpha \parallel \frac{1-e^{-\text{ad}(\beta)}}{\text{ad}(\beta)} \parallel RC_v^\beta \parallel C_v^{-\beta} \end{aligned}$$

Next we cancel the $C_v^{-\beta}$ at the right of every term, and a pair of repeating terms to get

$$\begin{aligned} & \beta \parallel RC_v^\beta \parallel \text{ad}_u^\alpha \parallel RC_v^\beta \parallel \text{div}_v \stackrel{?}{=} \beta \parallel \text{ad}_u^\alpha \parallel e^{-\text{ad}(\beta)} \parallel RC_v^\beta \parallel \text{div}_v \\ & + \beta \parallel \text{ad}_u^\alpha \parallel \frac{1-e^{-\text{ad}(\beta)}}{\text{ad}(\beta)} \parallel RC_v^\beta \parallel \text{ad}_v^\beta \parallel RC_v^\beta \parallel \text{div}_v \\ & - \beta \parallel \text{ad}_u^\alpha \parallel \frac{1-e^{-\text{ad}(\beta)}}{\text{ad}(\beta)} \parallel RC_v^\beta \parallel \text{div}_v \parallel \text{ad}_v^\beta \parallel RC_v^\beta \\ & - \beta \parallel RC_v^\beta \parallel \text{div}_v \parallel \text{ad}_v^{-\beta} \parallel \text{ad}_u^\alpha \parallel \frac{1-e^{-\text{ad}(\beta)}}{\text{ad}(\beta)} \parallel RC_v^\beta \end{aligned}$$

The two middle terms above differ only in the order of ad_v and div_v . So we apply Equation (37) again and get

$$\begin{aligned} & \beta \parallel RC_v^\beta \parallel \text{ad}_u^\alpha \parallel RC_v^\beta \parallel \text{div}_v \stackrel{?}{=} \beta \parallel \text{ad}_u^\alpha \parallel e^{-\text{ad}(\beta)} \parallel RC_v^\beta \parallel \text{div}_v \\ & + \beta \parallel RC_v^\beta \parallel \text{ad}_v^\beta \parallel \text{ad}_u^\alpha \parallel \frac{1-e^{-\text{ad}(\beta)}}{\text{ad}(\beta)} \parallel RC_v^\beta \parallel \text{div}_v - \beta \parallel RC_v^\beta \parallel \text{div}_v \parallel \text{ad}_v^\beta \parallel \text{ad}_u^\alpha \parallel \frac{1-e^{-\text{ad}(\beta)}}{\text{ad}(\beta)} \parallel RC_v^\beta \\ & + \left[\beta \parallel RC_v^\beta, \beta \parallel \text{ad}_u^\alpha \parallel \frac{1-e^{-\text{ad}(\beta)}}{\text{ad}(\beta)} \parallel RC_v^\beta \right] \parallel \text{div}_v - \beta \parallel RC_v^\beta \parallel \text{div}_v \parallel \text{ad}_v^{-\beta} \parallel \text{ad}_u^\alpha \parallel \frac{1-e^{-\text{ad}(\beta)}}{\text{ad}(\beta)} \parallel RC_v^\beta \end{aligned}$$

In the above, the two terms that do not end in div_v cancel each other. We then remove the div_v at the end of all remaining terms, thus making our quest only harder. Finally we note

that RC_v^β is a Lie algebra morphism, so we can pull it out of the bracket in the penultimate term, getting

$$\begin{aligned} \beta \parallel RC_v^\beta \parallel \text{ad}_u^\alpha \parallel RC_v^\beta \stackrel{?}{=} \beta \parallel \text{ad}_u^\alpha \parallel e^{-\text{ad}(\beta)} \parallel RC_v^\beta \\ + \beta \parallel RC_v^\beta \parallel \text{ad}_v^\beta \parallel \text{ad}_u^\alpha \parallel \frac{1-e^{-\text{ad}(\beta)}}{\text{ad}(\beta)} \parallel RC_v^\beta + \left[\beta, \beta \parallel \text{ad}_u^\alpha \parallel \frac{1-e^{-\text{ad}(\beta)}}{\text{ad}(\beta)} \right] \parallel RC_v^\beta \end{aligned}$$

The bracketing with β in the last term above cancels the $\text{ad}(\beta)$ denominator there, and then that term combines with the first term of the right hand side to yield

$$\beta \parallel RC_v^\beta \parallel \text{ad}_u^\alpha \parallel RC_v^\beta \stackrel{?}{=} \beta \parallel \text{ad}_u^\alpha \parallel RC_v^\beta + \beta \parallel RC_v^\beta \parallel \text{ad}_v^\beta \parallel \text{ad}_u^\alpha \parallel \frac{1-e^{-\text{ad}(\beta)}}{\text{ad}(\beta)} \parallel RC_v^\beta$$

We make our task harder again,

$$RC_v^\beta \parallel \text{ad}_u^\alpha \parallel RC_v^\beta \stackrel{?}{=} \text{ad}_u^\alpha \parallel RC_v^\beta + RC_v^\beta \parallel \text{ad}_v^\beta \parallel \text{ad}_u^\alpha \parallel \frac{1-e^{-\text{ad}(\beta)}}{\text{ad}(\beta)} \parallel RC_v^\beta$$

and then we both pre-compose and post-compose with the isomorphism $C_v^{-\beta}$, getting

$$\text{ad}_u^\alpha \parallel RC_v^\beta \parallel C_v^{-\beta} \stackrel{?}{=} C_v^{-\beta} \parallel \text{ad}_u^\alpha + \text{ad}_v^\beta \parallel \text{ad}_u^\alpha \parallel \frac{1-e^{-\text{ad}(\beta)}}{\text{ad}(\beta)} \parallel RC_v^\beta \parallel C_v^{-\beta}$$

The above is Equation (40), with α replaced by $-\alpha$, and hence it holds true. \square

Proof of Equation (21). As before, the equation clearly holds at $\gamma = 0$, so we take its radial derivative. That of the left hand side is

$$\gamma \parallel tm_w^{uv} \parallel RC_w^\gamma \parallel tm_w^{uv} \parallel \text{div}_w \parallel C_w^{-\gamma} \parallel tm_w^{uv}$$

Using Equation (15) and then Proposition 10.9, this becomes

$$\gamma \parallel RC_u^\gamma \parallel RC_v^\gamma \parallel RC_u^\gamma \parallel (\text{div}_u + \text{div}_v) \parallel tm_w^{uv} \parallel C_w^{-\gamma} \parallel tm_w^{uv}.$$

Now using the reverse of Equation (15), proven by reading the horizontal arrows within its proof backwards, this becomes

$$\gamma \parallel RC_u^\gamma \parallel RC_v^\gamma \parallel RC_u^\gamma \parallel (\text{div}_u + \text{div}_v) \parallel C_v^{-\gamma} \parallel RC_u^\gamma \parallel C_u^{-\gamma} \parallel tm_w^{uv}.$$

On the other hand, the radial variation of the right hand side of (21) is

$$\begin{aligned} \gamma \parallel RC_u^\gamma \parallel \text{div}_u \parallel C_u^{-\gamma} \parallel tm_w^{uv} + \gamma \parallel RC_u^\gamma \parallel RC_v^\gamma \parallel RC_u^\gamma \parallel \text{div}_v \parallel C_v^{-\gamma} \parallel RC_u^\gamma \parallel C_u^{-\gamma} \parallel tm_w^{uv} \\ + \gamma \parallel RC_u^\gamma \parallel \text{ad}_u^\gamma \parallel RC_u^\gamma \parallel \frac{1-e^{-\text{ad}(\gamma \parallel RC_u^\gamma)}}{\text{ad}(\gamma \parallel RC_u^\gamma)} \parallel RC_v^\gamma \parallel RC_u^\gamma \parallel \text{div}_v \parallel C_v^{-\gamma} \parallel RC_u^\gamma \parallel C_u^{-\gamma} \parallel t_w^{uv} \\ + \gamma \parallel RC_u^\gamma \parallel J_v \parallel \text{ad}_u^{-\gamma} \parallel RC_u^\gamma \parallel C_u^{-\gamma} \parallel t_w^{uv} \end{aligned}$$

Equating the last two formulae while eliminating the common term (the second term in each) and removing all trailing $C_u^{-\gamma} \parallel t_w^{uv}$'s (thus making the quest harder), we need to show

that

$$\begin{aligned} \gamma // RC_u^\gamma // RC_v^\gamma // RC_u^\gamma // \operatorname{div}_u // C_v^{-\gamma} // RC_u^\gamma &= \gamma // RC_u^\gamma // \operatorname{div}_u \\ &+ \gamma // RC_u^\gamma // \operatorname{ad}_u^\gamma // RC_u^\gamma // \frac{1 - e^{-\operatorname{ad}(\gamma // RC_u^\gamma)}}{\operatorname{ad}(\gamma // RC_u^\gamma)} // RC_v^\gamma // RC_u^\gamma // \operatorname{div}_v // C_v^{-\gamma} // RC_u^\gamma \\ &+ \gamma // RC_u^\gamma // J_v // \operatorname{ad}_u^{-\gamma} // RC_u^\gamma. \end{aligned}$$

Nicely enough, the above is Equation (38) with $\alpha = \beta = \gamma // RC_u^\gamma$. □

10.5. Notational Conventions and Glossary. For $n \in \mathbb{N}$ let \underline{n} denote some fixed set with n elements, say $\{1, 2, \dots, n\}$.

Often within this paper we use postfix notation for operator evaluations, so $f(x)$ may also be denoted $x // f$. Even better, we use $f // g$ for “composition done right”, meaning $f // g = g \circ f$, meaning that if $X \xrightarrow{f} Y \xrightarrow{g} Z$ then $X \xrightarrow{f // g} Z$ rather than the uglier (though equally correct) $X \xrightarrow{g \circ f} Z$. We hope that this notation will be adopted by others, to be used alongside and eventually instead of $g \circ f$, much as we hope that τ will be used alongside and eventually instead of the presently popular $\pi := \tau/2$. In L^AT_EX, `// = \sslash ∈ stmaryrd.sty`.

In the few paragraphs that follow, X is an arbitrary set. Though within this paper such X ’s will usually be finite, and their elements will thought of as “labels”. Hence if $f \in G^X$ is a function $f: X \rightarrow G$ where G is some other set, we think of f as a collection of elements of G labelled by the elements of X . We often write f_x to denote $f(x)$.

If $f \in G^X$ and $x \in X$, we let $f \setminus x$ denote the restricted function $f|_{X \setminus x}$ in which x is removed from the domain of f . In other words, $f \setminus x$ is “the collection f , with the element labelled x removed”. We often neglect to state the condition $x \in X$. Thus when writing $f \setminus x$ we implicitly assume that $x \in X$.

Likewise, we write $f \setminus \{x, y\}$ for “ f with x and y removed from its domain” and as before this includes the implicit assumption that $\{x, y\} \subset X$.

If $f_1: X_1 \rightarrow G$ and $f_2: X_2 \rightarrow G$ and X_1 and X_2 are disjoint, we denote by $f \cup g$ the obvious “union function” with domain $X_1 \cup X_2$ and range G . In fact, whenever we write $f \cup g$, we make the implicit assumption that the domains of f_1 and f_2 are disjoint.

In the spirit of “associative arrays” as they appear in various computer languages, we use the notation $(x \rightarrow a, y \rightarrow b, \dots)$ for “inline function definition”. Thus $()$ is the empty function, and if $f = (x \rightarrow a, y \rightarrow b)$, then the domain of f is $\{x, y\}$ and $f_x = a$ and $f_y = b$.

We denote by σ_y^x the operation that renames the key x in an associative array to y . Namely, if $f \in G^X$, $x \notin X$, and $y \notin X \setminus x$, then

$$\sigma_y^x f = (f \setminus x) \cup (y \rightarrow f_x).$$

Glossary of Notations. (Greek letters, then Latin, then symbols)

α, β, γ	Free Lie series	Sec. 4	δ	A map $u\mathcal{T}/v\mathcal{T}/w\mathcal{T} \rightarrow \mathcal{K}^{rbh}$	Sec. 2.2
$\alpha, \beta, \gamma, \delta$	Matrix parts	Sec. 9.4	$\delta\alpha, \delta\beta, \delta\gamma$	Infinitesimal free Lie series	Sec. 10.4
β	A repackaging of β	Sec. 9.4	ϵ_a	Units	Sec. 3.2
β_0	A reduction of M	Sec. 9.3	Π	The MMA “of groups”	Sec. 3.4

π	The fundamental invariant	Sec. 2.3	M	The “main” MMA	Sec. 5.2
π	The projection $\mathcal{K}_0^{r_{bh}} \rightarrow \mathcal{K}^{r_{bh}}$	Prop. 3.6	M_0	The MMA of trees	Sec. 4.2
ρ_{ux}^\pm	\pm -Hopf links in 4D	Ex. 2.2	MMA	Meta-monoid-action Def. 3.2, Sec. 10.3.4	
σ_y^x	Re-labelling	Sec. 10.5	m_u	Meridians	Sec. 2.3
τ	Tensorial interpretation map	Sec. 8.1	m_c^{ab}	Strand concatenation	Sec 3.2
ω	The wheels part of M/ζ	Sec.5	OC	Overcrossings commute	Fig. 3
ω	The scalar part in β/β_0	Sec. 9.3	\mathcal{P}^{bh}	Primitives of \mathcal{A}^{bh}	Sec. 7.3
Υ	Capping and sliding	Sec.10.2	R	Ring of c -stubs	Sec. 9.2
ζ	The main invariant	Sec. 5	R^r	R mod degree 1	Sec. 9.3
ζ_0	The tree-level invariant	Sec. 4	R1,R1',R2,R3	Reidemeister moves	Sec. 2.2, 7.1
ζ^β	A β -valued invariant	Sec. 9.4	RC_u^γ	Repeated C_u^γ / reverse $C_u^{-\gamma}$	Sec. 4.2
ζ^{β_0}	A β_0 -valued invariant	Sec. 9.3	rKBH	Ribbon knotted balloons&hoops	Def. 2.1
A	The matrix part in β/β_0	Sec. 9.3	S	Set of strand labels	Sec. 2.2
a, b, c	Strand labels	Sec. 2.2	T	Set of tail / balloon labels	Sec. 2
$\text{ad}_u^\gamma, \text{ad}_u\{\gamma\}$	Derivations of FL	Def. 10.5	te^u	Units	Ex. 2.2, Sec. 4.2,5.2
\mathcal{A}^{bh}	Space of arrow diagrams	Sec. 7.2	tha^{ux}	Tail by head action	Sec. 3,4,2,5.2
bch	Baker-Campbell-Hausdorff	Sec. 4.2	$t\eta^u$	Tail delete	Sec. 3,4,2,5.2
C_u^γ	Conjugating a generator	Sec. 4.2	tm_w^{uv}	Tail multiply	Sec. 3,4,2,5.2
CA	Circuit algebra	Sec. 7.1	$t\sigma_y^x$	Tail re-label	Sec. 3,4,2,5.2
CW	Cyclic words	Sec. 5.1	t, x, y, z	Coordinates	Sec. 2
CW^r	CW mod degree 1	Sec. 5.1	UC	Undercrossings commute	Fig. 3
c	A “sink” vertex	Sec. 9.1	u-tangle	A usual tangle	Sec. 2.2
c_u	A “ c -stub”	Sec. 9.1	$u\mathcal{T}$	All u-tangles	Sec. 2.2
div_u	The “divergence” $FL \rightarrow CW$	Sec. 5.1	u, v, w	Tail / balloon labels	Sec. 2
dm_c^{ab}	Double/diagonal multiplication	Sec. 3.2	v-tangle	A virtual tangle	Sec. 2.4
FA	Free associative algebra	Sec. 5.1	$v\mathcal{T}$	All v-tangles	Sec. 2.4
FL	Free Lie algebra	Sec. 4.2	w-tangle	A virtual tangle mod OC	Sec. 2.4
$\text{Fun}(X \rightarrow Y)$	Functions $X \rightarrow Y$	Sec. 8.1	$w\mathcal{T}$	All w-tangles	Sec. 2.4
H	Set of head / hoop labels	Sec. 2	x, y, z	Head / hoop labels	Sec. 2
$h\epsilon_x$	Units	Ex. 2.2, Sec. 4.2,5.2	Z^{bh}	An \mathcal{A}^{bh} -valued expansion	Sec. 7.4
$h\eta$	Head delete	Sec. 3,4,2,5.2	*	Merge operation	Sec. 3,4,2,5.2
hm_z^{xy}	Head multiply	Sec. 3,4,2,5.2	//	Composition done right	Sec. 10.5
$h\sigma_y^x$	Head re-label	Sec. 3,4,2,5.2	$x // f$	Postfix evaluation	Sec. 10.5
J_u	The “spice” $FL \rightarrow CW$	Sec. 5.1	$f \setminus x$	Entry removal	Sec. 10.5
$\mathcal{K}^{r_{bh}}$	All rKBHs	Def. 2.1	$x \rightarrow a$	Inline function definition	Sec. 10.5
$\mathcal{K}_0^{r_{bh}}$	Conjectured version of $\mathcal{K}^{r_{bh}}$	Sec. 3.3	\overline{uv}	“Top bracket form”	Sec. 6
l_{ux}	4D linking numbers	Sec. 10.1	\widehat{uv}	A cyclic word	Sec. 6
l_x	Longitudes	Sec. 2.3			

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REFERENCES

- [AT] A. Alekseev and C. Torossian, *The Kashiwara-Vergne conjecture and Drinfel'd's associators*, *Annals of Mathematics*, **175** (2012) 415–463, [arXiv:0802.4300](https://arxiv.org/abs/0802.4300).
- [BN1] D. Bar-Natan, *On the Vassiliev Knot Invariants*, *Topology* **34** (1995) 423–472.
- [BN2] D. Bar-Natan, *Vassiliev Homotopy String Link Invariants*, *J. of Knot Theory and its Ramifications* **4** (1995) 13–32.
- [BN3] D. Bar-Natan, *Finite Type Invariants*, in *Encyclopedia of Mathematical Physics*, (J.-P. Francoise, G. L. Naber and Tsou S. T., eds.) Elsevier, Oxford, 2006 (vol. 2 p. 340).
- [BN4] D. Bar-Natan, *Finite Type Invariants of W-Knotted Objects IV: Some Computations*, in preparation.
- [BND1] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I: Braids, Knots and the Alexander Polynomial*, <http://www.math.toronto.edu/drorbn/LOP.html#WK01>, [arXiv:1405.1956](https://arxiv.org/abs/1405.1956).
- [BND2] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects II: Tangles and the Kashiwara-Vergne Problem*, <http://www.math.toronto.edu/drorbn/LOP.html#WK02>, [arXiv:1405.1955](https://arxiv.org/abs/1405.1955).
- [BNS] D. Bar-Natan and S. Selmani, *Meta-Monoids, Meta-Bicrossed Products, and the Alexander Polynomial*, *J. of Knot Theory and its Ramifications* **22-10** (2013), [arXiv:1302.5689](https://arxiv.org/abs/1302.5689).
- [CS] J. S. Carter and M. Saito, *Knotted surfaces and their diagrams*, *Mathematical Surveys and Monographs* **55**, American Mathematical Society, Providence 1998.
- [CR] A. S. Cattaneo and C. A. Rossi, *Wilson Surfaces and Higher Dimensional Knot Invariants*, *Commun. in Math. Phys.* **256-3** (2005) 513–537, [arXiv:math-ph/0210037](https://arxiv.org/abs/math-ph/0210037).
- [DK] J. J. Duistermaat and J. A. C. Kolk, *Lie Groups*, Springer, 1999.
- [HL] N. Habegger and X-S. Lin, *The Classification of links up to link-homotopy*, *J. Amer. Math. Soc.* **3** (1990) 389–419.
- [HM] N. Habegger and G. Masbaum, *The Kontsevich integral and Milnor's invariants*, *Topology* **39-6** (2000) 1253-1289.
- [HKS] K. Habiro, T. Kanenobu, and A. Shima, *Finite Type Invariants of Ribbon 2-Knots*, in *Low Dimensional Topology*, (H. Nienka, ed.) *Cont. Math.* **233** (1999) 187–196.
- [HS] K. Habiro and A. Shima, *Finite Type Invariants of Ribbon 2-Knots, II*, *Topology and its Applications* **111-3** (2001) 265–287.
- [Ho1] J. Howie, *On the Asphericity of Ribbon Disc Complements*, *Trans. Amer. Math. Soc.* **289-1** (1985) 281–302.
- [Ho2] J. Howie, *Minimal Seifert Manifolds for Higher Ribbon Knots*, *Geometry and Topology Monographs* **1** 261–293, [arXiv:math/9810185](https://arxiv.org/abs/math/9810185).
- [KS] T. Kanenobu, A. Shima, *Two Filtrations of Ribbon 2-Knots*, *Topology and Appl.* **121** (2002) 143–168.
- [Kau] L. H. Kauffman, *Virtual Knot Theory*, *European J. Comb.* **20** (1999) 663–690, [arXiv:math.GT/9811028](https://arxiv.org/abs/math.GT/9811028).
- [Kaw] A. Kawachi, *A survey of knot theory*, Birkhäuser, Basel 1996.
- [KLW] P. Kirk, C. Livingston, and Z. Wang, *The Gassner Representation for String Links*, *Comm. Cont. Math.* **3** (2001) 87–136, [arXiv:math/9806035](https://arxiv.org/abs/math/9806035).
- [Kr] A. Kriker, *The lines of the Kontsevich integral and Rozansky's rationality conjecture*, [arXiv:math/0005284](https://arxiv.org/abs/math/0005284).
- [Kup] G. Kuperberg, *What is a Virtual Link?*, *Algebr. Geom. Topol.* **3** (2003) 587–591, [arXiv:math.GT/0208039](https://arxiv.org/abs/math.GT/0208039).

- [Lee] P. Lee, *Proof of a Conjectured Formula for the Alexander Invariant*, [arXiv:1209.0668](https://arxiv.org/abs/1209.0668).
- [Me] E. Meinrenken, *Lie Groups and Lie Algebras*, lecture notes, University of Toronto 2010, <http://www.math.toronto.edu/mein/teaching/lie.pdf>.
- [Mi] J. W. Milnor, *Link Groups*, *Annals of Math.* **59** (1954) 177–195.
- [Re] C. Reutenauer, *Free Lie Algebras*, Clarendon Press, Oxford 1993.
- [Ro] L. H. Rowen, *Polynomial Identities in Ring Theory*, Academic Press, New York 1980.
- [Sa] S. Satoh, *Virtual Knot Presentations of Ribbon Torus Knots*, *J. of Knot Theory and its Ramifications* **9-4** (2000) 531–542.
- [Wa1] T. Watanabe, *Clasper-Moves Among Ribbon 2-Knots Characterizing their Finite Type Invariants*, *J. of Knot Theory and its Ramifications* **15-9** (2006) 1163–1199.
- [Wa2] T. Watanabe, *Configuration Space Integrals for Long n -Knots, the Alexander Polynomial and Knot Space Cohomology*, *Alg. and Geom. Top.* **7** (2007) 47–92, [arXiv:math/0609742](https://arxiv.org/abs/math/0609742).
- [Web] D. Bar-Natan, *Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant* (self-reference). Web resources at <http://www.math.toronto.edu/~drorbn/papers/KBH/>.
- [Win1] B. Winter, *The Classification of Spun Torus Knots*, *J. of Knot Theory and its Ramifications* **18-9** (2009) 1287–1298, [arXiv:0711.1638](https://arxiv.org/abs/0711.1638).
- [Win2] B. Winter, *On Codimension Two Ribbon Embeddings*, [arXiv:0904.0684](https://arxiv.org/abs/0904.0684).

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