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<< \ c : / drorbn / projects / KAtlas / KnotTheory.m
Loading KnotTheory`...

<< \ c : / drorbn / projects / KAtlas / KnotTheoryData.m
Loading KnotTheoryData.m...

Lob[t_] := -NIntegrate[Log[Abs[2 Sin[u]]], {u, 0, t}];
Vol[z_] := Plus @@ (Lob[Arg[#]] & /@ {z, 1 - 1/z, 1/(1 - z)})

2 Vol[omega = 1/2 + Sqrt[3]/2 * I]
2.02988

Pachner[z_, w_] := {Vol[z], Vol[w], Vol[z] + Vol[w] +
  Vol[1/(z * w)] - Vol[z (1 - w) / (1 - z * w)] - Vol[w (1 - z) / (1 - z * w)]}

Pachner[Random[Real, {-1, 1}] + I * Random[Real, {-1, 1}],
  Random[Real, {-1, 1}] + I * Random[Real, {-1, 1}]]
{0.870435, -0.589215, -5.66219 * 10^-11}

n = 4;

eqns = (# == 1) & /@ (prods = {
  z11 z21 z31 z41, z12 z22 z32 z42, z13 z23 z33 z43,
  z12 z14, z34 z13 z31, z24 z42 z23, z21 z22 z44, z41 z43, z11 z33 z32,
  z32 ' z11 ' z41 ' z43 ' z13 ' z31 ' z21 ' z22 ',
  z33 ' z32 ' z22 ' z44 ' z14 ' z12 ' z42 ' z23 ',
  z34 ' z13 ' z43 ' z41 ' z11 ' z33 ' z23 ' z24 ',
  z24 ' z42 ' z32 ' z11 ' z21 ' z22 ' z12 ' z14 ',
  z41 ' z43 ' z33 ' z32 ' z42 ' z23 ' z13 ' z31 ',
  z43 ' z41 ' z31 ' z34 ' z44 ' z21 ' z11 ' z33 ',
  z21 ' z31 ' (z34 ' z44) ^ (-1)
}) /@ {(z_) ' => 1 - 1/z, (z_) ' => 1/(1 - z)}
{z11 z21 z31 z41 == 1, z12 z22 z32 z42 == 1, z13 z23 z33 z43 == 1,
  z12 z14 == 1, z13 z31 z34 == 1, z23 z24 z42 == 1, z21 z22 z44 == 1,
  z41 z43 == 1, z11 z32 z33 == 1,  $\frac{\left(1 - \frac{1}{z_{11}}\right) \left(1 - \frac{1}{z_{22}}\right) \left(1 - \frac{1}{z_{31}}\right) \left(1 - \frac{1}{z_{43}}\right)}{\left(1 - z_{13}\right) \left(1 - z_{21}\right) \left(1 - z_{32}\right) \left(1 - z_{41}\right)} == 1,$ 
 $\frac{\left(1 - \frac{1}{z_{12}}\right) \left(1 - \frac{1}{z_{23}}\right) \left(1 - \frac{1}{z_{32}}\right) \left(1 - \frac{1}{z_{44}}\right)}{\left(1 - z_{14}\right) \left(1 - z_{22}\right) \left(1 - z_{33}\right) \left(1 - z_{42}\right)} == 1,$ 
 $\frac{\left(1 - \frac{1}{z_{13}}\right) \left(1 - \frac{1}{z_{24}}\right) \left(1 - \frac{1}{z_{33}}\right) \left(1 - \frac{1}{z_{41}}\right)}{\left(1 - z_{11}\right) \left(1 - z_{23}\right) \left(1 - z_{34}\right) \left(1 - z_{43}\right)} == 1,$ 
 $\frac{\left(1 - \frac{1}{z_{12}}\right) \left(1 - \frac{1}{z_{21}}\right) \left(1 - \frac{1}{z_{24}}\right) \left(1 - \frac{1}{z_{32}}\right)}{\left(1 - z_{11}\right) \left(1 - z_{14}\right) \left(1 - z_{22}\right) \left(1 - z_{42}\right)} == 1,$ 
 $\frac{\left(1 - \frac{1}{z_{13}}\right) \left(1 - \frac{1}{z_{33}}\right) \left(1 - \frac{1}{z_{41}}\right) \left(1 - \frac{1}{z_{42}}\right)}{\left(1 - z_{23}\right) \left(1 - z_{31}\right) \left(1 - z_{32}\right) \left(1 - z_{43}\right)} == 1,$ 
 $\frac{\left(1 - \frac{1}{z_{11}}\right) \left(1 - \frac{1}{z_{31}}\right) \left(1 - \frac{1}{z_{43}}\right) \left(1 - \frac{1}{z_{44}}\right)}{\left(1 - z_{21}\right) \left(1 - z_{33}\right) \left(1 - z_{34}\right) \left(1 - z_{41}\right)} == 1,$ 
 $\frac{\left(1 - \frac{1}{z_{21}}\right) \left(1 - z_{34}\right)}{\left(1 - z_{31}\right) \left(1 - \frac{1}{z_{44}}\right)} == 1}$ 

```

```

1 - 1 / (1 - 1 / z) // Simplify

$$\frac{1}{1 - z}$$

vars = Union[Cases[eqns, _Symbol, Infinity]]
{z11, z12, z13, z14, z21, z22, z23, z24, z31, z32, z33, z34, z41, z42, z43, z44}

{
  sol =
  FindRoot @@ Join[{eqns}, ({#, Exp[Random[] Pi I]} & /@ vars), {MaxIterations -> 1000}],
  Max[Abs[({First /@ eqns} /. sol) - 1]],
  Plus @@ (Vol /@ (vars /. sol))
}

{{z11 -> -7.02527 - 1.73033 i, z12 -> 4.17529 - 9.61917 i, z13 -> 0.427557 - 0.764651 i,
  z14 -> 0.0379705 + 0.0874777 i, z21 -> -0.0419909 + 0.145974 i, z22 -> 0.0425348 + 0.0797777 i,
  z23 -> 7.26191 - 9.75456 i, z24 -> 0.358503 - 0.874039 i, z31 -> -2.35838 - 7.10734 i,
  z32 -> -5.39387 - 10.8514 i, z33 -> 0.00248685 - 0.0111311 i, z34 -> -0.149703 + 0.028706 i,
  z41 -> -0.119061 + 0.0242837 i, z42 -> -0.0448724 + 0.0745864 i,
  z43 -> -8.06363 - 1.64466 i, z44 -> -71.2244 - 15.1609 i}, 2.8409 x 10^-10, -2.02988}

SetAttributes[p, Orderless]; SetAttributes[{EQs, EQ}, {Orderless, Flat}]
im4[a_] := a + 1 /. 5 -> 1; dm4[a_] := a - 1 /. 0 -> 4
Expected volume for 5_2: 2.8281220883.
L = PD[Knot[4, 1]]
PD[X[4, 2, 5, 1], X[8, 6, 1, 5], X[6, 3, 7, 4], X[2, 7, 3, 8]]
n = Length[L]
4
dt = EQs @@ Flatten[
{
  Table[
    t[z[v, j], 3, {1, 2, 4}] ~ e ~ t[z[v, im4@j], 4, {2, 1, 3}],
    {v, n}, {j, 4}
  ],
  Table[e @@ Position[L, i], {i, 2 n}] /. e[x_, y_] -> {e[x, y], e[y, x]} /. {
    e[{v_, a_}, {u_, b_}] =>
      t[z[v, dm4@a], 2, {1, 3, 4}] ~ e ~ t[z[u, b], 1, {2, 3, 4}] /; OddQ[a + b],
    e[{v_, a_}, {u_, b_}] => t[z[v, dm4@a], 2, {1, 3, 4}] ~ e ~
      t[z[u, b], 1, {2, 4, 3}] /; EvenQ[a + b]
  }
} /. e[t[z_, a_, {c1_, c2_, c3_}], t[w_, b_, {d1_, d2_, d3_}]] => {
  t[z, a, c1] ~ p ~ t[w, b, d1], t[z, a, c2] ~ p ~ t[w, b, d2], t[z, a, c3] ~ p ~ t[w, b, d3]
}
] /.
t[z[v_, j_], a_, b_] => t[z[v, j]] ~ Join ~ (t[a, b] /. {3 -> 4, 4 -> 3}) /; EvenQ[j];

```

```

rels = List @@ Union[
  dt /. p[t[z_, a_, b_], t[w_, c_, d_]] =>
    EQs[t[z, a, b] ~to~ t[w, d, c], t[w, c, d] ~to~ t[z, b, a]] /. (
    EQs[xx_. * (t[x_, ab_] ~to~ t[y_, cd_]), zz_. * (t[y_, cd_] ~to~ t[z_, ef_])] ×
  => EQs[xx * y[cd] * zz * (t[x, ab] ~to~ t[z, ef])]
  ) /. (t[z_, ab_] ~to~ t[z_, ab_]) => z[ab] /.
  (w_z)[ab_] => Switch[Sort[{ab}],
    {1, 2} | {3, 4}, w[0],
    {1, 3} | {2, 4}, w[1],
    {1, 4} | {2, 3}, w[2]
  ]
] /. eq_Times => (EQ @@ eq);
merge = Position[Count[#, z[n, _][__]] & /@ rels, 4];
rels = Append[Delete[rels, merge], EQ @@ rels[[Join @@ merge]]];
rels = Append[rels, EQ @@ Product[
  z[k, 4][0] * z[k, 4][1] * z[k, 3][0] * z[k, 3][2] *
  If[L[[k, 4]] > L[[k, 2]] || L[[k, 2]] - L[[k, 4]] > 1,
    z[k, 1][1] * z[k, 4][2],
    z[k, 3][1] * z[k, 2][2]
  ],
  {k, n}
]
] /. z[i_, j_] => ToExpression["z" <> ToString[i] <> ToString[j]]
{EQ[z11[0], z12[0], z13[0], z14[0]],
EQ[z13[0], z23[0]], EQ[z21[0], z22[0], z23[0], z24[0]],
EQ[z14[0], z22[0], z34[0]], EQ[z31[0], z32[0], z33[0], z34[0]],
EQ[z12[1], z13[2], z21[1], z22[2], z23[1], z24[2], z31[2], z34[1]],
EQ[z11[1], z12[2], z13[1], z14[2], z23[2], z24[1], z33[2], z34[1]],
EQ[z11[0], z33[0], z41[0]], EQ[z32[0], z42[0]],
EQ[z21[2], z22[1], z31[1], z32[2], z33[1], z34[2], z41[2], z42[1]],
EQ[z21[0], z31[0], z43[0]], EQ[z11[2], z14[1], z31[1], z32[2],
z33[1], z34[2], z42[1], z43[2]], EQ[z12[0], z24[0], z44[0]],
EQ[z11[1], z12[2], z13[1], z14[2], z22[1], z23[2], z41[2], z44[1]],
EQ[z13[2], z14[1], z21[1], z22[2], z23[1], z24[2], z43[2], z44[1]],
EQ[z11[2], z12[1], z21[2], z24[1], z31[2], z32[1], z32[1], z33[2], z41[0], z41[1],
z41[1], z42[0], z42[2], z42[2], z43[0], z43[1], z43[1], z44[0], z44[2], z44[2]],
EQ[z12[2], z13[0], z13[1], z13[2], z14[0], z14[1], z22[2], z23[0],
z23[1], z23[2], z24[0], z24[1], z31[1], z33[0], z33[2], z34[0],
z34[1], z34[2], z41[1], z43[0], z43[2], z44[0], z44[1], z44[2]]}

```

Drop[rels, -2]

```
{EQ[z11[0], z12[0], z13[0], z14[0]],
  EQ[z13[0], z23[0]], EQ[z21[0], z22[0], z23[0], z24[0]],
  EQ[z14[0], z22[0], z34[0]], EQ[z31[0], z32[0], z33[0], z34[0]],
  EQ[z12[1], z13[2], z21[1], z22[2], z23[1], z24[2], z31[2], z34[1]],
  EQ[z11[1], z12[2], z13[1], z14[2], z23[2], z24[1], z33[2], z34[1]],
  EQ[z11[0], z33[0], z41[0]], EQ[z32[0], z42[0]],
  EQ[z21[2], z22[1], z31[1], z32[2], z33[1], z34[2], z41[2], z42[1]],
  EQ[z21[0], z31[0], z43[0]], EQ[z11[2], z14[1], z31[1], z32[2],
  z33[1], z34[2], z42[1], z43[2]], EQ[z12[0], z24[0], z44[0]],
  EQ[z11[1], z12[2], z13[1], z14[2], z22[1], z23[2], z41[2], z44[1]],
  EQ[z13[2], z14[1], z21[1], z22[2], z23[1], z24[2], z43[2], z44[1]]}
```

Length /@ rels

```
{4, 2, 4, 3, 4, 8, 8, 3, 2, 8, 3, 8, 3, 8, 8, 20, 24}
```

Plus @@ len /@ Length /@ rels

```
2 len[2] + 4 len[3] + 3 len[4] + 6 len[8] + len[20] + len[24]
```

Plus @@ len /@ Length /@ First /@ eqns

```
2 len[2] + 4 len[3] + 4 len[4] + 6 len[8]
```

pr[k_] := (k + 1) ~Mod~ 3; prpr[k_] := (k + 2) ~Mod~ 3;

```
Module[{t, at1, at2},
  For[s = Drop[rels, 0]; k = 1, k ≤ Length[s], ++k,
    Replace[s[[k]], {
      EQ[] => (s = Delete[s, k--]);
      EQ[z1_[p1_], z2_[p2_]] => (
        Print[StringForm["Doing bigon: k=`; s[[k]]=`; s=`", k, s[[k]], s]];
        t = Delete[s, k] //. {
          EQ[z1[(p1 + 1) ~Mod~ 3], z2[(p2 + 2) ~Mod~ 3]] → EQ[],
          EQ[z1[(p1 + 2) ~Mod~ 3], z2[(p2 + 1) ~Mod~ 3]] → EQ[]
        };
        If[Length[at1 = Position[t, z1]] ≠ 1 || Length[at2 = Position[t, z2]] ≠ 1,
          Print["Bad bigon", at1, " ", at2, "t is ", t];
          Continue[],
          at1 = at1[[1, 1]]; at2 = at2[[1, 1]];
          If[at1 == at2,
            Print["Equal ats", t[[at1]]];
            k = 0;
            s = t /. EQ[z1[p1], z2[p2]] → EQ[];
            Continue[],
            s = Append[Delete[t, {{at1}, {at2}}],
              EQ[t[[at1]], t[[at2]]] /. EQ[z1[p1], z2[p2]] → EQ[]];
            k = 0; Continue[]
          ]
        ]
      ],
  ),
```

```

EQ[z1_[p1_], z2_[p2_], z3_[p3_]] := (
  Print[StringForm["Doing triangle: k=``"; s[[k]]=``"; s=``", k, s[[k]], s]];
  {w1, w2} = Unique[{w, w}];
  t = Delete[s, k] //. {
    z3[p3] → EQ[w1[0], w2[0]],
    z2[p2] → EQ[w1[2], w2[1]], z1[p1] → EQ[w1[1], w2[2]],
    EQ[z3[p3 // pr], z3[p3 // prpr]] → EQ[-1, w1[2], w2[1], w1[1], w2[2]],
    EQ[z2[p2 // pr], z2[p2 // prpr]] → EQ[-1, w1[0], w2[0], w1[1], w2[2]],
    EQ[z1[p1 // pr], z1[p1 // prpr]] → EQ[-1, w1[0], w2[0], w1[2], w2[1]],
    EQ[z1[p1 // prpr], z2[p2 // pr]] → w1[0],
    EQ[z1[p1 // pr], z2[p2 // prpr]] → w2[0],
    EQ[z2[p2 // prpr], z3[p3 // pr]] → w1[1],
    EQ[z3[p3 // pr], z1[p1 // prpr]] → w2[1],
    EQ[z3[p3 // prpr], z1[p1 // pr]] → w1[2],
    EQ[z2[p2 // pr], z3[p3 // prpr]] → w2[2],
    EQ[z_ [0], z_ [1], z_ [2]] → EQ[-1], EQ[-1, -1] → EQ[]
  };
  If[! FreeQ[t, z1 | z2 | z3],
    Print[StringForm["Failed triangle; t=``"; problems at ``",
      t, Select[t, ! FreeQ[#, z1 | z2 | z3] &]]]; Break[],
    s = t; k = 0
  ];
  Continue[]
)
]; s
]

Doing bigon: k=2; s[[k]]=EQ[z13[0], z23[0]];
s={EQ[z11[0], z12[0], z13[0], z14[0]], EQ[z13[0], z23[0]], EQ[z21[0], z22[0], z23[0], z24[0]],
EQ[z14[0], z22[0], z34[0]], EQ[z31[0], z32[0], z33[0], z34[0]], EQ[z12[1], z13[2], z21[1], z22[
2], z23[1], z24[2], z31[2], z34[1]], EQ[z11[1], z12[2], z13[1], z14[2], z23[2], z24[1], z33[2
], z34[1]], EQ[z11[0], z33[0], z41[0]], EQ[z32[0], z42[0]], EQ[z21[2], z22[1], z31[1], z32[2]
, z33[1], z34[2], z41[2], z42[1]], EQ[z21[0], z31[0], z43[0]], EQ[z11[2], z14[1], z31[1], z32
[2], z33[1], z34[2], z42[1], z43[2]], EQ[z12[0], z24[0], z44[0]], EQ[z11[1], z12[2], z13[1],
z14[2], z22[1], z23[2], z41[2], z44[1]], EQ[z13[2], z14[1], z21[1], z22[2], z23[1], z24[2],
z43[2], z44[1]], EQ[z11[2], z12[1], z21[2], z24[1], z31[2], z32[1], z32[1], z33[2], z41[0],
z41[1], z41[1], z42[0], z42[2], z42[2], z43[0], z43[1], z43[1], z44[0], z44[2], z44[2]], EQ[
z12[2], z13[0], z13[1], z13[2], z14[0], z14[1], z22[2], z23[0], z23[1], z23[2], z24[0], z24[1]
, z31[1], z33[0], z33[2], z34[0], z34[1], z34[2], z41[1], z43[0], z43[2], z44[0], z44[1], z44[
2]]}

```

Bad bigon{{1, 3, 0}, {16, 2, 0}} at2\$3666t is {EQ[z11[0], z12[0], z13[0], z14[0]],
 EQ[z21[0], z22[0], z23[0], z24[0]], EQ[z14[0], z22[0], z34[0]],
 EQ[z31[0], z32[0], z33[0], z34[0]], EQ[z12[1], z21[1], z22[2], z24[2], z31[2], z34[1]],
 EQ[z11[1], z12[2], z14[2], z24[1], z33[2], z34[1]], EQ[z11[0], z33[0], z41[0]],
 EQ[z32[0], z42[0]], EQ[z21[2], z22[1], z31[1], z32[2], z33[1], z34[2], z41[2], z42[1]],
 EQ[z21[0], z31[0], z43[0]], EQ[z11[2], z14[1], z31[1], z32[2], z33[1], z34[2], z42[1], z43[2]],
 EQ[z12[0], z24[0], z44[0]], EQ[z11[1], z12[2], z14[2], z22[1], z41[2], z44[1]],
 EQ[z14[1], z21[1], z22[2], z24[2], z43[2], z44[1]],
 EQ[z11[2], z12[1], z21[2], z24[1], z31[2], z32[1], z32[1], z33[2], z41[0], z41[1],
 z41[1], z42[0], z42[2], z42[2], z43[0], z43[1], z43[1], z44[0], z44[2], z44[2]],
 EQ[z12[2], z13[0], z14[0], z14[1], z22[2], z23[0], z24[0], z24[1], z31[1], z33[0],
 z33[2], z34[0], z34[1], z34[2], z41[1], z43[0], z43[2], z44[0], z44[1], z44[2]]}

Doing triangle: k=4; s[[k]]=EQ[z14[0], z22[0], z34[0]];
 s={EQ[z11[0], z12[0], z13[0], z14[0]], EQ[z13[0], z23[0]], EQ[z21[0], z22[0], z23[0], z24[0]],
 EQ[z14[0], z22[0], z34[0]], EQ[z31[0], z32[0], z33[0], z34[0]], EQ[z12[1], z13[2], z21[1], z22[2],
 z23[1], z24[2], z31[2], z34[1]], EQ[z11[1], z12[2], z13[1], z14[2], z23[2], z24[1], z33[2],
 z34[1]], EQ[z11[0], z33[0], z41[0]], EQ[z32[0], z42[0]], EQ[z21[2], z22[1], z31[1], z32[2],
 z33[1], z34[2], z41[2], z42[1]], EQ[z21[0], z31[0], z43[0]], EQ[z11[2], z14[1], z31[1], z32[2],
 z33[1], z34[2], z42[1], z43[2]], EQ[z12[0], z24[0], z44[0]], EQ[z11[1], z12[2], z13[1],
 z14[2], z22[1], z23[2], z41[2], z44[1]], EQ[z13[2], z14[1], z21[1], z22[2], z23[1], z24[2],
 z43[2], z44[1]], EQ[z11[2], z12[1], z21[2], z24[1], z31[2], z32[1], z32[1], z33[2], z41[0],
 z41[1], z41[1], z42[0], z42[2], z42[2], z43[0], z43[1], z43[1], z44[0], z44[2], z44[2]], EQ[z12[2],
 z13[0], z13[1], z13[2], z14[0], z14[1], z22[2], z23[0], z23[1], z23[2], z24[0], z24[1],
 z31[1], z33[0], z33[2], z34[0], z34[1], z34[2], z41[1], z43[0], z43[2], z44[0], z44[1], z44[2]]}

Doing bigon: k=2; s[[k]]=EQ[z13[0], z23[0]];
 s={EQ[w\$3667[1], w\$3668[2], z11[0], z12[0], z13[0]], EQ[z13[0], z23[0]], EQ[w\$3667[2], w\$3668[1],
 z21[0], z23[0], z24[0]], EQ[w\$3667[0], w\$3668[0], z31[0], z32[0], z33[0]], EQ[w\$3667[1],
 z12[1], z13[2], z21[1], z23[1], z24[2], z31[2]], EQ[w\$3668[1], z11[1], z12[2], z13[1], z23[2],
 z24[1], z33[2]], EQ[z11[0], z33[0], z41[0]], EQ[z32[0], z42[0]], EQ[w\$3668[2], z21[2], z31[1],
 z32[2], z33[1], z41[2], z42[1]], EQ[z21[0], z31[0], z43[0]], EQ[w\$3667[2], z11[2], z31[1],
 z32[2], z33[1], z42[1], z43[2]], EQ[z12[0], z24[0], z44[0]], EQ[w\$3667[0], z11[1], z12[2], z13[1],
 z23[2], z41[2], z44[1]], EQ[w\$3668[0], z13[2], z21[1], z23[1], z24[2], z43[2], z44[1]],
 EQ[z11[2], z12[1], z21[2], z24[1], z31[2], z32[1], z32[1], z33[2], z41[0], z41[1], z41[1], z42[0],
 z42[2], z42[2], z43[0], z43[1], z43[1], z44[0], z44[2], z44[2]], EQ[w\$3667[1], w\$3668[0],
 w\$3668[2], z12[2], z24[0], z24[1], z31[1], z33[0], z33[2], z41[1], z43[0], z43[2]]}

Doing triangle: k=4; s[[k]]=EQ[z11[0], z33[0], z41[0]];
 s={EQ[w\$3667[0], w\$3668[0], z31[0], z32[0], z33[0]], EQ[w\$3667[1], z12[1], z21[1], z24[2], z31[2],
 EQ[w\$3668[1], z11[1], z12[2], z24[1], z33[2]], EQ[z11[0], z33[0], z41[0]], EQ[z32[0], z42[0]],
 EQ[w\$3668[2], z21[2], z31[1], z32[2], z33[1], z41[2], z42[1]], EQ[z21[0], z31[0], z43[0]],
 EQ[w\$3667[2], z11[2], z31[1], z32[2], z33[1], z42[1], z43[2]], EQ[z12[0], z24[0], z44[0]],
 EQ[w\$3667[0], z11[1], z12[2], z41[2], z44[1]], EQ[w\$3668[0], z21[1], z24[2], z43[2], z44[1]],
 EQ[z11[2], z12[1], z21[2], z24[1], z31[2], z32[1], z32[1], z33[2], z41[0], z41[1], z41[1], z42[0],
 z42[2], z42[2], z43[0], z43[1], z43[1], z44[0], z44[2], z44[2]], EQ[w\$3667[1], w\$3668[0],
 w\$3668[2], z12[2], z24[0], z24[1], z31[1], z33[0], z33[2], z41[1], z43[0], z43[2]],
 EQ[w\$3667[1], w\$3667[2], w\$3668[1], w\$3668[2], z11[0], z12[0], z21[0], z24[0]]}

Doing bigon: $k=4$; $s[[k]]=EQ[z32[0], z42[0]]$;

$$s = \{EQ[w\$3667[0], w\$3668[0], w\$3669[2], w\$3670[1], z31[0], z32[0]], EQ[w\$3667[1], z12[1], z21[1], z24[2], z31[2]], EQ[w\$3668[1], w\$3670[0], z12[2], z24[1]], EQ[z32[0], z42[0]], EQ[w\$3668[2], w\$3670[2], z21[2], z31[1], z32[2], z42[1]], EQ[z21[0], z31[0], z43[0]], EQ[w\$3667[2], w\$3669[0], z31[1], z32[2], z42[1], z43[2]], EQ[z12[0], z24[0], z44[0]], EQ[w\$3667[0], w\$3669[2], z12[2], z44[1]], EQ[w\$3668[0], z21[1], z24[2], z43[2], z44[1]], EQ[w\$3669[0], w\$3669[1], w\$3670[0], w\$3670[1], z12[1], z21[2], z24[1], z31[2], z32[1], z32[1], z42[0], z42[2], z42[2], z43[0], z43[1], z43[1], z44[0], z44[2], z44[2]], EQ[w\$3667[1], w\$3668[0], w\$3668[2], w\$3669[1], w\$3669[2], w\$3670[1], z12[2], z24[0], z24[1], z31[1], z43[0], z43[2]], EQ[w\$3667[1], w\$3667[2], w\$3668[1], w\$3668[2], w\$3669[1], w\$3670[2], z12[0], z21[0], z24[0]]\}$$

Doing triangle: $k=4$; $s[[k]]=EQ[z21[0], z31[0], z43[0]]$;

$$s = \{EQ[w\$3667[1], z12[1], z21[1], z24[2], z31[2]], EQ[w\$3668[1], w\$3670[0], z12[2], z24[1]], EQ[w\$3668[2], w\$3670[2], z21[2], z31[1]], EQ[z21[0], z31[0], z43[0]], EQ[w\$3667[2], w\$3669[0], z31[1], z43[2]], EQ[z12[0], z24[0], z44[0]], EQ[w\$3667[0], w\$3669[2], z12[2], z44[1]], EQ[w\$3668[0], z21[1], z24[2], z43[2], z44[1]], EQ[w\$3667[1], w\$3668[0], w\$3668[2], w\$3669[1], w\$3669[2], w\$3670[1], z12[2], z24[0], z24[1], z31[1], z43[0], z43[2]], EQ[w\$3667[1], w\$3667[2], w\$3668[1], w\$3668[2], w\$3669[1], w\$3670[2], z12[0], z21[0], z24[0]], EQ[w\$3667[0], w\$3668[0], w\$3669[0], w\$3669[1], w\$3669[2], w\$3670[0], w\$3670[1], w\$3670[1], z12[1], z21[2], z24[1], z31[0], z31[2], z43[0], z43[1], z43[1], z44[0], z44[2], z44[2]]\}$$

Doing triangle: $k=3$; $s[[k]]=EQ[w\$3668[2], w\$3670[2], w\$3671[0]]$;

$$s = \{EQ[w\$3667[1], w\$3672[0], z12[1], z24[2]], EQ[w\$3668[1], w\$3670[0], z12[2], z24[1]], EQ[w\$3668[2], w\$3670[2], w\$3671[0]], EQ[w\$3667[2], w\$3669[0], w\$3672[2]], EQ[z12[0], z24[0], z44[0]], EQ[w\$3667[0], w\$3669[2], z12[2], z44[1]], EQ[w\$3668[0], w\$3671[2], z24[2], z44[1]], EQ[w\$3667[1], w\$3668[0], w\$3668[2], w\$3669[1], w\$3669[2], w\$3670[1], w\$3671[0], w\$3672[0], w\$3672[2], z12[2], z24[0], z24[1]], EQ[w\$3667[1], w\$3667[2], w\$3668[1], w\$3668[2], w\$3669[1], w\$3670[2], w\$3671[1], w\$3672[2], z12[0], z24[0]], EQ[w\$3667[0], w\$3668[0], w\$3670[0], w\$3670[1], w\$3670[1], w\$3672[0], w\$3672[1], w\$3672[1], z12[1], z24[1], z44[0], z44[2], z44[2]]\}$$

Doing triangle: $k=2$; $s[[k]]=EQ[w\$3673[0], z12[2], z24[1]]$;

$$s = \{EQ[w\$3667[1], w\$3672[0], z12[1], z24[2]], EQ[w\$3673[0], z12[2], z24[1]], EQ[w\$3667[2], w\$3669[0], w\$3672[2]], EQ[z12[0], z24[0], z44[0]], EQ[w\$3667[0], w\$3669[2], z12[2], z44[1]], EQ[w\$3673[2], z24[2], z44[1]], EQ[w\$3667[1], w\$3669[1], w\$3669[2], w\$3672[0], w\$3672[2], w\$3673[0], w\$3673[1], w\$3674[0], w\$3674[0], w\$3674[0], w\$3674[2], z12[2], z24[0], z24[1]], EQ[w\$3667[1], w\$3667[2], w\$3669[1], w\$3672[2], w\$3673[1], w\$3673[2], w\$3674[1], w\$3674[1], w\$3674[2], z12[0], z24[0]], EQ[-1, w\$3667[0], w\$3672[0], w\$3672[1], w\$3672[1], w\$3673[0], w\$3673[1], w\$3674[0], w\$3674[0], w\$3674[2], z12[1], z24[1], z44[0], z44[2], z44[2]]\}$$

Doing triangle: $k=1$; $s[[k]]=EQ[w\$3667[1], w\$3672[0], w\$3675[1]]$;

$$s = \{EQ[w\$3667[1], w\$3672[0], w\$3675[1]], EQ[w\$3667[2], w\$3669[0], w\$3672[2]], EQ[w\$3676[2], z44[0]], EQ[w\$3667[0], w\$3669[2], w\$3675[2], w\$3676[1], z44[1]], EQ[w\$3676[1], z44[1]], EQ[w\$3667[1], w\$3669[1], w\$3669[2], w\$3672[0], w\$3672[2], w\$3674[0], w\$3674[0], w\$3674[2], w\$3675[2]], EQ[w\$3667[1], w\$3667[2], w\$3669[1], w\$3672[2], w\$3674[1], w\$3674[1], w\$3674[2], w\$3675[0], w\$3675[2]], EQ[-1, w\$3667[0], w\$3672[0], w\$3672[1], w\$3672[1], w\$3674[0], w\$3674[0], w\$3674[2], w\$3675[0], w\$3675[1], w\$3676[0], w\$3676[0], w\$3676[2], z44[0], z44[2], z44[2]]\}$$

```

Doing bigon: k=1; s[[k]]=EQ[w$3669[0], w$3678[0]];
s={EQ[w$3669[0], w$3678[0]], EQ[w$3676[2], z44[0]], EQ[w$3669[2], w$3676[1], w$3678[1], z44[1]]
, EQ[w$3676[1], z44[1]], EQ[w$3669[1], w$3669[2], w$3674[0], w$3674[0], w$3674[2], w$3677[1],
w$3677[1], w$3677[2], w$3678[1], w$3678[2]], EQ[w$3669[1], w$3674[1], w$3674[1], w$3674[2],
w$3677[1], w$3677[1], w$3677[2], w$3678[2]], EQ[w$3674[0], w$3674[0], w$3674[2], w$3676[0],
w$3676[0], w$3676[2], w$3677[0], w$3677[0], w$3677[2], z44[0], z44[2], z44[2]]}

Bad bigon{} 10t is {EQ[w$3676[2], z44[0]], EQ[w$3676[1], z44[1]], EQ[w$3676[1], z44[1]],
EQ[w$3674[0], w$3674[0], w$3674[2], w$3677[1], w$3677[1], w$3677[2]],
EQ[w$3674[1], w$3674[1], w$3674[2], w$3677[1], w$3677[1], w$3677[2]],
EQ[w$3674[0], w$3674[0], w$3674[2], w$3676[0], w$3676[0],
w$3676[2], w$3677[0], w$3677[0], w$3677[2], z44[0], z44[2], z44[2]]}

Doing bigon: k=2; s[[k]]=EQ[w$3676[2], z44[0]];
s={EQ[w$3669[0], w$3678[0]], EQ[w$3676[2], z44[0]], EQ[w$3669[2], w$3676[1], w$3678[1], z44[1]]
, EQ[w$3676[1], z44[1]], EQ[w$3669[1], w$3669[2], w$3674[0], w$3674[0], w$3674[2], w$3677[1],
w$3677[1], w$3677[2], w$3678[1], w$3678[2]], EQ[w$3669[1], w$3674[1], w$3674[1], w$3674[2],
w$3677[1], w$3677[1], w$3677[2], w$3678[2]], EQ[w$3674[0], w$3674[0], w$3674[2], w$3676[0],
w$3676[0], w$3676[2], w$3677[0], w$3677[0], w$3677[2], z44[0], z44[2], z44[2]]}

Equal atsEQ[w$3674[0], w$3674[0], w$3674[2], w$3676[2], w$3677[0], w$3677[0], w$3677[2], z44[0]]

Doing bigon: k=1; s[[k]]=EQ[w$3669[0], w$3678[0]];
s={EQ[w$3669[0], w$3678[0]], EQ[w$3669[2], w$3678[1]], EQ[], EQ[w$3669[1], w$3669[2], w$3674[0]
, w$3674[0], w$3674[2], w$3677[1], w$3677[1], w$3677[2], w$3678[1], w$3678[2]], EQ[w$3669[1],
w$3674[1], w$3674[1], w$3674[2], w$3677[1], w$3677[1], w$3677[2], w$3678[2]], EQ[w$3674[0],
w$3674[0], w$3674[2], w$3677[0], w$3677[0], w$3677[2]]}

Bad bigon{} 6t is
{EQ[], EQ[], EQ[w$3674[0], w$3674[0], w$3674[2], w$3677[1], w$3677[1], w$3677[2]],
EQ[w$3674[1], w$3674[1], w$3674[2], w$3677[1], w$3677[1], w$3677[2]],
EQ[w$3674[0], w$3674[0], w$3674[2], w$3677[0], w$3677[0], w$3677[2]]}

Doing bigon: k=2; s[[k]]=EQ[w$3669[2], w$3678[1]];
s={EQ[w$3669[0], w$3678[0]], EQ[w$3669[2], w$3678[1]], EQ[], EQ[w$3669[1], w$3669[2], w$3674[0]
, w$3674[0], w$3674[2], w$3677[1], w$3677[1], w$3677[2], w$3678[1], w$3678[2]], EQ[w$3669[1],
w$3674[1], w$3674[1], w$3674[2], w$3677[1], w$3677[1], w$3677[2], w$3678[2]], EQ[w$3674[0],
w$3674[0], w$3674[2], w$3677[0], w$3677[0], w$3677[2]]}

Equal ats
EQ[w$3669[2], w$3674[0], w$3674[0], w$3674[2], w$3677[1], w$3677[1], w$3677[2], w$3678[1]]
{EQ[w$3674[0], w$3674[0], w$3674[2], w$3677[1], w$3677[1], w$3677[2]],
EQ[w$3674[1], w$3674[1], w$3674[2], w$3677[1], w$3677[1], w$3677[2]],
EQ[w$3674[0], w$3674[0], w$3674[2], w$3677[0], w$3677[0], w$3677[2]]}

Length /@ s
{6, 6, 6}

```



```

eqns = (Times @@ # == 1) & /@ Rest[Reverse[s]] /.
  {w_[0] => w, w_[1] => 1 - 1/w, w_[2] => 1/(1 - w)}
{-
  (1 - 1/z11) z13 (1 - 1/z21) z23 (1 - 1/z31) z33 (1 - 1/z41) z43 (1 - 1/z51) z53
  (1 - z13) (1 - z23) (1 - z33) (1 - z43) (1 - z53)
  == 1,
  w$1696 w$1697 w$1698 w$1699 (1 - 1/z51)^2 z51 (1 - 1/z53)^2 z53
  (1 - w$1696) (1 - w$1697) (1 - w$1698) (1 - w$1699) (1 - z13) (1 - z21)^2 (1 - z43)
  w$1697 (1 - 1/w$1699) (1 - 1/z21) (1 - 1/z23) == 1, (1 - 1/w$1698) (1 - 1/z41) (1 - 1/z43) == 1,
  (1 - z11) (1 - z53) (1 - z33) (1 - z51)
  (1 - 1/w$1696) w$1698 (1 - 1/z21) (1 - 1/z23) == 1, z21 z53 == 1,
  (1 - z41) (1 - z53)
  (1 - 1/w$1697) (1 - 1/z11) (1 - 1/z13) == 1, z13 z31 z43 z51 == 1, w$1699 (1 - 1/z41) (1 - 1/z43) == 1,
  (1 - z33) (1 - z51) (1 - z23) (1 - z31)
  (1 - 1/z31) (1 - 1/z33) == 1, (1 - 1/z31) (1 - 1/z33) == 1, z11 z23 z33 z41 == 1,
  (1 - z11) (1 - z43) (1 - z13) (1 - z41)
  w$1696 (1 - 1/z11) (1 - 1/z13) == 1, (1 - 1/w$1697) (1 - 1/w$1698) z21 z23 == 1}
  (1 - z23) (1 - z31) (1 - w$1696) (1 - w$1699)
}

```

```
vars = Union[Cases[eqns, _Symbol, Infinity]]
```

```
{w$1696, w$1697, w$1698, w$1699, z11, z13, z21, z23, z31, z33, z41, z43, z51, z53}
```

```
{
  sol = FindRoot @@
    Join[{eqns}, ({#, Exp[Random[] Pi I/2]} & /@ vars), {MaxIterations -> 1000}],
  Max[Abs[(First /@ eqns) /. sol) - 1]],
  Plus @@ (Vol /@ (vars /. sol))
}
```

FindRoot::cvnwt: Newton's method failed to converge to the prescribed accuracy after 1000 iterations.

```
{
  {w$1696 -> 1.68063 - 1.07329 i, w$1697 -> 657 662. + 4.33095 x 10^6 i,
  w$1698 -> 653 890. + 772 596. i, w$1699 -> 1.68064 - 1.07329 i,
  z11 -> -890 189. - 6.29014 x 10^6 i, z13 -> 7.13397 x 10^6 + 3.22571 x 10^6 i,
  z21 -> 0.999998 - 6.76706 x 10^-7 i, z23 -> -0.688693 - 1.46105 i,
  z31 -> 0.745324 + 0.855922 i, z33 -> 1.05271 x 10^-14 - 3.20773 x 10^-15 i,
  z41 -> -936 924. - 1.12995 x 10^6 i, z43 -> 1.79445 x 10^6 - 229 187. i,
  z51 -> 4.20843 x 10^-15 - 9.39763 x 10^-15 i, z53 -> 1. - 5.30607 x 10^-7 i}, 5.23111, -0.98885}
```

```
sols = vars /. NSolve[eqns, vars]
```

\$Aborted

N[sols]

```
{ {0.5+0.866025 i, 0.5+0.866025 i}, {0.5-0.866025 i, 0.5+0.866025 i},
  {0.5+0.866025 i, 0.5-0.866025 i}, {0.5-0.866025 i, 0.5-0.866025 i},
  {-1.61803, -1.61803}, {0.618034, -1.61803}, {-1.61803, 0.618034}, {0.618034, 0.618034} }
```

(Plus @@ Vol /@ #) & /@ sols

```
NIntegrate::ncvb: NIntegrate failed to converge to prescribed accuracy after 7 recursive bisections in u
near u = 3.141592647481223`.
```

```
NIntegrate::ncvb: NIntegrate failed to converge to prescribed accuracy after 7 recursive bisections in u
near u = 3.141592647481223`.
```

```
NIntegrate::ncvb: NIntegrate failed to converge to prescribed accuracy after 7 recursive bisections in u
near u = 3.141592647481223`.
```

```
General::stop: Further output of NIntegrate::ncvb will be suppressed during this calculation.
```

```
{2.02988, 0., 0., -2.02988, -7.41074 × 10-15,
 -7.41074 × 10-15, -7.41074 × 10-15, -7.41074 × 10-15}
```

?? Solve

Solve[eqns, vars] attempts to solve an equation or set of equations for the variables vars. Solve[eqns, vars, elims] attempts to solve the equations for vars, eliminating the variables elims. More...

```
Attributes[Solve] = {Protected}
```

```
Options[Solve] = {InverseFunctions -> Automatic, MakeRules -> False, Method -> 3,
  Mode -> Generic, Sort -> True, VerifySolutions -> Automatic, WorkingPrecision -> ∞}
```

eqns ~Join~ ((# > 0) & /@ vars)

$$\left\{ \frac{\left(1 - \frac{1}{w_{173}}\right)^2 \left(1 - \frac{1}{w_{176}}\right)^2}{(1 - w_{173})(1 - w_{176})} == 1, \frac{w_{173}^2 \left(1 - \frac{1}{w_{176}}\right)^2}{(1 - w_{173})(1 - w_{176})} == 1, w_{173} > 0, w_{176} > 0 \right\}$$

Solve[eqns ~Join~ ((# > 0) & /@ vars), vars]

```
Solve::eqf: w$176 > 0 is not a well-formed equation.
```

```
Solve[ { {  $\frac{\left(1 - \frac{1}{w_{173}}\right)^2 \left(1 - \frac{1}{w_{176}}\right)^2}{(1 - w_{173})(1 - w_{176})} == 1, \frac{w_{173}^2 \left(1 - \frac{1}{w_{176}}\right)^2}{(1 - w_{173})(1 - w_{176})} == 1, w_{173} > 0, w_{176} > 0$  },
  {w$173, w$176} ]
```