

Expected volume for 5_2: 2.8281220883.

```
<< \ c : / drorbn / projects / KAtlas / KnotTheory.m
```

```
Loading KnotTheory`...
```

```
<< \ c : / drorbn / projects / KAtlas / KnotTheoryData.m
```

```
Loading KnotTheoryData.m...
```

```
Lob[t_] := -NIntegrate[Log[Abs[2 Sin[u]]], {u, 0, t}];
```

```
Vol[z_] := Plus @@ (Lob[Arg[#]] & /@ {z, 1 - 1/z, 1 / (1 - z)})
```

```
2 Vol[omega = 1/2 + Sqrt[3]/2 * I]
```

```
2.02988
```

```
Pachner[z_, w_] := {Vol[z], Vol[w], Vol[z] + Vol[w] +  
  Vol[1 / (z * w)] - Vol[z (1 - w) / (1 - z * w)] - Vol[w (1 - z) / (1 - z * w)]}
```

```
Pachner[Random[Real, {-1, 1}] + I * Random[Real, {-1, 1}],
```

```
  Random[Real, {-1, 1}] + I * Random[Real, {-1, 1}]]
```

```
{-0.842242, 0.0317351, -3.99565 × 10-11}
```

```
SetAttributes[p, Orderless]; SetAttributes[{EQs, EQ}, {Orderless, Flat}]
```

```

Rels[K_] := Rels[PD[K]];
Rels[pd_PD] := Module[
  {n, dt, z, e, u, v, j, im4, dm4, w, zz,
   xx, yy, ab, merge, rels, x, y, a, b, c, d, d1, d2, d3, eq},
  im4[a_] := a + 1 /. 5 -> 1; dm4[a_] := a - 1 /. 0 -> 4;
  n = Length[pd];
  dt = EQs @@ Flatten[
    {
      Table[
        t[z[v, j], 3, {1, 2, 4}] ~ e ~ t[z[v, im4@j], 4, {2, 1, 3}],
        {v, n}, {j, 4}
      ],
      Table[e @@ Position[pd, i], {i, 2 n}] /. e[x_, y_] -> {e[x, y], e[y, x]} /. {
        e[{v_, a_}, {u_, b_}] ->
          t[z[v, dm4@a], 2, {1, 3, 4}] ~ e ~ t[z[u, b], 1, {2, 3, 4}] /; OddQ[a + b],
        e[{v_, a_}, {u_, b_}] -> t[z[v, dm4@a], 2, {1, 3, 4}] ~ e ~
          t[z[u, b], 1, {2, 4, 3}] /; EvenQ[a + b]
      }
    ] /. e[t[z_, a_, {c1_, c2_, c3_}], t[w_, b_, {d1_, d2_, d3_}]] -> {
      t[z, a, c1] ~ p ~ t[w, b, d1], t[z, a, c2] ~ p ~ t[w, b, d2], t[z, a, c3] ~ p ~ t[w, b, d3]
    }
  ] /.
  t[z[v_, j_], a_, b_] -> t[z[v, j]] ~ Join ~ (t[a, b] /. {3 -> 4, 4 -> 3}) /; EvenQ[j];
  rels = List @@ Union[
    dt /. p[t[z_, a_, b_], t[w_, c_, d_]] ->
      EQs[t[z, a, b] ~ to ~ t[w, d, c], t[w, c, d] ~ to ~ t[z, b, a]] // . (
        EQs[xx_. * (t[x_, ab_] ~ to ~ t[y_, cd_]), zz_. *
          (t[y_, cd_] ~ to ~ t[z_, ef_])] x
      )
    -> EQs[xx * y[cd] * zz * (t[x, ab] ~ to ~ t[z, ef])]
      ) /. (t[z_, ab_] ~ to ~ t[z_, ab_]) -> z[ab] /.
      (w_z)[ab_] -> Switch[Sort[{ab}],
        {1, 2} | {3, 4}, w[0],
        {1, 3} | {2, 4}, w[1],
        {1, 4} | {2, 3}, w[2]
      ]
  ] /. eq_Times -> (EQ @@ eq);
  merge = Position[Count[#, z[n, _][__]] & /@ rels, 4];
  rels = Append[Delete[rels, merge], EQ @@ rels[[Join @@ merge]]];
  Append[rels, z1 ~ EQ ~ (EQ @@ Product[
    z[k, 4][0] * z[k, 4][1] * z[k, 3][0] * z[k, 3][2] *
      If[pd[[k, 4]] > pd[[k, 2]] || pd[[k, 2]] - pd[[k, 4]] > 1,
        z[k, 1][1] * z[k, 4][2],
        z[k, 3][1] * z[k, 2][2]
      ],
    {k, n}
  ])] // . {EQ[z_[0], z_[1], z_[2]] -> EQ[-1], EQ[-1, -1] -> EQ[]}
  ] /. z[i_, j_][k_] -> m[ToExpression["z" <> ToString[i] <> ToString[j]], k]
]

```

General::spell1: Possible spelling error: new symbol name "Rels" is similar to existing symbol "Reals".

General::spell1: Possible spelling error: new symbol name "rels" is similar to existing symbol "Rels".

rels = Rels[Knot[5, 2]]

```
{EQ[m[z11, 0], m[z12, 0], m[z13, 0], m[z14, 0]],
EQ[m[z21, 0], m[z22, 0], m[z23, 0], m[z24, 0]],
EQ[m[z14, 0], m[z34, 0]], EQ[m[z31, 0], m[z32, 0], m[z33, 0], m[z34, 0]],
EQ[m[z11, 1], m[z12, 2], m[z13, 1], m[z14, 2], m[z22, 1], m[z23, 2], m[z31, 2], m[z34, 1]],
EQ[m[z11, 0], m[z23, 0], m[z33, 0], m[z41, 0]], EQ[m[z32, 0], m[z42, 0]],
EQ[m[z13, 2], m[z14, 1], m[z31, 1], m[z32, 2], m[z33, 1], m[z34, 2], m[z41, 2], m[z42, 1]],
EQ[m[z11, 2], m[z14, 1], m[z31, 1], m[z32, 2], m[z33, 1], m[z34, 2], m[z42, 1], m[z43, 2]],
EQ[m[z41, 0], m[z42, 0], m[z43, 0], m[z44, 0]],
EQ[m[z23, 2], m[z24, 1], m[z31, 2], m[z32, 1], m[z41, 1], m[z42, 2], m[z43, 1], m[z44, 2]],
EQ[m[z13, 0], m[z31, 0], m[z43, 0], m[z51, 0]], EQ[m[z12, 0], m[z22, 0], m[z52, 0]],
EQ[m[z11, 1], m[z12, 2], m[z13, 1], m[z14, 2], m[z33, 2], m[z34, 1], m[z51, 2], m[z52, 1]],
EQ[m[z21, 0], m[z53, 0]], EQ[m[z21, 1], m[z22, 2], m[z23, 1], m[z24, 2],
m[z41, 2], m[z44, 1], m[z52, 1], m[z53, 2]], EQ[m[z24, 0], m[z44, 0], m[z54, 0]],
EQ[m[z32, 1], m[z33, 2], m[z41, 1], m[z42, 2], m[z43, 1], m[z44, 2], m[z51, 2], m[z54, 1]],
EQ[m[z11, 2], m[z12, 1], m[z21, 1], m[z22, 2], m[z23, 1], m[z24, 2], m[z53, 2], m[z54, 1]],
EQ[m[z12, 1], m[z13, 2], m[z21, 2], m[z21, 2], m[z22, 1], m[z24, 1],
m[z43, 2], m[z44, 1], m[z51, 0], m[z51, 1], m[z51, 1], m[z52, 0], m[z52, 2],
m[z52, 2], m[z53, 0], m[z53, 1], m[z53, 1], m[z54, 0], m[z54, 2], m[z54, 2]],
EQ[-1, z1, m[z11, 1], m[z13, 0], m[z13, 2], m[z21, 1], m[z23, 0], m[z23, 2], m[z31, 1],
m[z33, 0], m[z33, 2], m[z41, 1], m[z43, 0], m[z43, 2], m[z51, 1], m[z53, 0], m[z53, 2]]}
```

rels = Delete[rels, Position[rels, z1][[1, 1]]]

```
{EQ[m[z11, 0], m[z12, 0], m[z13, 0], m[z14, 0]],
EQ[m[z21, 0], m[z22, 0], m[z23, 0], m[z24, 0]],
EQ[m[z14, 0], m[z34, 0]], EQ[m[z31, 0], m[z32, 0], m[z33, 0], m[z34, 0]],
EQ[m[z11, 1], m[z12, 2], m[z13, 1], m[z14, 2], m[z22, 1], m[z23, 2], m[z31, 2], m[z34, 1]],
EQ[m[z11, 0], m[z23, 0], m[z33, 0], m[z41, 0]], EQ[m[z32, 0], m[z42, 0]],
EQ[m[z13, 2], m[z14, 1], m[z31, 1], m[z32, 2], m[z33, 1], m[z34, 2], m[z41, 2], m[z42, 1]],
EQ[m[z11, 2], m[z14, 1], m[z31, 1], m[z32, 2], m[z33, 1], m[z34, 2], m[z42, 1], m[z43, 2]],
EQ[m[z41, 0], m[z42, 0], m[z43, 0], m[z44, 0]],
EQ[m[z23, 2], m[z24, 1], m[z31, 2], m[z32, 1], m[z41, 1], m[z42, 2], m[z43, 1], m[z44, 2]],
EQ[m[z13, 0], m[z31, 0], m[z43, 0], m[z51, 0]], EQ[m[z12, 0], m[z22, 0], m[z52, 0]],
EQ[m[z11, 1], m[z12, 2], m[z13, 1], m[z14, 2], m[z33, 2], m[z34, 1], m[z51, 2], m[z52, 1]],
EQ[m[z21, 0], m[z53, 0]], EQ[m[z21, 1], m[z22, 2], m[z23, 1], m[z24, 2],
m[z41, 2], m[z44, 1], m[z52, 1], m[z53, 2]], EQ[m[z24, 0], m[z44, 0], m[z54, 0]],
EQ[m[z32, 1], m[z33, 2], m[z41, 1], m[z42, 2], m[z43, 1], m[z44, 2], m[z51, 2], m[z54, 1]],
EQ[m[z11, 2], m[z12, 1], m[z21, 1], m[z22, 2], m[z23, 1], m[z24, 2], m[z53, 2], m[z54, 1]],
EQ[m[z12, 1], m[z13, 2], m[z21, 2], m[z21, 2], m[z22, 1], m[z24, 1],
m[z43, 2], m[z44, 1], m[z51, 0], m[z51, 1], m[z51, 1], m[z52, 0], m[z52, 2],
m[z52, 2], m[z53, 0], m[z53, 1], m[z53, 1], m[z54, 0], m[z54, 2], m[z54, 2]]}
```

Length /@ rels

```
{4, 4, 2, 4, 8, 4, 2, 8, 8, 4, 8, 4, 3, 8, 2, 8, 3, 8, 8, 20}
```

Plus @@ len /@ Length /@ rels

```
2 len[2] + 4 len[3] + 3 len[4] + 6 len[8] + len[20]
```

```

RandomPermutation[L_List] := Last /@ Sort[{Random[], #} & /@ L];
$Log = True;
m[z_, k1_][k2_] := m[z, (k1 + k2) ~Mod~ 3];
RemoveTriangle[s_, k_Integer] :=
  (Replace[s[[k]], EQ[m[Z1_, p1_], m[Z2_, p2_], m[Z3_, p3_]] => (
    $Log && Print[StringForm["Doing triangle: k=``"; s[[k]]=``"; s=``", k, s[[k]], s]];
    z1 = m[Z1, p1]; z2 = m[Z2, p2]; z3 = m[Z3, p3];
    {w1, w2} = m[#, 0] & /@ Unique[{w, w}];
    t = Delete[s, k] //. {
      z3 -> EQ[w1, w2], z2 -> EQ[w1[2], w2[1]], z1 -> EQ[w1[1], w2[2]],
      EQ[z3[1], z3[2]] -> EQ[-1, w1[2], w2[1], w1[1], w2[2]],
      EQ[z2[1], z2[2]] -> EQ[-1, w1, w2, w1[1], w2[2]],
      EQ[z1[1], z1[2]] -> EQ[-1, w1, w2, w1[2], w2[1]],
      EQ[z1[2], z2[1]] -> w1, EQ[z1[1], z2[2]] -> w2, EQ[z2[2], z3[1]] -> w1[1],
      EQ[z3[1], z1[2]] -> w2[1],
      EQ[z3[2], z1[1]] -> w1[2], EQ[z2[1], z3[2]] -> w2[2],
      EQ[m[z_, 0], m[z_, 1], m[z_, 2]] -> EQ[-1], EQ[-1, -1] -> EQ[]
    };
    If[FreeQ[t, z123 = First[z1] | First[z2] | First[z3]],
      Return[t],
      $Log && Print[StringForm[
        "Failed triangle; t=``"; problems at ``", t, Select[t, !FreeQ[#, z123] &]]]
    ]
  )]; Failed);
Red[rels_List] := Module[
  {s = rels, ks, k, t, at, at1, at2},
  Label[Start]; s = RandomPermutation[DeleteCases[s, EQ[]]];
  For[k = 1, k <= Length[s], ++k, Replace[s[[k]], EQ[m[Z1_, p1_], m[Z2_, p2_]] => (
    $Log && Print[StringForm["Doing bigon: k=``"; s[[k]]=``"; s=``", k, s[[k]], s]];
    z1 = m[Z1, p1]; z2 = m[Z2, p2];
    t =
      s //. {EQ[z1, z2] -> EQ[], EQ[z1[1], z2[2]] -> EQ[], EQ[z1[2], z2[1]] -> EQ[]};
    at = Position[t, # // First] & /@ {z1, z2};
    If[(Length /@ at) == {0, 0}, s = t; Goto[Start]];
    If[(Length /@ at) != {1, 1},
      $Log && Print[StringForm["Bad bigon: at=``"; t=``", at, t]],
      at1 = at[[1, 1, 1]]; at2 = at[[2, 1, 1]];
      s = If[at1 == at2,
        $Log && Print["Equal ats ", t[[at1]]]; t /. EQ[z1, z2] -> EQ[],
        Append[Delete[t, {{at1}, {at2}}],
          EQ[t[[at1]], t[[at2]]] /. EQ[z1, z2] -> EQ[]
        ]];
      Goto[Start]
    ]
  )];
  For[k = 1, k <= Length[s], ++k,
    If[Failed != (t = RemoveTriangle[s, k]), s = t; Goto[Start]]];
  s
]

```

```
{s = Red[rels], Length /@ s, Length[s]}
```

```
Doing bigon: k=6; s[[k]] = EQ[m[z14, 0], m[z34, 0]];
```

```
s = {EQ[m[z21, 0], m[z22, 0], m[z23, 0], m[z24, 0]], EQ[m[z11, 0], m[z23, 0], m[z33, 0], m[z41, 0]],
EQ[m[z12, 1], m[z13, 2], m[z21, 2], m[z21, 2], m[z22, 1], m[z24, 1], m[z43, 2], m[z44, 1], m[z51,
0], m[z51, 1], m[z51, 1], m[z52, 0], m[z52, 2], m[z52, 2], m[z53, 0], m[z53, 1], m[z53, 1], m[
z54, 0], m[z54, 2], m[z54, 2]], EQ[m[z41, 0], m[z42, 0], m[z43, 0], m[z44, 0]], EQ[m[z11, 2], m
[z14, 1], m[z31, 1], m[z32, 2], m[z33, 1], m[z34, 2], m[z42, 1], m[z43, 2]], EQ[m[z14, 0], m[z34
, 0]], EQ[m[z24, 0], m[z44, 0], m[z54, 0]], EQ[m[z13, 2], m[z14, 1], m[z31, 1], m[z32, 2], m[
z33, 1], m[z34, 2], m[z41, 2], m[z42, 1]], EQ[m[z11, 1], m[z12, 2], m[z13, 1], m[z14, 2], m[z22,
1], m[z23, 2], m[z31, 2], m[z34, 1]], EQ[m[z13, 0], m[z31, 0], m[z43, 0], m[z51, 0]], EQ[m[z11,
1], m[z12, 2], m[z13, 1], m[z14, 2], m[z33, 2], m[z34, 1], m[z51, 2], m[z52, 1]], EQ[m[z11, 2],
m[z12, 1], m[z21, 1], m[z22, 2], m[z23, 1], m[z24, 2], m[z53, 2], m[z54, 1]], EQ[m[z31, 0], m[z32
, 0], m[z33, 0], m[z34, 0]], EQ[m[z11, 0], m[z12, 0], m[z13, 0], m[z14, 0]], EQ[m[z32, 0], m[
z42, 0]], EQ[m[z21, 1], m[z22, 2], m[z23, 1], m[z24, 2], m[z41, 2], m[z44, 1], m[z52, 1], m[z53,
2]], EQ[m[z21, 0], m[z53, 0]], EQ[m[z23, 2], m[z24, 1], m[z31, 2], m[z32, 1], m[z41, 1], m[z42,
2], m[z43, 1], m[z44, 2]], EQ[m[z12, 0], m[z22, 0], m[z52, 0]], EQ[m[z32, 1], m[z33, 2], m[z41,
1], m[z42, 2], m[z43, 1], m[z44, 2], m[z51, 2], m[z54, 1]]}
```

```
Doing bigon: k=7; s[[k]] = EQ[m[z32, 0], m[z42, 0]];
```

```
s = {EQ[m[z21, 0], m[z22, 0], m[z23, 0], m[z24, 0]], EQ[m[z12, 0], m[z22, 0], m[z52, 0]], EQ[m[z11,
0], m[z12, 0], m[z13, 0], m[z31, 0], m[z32, 0], m[z33, 0]], EQ[m[z11, 2], m[z31, 1], m[z32, 2],
m[z33, 1], m[z42, 1], m[z43, 2]], EQ[m[z41, 0], m[z42, 0], m[z43, 0], m[z44, 0]], EQ[m[z13, 2], m
[z31, 1], m[z32, 2], m[z33, 1], m[z41, 2], m[z42, 1]], EQ[m[z32, 0], m[z42, 0]], EQ[m[z12, 1],
m[z13, 2], m[z21, 2], m[z21, 2], m[z22, 1], m[z24, 1], m[z43, 2], m[z44, 1], m[z51, 0], m[z51, 1]
, m[z51, 1], m[z52, 0], m[z52, 2], m[z52, 2], m[z53, 0], m[z53, 1], m[z53, 1], m[z54, 0], m[z54,
2], m[z54, 2]], EQ[m[z21, 0], m[z53, 0]], EQ[m[z11, 2], m[z12, 1], m[z21, 1], m[z22, 2], m[z23,
1], m[z24, 2], m[z53, 2], m[z54, 1]], EQ[m[z21, 1], m[z22, 2], m[z23, 1], m[z24, 2], m[z41, 2],
m[z44, 1], m[z52, 1], m[z53, 2]], EQ[m[z11, 1], m[z12, 2], m[z13, 1], m[z22, 1], m[z23, 2], m[z31
, 2]], EQ[m[z11, 1], m[z12, 2], m[z13, 1], m[z33, 2], m[z51, 2], m[z52, 1]], EQ[m[z24, 0], m[
z44, 0], m[z54, 0]], EQ[m[z11, 0], m[z23, 0], m[z33, 0], m[z41, 0]], EQ[m[z23, 2], m[z24, 1], m
[z31, 2], m[z32, 1], m[z41, 1], m[z42, 2], m[z43, 1], m[z44, 2]], EQ[m[z32, 1], m[z33, 2], m[z41
, 1], m[z42, 2], m[z43, 1], m[z44, 2], m[z51, 2], m[z54, 1]], EQ[m[z13, 0], m[z31, 0], m[z43, 0]
, m[z51, 0]]}
```

```
Doing bigon: k=1; s[[k]] = EQ[m[z21, 0], m[z53, 0]];
```

```
s = {EQ[m[z21, 0], m[z53, 0]], EQ[m[z24, 0], m[z44, 0], m[z54, 0]], EQ[m[z12, 0], m[z22, 0], m[z52,
0]], EQ[m[z23, 2], m[z24, 1], m[z31, 2], m[z41, 1], m[z43, 1], m[z44, 2]], EQ[m[z13, 2], m[z31,
1], m[z33, 1], m[z41, 2]], EQ[m[z13, 0], m[z31, 0], m[z43, 0], m[z51, 0]], EQ[m[z12, 1], m[z13,
2], m[z21, 2], m[z21, 2], m[z22, 1], m[z24, 1], m[z43, 2], m[z44, 1], m[z51, 0], m[z51, 1], m[
z51, 1], m[z52, 0], m[z52, 2], m[z52, 2], m[z53, 0], m[z53, 1], m[z53, 1], m[z54, 0], m[z54, 2],
m[z54, 2]], EQ[m[z21, 1], m[z22, 2], m[z23, 1], m[z24, 2], m[z41, 2], m[z44, 1], m[z52, 1], m[z53
, 2]], EQ[m[z11, 2], m[z12, 1], m[z21, 1], m[z22, 2], m[z23, 1], m[z24, 2], m[z53, 2], m[z54, 1]
], EQ[m[z11, 2], m[z31, 1], m[z33, 1], m[z43, 2]], EQ[m[z11, 0], m[z23, 0], m[z33, 0], m[z41, 0]
], EQ[m[z11, 1], m[z12, 2], m[z13, 1], m[z22, 1], m[z23, 2], m[z31, 2]], EQ[m[z11, 1], m[z12,
2], m[z13, 1], m[z33, 2], m[z51, 2], m[z52, 1]], EQ[m[z11, 0], m[z12, 0], m[z13, 0], m[z31, 0],
m[z33, 0], m[z41, 0], m[z43, 0], m[z44, 0]], EQ[m[z33, 2], m[z41, 1], m[z43, 1], m[z44, 2], m[z51
, 2], m[z54, 1]], EQ[m[z21, 0], m[z22, 0], m[z23, 0], m[z24, 0]]}
```

Doing triangle: k=7; s[[k]]=EQ[m[z12, 0], m[z22, 0], m[z52, 0]];

s={EQ[m[z12, 1], m[z13, 2], m[z22, 0], m[z22, 1], m[z23, 0], m[z24, 0], m[z24, 1], m[z43, 2], m[z44, 1], m[z51, 0], m[z51, 1], m[z51, 1], m[z52, 0], m[z52, 2], m[z52, 2], m[z54, 0], m[z54, 2], m[z54, 2]], EQ[m[z13, 2], m[z31, 1], m[z33, 1], m[z41, 2]], EQ[m[z11, 1], m[z12, 2], m[z13, 1], m[z33, 2], m[z51, 2], m[z52, 1]], EQ[m[z11, 1], m[z12, 2], m[z13, 1], m[z22, 1], m[z23, 2], m[z31, 2]], EQ[m[z33, 2], m[z41, 1], m[z43, 1], m[z44, 2], m[z51, 2], m[z54, 1]], EQ[m[z22, 2], m[z23, 1], m[z24, 2], m[z41, 2], m[z44, 1], m[z52, 1]], EQ[m[z12, 0], m[z22, 0], m[z52, 0]], EQ[m[z24, 0], m[z44, 0], m[z54, 0]], EQ[m[z23, 2], m[z24, 1], m[z31, 2], m[z41, 1], m[z43, 1], m[z44, 2]], EQ[m[z11, 0], m[z12, 0], m[z13, 0], m[z31, 0], m[z33, 0], m[z41, 0], m[z43, 0], m[z44, 0]], EQ[m[z13, 0], m[z31, 0], m[z43, 0], m[z51, 0]], EQ[m[z11, 0], m[z23, 0], m[z33, 0], m[z41, 0]], EQ[m[z11, 2], m[z12, 1], m[z22, 2], m[z23, 1], m[z24, 2], m[z54, 1]], EQ[m[z11, 2], m[z31, 1], m[z33, 1], m[z43, 2]]}

Doing triangle: k=5; s[[k]]=EQ[m[z24, 0], m[z44, 0], m[z54, 0]];

s={EQ[m[w\$53, 1], m[w\$54, 2], m[z11, 0], m[z13, 0], m[z31, 0], m[z33, 0], m[z41, 0], m[z43, 0], m[z44, 0]], EQ[m[z13, 2], m[z31, 1], m[z33, 1], m[z41, 2]], EQ[m[w\$53, 0], m[z11, 1], m[z13, 1], m[z23, 2], m[z31, 2]], EQ[m[z23, 2], m[z24, 1], m[z31, 2], m[z41, 1], m[z43, 1], m[z44, 2]], EQ[m[z24, 0], m[z44, 0], m[z54, 0]], EQ[m[z33, 2], m[z41, 1], m[z43, 1], m[z44, 2], m[z51, 2], m[z54, 1]], EQ[m[z11, 2], m[z31, 1], m[z33, 1], m[z43, 2]], EQ[m[w\$54, 1], m[z11, 1], m[z13, 1], m[z33, 2], m[z51, 2]], EQ[-1, m[w\$53, 0], m[w\$53, 2], m[w\$53, 2], m[z13, 2], m[z23, 0], m[z24, 0], m[z24, 1], m[z43, 2], m[z44, 1], m[z51, 0], m[z51, 1], m[z51, 1], m[z54, 0], m[z54, 2], m[z54, 2]], EQ[m[w\$53, 1], m[z23, 1], m[z24, 2], m[z41, 2], m[z44, 1]], EQ[m[z11, 0], m[z23, 0], m[z33, 0], m[z41, 0]], EQ[m[w\$54, 0], m[z11, 2], m[z23, 1], m[z24, 2], m[z54, 1]], EQ[m[z13, 0], m[z31, 0], m[z43, 0], m[z51, 0]]}

{EQ[m[z13, 0], m[z31, 0], m[z43, 0], m[z51, 0]], EQ[m[z11, 0], m[z23, 0], m[z33, 0], m[z41, 0]], EQ[m[z13, 2], m[z31, 1], m[z33, 1], m[z41, 2]], EQ[m[w\$53, 1], m[w\$55, 0], m[z23, 1], m[z41, 2]], EQ[m[w\$54, 0], m[w\$56, 1], m[z11, 2], m[z23, 1]], EQ[m[w\$54, 1], m[z11, 1], m[z13, 1], m[z33, 2], m[z51, 2]], EQ[m[w\$53, 0], m[z11, 1], m[z13, 1], m[z23, 2], m[z31, 2]], EQ[m[z11, 2], m[z31, 1], m[z33, 1], m[z43, 2]], EQ[m[w\$53, 0], m[w\$53, 2], m[w\$53, 2], m[w\$56, 0], m[w\$56, 2], m[w\$56, 2], m[z13, 2], m[z23, 0], m[z43, 2], m[z51, 0], m[z51, 1], m[z51, 1]], EQ[m[w\$56, 0], m[z23, 2], m[z31, 2], m[z41, 1], m[z43, 1]], EQ[m[w\$53, 1], m[w\$54, 2], m[w\$55, 2], m[w\$56, 1], m[z11, 0], m[z13, 0], m[z31, 0], m[z33, 0], m[z41, 0], m[z43, 0]], EQ[m[w\$55, 1], m[z33, 2], m[z41, 1], m[z43, 1], m[z51, 2]]}, {4, 4, 4, 4, 4, 5, 5, 4, 12, 5, 10, 5}, 12}

Length /@ rels

{4, 2, 4, 3, 4, 8, 8, 3, 2, 8, 3, 8, 3, 8, 8, 20}

RandomPermutation[

Flatten@Table[T[k, i, j], {k, Length[s]}, {i, 2, Length[s[[k]]}], {j, i}]]

{T[2, 4, 2], T[2, 4, 4], T[1, 6, 3], T[1, 6, 5], T[1, 5, 1], T[1, 6, 1], T[1, 5, 2], T[2, 4, 3], T[1, 4, 4], T[2, 4, 1], T[1, 5, 5], T[2, 6, 6], T[1, 2, 1], T[1, 4, 2], T[1, 5, 4], T[1, 6, 4], T[1, 4, 1], T[1, 2, 2], T[1, 3, 2], T[1, 6, 2], T[1, 3, 3], T[2, 6, 5], T[2, 3, 3], T[2, 6, 4], T[1, 3, 1], T[2, 3, 1], T[2, 3, 2], T[2, 5, 3], T[2, 6, 3], T[2, 2, 2], T[1, 4, 3], T[2, 6, 1], T[2, 5, 5], T[1, 6, 6], T[2, 5, 2], T[2, 6, 2], T[2, 2, 1], T[2, 5, 1], T[1, 5, 3], T[2, 5, 4]}

```
eqns = (Times @@ # == 1) & /@ Rest[Reverse[s]] /.
  {m[w_, 0] => w, m[w_, 1] => 1 - 1/w, m[w_, 2] => 1/(1 - w)}
  {
    
$$\left\{ \frac{\left(1 - \frac{1}{w\$20}\right) \left(1 - \frac{1}{w\$22}\right)^2 w\$22}{(1 - w\$20)^2} == 1 \right\}$$

  }
```

```
vars = Complement[Union[Cases[eqns, _Symbol, Infinity]], {z1}]
{w$20, w$22}
```

```
{
  sol = FindRoot @@ Join[{eqns /. z1 -> 1},
    ({#, Exp[Random[] Pi I/2]} & /@ vars), {MaxIterations -> 1000}],
  Max[Abs[({(First /@ (eqns /. z1 -> 1)) /. sol) - 1}],
  Plus @@ (Vol /@ (vars /. sol))
}
```

FindRoot::frnum: Function {-0.950343 + 0.0208871 i} is not a length 2 list of numbers at {w\$20, w\$22} = {0.709676 + 0.704528 i, 0.979475 + 0.201566 i}.

FindRoot::frnum: Function {-0.950343 + 0.0208871 i} is not a length 2 list of numbers at {w\$20, w\$22} = {0.709676 + 0.704528 i, 0.979475 + 0.201566 i}.

FindRoot::frnum: Function {-0.950343 + 0.0208871 i} is not a length 2 list of numbers at {w\$20, w\$22} = {0.709676 + 0.704528 i, 0.979475 + 0.201566 i}.

General::stop: Further output of FindRoot::frnum will be suppressed during this calculation.

ReplaceAll::reps: {FindRoot[$\left\{ \frac{\left(1 - \frac{1}{w\$20}\right) \left(1 - \frac{1}{w\$22}\right)^2 w\$22}{(1 - w\$20)^2} == 1 \right\}$, {w\$20, 0.709676 + 0.704528 i}, {<<1>>}, MaxIterations->1000]} is neither a list of replacement rules nor a valid dispatch table, and so cannot be used for replacing.

ReplaceAll::reps: {FindRoot[$\left\{ \frac{\left(1 - \frac{1}{w\$20}\right) \left(1 - \frac{1}{w\$22}\right)^2 w\$22}{(1 - w\$20)^2} == 1 \right\}$, {w\$20, 0.709676 + 0.704528 i}, {<<1>>}, MaxIterations->1000]} is neither a list of replacement rules nor a valid dispatch table, and so cannot be used for replacing.

ReplaceAll::reps: {FindRoot[$\left\{ \frac{\left(1 - \frac{1}{w\$20}\right) \left(1 - \frac{1}{w\$22}\right)^2 w\$22}{(1 - w\$20)^2} == 1 \right\}$, {w\$20, 0.709676 + 0.704528 i}, {<<1>>}, MaxIterations->1000]} is neither a list of replacement rules nor a valid dispatch table, and so cannot be used for replacing.

General::stop: Further output of ReplaceAll::reps will be suppressed during this calculation.

NIntegrate::nlim: u = {Arg[w\$20], Arg[w\$22]} is not a valid limit of integration.

NIntegrate::nlim: u = {Arg[1 - $\frac{1}{w\$20}$], Arg[1 - $\frac{1}{w\$22}$]} is not a valid limit of integration.

NIntegrate::nlim: u = {Arg[$\frac{1}{1 - 1.w\$20}$], Arg[$\frac{1}{1 - 1.w\$22}$]} is not a valid limit of integration.

General::stop: Further output of NIntegrate::nlim will be suppressed during this calculation.

```

{FindRoot[{{(1 - 1/w$20) (1 - 1/w$22)^2 w$22 / (1 - w$20)^2 == 1},
  {w$20, 0.709676 + 0.704528 i}, {w$22, 0.979475 + 0.201566 i}, MaxIterations -> 1000},
Abs[-1 + ( (1 - 1/w$20) (1 - 1/w$22)^2 w$22 / (1 - w$20)^2 ) /. FindRoot[{{(1 - 1/w$20) (1 - 1/w$22)^2 w$22 / (1 - w$20)^2 == 1},
  {w$20, 0.709676 + 0.704528 i}, {w$22, 0.979475 + 0.201566 i}, MaxIterations -> 1000}]]],
-NIntegrate[Log[Abs[2 Sin[u]]], {u, 0,
  Arg[1 - 1 / FindRoot[{{(1 - 1/w$20) (1 - 1/w$22)^2 w$22 / (1 - w$20)^2 == 1}, {w$20, 0.709676 + 0.704528 i},
  {w$22, 0.979475 + 0.201566 i}, MaxIterations -> 1000}]]] - NIntegrate[
  Log[Abs[2 Sin[u]]], {u, 0, Arg[1 / (1 - FindRoot[{{(1 - 1/w$20) (1 - 1/w$22)^2 w$22 / (1 - w$20)^2 == 1}, {w$20,
  0.709676 + 0.704528 i}, {w$22, 0.979475 + 0.201566 i}, MaxIterations -> 1000}]]]}] -
NIntegrate[Log[Abs[2 Sin[u]]], {u, 0, Arg[FindRoot[{{(1 - 1/w$20) (1 - 1/w$22)^2 w$22 / (1 - w$20)^2 == 1},
  {w$20, 0.709676 + 0.704528 i}, {w$22, 0.979475 + 0.201566 i}, MaxIterations -> 1000}]]]}] -
NIntegrate[Log[Abs[2 Sin[u]]], {u, 0, {Arg[1 - 1/w$20], Arg[1 - 1/w$22]}]}] -
NIntegrate[Log[Abs[2 Sin[u]]],
  {u, 0, {Arg[1/(1 - w$20)], Arg[1/(1 - w$22)]}}]}] -
NIntegrate[Log[Abs[2 Sin[u]]], {u, 0, {Arg[w$20], Arg[w$22]}]}]

```

sols = vars /. Solve[eqns /. z1 -> 1, vars]

General::spell1: Possible spelling error: new symbol name "sols" is similar to existing symbol "sol".

Solve::svars: Equations may not give solutions for all "solve" variables.

$$\left\{ \left\{ \frac{\sqrt{w} - \sqrt{4 - 7w + 4w^2}}{2\sqrt{w}}, w \right\}, \left\{ \frac{\sqrt{w} + \sqrt{4 - 7w + 4w^2}}{2\sqrt{w}}, w \right\} \right\}$$

N[sols]

$$\left\{ \left\{ \frac{0.5 \left(\sqrt{w\$22} - 1. \sqrt{4. - 7. w\$22 + 4. w\$22^2} \right)}{\sqrt{w\$22}}, w\$22 \right\}, \right. \\ \left. \left\{ \frac{0.5 \left(\sqrt{w\$22} + \sqrt{4. - 7. w\$22 + 4. w\$22^2} \right)}{\sqrt{w\$22}}, w\$22 \right\} \right\}$$

(Plus @@ Vol /@ #) & /@ sols

NIntegrate::nlim: u = Arg $\left[\frac{\sqrt{w\$22} - 1. \sqrt{4. - 7. w\$22 + 4. w\$22^2}}{\sqrt{w\$22}}\right]$ is not a valid limit of integration.

NIntegrate::nlim: u = Arg $\left[1. - \frac{2. \sqrt{w\$22}}{\sqrt{w\$22} - 1. \sqrt{4. - 7. w\$22 + 4. w\$22^2}}\right]$ is not a valid limit of integration.

NIntegrate::nlim: u = Arg $\left[\frac{1}{1. - \frac{0.5 \left(\sqrt{w\$22} - 1. \sqrt{4. - 7. w\$22 + 4. w\$22^2} \right)}{\sqrt{w\$22}}}\right]$ is not a valid limit of integration.

General::stop: Further output of NIntegrate::nlim will be suppressed during this calculation.

$$\left\{ -\text{NIntegrate}\left[\text{Log}\left[\text{Abs}\left[2 \text{Sin}\left[u\right]\right]\right], \left\{u, 0, \text{Arg}\left[1 - \frac{1}{w\$22}\right]\right\}\right] - \right. \\ \text{NIntegrate}\left[\text{Log}\left[\text{Abs}\left[2 \text{Sin}\left[u\right]\right]\right], \left\{u, 0, \text{Arg}\left[\frac{1}{1 - w\$22}\right]\right\}\right] - \\ \text{NIntegrate}\left[\text{Log}\left[\text{Abs}\left[2 \text{Sin}\left[u\right]\right]\right], \left\{u, 0, \text{Arg}\left[w\$22\right]\right\}\right] - \\ \text{NIntegrate}\left[\text{Log}\left[\text{Abs}\left[2 \text{Sin}\left[u\right]\right]\right], \left\{u, 0, \text{Arg}\left[\frac{\sqrt{w\$22} - \sqrt{4 - 7 w\$22 + 4 w\$22^2}}{\sqrt{w\$22}}\right]\right\}\right] - \\ \text{NIntegrate}\left[\text{Log}\left[\text{Abs}\left[2 \text{Sin}\left[u\right]\right]\right], \left\{u, 0, \text{Arg}\left[1 - \frac{2 \sqrt{w\$22}}{\sqrt{w\$22} - \sqrt{4 - 7 w\$22 + 4 w\$22^2}}\right]\right\}\right] - \\ \text{NIntegrate}\left[\text{Log}\left[\text{Abs}\left[2 \text{Sin}\left[u\right]\right]\right], \left\{u, 0, \text{Arg}\left[\frac{1}{1 - \frac{\sqrt{w\$22} - \sqrt{4 - 7 w\$22 + 4 w\$22^2}}{2 \sqrt{w\$22}}}\right]\right\}\right], \\ -\text{NIntegrate}\left[\text{Log}\left[\text{Abs}\left[2 \text{Sin}\left[u\right]\right]\right], \left\{u, 0, \text{Arg}\left[1 - \frac{1}{w\$22}\right]\right\}\right] - \\ \text{NIntegrate}\left[\text{Log}\left[\text{Abs}\left[2 \text{Sin}\left[u\right]\right]\right], \left\{u, 0, \text{Arg}\left[\frac{1}{1 - w\$22}\right]\right\}\right] - \\ \text{NIntegrate}\left[\text{Log}\left[\text{Abs}\left[2 \text{Sin}\left[u\right]\right]\right], \left\{u, 0, \text{Arg}\left[w\$22\right]\right\}\right] - \\ \text{NIntegrate}\left[\text{Log}\left[\text{Abs}\left[2 \text{Sin}\left[u\right]\right]\right], \left\{u, 0, \text{Arg}\left[\frac{\sqrt{w\$22} + \sqrt{4 - 7 w\$22 + 4 w\$22^2}}{\sqrt{w\$22}}\right]\right\}\right] - \\ \text{NIntegrate}\left[\text{Log}\left[\text{Abs}\left[2 \text{Sin}\left[u\right]\right]\right], \left\{u, 0, \text{Arg}\left[1 - \frac{2 \sqrt{w\$22}}{\sqrt{w\$22} + \sqrt{4 - 7 w\$22 + 4 w\$22^2}}\right]\right\}\right] - \\ \left. \text{NIntegrate}\left[\text{Log}\left[\text{Abs}\left[2 \text{Sin}\left[u\right]\right]\right], \left\{u, 0, \text{Arg}\left[\frac{1}{1 - \frac{\sqrt{w\$22} + \sqrt{4 - 7 w\$22 + 4 w\$22^2}}{2 \sqrt{w\$22}}}\right]\right\}\right] \right\}$$