

Loading Pre-Computed Data

```
In[*]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\HigherRank"];
Once[<< KnotTheory`];
<< Rot.m
```

Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.

Read more at <http://katlas.org/wiki/KnotTheory>.

Loading Rot.m from <http://drorbn.net/AP/Projects/HigherRank> to compute rotation numbers.

```
In[*]:= CCF[ε_] := ExpandDenominator@ExpandNumerator@Together[ε];
CCF[ε_] := Factor[ε];
CF[ε_List] := CF/@ε;
CF[ε_] := Module[{vs = Cases[ε, (x | p | π | g)_, ∞] ∪ {x, p, ε}, ps, c},
  Total[CoefficientRules[Expand[ε], vs] /. (ps_ -> c_) -> CCF[c] (Times @@ vs^ps)]];
```

```
In[*]:= {
  {r0,ppx[1, i_, j_], r0,ppx[-1, i_, j_]},
  {r1,ppx[1, i_, j_], r1,ppx[-1, i_, j_]},
  {r1,rest[1, i_, j_], r1,rest[-1, i_, j_]},
  γ1[φ_, k_]
} = CF[Plus[
  {T1, 1 - T1 T2, T1 (1 - T1 T2), T1 (1 - T1 T2)} * Get["px-data.m"],
  -{{0, 0},
    {0, 0},
    {1/2 + T3 x3,i (p3,i+1 - p3,j+1), -1/2 - T3^-1 x3,i (p3,i+1 - p3,j+1)},
    φ/2
  }],
  {{0, 0},
  {0, 0},
  {α (1 + x3,j (p3,j+1 - p3,j)) +
    β (1 + x3,i (p3,i+1 - p3,i) + (T3 - 1) x3,i (p3,i+1 - p3,j+1)) + γ x3,i (p3,j+1 - p3,j) +
    δ (x3,j (p3,i+1 - p3,i) + (T3 - 1) x3,j (p3,i+1 - p3,j+1)) /. {β -> 1, γ -> -1, δ -> 0, α -> 0},
  α (1 + x3,j (p3,j+1 - p3,j)) + β (1 + x3,i (p3,i+1 - p3,i) + (T3^-1 - 1) x3,i (p3,i+1 - p3,j+1)) +
  γ x3,i (p3,j+1 - p3,j) + δ (x3,j (p3,i+1 - p3,i) + (T3^-1 - 1) x3,j (p3,i+1 - p3,j+1)) /.
  {β -> -1, γ -> 1, δ -> 0, α -> 0}},
  0
}
]]
```

Out[*]=

$$\begin{aligned}
& \left\{ T_1 p_{3,j} x_{1,i} x_{2,i} - p_{3,j} x_{1,j} x_{2,i}, -\frac{p_{3,j} x_{1,i} x_{2,i}}{T_1 T_2} + \frac{p_{3,j} x_{1,j} x_{2,i}}{T_2} \right\}, \\
& \left\{ \frac{(1 - T_1 T_2) p_{1,j} p_{2,i} x_{3,i} + (-1 + T_1 T_2) p_{1,j} p_{2,j} x_{3,i},}{T_1} \right. \\
& \quad \left. \frac{(-1 + T_1 T_2) p_{1,j} p_{2,i} x_{3,i}}{T_1} - \frac{(-1 + T_1 T_2) p_{1,j} p_{2,j} x_{3,i}}{T_1} \right\}, \\
& \left\{ \frac{1}{2} - T_1 T_2 p_{1,j} p_{2,j} x_{1,i} x_{2,i} + \frac{p_{1,j} p_{2,i} x_{1,j} x_{2,i}}{-1 + T_1} + \frac{(-1 + T_1 T_2) p_{1,j} p_{2,j} x_{1,j} x_{2,i}}{-1 + T_1} - \frac{T_1 p_{1,i} p_{2,j} x_{1,i} x_{2,j}}{-1 + T_1} - \right. \\
& \quad p_{3,i} x_{3,i} + T_1 (-1 + T_1 T_2) p_{1,j} p_{3,j} x_{1,i} x_{3,i} - \frac{p_{1,j} p_{3,i} x_{1,j} x_{3,i}}{-1 + T_1} - \frac{T_1 (-1 + T_1 T_2) p_{1,j} p_{3,j} x_{1,j} x_{3,i}}{-1 + T_1} + \\
& \quad T_2 (-1 + T_1 T_2) p_{2,j} p_{3,j} x_{2,i} x_{3,i} + p_{2,j} p_{3,i} x_{2,j} x_{3,i} + \frac{T_1 p_{1,i} p_{3,j} x_{1,i} x_{3,j}}{-1 + T_1} - T_2 p_{2,j} p_{3,j} x_{2,i} x_{3,j}, \\
& \quad \frac{1}{2} \frac{p_{1,j} p_{2,i} x_{1,i} x_{2,i}}{T_1} + \frac{T_1 (-1 + T_2) p_{1,i} p_{2,j} x_{1,i} x_{2,i}}{(-1 + T_1) T_2} - \frac{(-T_1 - T_2 + T_1 T_2) p_{1,j} p_{2,j} x_{1,i} x_{2,i}}{T_1 T_2} - \\
& \quad \frac{p_{1,j} p_{2,i} x_{1,j} x_{2,i}}{-1 + T_1} - p_{1,j} p_{2,j} x_{1,j} x_{2,i} + \frac{T_1 p_{1,i} p_{2,j} x_{1,i} x_{2,j}}{-1 + T_1} - p_{1,j} p_{2,j} x_{1,i} x_{2,j} + p_{3,i} x_{3,i} + \\
& \quad \frac{p_{1,j} p_{3,i} x_{1,i} x_{3,i}}{T_1} - \frac{(-1 + T_1 T_2) p_{1,i} p_{3,j} x_{1,i} x_{3,i}}{(-1 + T_1) T_2} + \frac{(-1 + T_1 T_2) p_{1,j} p_{3,j} x_{1,i} x_{3,i}}{T_1 T_2} + \\
& \quad \frac{p_{1,j} p_{3,i} x_{1,j} x_{3,i}}{-1 + T_1} - \frac{(-1 + T_2) p_{2,j} p_{3,i} x_{2,i} x_{3,i}}{T_2} - \frac{(-1 + T_1 T_2) p_{2,i} p_{3,j} x_{2,i} x_{3,i}}{T_1 T_2} + \\
& \quad \frac{(-1 + 2 T_2) (-1 + T_1 T_2) p_{2,j} p_{3,j} x_{2,i} x_{3,i}}{T_1 T_2^2} - p_{2,j} p_{3,i} x_{2,j} x_{3,i} + \frac{(-1 + T_1 T_2) p_{2,j} p_{3,j} x_{2,j} x_{3,i}}{T_1 T_2} - \\
& \quad \left. \frac{T_1 p_{1,i} p_{3,j} x_{1,i} x_{3,j}}{-1 + T_1} + p_{1,j} p_{3,j} x_{1,i} x_{3,j} + p_{2,j} p_{3,j} x_{2,i} x_{3,j} \right\}, -\frac{\varphi}{2} + \varphi p_{3,k} x_{3,k} \}
\end{aligned}$$

$$\begin{aligned}
In[*] := & \mathbf{q}[s_-, i_-, j_-] := \text{Sum}[\\
& \quad \mathbf{x}_{v,i} (p_{v,i^*} - p_{v,i}) + \mathbf{x}_{v,j} (p_{v,j^*} - p_{v,j}) + (T_v^S - 1) \mathbf{x}_{v,i} (p_{v,i^*} - p_{v,j^*}), \\
& \quad \{v, 3\}]; \\
& \mathcal{L}[\mathbf{X}_{i_-, j_-}[s_-]] := \\
& \quad T_3^S \mathbb{E}[\mathbf{q}[s, i, j] + \mathbf{r}_{\emptyset, \text{ppx}}[s, i, j] + \mathbf{e} \mathbf{r}_{1, \text{ppx}}[s, i, j] + \mathbf{e} \mathbf{r}_{1, \text{rest}}[s, i, j] + \mathbf{O}[\epsilon]^2]; \\
& \mathcal{L}[\mathbf{C}_{k_-}[\varphi_-]] := T_3^\varphi \mathbb{E}[\text{Sum}[\mathbf{x}_{v,k} (p_{v,k^*} - p_{v,k}), \{v, 3\}] + \mathbf{e} \gamma_1[\varphi, k] + \mathbf{O}[\epsilon]^2]; \\
& \mathcal{L}[\mathbf{X}_{i,j}[1]] \\
& \mathcal{L}[\mathbf{X}_{i,j}[-1]] \\
& \mathcal{L}[\mathbf{C}_i[\varphi]]
\end{aligned}$$

Out[*]=

$$\begin{aligned}
& T_1 T_2 \mathbb{E} \left[\left((-p_{1,i} + p_{1,i^+}) x_{1,i} + (-1 + T_1) (p_{1,i^+} - p_{1,j^+}) x_{1,i} + (-p_{1,j} + p_{1,j^+}) x_{1,j} + (-p_{2,i} + p_{2,i^+}) x_{2,i} + \right. \right. \\
& \quad (-1 + T_2) (p_{2,i^+} - p_{2,j^+}) x_{2,i} + T_1 p_{3,j} x_{1,i} x_{2,i} - p_{3,j} x_{1,j} x_{2,i} + (-p_{2,j} + p_{2,j^+}) x_{2,j} + \\
& \quad \left. (-p_{3,i} + p_{3,i^+}) x_{3,i} + (-1 + T_1 T_2) (p_{3,i^+} - p_{3,j^+}) x_{3,i} + (-p_{3,j} + p_{3,j^+}) x_{3,j} \right) + \\
& \quad \left(\frac{1}{2} - T_1 T_2 p_{1,j} p_{2,j} x_{1,i} x_{2,i} + \frac{p_{1,j} p_{2,i} x_{1,j} x_{2,i}}{-1 + T_1} + \frac{(-1 + T_1 T_2) p_{1,j} p_{2,j} x_{1,j} x_{2,i}}{-1 + T_1} - \right. \\
& \quad \frac{T_1 p_{1,i} p_{2,j} x_{1,i} x_{2,j}}{-1 + T_1} + (1 - T_1 T_2) p_{1,j} p_{2,i} x_{3,i} + (-1 + T_1 T_2) p_{1,j} p_{2,j} x_{3,i} - \\
& \quad p_{3,i} x_{3,i} + T_1 (-1 + T_1 T_2) p_{1,j} p_{3,j} x_{1,i} x_{3,i} - \frac{p_{1,j} p_{3,i} x_{1,j} x_{3,i}}{-1 + T_1} - \\
& \quad \frac{T_1 (-1 + T_1 T_2) p_{1,j} p_{3,j} x_{1,j} x_{3,i}}{-1 + T_1} + T_2 (-1 + T_1 T_2) p_{2,j} p_{3,j} x_{2,i} x_{3,i} + \\
& \quad \left. p_{2,j} p_{3,i} x_{2,j} x_{3,i} + \frac{T_1 p_{1,i} p_{3,j} x_{1,i} x_{3,j}}{-1 + T_1} - T_2 p_{2,j} p_{3,j} x_{2,i} x_{3,j} \right) \epsilon + O[\epsilon]^2 \Big]
\end{aligned}$$

Out[*]=

$$\begin{aligned}
& \frac{1}{T_1 T_2} \mathbb{E} \left[\left((-p_{1,i} + p_{1,i^+}) x_{1,i} + \left(-1 + \frac{1}{T_1} \right) (p_{1,i^+} - p_{1,j^+}) x_{1,i} + (-p_{1,j} + p_{1,j^+}) x_{1,j} + (-p_{2,i} + p_{2,i^+}) x_{2,i} + \right. \right. \\
& \quad \left. \left(-1 + \frac{1}{T_2} \right) (p_{2,i^+} - p_{2,j^+}) x_{2,i} - \frac{p_{3,j} x_{1,i} x_{2,i}}{T_1 T_2} + \frac{p_{3,j} x_{1,j} x_{2,i}}{T_2} + (-p_{2,j} + p_{2,j^+}) x_{2,j} + \right. \\
& \quad \left. (-p_{3,i} + p_{3,i^+}) x_{3,i} + \left(-1 + \frac{1}{T_1 T_2} \right) (p_{3,i^+} - p_{3,j^+}) x_{3,i} + (-p_{3,j} + p_{3,j^+}) x_{3,j} \right) + \\
& \quad \left(-\frac{1}{2} - \frac{p_{1,j} p_{2,i} x_{1,i} x_{2,i}}{T_1} + \frac{T_1 (-1 + T_2) p_{1,i} p_{2,j} x_{1,i} x_{2,i}}{(-1 + T_1) T_2} - \frac{(-T_1 - T_2 + T_1 T_2) p_{1,j} p_{2,j} x_{1,i} x_{2,i}}{T_1 T_2} - \right. \\
& \quad \frac{p_{1,j} p_{2,i} x_{1,j} x_{2,i}}{-1 + T_1} - p_{1,j} p_{2,j} x_{1,j} x_{2,i} + \frac{T_1 p_{1,i} p_{2,j} x_{1,i} x_{2,j}}{-1 + T_1} - p_{1,j} p_{2,j} x_{1,i} x_{2,j} + \\
& \quad \frac{(-1 + T_1 T_2) p_{1,j} p_{2,i} x_{3,i}}{T_1} - \frac{(-1 + T_1 T_2) p_{1,j} p_{2,j} x_{3,i}}{T_1} + p_{3,i} x_{3,i} + \\
& \quad \frac{p_{1,j} p_{3,i} x_{1,i} x_{3,i}}{T_1} - \frac{(-1 + T_1 T_2) p_{1,i} p_{3,j} x_{1,i} x_{3,i}}{(-1 + T_1) T_2} + \frac{(-1 + T_1 T_2) p_{1,j} p_{3,j} x_{1,i} x_{3,i}}{T_1 T_2} + \\
& \quad \frac{p_{1,j} p_{3,i} x_{1,j} x_{3,i}}{-1 + T_1} - \frac{(-1 + T_2) p_{2,j} p_{3,i} x_{2,i} x_{3,i}}{T_2} - \frac{(-1 + T_1 T_2) p_{2,i} p_{3,j} x_{2,i} x_{3,i}}{T_1 T_2} + \\
& \quad \frac{(-1 + 2 T_2) (-1 + T_1 T_2) p_{2,j} p_{3,j} x_{2,i} x_{3,i}}{T_1 T_2^2} - p_{2,j} p_{3,i} x_{2,j} x_{3,i} + \frac{(-1 + T_1 T_2) p_{2,j} p_{3,j} x_{2,j} x_{3,i}}{T_1 T_2} - \\
& \quad \left. \frac{T_1 p_{1,i} p_{3,j} x_{1,i} x_{3,j}}{-1 + T_1} + p_{1,j} p_{3,j} x_{1,i} x_{3,j} + p_{2,j} p_{3,j} x_{2,i} x_{3,j} \right) \epsilon + O[\epsilon]^2 \Big]
\end{aligned}$$

Out[*]=

$$(T_1 T_2)^\varphi \mathbb{E} \left[\left((-p_{1,i} + p_{1,i^+}) x_{1,i} + (-p_{2,i} + p_{2,i^+}) x_{2,i} + (-p_{3,i} + p_{3,i^+}) x_{3,i} \right) + \left(-\frac{\varphi}{2} + \varphi p_{3,i} x_{3,i} \right) \epsilon + O[\epsilon]^2 \right]$$

$$In[*]:= \{p^*, x^*, \pi^*, \xi^*\} = \{\pi, \xi, p, x\}; \quad (u_{-i-})^* := (u^*)_i;$$

```
In[*]:= Zip[_][_]:= ;
Zip[_][_]:= (Collect[_ // Zip[_], _] /. f_.*_<sup>d</sup> -> (D[f, {*_*, d}])) /. *_* -> 0
```

```
In[*]:= px2g[_]:= CF@Module[{ps, xs, Q, alpha, beta},
  ps = Union[Cases[_ , p_ , infinity]]; xs = Union[Cases[_ , x_ , infinity]];
  Q = Sum[p0* x0* g_{p0[[2]], x0[[2]], p0[[3]], x0[[3]}, {p0, ps}, {x0, xs}];
  Expand[Zip_{ps union xs}[_ e^Q] /. g_{alpha, beta, i, j} -> If[alpha === beta, g_{alpha, i, j}, 0]]
]
```

```
In[*]:= px2g[p_{2,j}^2 x_{2,i} x_{2,j}]
```

```
Out[*]=
2 g_{2,j,i} g_{2,j,j}
```

```
In[*]:= R1[1, i_ , j_] = px2g[r_{1,rest}[1, i, j]]
```

```
Out[*]=
1/2 + g_{1,j,j} g_{2,i,i} / (-1 + T1) - T1 T2 g_{1,j,i} g_{2,j,i} + (-1 + T1 T2) g_{1,j,j} g_{2,j,i} / (-1 + T1) -
T1 g_{1,i,i} g_{2,j,j} / (-1 + T1) - g_{3,i,i} - g_{1,j,j} g_{3,i,i} / (-1 + T1) + g_{2,j,j} g_{3,i,i} + T1 (-1 + T1 T2) g_{1,j,i} g_{3,j,i} -
T1 (-1 + T1 T2) g_{1,j,j} g_{3,j,i} / (-1 + T1) + T2 (-1 + T1 T2) g_{2,j,i} g_{3,j,i} + T1 g_{1,i,i} g_{3,j,j} / (-1 + T1) - T2 g_{2,j,i} g_{3,j,j}
```

```
In[*]:= R1[-1, i_ , j_] = px2g[r_{1,rest}[-1, i, j]]
```

```
Out[*]=
1/2 - g_{1,j,i} g_{2,i,i} / T1 - g_{1,j,j} g_{2,i,i} / (-1 + T1) + T1 (-1 + T2) g_{1,i,i} g_{2,j,i} / ((-1 + T1) T2) -
(-T1 - T2 + T1 T2) g_{1,j,i} g_{2,j,i} / (T1 T2) - g_{1,j,j} g_{2,j,i} + T1 g_{1,i,i} g_{2,j,j} / (-1 + T1) - g_{1,j,i} g_{2,j,j} + g_{3,i,i} +
g_{1,j,i} g_{3,i,i} / T1 + g_{1,j,j} g_{3,i,i} / (-1 + T1) - (-1 + T2) g_{2,j,i} g_{3,i,i} / T2 - g_{2,j,j} g_{3,i,i} - (-1 + T1 T2) g_{1,i,i} g_{3,j,i} / ((-1 + T1) T2) +
(-1 + T1 T2) g_{1,j,i} g_{3,j,i} / (T1 T2) - (-1 + T1 T2) g_{2,i,i} g_{3,j,i} / (T1 T2) + (-1 + 2 T2) (-1 + T1 T2) g_{2,j,i} g_{3,j,i} / (T1 T2^2) +
(-1 + T1 T2) g_{2,j,j} g_{3,j,i} / (T1 T2) - T1 g_{1,i,i} g_{3,j,j} / (-1 + T1) + g_{1,j,i} g_{3,j,j} + g_{2,j,i} g_{3,j,j}
```

```
In[*]:= px2g[r_{0,pxx}[1, i0, j0] r_{1,ppx}[1, i1, j1]]
```

```
Out[*]=
-T1 (-1 + T1 T2) g_{1,j1,i0} g_{2,i1,i0} g_{3,j0,i1} + (-1 + T1 T2) g_{1,j1,j0} g_{2,i1,i0} g_{3,j0,i1} +
T1 (-1 + T1 T2) g_{1,j1,i0} g_{2,j1,i0} g_{3,j0,i1} + (1 - T1 T2) g_{1,j1,j0} g_{2,j1,i0} g_{3,j0,i1}
```

$$\begin{aligned}
In[*]:= & \theta[\{1, i0_, j0_ \}, \{1, i1_, j1_ \}] = px2g[r_{\theta, pxx}[1, i0, j0] r_{1, ppx}[1, i1, j1]] \\
& \theta[\{1, i0_, j0_ \}, \{-1, i1_, j1_ \}] = px2g[r_{\theta, pxx}[1, i0, j0] r_{1, ppx}[-1, i1, j1]] \\
& \theta[\{-1, i0_, j0_ \}, \{1, i1_, j1_ \}] = px2g[r_{\theta, pxx}[-1, i0, j0] r_{1, ppx}[1, i1, j1]] \\
& \theta[\{-1, i0_, j0_ \}, \{-1, i1_, j1_ \}] = px2g[r_{\theta, pxx}[-1, i0, j0] r_{1, ppx}[-1, i1, j1]]
\end{aligned}$$

$$\begin{aligned}
Out[*]= & -T_1 (-1 + T_1 T_2) g_{1, j1, i0} g_{2, i1, i0} g_{3, j0, i1} + (-1 + T_1 T_2) g_{1, j1, j0} g_{2, i1, i0} g_{3, j0, i1} + \\
& T_1 (-1 + T_1 T_2) g_{1, j1, i0} g_{2, j1, i0} g_{3, j0, i1} + (1 - T_1 T_2) g_{1, j1, j0} g_{2, j1, i0} g_{3, j0, i1}
\end{aligned}$$

$$\begin{aligned}
Out[*]= & (-1 + T_1 T_2) g_{1, j1, i0} g_{2, i1, i0} g_{3, j0, i1} - \frac{(-1 + T_1 T_2) g_{1, j1, j0} g_{2, i1, i0} g_{3, j0, i1}}{T_1} + \\
& (1 - T_1 T_2) g_{1, j1, i0} g_{2, j1, i0} g_{3, j0, i1} + \frac{(-1 + T_1 T_2) g_{1, j1, j0} g_{2, j1, i0} g_{3, j0, i1}}{T_1}
\end{aligned}$$

$$\begin{aligned}
Out[*]= & \frac{(-1 + T_1 T_2) g_{1, j1, i0} g_{2, i1, i0} g_{3, j0, i1}}{T_1 T_2} - \frac{(-1 + T_1 T_2) g_{1, j1, j0} g_{2, i1, i0} g_{3, j0, i1}}{T_2} - \\
& \frac{(-1 + T_1 T_2) g_{1, j1, i0} g_{2, j1, i0} g_{3, j0, i1}}{T_1 T_2} + \frac{(-1 + T_1 T_2) g_{1, j1, j0} g_{2, j1, i0} g_{3, j0, i1}}{T_2}
\end{aligned}$$

$$\begin{aligned}
Out[*]= & - \frac{(-1 + T_1 T_2) g_{1, j1, i0} g_{2, i1, i0} g_{3, j0, i1}}{T_1^2 T_2} + \frac{(-1 + T_1 T_2) g_{1, j1, j0} g_{2, i1, i0} g_{3, j0, i1}}{T_1 T_2} + \\
& \frac{(-1 + T_1 T_2) g_{1, j1, i0} g_{2, j1, i0} g_{3, j0, i1}}{T_1^2 T_2} - \frac{(-1 + T_1 T_2) g_{1, j1, j0} g_{2, j1, i0} g_{3, j0, i1}}{T_1 T_2}
\end{aligned}$$

$$In[*]:= CF[\theta[\{1, i0, j0\}, \{1, i1, j1\}] + (\theta[\{-1, i0, j0\}, \{-1, i1, j1\}] /. T_{i_} \Rightarrow T_{i_}^{-1})]$$

$$Out[*]= 0$$

$$In[*]:= CF[\theta[\{1, i0, j0\}, \{-1, i1, j1\}] + (\theta[\{-1, i0, j0\}, \{1, i1, j1\}] /. T_{i_} \Rightarrow T_{i_}^{-1})]$$

$$Out[*]= 0$$

$$In[*]:= T_1[\varphi_, k_] = px2g[\gamma_1[\varphi, k]]$$

$$Out[*]= -\frac{\varphi}{2} + \varphi g_{3, k, k}$$

The Programs

```

In[*]:= T3 = T1 T2;
Theta[K_] := Theta[K] = Module[{Cs, phi, n, A, s, i, j, k, Delta, G, v, alpha, beta, gEval, Y, yEval, c, z},
  {Cs, phi} = Rot[K]; n = Length[Cs];
  A = IdentityMatrix[2 n + 1];
  Cases[Cs, {s_, i_, j_} >=> (A[[{i, j}, {i + 1, j + 1}]] += (
    
$$\begin{pmatrix} -T^s & T^s - 1 \\ 0 & -1 \end{pmatrix}$$

  ))];
  Delta = T^(-Total[phi]-Total[Cs[[All,1]]])/2 Det[A];
  G = Inverse[A]; gEval[epsilon_] := CCF[epsilon /. gv_, alpha_, beta_ >=> (G[[alpha, beta]] /. T -> Tv)];
  z = gEval[Sum[Cs[[k1]], Cs[[k2]]], {k1, 1, n}, {k2, 1, n}];
  z += gEval[Sum[R1 @@ Cs[[k]], {k, 1, n}];
  z += gEval[Sum[T1[[phi[[k]], k]], {k, 1, 2 n}];
  {Delta, (Delta /. T -> T1) (Delta /. T -> T2) (Delta /. T -> T3) z} // CCF
];

```

```

In[*]:= ThetaT1,T2[K_] := ThetaT1,T2[K] = Module[{Cs, phi, n, A, s, i, j, k, Delta, G, gEval, Y, yEval, c, z = 0},
  {Cs, phi} = Rot[K]; n = Length[Cs];
  temp0 = PrintTemporary["At work, n=", n];
  A = IdentityMatrix[2 n + 1];
  Cases[Cs, {s_, i_, j_} >=> (A[[{i, j}, {i + 1, j + 1}]] += (
    
$$\begin{pmatrix} -T^s & T^s - 1 \\ 0 & -1 \end{pmatrix}$$

  ))];
  Delta[0] := Delta[0] = T^(-Total[phi]-Total[Cs[[All,1]]])/2 Det[A];
  G[0] := G[0] = Inverse[A];
  {Delta[1], G[1]} = If[NumberQ@T1,
    {Det[A /. T -> T1], Inverse[A /. T -> T1]}, {Delta[0], G[0]} /. T -> T1];
  temp = PrintTemporary@"Done with {Delta[1],G[1]}.";
  {Delta[2], G[2]} = If[NumberQ@T2,
    {Det[A /. T -> T2], Inverse[A /. T -> T2]}, {Delta[0], G[0]} /. T -> T2];
  NotebookDelete[temp]; temp = PrintTemporary@"Done with {Delta[2],G[2]}.";
  {Delta[3], G[3]} = If[NumberQ[T1 T2],
    {Det[A /. T -> T1 T2], Inverse[A /. T -> T1 T2]}, {Delta[0], G[0]} /. T -> T1 T2];
  NotebookDelete[temp]; temp = PrintTemporary@"Done with {Delta[3],G[3]}.";
  gEval[epsilon_] := CCF[epsilon /. {T1 -> T1, T2 -> T2, gv_, alpha_, beta_ >=> G[v] [[alpha, beta]]];
  Do[z += gEval[Theta[Cs[[k1]], Cs[[k2]]], {k1, n}, {k2, n}];
  Do[z += gEval[R1 @@ Cs[[k]], {k, n}];
  Do[z += gEval[T1[[phi[[k]], k]], {k, 2 n}];
  NotebookDelete[temp0]; NotebookDelete[temp];
  {{Delta[1], Delta[2], Delta[3]}, Delta[1] Delta[2] Delta[3] z} // CCF
];

```

```

In[*]:= TestSymmetries[K_] := Module[{e0, e1},
  {e0, e1} = {e[K][[2]], e[Mirror@K][[2]]};
  Simplify@And[
    e0 == (e0 /. {T1 -> T2, T2 -> T1}),
    e0 == -e1,
    e0 == (e0 /. T_i_ -> T_i^-1),
    e0 == (e0 /. T2 -> T1^-1 T2^-1)
  ]
]

```

```

In[*]:= hex = Table[{Cos[α], Sin[α]} / Cos[2 π / 12] / 2, {α, 2 π / 12, 2 π, 2 π / 6}];
PolyPlot__[0] = Graphics[{}];
PolyPlot_T1_,T2_[p_] := PolyPlot_Hexagon,T1,T2[p]
PolyPlot_shape_,T1_,T2_[p_] := Module[{crs, m1, m2, maxc, minc, s},
  crs = CoefficientRules[T1^m1=-Exponent[p,T1,Min] T2^m2=-Exponent[p,T2,Min] p, {T1, T2}];
  maxc = Max@Abs[Last /@ crs];
  minc = Min@Select[Abs[Last /@ crs], # > 0 &];
  If[minc == maxc,
    s[_] = 0,
    s[c_] := s[c] =
      N[Interpolation[{{Log@minc, 1}, {Log@maxc, 0}}, InterpolationOrder -> 1][Log@c]]
  ];
  Graphics[crs /. ({x1_, x2_} -> c_) -> {
    If[c == 0, White, Lighter[If[c > 0, Red, Blue], 0.88 s[Abs@c]]],
    Switch[shape,
      Disk, Disk[{{1, -1/2}, {0, sqrt[3]/2}} . {x1 + m1, x2 + m2}, 0.5],
      Hexagon, Polygon[{{1, -1/2}, {0, sqrt[3]/2}} . {x1 + m1, x2 + m2} + #] & /@ hex]
  ]
]
]

```

Sporadic Testing

```
In[*]:= K = Knot[3, 1]; Timing[Expand[Theta[K]]]
TestSymmetries[K]
```

 KnotTheory: Loading precomputed data in PD4Knots`.

Out[*]=

$$\left\{ \theta., \left\{ -1 + \frac{1}{T_1} + T_1, -\frac{1}{T_1^2} - T_1^2 - \frac{1}{T_2^2} - \frac{1}{T_1^2 T_2^2} + \frac{1}{T_1 T_2^2} + \frac{1}{T_1^2 T_2} + \frac{T_1}{T_2} + \frac{T_2}{T_1} + T_1^2 T_2 - T_2^2 + T_1 T_2^2 - T_1^2 T_2^2 \right\} \right\}$$

Out[*]=

True

```
In[*]:= K = Knot[8, 19]; Timing[Expand[Theta[K]]]
TestSymmetries[K]
```

Out[*]=

$$\left\{ \theta.015625, \left\{ 1 + \frac{1}{T_1^3} - \frac{1}{T_1^2} - T_1^2 + T_1^3, \frac{3}{T_1^6} - \frac{3}{T_1^4} + \frac{4}{T_1^3} - \frac{1}{T_1^2} - T_1^2 + 4 T_1^3 - 3 T_1^4 + 3 T_1^6 + \frac{3}{T_2^6} + \frac{3}{T_1^6 T_2^6} - \frac{3}{T_1^5 T_2^6} + \frac{3}{T_1^3 T_2^6} - \frac{3}{T_1 T_2^6} - \frac{3}{T_1^6 T_2^5} + \frac{3}{T_1^4 T_2^5} - \frac{3}{T_1^3 T_2^5} - \frac{3}{T_1^2 T_2^5} + \frac{3}{T_1 T_2^5} - \frac{3 T_1}{T_2^5} - \frac{3}{T_2^4} + \frac{3}{T_1^5 T_2^4} - \frac{3}{T_1^4 T_2^4} + \frac{3}{T_1^2 T_2^4} + \frac{3 T_1}{T_2^4} + \frac{4}{T_2^3} + \frac{3}{T_1^6 T_2^3} - \frac{3}{T_1^5 T_2^3} + \frac{4}{T_1^3 T_2^3} - \frac{2}{T_1^2 T_2^3} - \frac{2}{T_1 T_2^3} - \frac{3 T_1^2}{T_2^3} + \frac{3 T_1^3}{T_2^2} - \frac{1}{T_2^2} - \frac{3}{T_1^5 T_2^2} + \frac{3}{T_1^4 T_2^2} - \frac{2}{T_1^3 T_2^2} - \frac{1}{T_1^2 T_2^2} + \frac{1}{T_1 T_2^2} - \frac{2 T_1}{T_2^2} + \frac{3 T_1^2}{T_2^2} - \frac{3 T_1^3}{T_2^2} - \frac{3}{T_1^6 T_2} + \frac{3}{T_1^5 T_2} - \frac{2}{T_1^3 T_2} + \frac{1}{T_1^2 T_2} + \frac{T_1}{T_2} - \frac{2 T_1^2}{T_2} + \frac{3 T_1^4}{T_2} - \frac{3 T_1^5}{T_2} - \frac{3 T_2}{T_1^5} + \frac{3 T_2}{T_1^4} - \frac{2 T_2}{T_1^3} + \frac{T_2}{T_1^2} + T_1^2 T_2 - 2 T_1^3 T_2 + 3 T_1^5 T_2 - 3 T_1^6 T_2 - T_2^2 - \frac{3 T_2^2}{T_1^3} + \frac{3 T_2^2}{T_1^2} - \frac{2 T_2^2}{T_1} + T_1 T_2^2 - T_1^2 T_2^2 - 2 T_1^3 T_2^2 + 3 T_1^4 T_2^2 - 3 T_1^5 T_2^2 + 4 T_2^3 + \frac{3 T_2^3}{T_1^3} - \frac{3 T_2^3}{T_1^2} - 2 T_1 T_2^3 - 2 T_1^2 T_2^3 + 4 T_1^3 T_2^3 - 3 T_1^5 T_2^3 + 3 T_1^6 T_2^3 - 3 T_2^4 + \frac{3 T_2^4}{T_1} + 3 T_1^2 T_2^4 - 3 T_1^4 T_2^4 + 3 T_1^5 T_2^4 - \frac{3 T_2^5}{T_1} + 3 T_1 T_2^5 - 3 T_1^2 T_2^5 - 3 T_1^3 T_2^5 + 3 T_1^4 T_2^5 - 3 T_1^6 T_2^5 + 3 T_2^6 - 3 T_1 T_2^6 + 3 T_1^3 T_2^6 - 3 T_1^5 T_2^6 + 3 T_1^6 T_2^6 \right\} \right\}$$

Out[*]=

True

```
In[*]:= Timing[Expand@Theta_{T1, T2}[Knot[3, 1]]]
```

Out[*]=

$$\left\{ \theta., \left\{ \left\{ -1 + \frac{1}{T_1} + T_1, -1 + \frac{1}{T_2} + T_2, -1 + \frac{1}{T_1 T_2} + T_1 T_2 \right\}, -\frac{1}{T_1^2} - T_1^2 - \frac{1}{T_2^2} - \frac{1}{T_1^2 T_2^2} + \frac{1}{T_1 T_2^2} + \frac{1}{T_1^2 T_2} + \frac{T_1}{T_2} + \frac{T_2}{T_1} + T_1^2 T_2 - T_2^2 + T_1 T_2^2 - T_1^2 T_2^2 \right\} \right\}$$


```
In[*]:= K = Knot[4, 1]; Timing[Theta[K]]
TestSymmetries[K]
```

Out[*]=


$$\left\{ \theta., \left\{ -\frac{1 - 3T + T^2}{T}, \theta \right\} \right\}$$

Out[*]=

True

```
In[*]:= K = Knot["K11n34"]; Timing[Theta[K]]
TestSymmetries[K]
```

 KnotTheory: Loading precomputed data in DTCode4KnotsTo11`.

 KnotTheory: The GaussCode to PD conversion was written by Siddarth Sankaran at the University of Toronto in the summer of 2005.

Out[*]=

$$\left\{ \theta., \left\{ 1, -\frac{1}{T_1^6 T_2^6} \left(T_1^2 - 2 T_1^3 + T_1^4 - 2 T_1 T_2 + 2 T_1^2 T_2 + 2 T_1^5 T_2 - 2 T_1^6 T_2 + T_2^2 + 2 T_1 T_2^2 - 2 T_1^2 T_2^2 - 2 T_1^4 T_2^2 - 2 T_1^6 T_2^2 + 2 T_1^7 T_2^2 + T_1^8 T_2^2 - 2 T_2^3 + T_1^4 T_2^3 + T_1^5 T_2^3 - 2 T_1^9 T_2^3 + T_2^4 - 2 T_1^2 T_2^4 + T_1^3 T_2^4 + 2 T_1^4 T_2^4 + 2 T_1^6 T_2^4 + T_1^7 T_2^4 - 2 T_1^8 T_2^4 + T_1^{10} T_2^4 + 2 T_1 T_2^5 + T_1^3 T_2^5 - 4 T_1^5 T_2^5 - 4 T_1^6 T_2^5 + T_1^8 T_2^5 + 2 T_1^{10} T_2^5 - 2 T_1 T_2^6 - 2 T_1^2 T_2^6 + 2 T_1^4 T_2^6 - 4 T_1^5 T_2^6 + 12 T_1^6 T_2^6 - 4 T_1^7 T_2^6 + 2 T_1^8 T_2^6 - 2 T_1^{10} T_2^6 - 2 T_1^{11} T_2^6 + 2 T_1^2 T_2^7 + T_1^4 T_2^7 - 4 T_1^6 T_2^7 - 4 T_1^7 T_2^7 + T_1^9 T_2^7 + 2 T_1^{11} T_2^7 + T_1^2 T_2^8 - 2 T_1^4 T_2^8 + T_1^5 T_2^8 + 2 T_1^6 T_2^8 + 2 T_1^8 T_2^8 + T_1^9 T_2^8 - 2 T_1^{10} T_2^8 + T_1^{12} T_2^8 - 2 T_1^3 T_2^9 + T_1^7 T_2^9 + T_1^8 T_2^9 - 2 T_1^{12} T_2^9 + T_1^4 T_2^{10} + 2 T_1^5 T_2^{10} - 2 T_1^6 T_2^{10} - 2 T_1^8 T_2^{10} - 2 T_1^{10} T_2^{10} + 2 T_1^{11} T_2^{10} + T_1^{12} T_2^{10} - 2 T_1^6 T_2^{11} + 2 T_1^7 T_2^{11} + 2 T_1^{10} T_2^{11} - 2 T_1^{11} T_2^{11} + T_1^8 T_2^{12} - 2 T_1^9 T_2^{12} + T_1^{10} T_2^{12} \right) \right\} \right\}$$

Out[*]=

True

```
In[*]:= K = Knot["K11n42"]; Timing[Theta[K]]
TestSymmetries[K]
```

Out[*]=

$$\left\{ \theta.03125, \left\{ 1, \frac{1}{T_1^3 T_2^3} \left(T_1 + T_1^2 + T_2 - 2 T_1 T_2 - 2 T_1^2 T_2 - 2 T_1^3 T_2 + T_1^4 T_2 + T_2^2 - 2 T_1 T_2^2 + 2 T_1^2 T_2^2 + 2 T_1^3 T_2^2 - 2 T_1^4 T_2^2 + T_1^5 T_2^2 - 2 T_1 T_2^3 + 2 T_1^2 T_2^3 + 2 T_1^4 T_2^3 - 2 T_1^5 T_2^3 + T_1 T_2^4 - 2 T_1^2 T_2^4 + 2 T_1^3 T_2^4 + 2 T_1^4 T_2^4 - 2 T_1^5 T_2^4 + T_1^6 T_2^4 + T_1^2 T_2^5 - 2 T_1^3 T_2^5 - 2 T_1^4 T_2^5 - 2 T_1^5 T_2^5 + T_1^6 T_2^5 + T_1^4 T_2^6 + T_1^5 T_2^6 \right) \right\} \right\}$$

Out[*]=

True

```
In[*]:= PD[GST48] = PD[X[1, 15, 2, 14], X[29, 2, 30, 3], X[40, 4, 41, 3],
  X[4, 44, 5, 43], X[5, 26, 6, 27], X[95, 7, 96, 6], X[7, 1, 8, 96], X[8, 14, 9, 13],
  X[28, 9, 29, 10], X[41, 11, 42, 10], X[11, 43, 12, 42], X[12, 27, 13, 28],
  X[15, 31, 16, 30], X[61, 16, 62, 17], X[72, 17, 73, 18], X[83, 18, 84, 19],
  X[34, 20, 35, 19], X[20, 89, 21, 90], X[92, 21, 93, 22], X[22, 79, 23, 80],
  X[23, 68, 24, 69], X[24, 57, 25, 58], X[56, 25, 57, 26], X[31, 63, 32, 62],
  X[32, 74, 33, 73], X[33, 85, 34, 84], X[35, 50, 36, 51], X[81, 37, 82, 36],
  X[70, 38, 71, 37], X[59, 39, 60, 38], X[54, 39, 55, 40], X[55, 45, 56, 44],
  X[45, 59, 46, 58], X[46, 70, 47, 69], X[47, 81, 48, 80], X[91, 49, 92, 48],
  X[49, 91, 50, 90], X[82, 52, 83, 51], X[71, 53, 72, 52], X[60, 54, 61, 53],
  X[74, 63, 75, 64], X[85, 64, 86, 65], X[65, 76, 66, 77], X[66, 87, 67, 88],
  X[94, 67, 95, 68], X[86, 75, 87, 76], X[77, 88, 78, 89], X[93, 78, 94, 79]];
```

```
In[*]:= K = GST48; AbsoluteTiming[Short@θ[K]]
TestSymmetries[K]
```

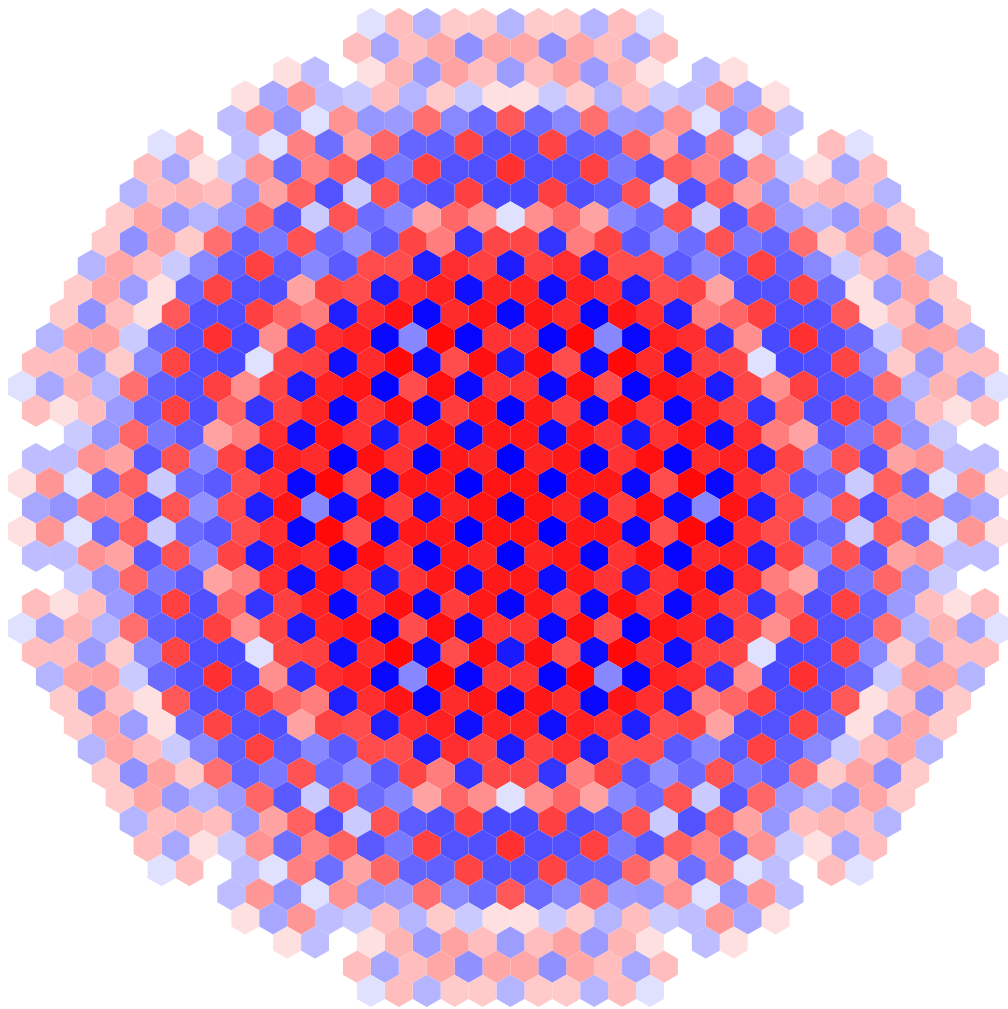
Out[*]=

$$\left\{ 13.5155, \left\{ -\frac{(-1 + 2T - T^2 - T^3 + 2T^4 - T^5 + T^8)(-1 + \langle\langle 7 \rangle\rangle + T^8)}{T^8}, \frac{\langle\langle 1764 \rangle\rangle + T_1^{\langle\langle 2 \rangle\rangle} \langle\langle 1 \rangle\rangle}{T_1^{20} T_2^{20}} \right\} \right\}$$

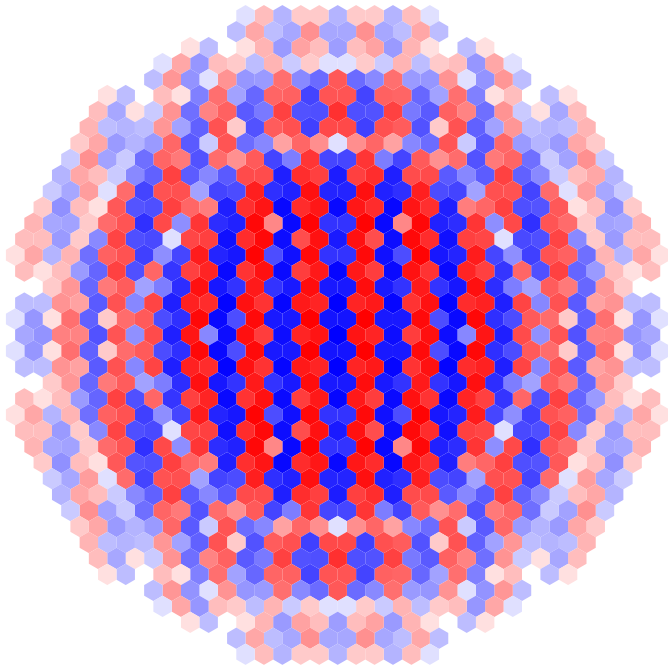
Out[*]=

True

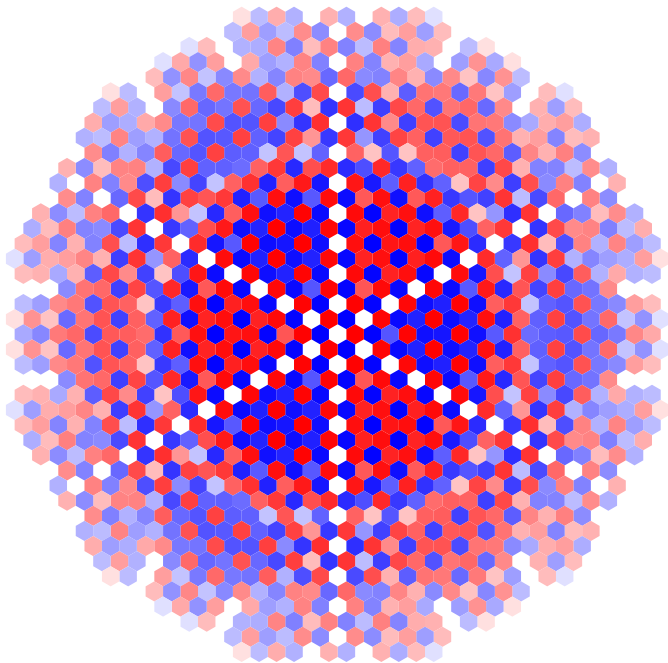
```
In[ ]:= PolyPlotT1,T2[- $\theta$ [GST48][[2]]]  
Out[ ]=
```



```
In[*]:= PolyPlotT1,T2[- $\theta$ [GST48][[2]] /. {T1 → T1, T2 → -T2}]
Out[*]=
```



```
In[*]:= PolyPlotT1,T2[ $\theta$ [GST48][[2]] (T1 + T2 - T3 - T1-1 - T2-1 + T3-1)]
Out[*]=
```



```
In[*]:= AbsoluteTiming[ $\theta$ T1,T2[GST48];]
Out[*]= {60.1527, Null}
```

In[*]:= AbsoluteTiming[$\theta_{22/7,34/21}$ [GST48]]

Out[*]=

$$\left\{ 0.440852, \left\{ \left\{ -\frac{1\,422\,357\,287\,561\,349\,859\,889}{10\,190\,414\,377\,180\,576}, -\frac{486\,885\,265\,100\,293\,177\,259\,569}{15\,915\,006\,754\,796\,041\,036\,704}, \right. \right. \\ \left. \left. -\frac{6\,215\,902\,990\,719\,340\,337\,664\,427\,997\,383\,765\,280\,900\,656\,009}{162\,180\,513\,646\,999\,558\,542\,864\,476\,199\,651\,861\,504} \right\}, \right. \\ \left. 21\,304\,335\,657\,502\,800\,961\,104\,521\,150\,882\,906\,491\,585\,928\,445\,141\,602\,977\,673\,772\,524\,921\,333\,287\,756 \ ; \right. \\ \left. 546\,740\,936\,248\,585\,046\,107\,073\,499 \ / \right. \\ \left. 4\,728\,039\,585\,290\,312\,086\,302\,386\,002\,441\,018\,010\,474\,726\,543\,601\,178\,518\,697\,564\,845\,087\,356\,778\,405 \ ; \right. \\ \left. 506\,577\,334\,272 \right\} \}$$

Systematic Testing

In[*]:= DuplicateFreeQ[θ /@ AllKnots[{3, 10}]]

Out[*]=

True

In[*]:= Total[TestSymmetries /@ AllKnots[{3, 10}]]

Out[*]=

249 True

In[*]:= DuplicateFreeQ[θ /@ AllKnots[{3, 12}]]

 KnotTheory: Loading precomputed data in KnotTheory/12A.dts.

 KnotTheory: Loading precomputed data in KnotTheory/12N.dts.


Out[*]=

False

In[*]:= tab11 = Table[K \rightarrow θ @K, {K, AllKnots[{3, 11}]}]

Out[*]=

$$\left\{ \text{Knot}[3, 1] \rightarrow \left\{ \frac{1-T+T^2}{T}, -\frac{1-T_1+T_1^2-T_2-T_1^3-T_2+T_1^4-T_1^3-T_2-T_1^4-T_2+T_1^4-T_1^3-T_2-T_1^4-T_2+T_1^4}{T_1^2 T_2^2} \right\}, \text{Knot}[4, 1] \rightarrow \left\{ -\frac{1-3T+T^2}{T}, \emptyset \right\}, \right. \\ \left. \text{Knot}[5, 1] \rightarrow \left\{ \frac{1-T+T^2-T^3+T^4}{T^2}, -\frac{\dots 53 \dots + 2 T_1^3 T_2^2}{T_1^4 T_2^2} \right\}, \dots 795 \dots, \text{Knot}[11, \text{NonAlternating}, 183] \rightarrow \left\{ \frac{\dots 1 \dots}{T^3}, \dots 1 \dots \right\}, \right. \\ \left. \text{Knot}[11, \text{NonAlternating}, 184] \rightarrow \left\{ \frac{(1-T+T^2)(2-7T+11T^2-7T^3+2T^4)}{T^3}, \frac{9-41T_1+92T_1^2-115T_1^3+\dots 166 \dots +92T_1^9 T_2^{12}-41T_1^{11} T_2^{12}+9T_1^{12} T_2^{12}}{T_1^9 T_2^9} \right\}, \right. \\ \left. \text{Knot}[11, \text{NonAlternating}, 185] \rightarrow \left\{ -\frac{(1-3T+T^2)(1-T+T^2)(2-3T+2T^2)}{T^3}, \right. \right. \\ \left. \left. -\frac{1}{T_1^9 T_2^9} \left(17-93T_1+202T_1^2-261T_1^3+202T_1^4-93T_1^5+17T_1^6-93T_2+416T_1T_2-593T_1^2T_2+321T_1^3T_2+\dots 153 \dots + \right. \right. \right. \\ \left. \left. \left. 416T_1^{11}T_2^{11}-93T_1^{12}T_2^{11}+17T_1^6T_2^{12}-93T_1^7T_2^{12}+202T_1^8T_2^{12}-261T_1^9T_2^{12}+202T_1^{10}T_2^{12}-93T_1^{11}T_2^{12}+17T_1^{12}T_2^{12} \right) \right\} \right\}$$

Full expression not available (original memory size: 33.2 MB) 

In[*]:= Gather[tab11, Last[#1] === Last[#2] &]

Out[*]=

$$\left\{ \left\{ \text{Knot}[3, 1] \rightarrow \left\{ \frac{1-T+T^2}{T}, -\frac{1-T_1+T_1^2-T_2-T_1^3 T_2+T_2^2+T_1^4 T_2^2-T_1 T_2^3-T_1^2 T_2^3+T_1^3 T_2^3-T_1^4 T_2^3+T_1^4 T_2^4}{T_1^2 T_2^2} \right\} \right\}, \left\{ \text{Knot}[4, 1] \rightarrow \left\{ -\frac{1-3T+T^2}{T}, \emptyset \right\} \right\}, \right.$$

$$\left. \left\{ \text{Knot}[5, 1] \rightarrow \left\{ \frac{1-T+T^2-T^3+T^4}{T^2}, -\frac{\dots 53 \dots +2 T_1^8 T_2^8}{T_1^4 T_2^4} \right\} \right\}, \dots 792 \dots, \left\{ \text{Knot}[11, \text{NonAlternating}, 183] \rightarrow \left\{ -\frac{\dots 1 \dots}{T^3}, \dots 1 \dots \right\} \right\}, \right.$$

$$\left\{ \text{Knot}[11, \text{NonAlternating}, 184] \rightarrow \left\{ \frac{(1-T+T^2)(2-7T+11T^2-7T^3+2T^4)}{T^3}, \frac{9-41 T_1+\dots 169 \dots +92 T_1^{10} T_2^{12}-41 T_1^{11} T_2^{12}+9 T_1^{12} T_2^{12}}{T_1^5 T_2^5} \right\} \right\},$$

$$\left\{ \text{Knot}[11, \text{NonAlternating}, 185] \rightarrow \left\{ -\frac{(1-3T+T^2)(1-T+T^2)(2-3T+2T^2)}{T^3}, \right.$$

$$\left. -\frac{1}{T_1^5 T_2^5} (17-93 T_1+202 T_1^2-261 T_1^3+202 T_1^4-93 T_1^5+17 T_1^6-93 T_2+416 T_1 T_2-593 T_1^2 T_2+321 T_1^3 T_2+\dots 153 \dots + \right.$$

$$\left. 416 T_1^{11} T_2^{11}-93 T_1^{12} T_2^{11}+17 T_1^6 T_2^{12}-93 T_1^7 T_2^{12}+202 T_1^8 T_2^{12}-261 T_1^9 T_2^{12}+202 T_1^{10} T_2^{12}-93 T_1^{11} T_2^{12}+17 T_1^{12} T_2^{12}) \right\} \right\}$$

Full expression not available (original memory size: 33.2 MB)

In[*]:= Select[Gather[tab11, Last[#1] === Last[#2] &], Length[#] > 1 &]

Out[*]=

$$\left\{ \left\{ \text{Knot}[11, \text{Alternating}, 44] \rightarrow \right.$$

$$\left. \left\{ \frac{(1-T+T^2)^2(1-3T+5T^2-3T^3+T^4)}{T^4}, -\frac{1}{T_1^6 T_2^6} 2(1-T_1+T_1^2)(1-T_2+T_2^2)(1-T_1 T_2+T_1^2 T_2^2) \right. \right.$$

$$\left. \left. \left(T_1-2 T_1^2+T_1^3+T_2-5 T_1 T_2+5 T_1^2 T_2+5 T_1^3 T_2-5 T_1^4 T_2+T_1^5 T_2-2 T_2^2+5 T_1 T_2^2+5 T_1^2 T_2^2-26 T_1^3 T_2^2+ \right. \right.$$

$$\left. \left. 5 T_1^4 T_2^2+5 T_1^5 T_2^2-2 T_1^6 T_2^2+T_2^3+5 T_1 T_2^3-26 T_1^2 T_2^3+32 T_1^3 T_2^3+32 T_1^4 T_2^3-26 T_1^5 T_2^3+5 T_1^6 T_2^3+T_1^7 T_2^3- \right. \right.$$

$$\left. \left. 5 T_1 T_2^4+5 T_1^2 T_2^4+32 T_1^3 T_2^4-96 T_1^4 T_2^4+32 T_1^5 T_2^4+5 T_1^6 T_2^4-5 T_1^7 T_2^4+T_1 T_2^5+5 T_1^2 T_2^5-26 T_1^3 T_2^5+ \right. \right.$$

$$\left. \left. 32 T_1^4 T_2^5+32 T_1^5 T_2^5-26 T_1^6 T_2^5+5 T_1^7 T_2^5+T_1^8 T_2^5-2 T_1 T_2^6+5 T_1^3 T_2^6+5 T_1^4 T_2^6-26 T_1^5 T_2^6+5 T_1^6 T_2^6+ \right. \right.$$

$$\left. \left. 5 T_1^7 T_2^6-2 T_1^8 T_2^6+T_1^3 T_2^7-5 T_1^4 T_2^7+5 T_1^5 T_2^7+5 T_1^6 T_2^7-5 T_1^7 T_2^7+T_1^8 T_2^7+T_1^5 T_2^8-2 T_1^6 T_2^8+T_1^7 T_2^8) \right\} \right\},$$

$$\text{Knot}[11, \text{Alternating}, 47] \rightarrow \left\{ \frac{(1-T+T^2)^2(1-3T+5T^2-3T^3+T^4)}{T^4}, \right.$$

$$\left. -\frac{1}{T_1^6 T_2^6} 2(1-T_1+T_1^2)(1-T_2+T_2^2)(1-T_1 T_2+T_1^2 T_2^2) \right.$$

$$\left. \left. \left(T_1-2 T_1^2+T_1^3+T_2-5 T_1 T_2+5 T_1^2 T_2+5 T_1^3 T_2-5 T_1^4 T_2+T_1^5 T_2-2 T_2^2+5 T_1 T_2^2+5 T_1^2 T_2^2-26 T_1^3 T_2^2+ \right. \right.$$

$$\left. \left. 5 T_1^4 T_2^2+5 T_1^5 T_2^2-2 T_1^6 T_2^2+T_2^3+5 T_1 T_2^3-26 T_1^2 T_2^3+32 T_1^3 T_2^3+32 T_1^4 T_2^3-26 T_1^5 T_2^3+5 T_1^6 T_2^3+T_1^7 T_2^3- \right. \right.$$

$$\left. \left. 5 T_1 T_2^4+5 T_1^2 T_2^4+32 T_1^3 T_2^4-96 T_1^4 T_2^4+32 T_1^5 T_2^4+5 T_1^6 T_2^4-5 T_1^7 T_2^4+T_1 T_2^5+5 T_1^2 T_2^5-26 T_1^3 T_2^5+ \right. \right.$$

$$\left. \left. 32 T_1^4 T_2^5+32 T_1^5 T_2^5-26 T_1^6 T_2^5+5 T_1^7 T_2^5+T_1^8 T_2^5-2 T_1 T_2^6+5 T_1^3 T_2^6+5 T_1^4 T_2^6-26 T_1^5 T_2^6+5 T_1^6 T_2^6+ \right. \right.$$

$$\left. \left. 5 T_1^7 T_2^6-2 T_1^8 T_2^6+T_1^3 T_2^7-5 T_1^4 T_2^7+5 T_1^5 T_2^7+5 T_1^6 T_2^7-5 T_1^7 T_2^7+T_1^8 T_2^7+T_1^5 T_2^8-2 T_1^6 T_2^8+T_1^7 T_2^8) \right\} \right\},$$

$$\left\{ \text{Knot}[11, \text{Alternating}, 57] \rightarrow \left\{ -\frac{(1-T+T^2)^2(1-3T+3T^2-3T^3+T^4)}{T^4}, \right.$$

$$\left. \frac{1}{T_1^8 T_2^8} (1-T_1+T_1^2)(1-T_2+T_2^2)(1-T_1 T_2+T_1^2 T_2^2) \right.$$

$$\left. \left. \left(1-4 T_1+7 T_1^2-9 T_1^3+7 T_1^4-4 T_1^5+T_1^6-4 T_2+12 T_1 T_2-12 T_1^2 T_2+8 T_1^3 T_2+8 T_1^4 T_2-12 T_1^5 T_2+ \right. \right.$$

$$\left. \left. 12 T_1^6 T_2-4 T_1^7 T_2+7 T_2^2-12 T_1 T_2^2-8 T_1^2 T_2^2+25 T_1^3 T_2^2-52 T_1^4 T_2^2+25 T_1^5 T_2^2-8 T_1^6 T_2^2-12 T_1^7 T_2^2+ \right. \right.$$

$$\left. \left. 7 T_1^8 T_2^2-9 T_2^3+8 T_1 T_2^3+25 T_1^2 T_2^3-32 T_1^3 T_2^3+37 T_1^4 T_2^3+37 T_1^5 T_2^3-32 T_1^6 T_2^3+25 T_1^7 T_2^3+8 T_1^8 T_2^3- \right. \right.$$

$$\left. \left. 9 T_1^9 T_2^3+7 T_2^4+8 T_1 T_2^4-52 T_1^2 T_2^4+37 T_1^3 T_2^4-6 T_1^4 T_2^4-68 T_1^5 T_2^4-6 T_1^6 T_2^4+37 T_1^7 T_2^4-52 T_1^8 T_2^4+ \right. \right.$$

$$\left. \left. 8 T_1^9 T_2^4+7 T_1^{10} T_2^4-4 T_2^5-12 T_1 T_2^5+25 T_1^2 T_2^5+37 T_1^3 T_2^5-68 T_1^4 T_2^5+66 T_1^5 T_2^5+66 T_1^6 T_2^5-68 T_1^7 T_2^5+ \right. \right.$$

$$\left. \left. 37 T_1^8 T_2^5+25 T_1^9 T_2^5-12 T_1^{10} T_2^5-4 T_1^{11} T_2^5+T_2^6+12 T_1 T_2^6-8 T_1^2 T_2^6-32 T_1^3 T_2^6-6 T_1^4 T_2^6+66 T_1^5 T_2^6- \right. \right.$$

$$\begin{aligned} & 156 T_1^6 T_2^6 + 66 T_1^7 T_2^6 - 6 T_1^8 T_2^6 - 32 T_1^9 T_2^6 - 8 T_1^{10} T_2^6 + 12 T_1^{11} T_2^6 + T_1^{12} T_2^6 - 4 T_1 T_2^7 - 12 T_1^2 T_2^7 + \\ & 25 T_1^3 T_2^7 + 37 T_1^4 T_2^7 - 68 T_1^5 T_2^7 + 66 T_1^6 T_2^7 + 66 T_1^7 T_2^7 - 68 T_1^8 T_2^7 + 37 T_1^9 T_2^7 + 25 T_1^{10} T_2^7 - 12 T_1^{11} T_2^7 - \\ & 4 T_1^{12} T_2^7 + 7 T_1^2 T_2^8 + 8 T_1^3 T_2^8 - 52 T_1^4 T_2^8 + 37 T_1^5 T_2^8 - 6 T_1^6 T_2^8 - 68 T_1^7 T_2^8 - 6 T_1^8 T_2^8 + 37 T_1^9 T_2^8 - 52 T_1^{10} T_2^8 + \\ & 8 T_1^{11} T_2^8 + 7 T_1^{12} T_2^8 - 9 T_1^3 T_2^9 + 8 T_1^4 T_2^9 + 25 T_1^5 T_2^9 - 32 T_1^6 T_2^9 + 37 T_1^7 T_2^9 + 37 T_1^8 T_2^9 - 32 T_1^9 T_2^9 + \\ & 25 T_1^{10} T_2^9 + 8 T_1^{11} T_2^9 - 9 T_1^{12} T_2^9 + 7 T_1^4 T_2^{10} - 12 T_1^5 T_2^{10} - 8 T_1^6 T_2^{10} + 25 T_1^7 T_2^{10} - 52 T_1^8 T_2^{10} + 25 T_1^9 T_2^{10} - \\ & 8 T_1^{10} T_2^{10} - 12 T_1^{11} T_2^{10} + 7 T_1^{12} T_2^{10} - 4 T_1^5 T_2^{11} + 12 T_1^6 T_2^{11} - 12 T_1^7 T_2^{11} + 8 T_1^8 T_2^{11} + 8 T_1^9 T_2^{11} - 12 T_1^{10} T_2^{11} + \\ & 12 T_1^{11} T_2^{11} - 4 T_1^{12} T_2^{11} + T_1^6 T_2^{12} - 4 T_1^7 T_2^{12} + 7 T_1^8 T_2^{12} - 9 T_1^9 T_2^{12} + 7 T_1^{10} T_2^{12} - 4 T_1^{11} T_2^{12} + T_1^{12} T_2^{12} \} , \end{aligned}$$

$$\text{Knot}[11, \text{Alternating}, 231] \rightarrow \left\{ -\frac{(1 - T + T^2)^2 (1 - 3T + 3T^2 - 3T^3 + T^4)}{T^4}, \right.$$

$$\left. \frac{1}{T_1^8 T_2^8} (1 - T_1 + T_1^2) (1 - T_2 + T_2^2) (1 - T_1 T_2 + T_1^2 T_2^2) \right.$$

$$\begin{aligned} & (1 - 4 T_1 + 7 T_1^2 - 9 T_1^3 + 7 T_1^4 - 4 T_1^5 + T_1^6 - 4 T_2 + 12 T_1 T_2 - 12 T_1^2 T_2 + 8 T_1^3 T_2 + 8 T_1^4 T_2 - 12 T_1^5 T_2 + \\ & 12 T_1^6 T_2 - 4 T_1^7 T_2 + 7 T_2^2 - 12 T_1 T_2^2 - 8 T_1^2 T_2^2 + 25 T_1^3 T_2^2 - 52 T_1^4 T_2^2 + 25 T_1^5 T_2^2 - 8 T_1^6 T_2^2 - 12 T_1^7 T_2^2 + \\ & 7 T_1^8 T_2^2 - 9 T_2^3 + 8 T_1 T_2^3 + 25 T_1^2 T_2^3 - 32 T_1^3 T_2^3 + 37 T_1^4 T_2^3 + 37 T_1^5 T_2^3 - 32 T_1^6 T_2^3 + 25 T_1^7 T_2^3 + 8 T_1^8 T_2^3 - \\ & 9 T_1^9 T_2^3 + 7 T_2^4 + 8 T_1 T_2^4 - 52 T_1^2 T_2^4 + 37 T_1^3 T_2^4 - 6 T_1^4 T_2^4 - 68 T_1^5 T_2^4 - 6 T_1^6 T_2^4 + 37 T_1^7 T_2^4 - 52 T_1^8 T_2^4 + \\ & 8 T_1^9 T_2^4 + 7 T_1^{10} T_2^4 - 4 T_2^5 - 12 T_1 T_2^5 + 25 T_1^2 T_2^5 + 37 T_1^3 T_2^5 - 68 T_1^4 T_2^5 + 66 T_1^5 T_2^5 + 66 T_1^6 T_2^5 - 68 T_1^7 T_2^5 + \\ & 37 T_1^8 T_2^5 + 25 T_1^9 T_2^5 - 12 T_1^{10} T_2^5 - 4 T_1^{11} T_2^5 + T_2^6 + 12 T_1 T_2^6 - 8 T_1^2 T_2^6 - 32 T_1^3 T_2^6 - 6 T_1^4 T_2^6 + 66 T_1^5 T_2^6 - \\ & 156 T_1^6 T_2^6 + 66 T_1^7 T_2^6 - 6 T_1^8 T_2^6 - 32 T_1^9 T_2^6 - 8 T_1^{10} T_2^6 + 12 T_1^{11} T_2^6 + T_1^{12} T_2^6 - 4 T_1 T_2^7 - 12 T_1^2 T_2^7 + \\ & 25 T_1^3 T_2^7 + 37 T_1^4 T_2^7 - 68 T_1^5 T_2^7 + 66 T_1^6 T_2^7 + 66 T_1^7 T_2^7 - 68 T_1^8 T_2^7 + 37 T_1^9 T_2^7 + 25 T_1^{10} T_2^7 - 12 T_1^{11} T_2^7 - \\ & 4 T_1^{12} T_2^7 + 7 T_1^2 T_2^8 + 8 T_1^3 T_2^8 - 52 T_1^4 T_2^8 + 37 T_1^5 T_2^8 - 6 T_1^6 T_2^8 - 68 T_1^7 T_2^8 - 6 T_1^8 T_2^8 + 37 T_1^9 T_2^8 - 52 T_1^{10} T_2^8 + \\ & 8 T_1^{11} T_2^8 + 7 T_1^{12} T_2^8 - 9 T_1^3 T_2^9 + 8 T_1^4 T_2^9 + 25 T_1^5 T_2^9 - 32 T_1^6 T_2^9 + 37 T_1^7 T_2^9 + 37 T_1^8 T_2^9 - 32 T_1^9 T_2^9 + \\ & 25 T_1^{10} T_2^9 + 8 T_1^{11} T_2^9 - 9 T_1^{12} T_2^9 + 7 T_1^4 T_2^{10} - 12 T_1^5 T_2^{10} - 8 T_1^6 T_2^{10} + 25 T_1^7 T_2^{10} - 52 T_1^8 T_2^{10} + 25 T_1^9 T_2^{10} - \\ & 8 T_1^{10} T_2^{10} - 12 T_1^{11} T_2^{10} + 7 T_1^{12} T_2^{10} - 4 T_1^5 T_2^{11} + 12 T_1^6 T_2^{11} - 12 T_1^7 T_2^{11} + 8 T_1^8 T_2^{11} + 8 T_1^9 T_2^{11} - 12 T_1^{10} T_2^{11} + \\ & 12 T_1^{11} T_2^{11} - 4 T_1^{12} T_2^{11} + T_1^6 T_2^{12} - 4 T_1^7 T_2^{12} + 7 T_1^8 T_2^{12} - 9 T_1^9 T_2^{12} + 7 T_1^{10} T_2^{12} - 4 T_1^{11} T_2^{12} + T_1^{12} T_2^{12} \} \} , \end{aligned}$$

$$\left\{ \text{Knot}[11, \text{NonAlternating}, 73] \rightarrow \left\{ \frac{(1 - T + T^2)^2}{T^2}, \right.$$

$$\left. - \frac{2 (1 - T_1 + T_1^2) (1 - T_2 + T_2^2) (1 - T_1 T_2 + T_1^2 T_2^2) (1 + T_1 + T_2 - 6 T_1 T_2 + T_1^2 T_2 + T_1 T_2^2 + T_1^2 T_2^2)}{T_1^3 T_2^3} \right\} ,$$

$$\text{Knot}[11, \text{NonAlternating}, 74] \rightarrow$$

$$\left\{ \frac{(1 - T + T^2)^2}{T^2}, \right. \\ \left. - \frac{2 (1 - T_1 + T_1^2) (1 - T_2 + T_2^2) (1 - T_1 T_2 + T_1^2 T_2^2) (1 + T_1 + T_2 - 6 T_1 T_2 + T_1^2 T_2 + T_1 T_2^2 + T_1^2 T_2^2)}{T_1^3 T_2^3} \right\} \} \}$$

In[*]:= `tab12 = Table[K -> e@K, {K, AllKnots[{3, 12]}]}`

 KnotTheory: Loading precomputed data in KnotTheory/12A.dts.

 KnotTheory: Loading precomputed data in KnotTheory/12N.dts.

Out[]=

$$\left\{ \text{Knot}[3, 1] \rightarrow \left\{ \frac{1-T+T^2}{T}, -\frac{1-T_1+T_1^2-T_2-T_1^3 T_2+T_1^4 T_2^2-T_1 T_2^3-T_1^4 T_2^3+T_1^5 T_2^4-T_1^4 T_2^4+T_1^4 T_2^4}{T_1^2 T_2^2} \right\}, \text{Knot}[4, 1] \rightarrow \left\{ -\frac{1-3T+T^2}{T}, 0 \right\}, \right.$$

$$\text{Knot}[5, 1] \rightarrow \left\{ \frac{1-T+T^2-T^3+T^4}{T^2}, -\frac{\dots 53 \dots + 2 T_1^8 T_2^8}{T_1^4 T_2^4} \right\}, \dots 2971 \dots, \text{Knot}[12, \text{NonAlternating}, 886] \rightarrow \left\{ \frac{\dots 1 \dots}{T^3}, \dots 1 \dots \right\},$$

$$\text{Knot}[12, \text{NonAlternating}, 887] \rightarrow \left\{ \frac{1-6T+16T^2-25T^3+29T^4-25T^5+16T^6-6T^7+T^8}{T^4}, \frac{2-12T_1+\dots 327 \dots + 2 T_1^{16} T_2^{16}}{T_1^8 T_2^8} \right\},$$

$$\text{Knot}[12, \text{NonAlternating}, 888] \rightarrow \left\{ \frac{(1-T+T^2)^2 (1+T-2T^2+T^3-2T^4+T^5+T^6)}{T^5}, \frac{1}{T_1^{10} T_2^{10}} \right.$$

$$\left. \left((1-T_1+T_1^2) (1-T_2+T_2^2) (\dots 1 \dots) (5-10T_1^2+20T_1^3-25T_1^4+20T_1^5-10T_1^6+5T_1^8-10T_2^2+11T_1^2 T_2^2-39T_1^3 T_2^2 + \dots 208 \dots + 11T_1^{14} T_2^{14} - 10T_1^{16} T_2^{14} + 5T_1^8 T_2^{16} - 10T_1^{10} T_2^{16} + 20T_1^{11} T_2^{16} - 25T_1^{12} T_2^{16} + 20T_1^{13} T_2^{16} - 10T_1^{14} T_2^{16} + 5T_1^{16} T_2^{16}) \right\} \right\}$$

Full expression not available (original memory size: 150.5 MB)

In[]:= **dup12 = Map[First, Select[Gather[tab12, Last[#1] === Last[#2] &], Length[#] > 1 &], {2}]**

Out[]=

- {Knot[10, 106], Knot[12, NonAlternating, 369]},
- {Knot[11, Alternating, 44], Knot[11, Alternating, 47]},
- {Knot[11, Alternating, 57], Knot[11, Alternating, 231]},
- {Knot[11, NonAlternating, 73], Knot[11, NonAlternating, 74]},
- {Knot[12, Alternating, 30], Knot[12, Alternating, 33]},
- {Knot[12, Alternating, 122], Knot[12, Alternating, 182]},
- {Knot[12, Alternating, 164], Knot[12, Alternating, 166]},
- {Knot[12, Alternating, 167], Knot[12, Alternating, 692]},
- {Knot[12, Alternating, 273], Knot[12, Alternating, 890]},
- {Knot[12, Alternating, 341], Knot[12, Alternating, 627]},
- {Knot[12, Alternating, 427], Knot[12, Alternating, 435], Knot[12, Alternating, 990]},
- {Knot[12, Alternating, 458], Knot[12, Alternating, 887]},
- {Knot[12, Alternating, 510], Knot[12, Alternating, 821]},
- {Knot[12, NonAlternating, 56], Knot[12, NonAlternating, 57]},
- {Knot[12, NonAlternating, 60], Knot[12, NonAlternating, 61]},
- {Knot[12, NonAlternating, 62], Knot[12, NonAlternating, 66]},
- {Knot[12, NonAlternating, 144], Knot[12, NonAlternating, 507]},
- {Knot[12, NonAlternating, 313], Knot[12, NonAlternating, 430]}

In[]:= **Length /@ dup12**

Out[]=

- {2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 2, 2, 2, 2, 2, 2, 2}

In[]:= **Total[(Length /@ dup12) - 1]**

Out[]=

19

In[]:= **Length /@ Select[Gather[tab12 /. {T1 -> 22 / 7, T2 -> 13 / 21}, Last[#1] === Last[#2] &], Length[#] > 1 &]**

Out[]=

- {2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 2, 2, 2, 2, 2, 2, 2}

In[]:= **Put[tab12 /. {T1 -> T1, T2 -> T2}, "Data12.m"]**

Ribbon Knots

In[*]:= Table[K → θ [K],

{K, {Knot[6, 1], Knot[8, 8], Knot[8, 9], Knot[8, 20], Knot[9, 27], Knot[9, 41],
Knot[9, 46], Knot[10, 3], Knot[10, 22], Knot[10, 35], Knot[10, 42], Knot[10, 48],
Knot[10, 75], Knot[10, 87], Knot[10, 99], Knot[10, 123], Knot[10, 129],
Knot[10, 137], Knot[10, 140], Knot[10, 153], Knot[10, 155]}}

Out[*]=

$$\left\{ \text{Knot}[6, 1] \rightarrow \left\{ -\frac{(-2 + T)(-1 + 2T)}{T}, \frac{1}{T_1^2 T_2^2} \left(1 - 3T_1 + T_1^2 - 3T_2 + 6T_1 T_2 + 6T_1^2 T_2 - 3T_1^3 T_2 + T_2^2 + 6T_1 T_2^2 - 24T_1^2 T_2^2 + 6T_1^3 T_2^2 + T_1^4 T_2^2 - 3T_1 T_2^3 + 6T_1^2 T_2^3 + 6T_1^3 T_2^3 - 3T_1^4 T_2^3 + T_1^2 T_2^4 - 3T_1^3 T_2^4 + T_1^4 T_2^4 \right) \right\}, \right.$$

$$\text{Knot}[8, 8] \rightarrow \left\{ \frac{(2 - 2T + T^2)(1 - 2T + 2T^2)}{T^2}, \frac{1}{T_1^4 T_2^2} \left(1 - 3T_1 + 5T_1^2 - 3T_1^3 + T_1^4 - 3T_2 + 6T_1 T_2 - 6T_1^2 T_2 - 6T_1^3 T_2 + 6T_1^4 T_2 - 3T_1^5 T_2 + 5T_2^2 - 6T_1 T_2^2 + 9T_1^2 T_2^2 + 5T_1^3 T_2^2 + 9T_1^4 T_2^2 - 6T_1^5 T_2^2 + 5T_1^6 T_2^2 - 3T_2^3 - 6T_1 T_2^3 + 5T_1^2 T_2^3 - 18T_1^3 T_2^3 - 18T_1^4 T_2^3 + 5T_1^5 T_2^3 - 6T_1^6 T_2^3 - 3T_1^7 T_2^3 + T_2^4 + 6T_1 T_2^4 + 9T_1^2 T_2^4 - 18T_1^3 T_2^4 + 60T_1^4 T_2^4 - 18T_1^5 T_2^4 + 9T_1^6 T_2^4 + 6T_1^7 T_2^4 + T_2^5 + 8T_1 T_2^5 - 3T_1^2 T_2^5 - 6T_1^3 T_2^5 + 5T_1^4 T_2^5 - 18T_1^5 T_2^5 - 18T_1^6 T_2^5 + 5T_1^7 T_2^5 - 6T_1^8 T_2^5 + 5T_1^9 T_2^5 - 6T_1^{10} T_2^5 + 9T_1^{11} T_2^5 - 6T_1^{12} T_2^5 + 5T_1^{13} T_2^5 - 3T_2^6 + 6T_1 T_2^6 - 6T_1^2 T_2^6 - 6T_1^3 T_2^6 + 6T_1^4 T_2^6 - 3T_1^5 T_2^6 + T_2^7 + 4T_1 T_2^7 - 3T_1^2 T_2^7 + T_1^3 T_2^7 - 3T_1^4 T_2^7 + T_1^5 T_2^7 \right) \right\},$$

$$\text{Knot}[8, 9] \rightarrow \left\{ -\frac{(-1 + T - 2T^2 + T^3)(-1 + 2T - T^2 + T^3)}{T^3}, \emptyset \right\},$$

$$\text{Knot}[8, 20] \rightarrow \left\{ \frac{(1 - T + T^2)^2}{T^2}, -\frac{1}{T_1^2 T_2^2} 2 \left(3 - 4T_1 + 3T_1^2 - 4T_2 + T_1 T_2 + T_1^2 T_2 - 4T_1^3 T_2 + 3T_2^2 + T_1 T_2^2 + T_1^3 T_2^2 + 3T_1^4 T_2^2 - 4T_1 T_2^3 + T_1^2 T_2^3 + T_1^3 T_2^3 - 4T_1^4 T_2^3 + 3T_1^5 T_2^3 - 4T_1^6 T_2^3 + 3T_1^7 T_2^3 \right) \right\},$$

$$\text{Knot}[9, 27] \rightarrow \left\{ -\frac{(-1 + 2T - 3T^2 + T^3)(-1 + 3T - 2T^2 + T^3)}{T^3}, \right.$$

$$-\frac{1}{T_1^4 T_2^4} \left(1 - T_1 + T_1^2 - T_1^3 + T_1^4 - T_2 - 8T_1 T_2 + 4T_1^2 T_2 + 4T_1^3 T_2 - 8T_1^4 T_2 - T_1^5 T_2 + T_2^2 + 4T_1 T_2^2 + 49T_1^2 T_2^2 - 67T_1^3 T_2^2 + 49T_1^4 T_2^2 + 4T_1^5 T_2^2 + T_1^6 T_2^2 - T_2^3 + 4T_1 T_2^3 - 67T_1^2 T_2^3 + 20T_1^3 T_2^3 + 20T_1^4 T_2^3 - 67T_1^5 T_2^3 + 4T_1^6 T_2^3 - T_1^7 T_2^3 + T_2^4 - 8T_1 T_2^4 + 49T_1^2 T_2^4 + 20T_1^3 T_2^4 - 12T_1^4 T_2^4 + 20T_1^5 T_2^4 + 49T_1^6 T_2^4 - 8T_1^7 T_2^4 + T_1^8 T_2^4 - T_1 T_2^5 + 4T_1^2 T_2^5 - 67T_1^3 T_2^5 + 20T_1^4 T_2^5 + 20T_1^5 T_2^5 - 67T_1^6 T_2^5 + 4T_1^7 T_2^5 - T_1^8 T_2^5 + T_1^9 T_2^5 + 4T_1^{10} T_2^5 - 67T_1^{11} T_2^5 + 49T_1^{12} T_2^5 + 4T_1^{13} T_2^5 - T_1^{14} T_2^5 - 8T_1^{15} T_2^5 + 4T_1^{16} T_2^5 + 4T_1^{17} T_2^5 - 8T_1^{18} T_2^5 - T_1^{19} T_2^5 + T_1^{20} T_2^5 - T_1^{21} T_2^5 + T_1^{22} T_2^5 \right), \text{Knot}[9, 41] \rightarrow$$

$$\left\{ \frac{(3 - 3T + T^2)(1 - 3T + 3T^2)}{T^2}, -\frac{1}{T_1^4 T_2^4} \left(3 - 15T_1 + 27T_1^2 - 15T_1^3 + 3T_1^4 - 15T_2 + 58T_1 T_2 - 56T_1^2 T_2 - 56T_1^3 T_2 + 58T_1^4 T_2 - 15T_1^5 T_2 + 27T_2^2 - 56T_1 T_2^2 - 81T_1^2 T_2^2 + 333T_1^3 T_2^2 - 81T_1^4 T_2^2 - 56T_1^5 T_2^2 + 27T_1^6 T_2^2 - 15T_2^3 - 56T_1 T_2^3 + 333T_1^2 T_2^3 - 396T_1^3 T_2^3 - 396T_1^4 T_2^3 + 333T_1^5 T_2^3 - 56T_1^6 T_2^3 - 15T_1^7 T_2^3 + 3T_2^4 + 58T_1 T_2^4 - 81T_1^2 T_2^4 - 396T_1^3 T_2^4 + 1188T_1^4 T_2^4 - 396T_1^5 T_2^4 - 81T_1^6 T_2^4 + 58T_1^7 T_2^4 + 3T_1^8 T_2^4 - 15T_1 T_2^5 - 56T_1^2 T_2^5 + 333T_1^3 T_2^5 - 396T_1^4 T_2^5 - 396T_1^5 T_2^5 + 333T_1^6 T_2^5 - 56T_1^7 T_2^5 - 15T_1^8 T_2^5 + \right.$$

$$27 T_1^2 T_2^6 - 56 T_1^3 T_2^6 - 81 T_1^4 T_2^6 + 333 T_1^5 T_2^6 - 81 T_1^6 T_2^6 - 56 T_1^7 T_2^6 + 27 T_1^8 T_2^6 - 15 T_1^3 T_2^7 + 58 T_1^4 T_2^7 - 56 T_1^5 T_2^7 - 56 T_1^6 T_2^7 + 58 T_1^7 T_2^7 - 15 T_1^8 T_2^7 + 3 T_1^4 T_2^8 - 15 T_1^5 T_2^8 + 27 T_1^6 T_2^8 - 15 T_1^7 T_2^8 + 3 T_1^8 T_2^8 \},$$

$$\text{Knot}[9, 46] \rightarrow \left\{ -\frac{(-2+T)(-1+2T)}{T}, \frac{1}{T_1^2 T_2^2} 3(1-3T_1+T_1^2-3T_2+6T_1 T_2+6T_1^2 T_2-3T_1^3 T_2+T_2^2+6T_1 T_2^2-24T_1^2 T_2^2+6T_1^3 T_2^2+T_1^4 T_2^2-3T_1 T_2^3+6T_1^2 T_2^3+6T_1^3 T_2^3-3T_1^4 T_2^3+T_1^2 T_2^4-3T_1^3 T_2^4+T_1^4 T_2^4) \right\},$$

$$\text{Knot}[10, 3] \rightarrow \left\{ -\frac{(-3+2T)(-2+3T)}{T}, \frac{1}{T_1^2 T_2^2} (45-101T_1+45T_1^2-101T_2+126T_1 T_2+126T_1^2 T_2-101T_1^3 T_2+45T_1^4 T_2-101T_1 T_2^3+126T_1^2 T_2^3+126T_1^3 T_2^3-101T_1^4 T_2^3+45T_1^2 T_2^4-101T_1^3 T_2^4+45T_1^4 T_2^4) \right\},$$

$$\text{Knot}[10, 22] \rightarrow \left\{ -\frac{(-2+2T-2T^2+T^3)(-1+2T-2T^2+2T^3)}{T^3}, \right.$$

$$-\frac{1}{T_1^6 T_2^6} (1-3T_1+5T_1^2-7T_1^3+5T_1^4-3T_1^5+T_1^6-3T_2+6T_1 T_2-6T_1^2 T_2+6T_1^3 T_2+6T_1^4 T_2-6T_1^5 T_2+6T_1^6 T_2-3T_1^7 T_2+5T_1^8 T_2-6T_1 T_2^2+3T_1^2 T_2^2-9T_1^3 T_2^2-5T_1^4 T_2^2-9T_1^5 T_2^2+3T_1^6 T_2^2-6T_1^7 T_2^2+5T_1^8 T_2^2-7T_1^3 T_2^3+6T_1 T_2^3-9T_1^2 T_2^3+30T_1^3 T_2^3+4T_1^4 T_2^3+4T_1^5 T_2^3+30T_1^6 T_2^3-9T_1^7 T_2^3+6T_1^8 T_2^3-7T_1^9 T_2^3+5T_1^4 T_2^4+6T_1 T_2^4-5T_1^2 T_2^4+4T_1^3 T_2^4-89T_1^4 T_2^4+63T_1^5 T_2^4-89T_1^6 T_2^4+4T_1^7 T_2^4-5T_1^8 T_2^4+6T_1^9 T_2^4+5T_1^{10} T_2^4-3T_2^5-6T_1 T_2^5-9T_1^2 T_2^5+4T_1^3 T_2^5+63T_1^4 T_2^5+22T_1^5 T_2^5+22T_1^6 T_2^5+63T_1^7 T_2^5+4T_1^8 T_2^5-9T_1^9 T_2^5-6T_1^{10} T_2^5-3T_1^{11} T_2^5+T_2^6+6T_1 T_2^6+3T_1^2 T_2^6+30T_1^3 T_2^6-89T_1^4 T_2^6+22T_1^5 T_2^6-108T_1^6 T_2^6+22T_1^7 T_2^6-89T_1^8 T_2^6+30T_1^9 T_2^6+3T_1^{10} T_2^6+6T_1^{11} T_2^6+T_1^{12} T_2^6-3T_1 T_2^7-6T_1^2 T_2^7-9T_1^3 T_2^7+4T_1^4 T_2^7+63T_1^5 T_2^7+22T_1^6 T_2^7+22T_1^7 T_2^7+63T_1^8 T_2^7+4T_1^9 T_2^7-9T_1^{10} T_2^7-6T_1^{11} T_2^7-3T_1^{12} T_2^7+5T_1^2 T_2^8+6T_1^3 T_2^8-5T_1^4 T_2^8+4T_1^5 T_2^8-89T_1^6 T_2^8+63T_1^7 T_2^8-89T_1^8 T_2^8+4T_1^9 T_2^8-5T_1^{10} T_2^8+6T_1^{11} T_2^8+5T_1^{12} T_2^8-7T_1^3 T_2^9+6T_1^4 T_2^9-9T_1^5 T_2^9+30T_1^6 T_2^9+4T_1^7 T_2^9+4T_1^8 T_2^9+30T_1^9 T_2^9-9T_1^{10} T_2^9+6T_1^{11} T_2^9-7T_1^{12} T_2^9+5T_1^4 T_2^{10}-6T_1^5 T_2^{10}+3T_1^6 T_2^{10}-9T_1^7 T_2^{10}-5T_1^8 T_2^{10}-9T_1^9 T_2^{10}+3T_1^{10} T_2^{10}-6T_1^{11} T_2^{10}+5T_1^{12} T_2^{10}-3T_1^5 T_2^{11}+6T_1^6 T_2^{11}-6T_1^7 T_2^{11}+6T_1^8 T_2^{11}+6T_1^9 T_2^{11}-6T_1^{10} T_2^{11}+6T_1^{11} T_2^{11}-3T_1^{12} T_2^{11}+T_1^6 T_2^{12}-3T_1^7 T_2^{12}+5T_1^8 T_2^{12}-7T_1^9 T_2^{12}+5T_1^{10} T_2^{12}-3T_1^{11} T_2^{12}+T_1^{12} T_2^{12}) \},$$

$$\text{Knot}[10, 35] \rightarrow \left\{ \frac{(2-4T+T^2)(1-4T+2T^2)}{T^2}, \frac{1}{T_1^4 T_2^4} (1-7T_1+13T_1^2-7T_1^3+T_1^4-7T_2+42T_1 T_2-42T_1^2 T_2-42T_1^3 T_2+42T_1^4 T_2-7T_1^5 T_2+13T_2^2-42T_1 T_2^2-148T_1^2 T_2^2+426T_1^3 T_2^2-148T_1^4 T_2^2-42T_1^5 T_2^2+13T_1^6 T_2^2-7T_2^3-42T_1 T_2^3+426T_1^2 T_2^3-468T_1^3 T_2^3-468T_1^4 T_2^3+426T_1^5 T_2^3-42T_1^6 T_2^3-7T_1^7 T_2^3+T_2^4+42T_1 T_2^4-148T_1^2 T_2^4-468T_1^3 T_2^4+1392T_1^4 T_2^4-468T_1^5 T_2^4-148T_1^6 T_2^4+42T_1^7 T_2^4+T_1^8 T_2^4-7T_1 T_2^5-42T_1^2 T_2^5+426T_1^3 T_2^5-468T_1^4 T_2^5-468T_1^5 T_2^5+426T_1^6 T_2^5-42T_1^7 T_2^5-7T_1^8 T_2^5+13T_1^9 T_2^5-42T_1^3 T_2^6-148T_1^4 T_2^6+426T_1^5 T_2^6-148T_1^6 T_2^6-42T_1^7 T_2^6+13T_1^8 T_2^6-7T_1^3 T_2^7+42T_1^4 T_2^7-42T_1^5 T_2^7+42T_1^6 T_2^7-7T_1^8 T_2^7+T_1^8 T_2^8-7T_1^5 T_2^8+13T_1^6 T_2^8-7T_1^7 T_2^8+T_1^8 T_2^8) \},$$

$$\text{Knot}[10, 42] \rightarrow \left\{ -\frac{(-1+3T-4T^2+T^3)(-1+4T-3T^2+T^3)}{T^3}, \right.$$

$$-\frac{1}{T_1^4 T_2^4} (6-24T_1+38T_1^2-24T_1^3+6T_1^4-24T_2+72T_1 T_2-54T_1^2 T_2-54T_1^3 T_2+72T_1^4 T_2-24T_1^5 T_2+38T_2^2-54T_1 T_2^2-109T_1^2 T_2^2+279T_1^3 T_2^2-109T_1^4 T_2^2-54T_1^5 T_2^2+38T_1^6 T_2^2-24T_2^3-54T_1 T_2^3+279T_1^2 T_2^3-222T_1^3 T_2^3-222T_1^4 T_2^3+279T_1^5 T_2^3-54T_1^6 T_2^3-24T_1^7 T_2^3+6T_2^4+72T_1 T_2^4-109T_1^2 T_2^4-222T_1^3 T_2^4+552T_1^4 T_2^4-222T_1^5 T_2^4-109T_1^6 T_2^4+72T_1^7 T_2^4+6T_1^8 T_2^4-24T_1 T_2^5-54T_1^2 T_2^5+279T_1^3 T_2^5-222T_1^4 T_2^5-222T_1^5 T_2^5+279T_1^6 T_2^5-54T_1^7 T_2^5-24T_1^8 T_2^5+38T_1^9 T_2^5-24T_1^2 T_2^6-54T_1^3 T_2^6+109T_1^4 T_2^6-109T_1^5 T_2^6+54T_1^6 T_2^6-24T_2^7-54T_1 T_2^7-109T_1^2 T_2^7+279T_1^3 T_2^7-109T_1^4 T_2^7-54T_1^5 T_2^7+38T_1^6 T_2^7-24T_1^7 T_2^7+6T_2^8-54T_1 T_2^8-109T_1^2 T_2^8+279T_1^3 T_2^8-109T_1^4 T_2^8-54T_1^5 T_2^8+38T_1^6 T_2^8-24T_1^7 T_2^8+6T_1^8 T_2^8)$$

$$\begin{aligned}
 & 54 T_1^3 T_2^6 - 109 T_1^4 T_2^6 + 279 T_1^5 T_2^6 - 109 T_1^6 T_2^6 - 54 T_1^7 T_2^6 + 38 T_1^8 T_2^6 - 24 T_1^3 T_2^7 + 72 T_1^4 T_2^7 - \\
 & 54 T_1^5 T_2^7 - 54 T_1^6 T_2^7 + 72 T_1^7 T_2^7 - 24 T_1^8 T_2^7 + 6 T_1^4 T_2^8 - 24 T_1^5 T_2^8 + 38 T_1^6 T_2^8 - 24 T_1^7 T_2^8 + 6 T_1^8 T_2^8 \} , \\
 \text{Knot [10, 48]} \rightarrow & \left\{ \frac{(1 - T + 2 T^2 - 2 T^3 + T^4) (1 - 2 T + 2 T^2 - T^3 + T^4)}{T^4}, \right. \\
 & - \frac{1}{T_1^6 T_2^6} (1 - 2 T_1 + 2 T_1^2 - T_1^3 + 2 T_1^4 - 2 T_1^5 + T_1^6 - 2 T_2 + 2 T_1 T_2 + T_1^2 T_2 - 3 T_1^3 T_2 - 3 T_1^4 T_2 + T_1^5 T_2 + 2 T_1^6 T_2 - \\
 & 2 T_1^7 T_2 + 2 T_1^2 T_2^2 + T_1 T_2^2 - 9 T_1^2 T_2^2 + 9 T_1^3 T_2^2 + T_1^4 T_2^2 + 9 T_1^5 T_2^2 - 9 T_1^6 T_2^2 + T_1^7 T_2^2 + 2 T_1^8 T_2^2 - T_2^3 - 3 T_1 T_2^3 + \\
 & 9 T_1^2 T_2^3 + 2 T_1^3 T_2^3 - 10 T_1^4 T_2^3 - 10 T_1^5 T_2^3 + 2 T_1^6 T_2^3 + 9 T_1^7 T_2^3 - 3 T_1^8 T_2^3 - T_1^9 T_2^3 + 2 T_2^4 - 3 T_1 T_2^4 + T_1^2 T_2^4 - \\
 & 10 T_1^3 T_2^4 - 5 T_1^4 T_2^4 + 29 T_1^5 T_2^4 - 5 T_1^6 T_2^4 - 10 T_1^7 T_2^4 + T_1^8 T_2^4 - 3 T_1^9 T_2^4 + 2 T_1^{10} T_2^4 - 2 T_2^5 + T_1 T_2^5 + 9 T_1^2 T_2^5 - \\
 & 10 T_1^3 T_2^5 + 29 T_1^4 T_2^5 - 22 T_1^5 T_2^5 - 22 T_1^6 T_2^5 + 29 T_1^7 T_2^5 - 10 T_1^8 T_2^5 + 9 T_1^9 T_2^5 + T_1^{10} T_2^5 - 2 T_1^{11} T_2^5 + T_2^6 + \\
 & 2 T_1 T_2^6 - 9 T_1^2 T_2^6 + 2 T_1^3 T_2^6 - 5 T_1^4 T_2^6 - 22 T_1^5 T_2^6 + 48 T_1^6 T_2^6 - 22 T_1^7 T_2^6 - 5 T_1^8 T_2^6 + 2 T_1^9 T_2^6 - 9 T_1^{10} T_2^6 + \\
 & 2 T_1^{11} T_2^6 + T_2^7 - 2 T_1 T_2^7 + T_1^2 T_2^7 + 9 T_1^3 T_2^7 - 10 T_1^4 T_2^7 + 29 T_1^5 T_2^7 - 22 T_1^6 T_2^7 - 22 T_1^7 T_2^7 + 29 T_1^8 T_2^7 - \\
 & 10 T_1^9 T_2^7 + 9 T_1^{10} T_2^7 + T_1^{11} T_2^7 - 2 T_1^{12} T_2^7 + 2 T_1^2 T_2^8 - 3 T_1^3 T_2^8 + T_1^4 T_2^8 - 10 T_1^5 T_2^8 - 5 T_1^6 T_2^8 + 29 T_1^7 T_2^8 - \\
 & 5 T_1^8 T_2^8 - 10 T_1^9 T_2^8 + T_1^{10} T_2^8 - 3 T_1^{11} T_2^8 + 2 T_1^{12} T_2^8 - T_2^9 - 3 T_1 T_2^9 + 9 T_1^2 T_2^9 + 2 T_1^6 T_2^9 - 10 T_1^7 T_2^9 - \\
 & 10 T_1^8 T_2^9 + 2 T_1^9 T_2^9 + 9 T_1^{10} T_2^9 - 3 T_1^{11} T_2^9 - T_1^{12} T_2^9 + 2 T_1^4 T_2^{10} + T_1^5 T_2^{10} - 9 T_1^6 T_2^{10} + 9 T_1^7 T_2^{10} + T_1^8 T_2^{10} + \\
 & 9 T_1^9 T_2^{10} - 9 T_1^{10} T_2^{10} + T_1^{11} T_2^{10} + 2 T_1^{12} T_2^{10} - 2 T_1^5 T_2^{11} + 2 T_1^6 T_2^{11} + T_1^7 T_2^{11} - 3 T_1^8 T_2^{11} - 3 T_1^9 T_2^{11} + T_1^{10} T_2^{11} + \\
 & 2 T_1^{11} T_2^{11} - 2 T_1^{12} T_2^{11} + T_1^6 T_2^{12} - 2 T_1^7 T_2^{12} + 2 T_1^8 T_2^{12} - T_1^9 T_2^{12} + 2 T_1^{10} T_2^{12} - 2 T_1^{11} T_2^{12} + T_1^{12} T_2^{12}) \} ,
 \end{aligned}$$

$$\begin{aligned}
 \text{Knot [10, 75]} \rightarrow & \left\{ - \frac{(-1 + 3 T - 4 T^2 + T^3) (-1 + 4 T - 3 T^2 + T^3)}{T^3}, \right. \\
 & - \frac{1}{T_1^4 T_2^4} (2 - 8 T_1 + 16 T_1^2 - 8 T_1^3 + 2 T_1^4 - 8 T_2 + 18 T_1 T_2 - 32 T_1^2 T_2 - 32 T_1^3 T_2 + 18 T_1^4 T_2 - \\
 & 8 T_1^5 T_2 + 16 T_2^2 - 32 T_1 T_2^2 + 75 T_1^2 T_2^2 + 75 T_1^3 T_2^2 + 75 T_1^4 T_2^2 - 32 T_1^5 T_2^2 + 16 T_1^6 T_2^2 - 8 T_2^3 - \\
 & 32 T_1 T_2^3 + 75 T_1^2 T_2^3 - 256 T_1^3 T_2^3 - 256 T_1^4 T_2^3 + 75 T_1^5 T_2^3 - 32 T_1^6 T_2^3 - 8 T_1^7 T_2^3 + 2 T_2^4 + 18 T_1 T_2^4 + \\
 & 75 T_1^2 T_2^4 - 256 T_1^3 T_2^4 + 900 T_1^4 T_2^4 - 256 T_1^5 T_2^4 + 75 T_1^6 T_2^4 + 18 T_1^7 T_2^4 + 2 T_1^8 T_2^4 - 8 T_1 T_2^5 - \\
 & 32 T_1^2 T_2^5 + 75 T_1^3 T_2^5 - 256 T_1^4 T_2^5 - 256 T_1^5 T_2^5 + 75 T_1^6 T_2^5 - 32 T_1^7 T_2^5 - 8 T_1^8 T_2^5 + 16 T_1^2 T_2^6 - \\
 & 32 T_1^3 T_2^6 + 75 T_1^4 T_2^6 + 75 T_1^5 T_2^6 + 75 T_1^6 T_2^6 - 32 T_1^7 T_2^6 + 16 T_1^8 T_2^6 - 8 T_1^3 T_2^7 + 18 T_1^4 T_2^7 - \\
 & 32 T_1^5 T_2^7 - 32 T_1^6 T_2^7 + 18 T_1^7 T_2^7 - 8 T_1^8 T_2^7 + 2 T_1^4 T_2^8 - 8 T_1^5 T_2^8 + 16 T_1^6 T_2^8 - 8 T_1^7 T_2^8 + 2 T_1^8 T_2^8) \} ,
 \end{aligned}$$

$$\begin{aligned}
 \text{Knot [10, 87]} \rightarrow & \left\{ - \frac{(-2 + T) (-1 + 2 T) (1 - T + T^2)^2}{T^3}, \right. \\
 & - \frac{1}{T_1^6 T_2^6} (1 - 4 T_1 + 8 T_1^2 - 11 T_1^3 + 8 T_1^4 - 4 T_1^5 + T_1^6 - 4 T_2 + 12 T_1 T_2 - 16 T_1^2 T_2 + 12 T_1^3 T_2 + 12 T_1^4 T_2 - \\
 & 16 T_1^5 T_2 + 12 T_1^6 T_2 - 4 T_1^7 T_2 + 8 T_2^2 - 16 T_1 T_2^2 + 12 T_1^2 T_2^2 - 13 T_1^3 T_2^2 - 18 T_1^4 T_2^2 - 13 T_1^5 T_2^2 + \\
 & 12 T_1^6 T_2^2 - 16 T_1^7 T_2^2 + 8 T_1^8 T_2^2 - 11 T_2^3 + 12 T_1 T_2^3 - 13 T_1^2 T_2^3 + 62 T_1^3 T_2^3 + 9 T_1^4 T_2^3 + 9 T_1^5 T_2^3 + \\
 & 62 T_1^6 T_2^3 - 13 T_1^7 T_2^3 + 12 T_1^8 T_2^3 - 11 T_1^9 T_2^3 + 8 T_2^4 + 12 T_1 T_2^4 - 18 T_1^2 T_2^4 + 9 T_1^3 T_2^4 - 296 T_1^4 T_2^4 + \\
 & 290 T_1^5 T_2^4 - 296 T_1^6 T_2^4 + 9 T_1^7 T_2^4 - 18 T_1^8 T_2^4 + 12 T_1^9 T_2^4 + 8 T_1^{10} T_2^4 - 4 T_2^5 - 16 T_1 T_2^5 - 13 T_1^2 T_2^5 + \\
 & 9 T_1^3 T_2^5 + 290 T_1^4 T_2^5 - 32 T_1^5 T_2^5 - 32 T_1^6 T_2^5 + 290 T_1^7 T_2^5 + 9 T_1^8 T_2^5 - 13 T_1^9 T_2^5 - 16 T_1^{10} T_2^5 - 4 T_1^{11} T_2^5 + \\
 & T_2^6 + 12 T_1 T_2^6 + 12 T_1^2 T_2^6 + 62 T_1^3 T_2^6 - 296 T_1^4 T_2^6 - 32 T_1^5 T_2^6 - 72 T_1^6 T_2^6 - 32 T_1^7 T_2^6 - 296 T_1^8 T_2^6 + \\
 & 62 T_1^9 T_2^6 + 12 T_1^{10} T_2^6 + 12 T_1^{11} T_2^6 + T_2^7 - 4 T_1 T_2^7 - 16 T_1^2 T_2^7 - 13 T_1^3 T_2^7 + 9 T_1^4 T_2^7 + 290 T_1^5 T_2^7 - \\
 & 32 T_1^6 T_2^7 - 32 T_1^7 T_2^7 + 290 T_1^8 T_2^7 + 9 T_1^9 T_2^7 - 13 T_1^{10} T_2^7 - 16 T_1^{11} T_2^7 - 4 T_1^{12} T_2^7 + 8 T_1^2 T_2^8 + 12 T_1^3 T_2^8 - \\
 & 18 T_1^4 T_2^8 + 9 T_1^5 T_2^8 - 296 T_1^6 T_2^8 + 290 T_1^7 T_2^8 - 296 T_1^8 T_2^8 + 9 T_1^9 T_2^8 - 18 T_1^{10} T_2^8 + 12 T_1^{11} T_2^8 + 8 T_1^{12} T_2^8 - \\
 & 11 T_1^3 T_2^9 + 12 T_1^4 T_2^9 - 13 T_1^5 T_2^9 + 62 T_1^6 T_2^9 + 9 T_1^7 T_2^9 + 9 T_1^8 T_2^9 + 62 T_1^9 T_2^9 - 13 T_1^{10} T_2^9 + 12 T_1^{11} T_2^9 - \\
 & 11 T_1^{12} T_2^9 + 8 T_1^4 T_2^{10} - 16 T_1^5 T_2^{10} + 12 T_1^6 T_2^{10} - 13 T_1^7 T_2^{10} - 18 T_1^8 T_2^{10} - 13 T_1^9 T_2^{10} + 12 T_1^{10} T_2^{10} - \\
 & 16 T_1^{11} T_2^{10} + 8 T_1^{12} T_2^{10} - 4 T_1^5 T_2^{11} + 12 T_1^6 T_2^{11} - 16 T_1^7 T_2^{11} + 12 T_1^8 T_2^{11} + 12 T_1^9 T_2^{11} - 16 T_1^{10} T_2^{11} +
 \end{aligned}$$

$$\begin{aligned}
& \left. 12 T_1^{11} T_2^{11} - 4 T_1^{12} T_2^{11} + T_1^6 T_2^{12} - 4 T_1^7 T_2^{12} + 8 T_1^8 T_2^{12} - 11 T_1^9 T_2^{12} + 8 T_1^{10} T_2^{12} - 4 T_1^{11} T_2^{12} + T_1^{12} T_2^{12} \right\}, \\
\text{Knot [10, 99]} & \rightarrow \left\{ \frac{(1 - T + T^2)^4}{T^4}, \emptyset \right\}, \text{Knot [10, 123]} \rightarrow \\
& \left\{ \frac{(1 - 3T + 3T^2 - 3T^3 + T^4)^2}{T^4}, \right. \\
& \left. \emptyset \right\}, \\
\text{Knot [10, 129]} & \rightarrow \left\{ \frac{(2 - 2T + T^2)(1 - 2T + 2T^2)}{T^2}, \right. \\
& \frac{1}{T_1^4 T_2^4} (1 - 2T_1 + 3T_1^2 - 2T_1^3 + T_1^4 - 2T_2 + 4T_1 T_2 - 2T_1^2 T_2 - 2T_1^3 T_2 + 4T_1^4 T_2 - 2T_1^5 T_2 + 3T_2^2 - 2T_1 T_2^2 - \\
& 31T_1^2 T_2^2 + 43T_1^3 T_2^2 - 31T_1^4 T_2^2 - 2T_1^5 T_2^2 + 3T_1^6 T_2^2 - 2T_2^3 - 2T_1 T_2^3 + 43T_1^2 T_2^3 - 14T_1^3 T_2^3 - 14T_1^4 T_2^3 + \\
& 43T_1^5 T_2^3 - 2T_1^6 T_2^3 - 2T_1^7 T_2^3 + T_2^4 + 4T_1 T_2^4 - 31T_1^2 T_2^4 - 14T_1^3 T_2^4 + 12T_1^4 T_2^4 - 14T_1^5 T_2^4 - 31T_1^6 T_2^4 + 4T_1^7 T_2^4 + \\
& T_1^8 T_2^4 - 2T_1 T_2^5 - 2T_1^2 T_2^5 + 43T_1^3 T_2^5 - 14T_1^4 T_2^5 - 14T_1^5 T_2^5 + 43T_1^6 T_2^5 - 2T_1^7 T_2^5 - 2T_1^8 T_2^5 + 3T_1^2 T_2^6 - \\
& 2T_1^3 T_2^6 - 31T_1^4 T_2^6 + 43T_1^5 T_2^6 - 31T_1^6 T_2^6 - 2T_1^7 T_2^6 + 3T_1^8 T_2^6 - 2T_1^3 T_2^7 + 4T_1^4 T_2^7 - 2T_1^5 T_2^7 - 2T_1^6 T_2^7 + \\
& 4T_1^7 T_2^7 - 2T_1^8 T_2^7 + T_1^4 T_2^8 - 2T_1^5 T_2^8 + 3T_1^6 T_2^8 - 2T_1^7 T_2^8 + T_1^8 T_2^8) \left. \right\}, \text{Knot [10, 137]} \rightarrow \left\{ \frac{(1 - 3T + T^2)^2}{T^2}, \right. \\
& \left. - \frac{2(1 - 3T_1 + T_1^2)(1 - 3T_2 + T_2^2)(1 - 3T_1 T_2 + T_1^2 T_2^2)(1 + T_1 + T_2 - 6T_1 T_2 + T_1^2 T_2 + T_1 T_2^2 + T_1^2 T_2^2)}{T_1^3 T_2^3} \right\}, \\
\text{Knot [10, 140]} & \rightarrow \left\{ \frac{(1 - T + T^2)^2}{T^2}, \right. \\
& - \frac{1}{T_1^2 T_2^2} 4(3 - 4T_1 + 3T_1^2 - 4T_2 + T_1 T_2 + T_1^2 T_2 - 4T_1^3 T_2 + 3T_2^2 + T_1 T_2^2 + T_1^3 T_2^2 + \\
& 3T_1^4 T_2^2 - 4T_1 T_2^3 + T_1^2 T_2^3 + T_1^3 T_2^3 - 4T_1^4 T_2^3 + 3T_1^2 T_2^4 - 4T_1^3 T_2^4 + 3T_1^4 T_2^4) \left. \right\}, \\
\text{Knot [10, 153]} & \rightarrow \left\{ \frac{(1 - T + T^3)(1 - T^2 + T^3)}{T^3}, \right. \\
& - \frac{1}{T_1^6 T_2^6} (1 - T_1 - T_1^2 + 3T_1^3 - T_1^4 - T_1^5 + T_1^6 - T_2 - 2T_1 T_2 + 5T_1^2 T_2 - 4T_1^3 T_2 - 4T_1^4 T_2 + 5T_1^5 T_2 - 2T_1^6 T_2 - \\
& T_1^7 T_2 - T_2^2 + 5T_1 T_2^2 - 2T_1^2 T_2^2 - 4T_1^3 T_2^2 + 10T_1^4 T_2^2 - 4T_1^5 T_2^2 - 2T_1^6 T_2^2 + 5T_1^7 T_2^2 - T_1^8 T_2^2 + 3T_2^3 - 4T_1 T_2^3 - \\
& 4T_1^2 T_2^3 + 10T_1^3 T_2^3 - 6T_1^4 T_2^3 - 6T_1^5 T_2^3 + 10T_1^6 T_2^3 - 4T_1^7 T_2^3 - 4T_1^8 T_2^3 + 3T_1^9 T_2^3 - T_2^4 - 4T_1 T_2^4 + 10T_1^2 T_2^4 - \\
& 6T_1^3 T_2^4 - 10T_1^4 T_2^4 + 18T_1^5 T_2^4 - 10T_1^6 T_2^4 - 6T_1^7 T_2^4 + 10T_1^8 T_2^4 - 4T_1^9 T_2^4 - T_1^{10} T_2^4 - T_2^5 + 5T_1 T_2^5 - \\
& 4T_1^2 T_2^5 - 6T_1^3 T_2^5 + 18T_1^4 T_2^5 - 10T_1^5 T_2^5 - 10T_1^6 T_2^5 + 18T_1^7 T_2^5 - 6T_1^8 T_2^5 - 4T_1^9 T_2^5 + 5T_1^{10} T_2^5 - T_1^{11} T_2^5 + \\
& T_2^6 - 2T_1 T_2^6 - 2T_1^2 T_2^6 + 10T_1^3 T_2^6 - 10T_1^4 T_2^6 - 10T_1^5 T_2^6 + 24T_1^6 T_2^6 - 10T_1^7 T_2^6 - 10T_1^8 T_2^6 + 10T_1^9 T_2^6 - \\
& 2T_1^{10} T_2^6 - 2T_1^{11} T_2^6 + T_1^{12} T_2^6 - T_1 T_2^7 + 5T_1^2 T_2^7 - 4T_1^3 T_2^7 - 6T_1^4 T_2^7 + 18T_1^5 T_2^7 - 10T_1^6 T_2^7 - 10T_1^7 T_2^7 + \\
& 18T_1^8 T_2^7 - 6T_1^9 T_2^7 - 4T_1^{10} T_2^7 + 5T_1^{11} T_2^7 - T_1^{12} T_2^7 - T_1^2 T_2^8 - 4T_1^3 T_2^8 + 10T_1^4 T_2^8 - 6T_1^5 T_2^8 - 10T_1^6 T_2^8 + \\
& 18T_1^7 T_2^8 - 10T_1^8 T_2^8 - 6T_1^9 T_2^8 + 10T_1^{10} T_2^8 - 4T_1^{11} T_2^8 - T_1^{12} T_2^8 + 3T_1^3 T_2^9 - 4T_1^4 T_2^9 - 4T_1^5 T_2^9 + 10T_1^6 T_2^9 - \\
& 6T_1^7 T_2^9 - 6T_1^8 T_2^9 + 10T_1^9 T_2^9 - 4T_1^{10} T_2^9 - 4T_1^{11} T_2^9 + 3T_1^{12} T_2^9 - T_1^4 T_2^{10} + 5T_1^5 T_2^{10} - 2T_1^6 T_2^{10} - 4T_1^7 T_2^{10} + \\
& 10T_1^8 T_2^{10} - 4T_1^9 T_2^{10} - 2T_1^{10} T_2^{10} + 5T_1^{11} T_2^{10} - T_1^{12} T_2^{10} - T_1^5 T_2^{11} - 2T_1^6 T_2^{11} + 5T_1^7 T_2^{11} - 4T_1^8 T_2^{11} - 4T_1^9 T_2^{11} + \\
& 5T_1^{10} T_2^{11} - 2T_1^{11} T_2^{11} - T_1^{12} T_2^{11} + T_1^6 T_2^{12} - T_1^7 T_2^{12} - T_1^8 T_2^{12} + 3T_1^9 T_2^{12} - T_1^{10} T_2^{12} - T_1^{11} T_2^{12} + T_1^{12} T_2^{12}) \left. \right\}, \\
\text{Knot [10, 155]} & \rightarrow \left\{ - \frac{(-1 + T - 2T^2 + T^3)(-1 + 2T - T^2 + T^3)}{T^3}, \right.
\end{aligned}$$

$$\begin{aligned}
 & - \frac{1}{T_1^4 T_2^4} 2 \left(1 - 4 T_1 + 5 T_1^2 - 4 T_1^3 + T_1^4 - 4 T_2 + 11 T_1 T_2 - 3 T_1^2 T_2 - 3 T_1^3 T_2 + 11 T_1^4 T_2 - 4 T_1^5 T_2 + 5 T_2^2 - \right. \\
 & \quad 3 T_1 T_2^2 - 26 T_1^2 T_2^2 + 24 T_1^3 T_2^2 - 26 T_1^4 T_2^2 - 3 T_1^5 T_2^2 + 5 T_1^6 T_2^2 - 4 T_2^3 - 3 T_1 T_2^3 + 24 T_1^2 T_2^3 + 4 T_1^3 T_2^3 + \\
 & \quad 4 T_1^4 T_2^3 + 24 T_1^5 T_2^3 - 3 T_1^6 T_2^3 - 4 T_1^7 T_2^3 + T_2^4 + 11 T_1 T_2^4 - 26 T_1^2 T_2^4 + 4 T_1^3 T_2^4 - 30 T_1^4 T_2^4 + 4 T_1^5 T_2^4 - \\
 & \quad 26 T_1^6 T_2^4 + 11 T_1^7 T_2^4 + T_1^8 T_2^4 - 4 T_1 T_2^5 - 3 T_1^2 T_2^5 + 24 T_1^3 T_2^5 + 4 T_1^4 T_2^5 + 4 T_1^5 T_2^5 + 24 T_1^6 T_2^5 - 3 T_1^7 T_2^5 - \\
 & \quad 4 T_1^8 T_2^5 + 5 T_1^2 T_2^6 - 3 T_1^3 T_2^6 - 26 T_1^4 T_2^6 + 24 T_1^5 T_2^6 - 26 T_1^6 T_2^6 - 3 T_1^7 T_2^6 + 5 T_1^8 T_2^6 - 4 T_1^3 T_2^7 + \\
 & \quad \left. \left. 11 T_1^4 T_2^7 - 3 T_1^5 T_2^7 - 3 T_1^6 T_2^7 + 11 T_1^7 T_2^7 - 4 T_1^8 T_2^7 + T_1^4 T_2^8 - 4 T_1^5 T_2^8 + 5 T_1^6 T_2^8 - 4 T_1^7 T_2^8 + T_1^8 T_2^8 \right) \right\}
 \end{aligned}$$

```

In[*]:= DunfieldKnots = ReadList["../../People/Dunfield/nmd_random_knots"] /. k_Integer -> k + 1;
DK[n_] := DunfieldKnots[[n - 2]]

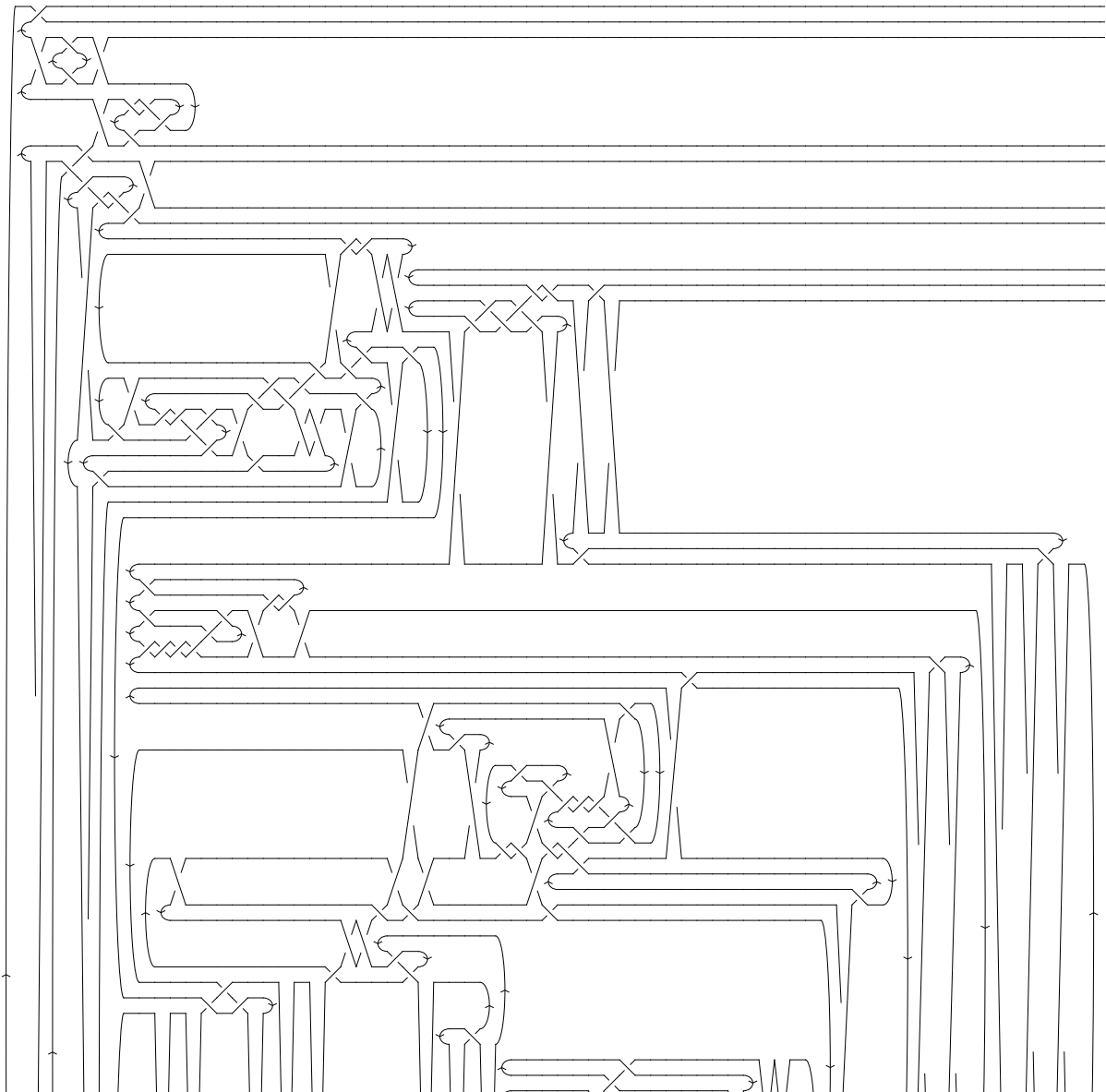
```

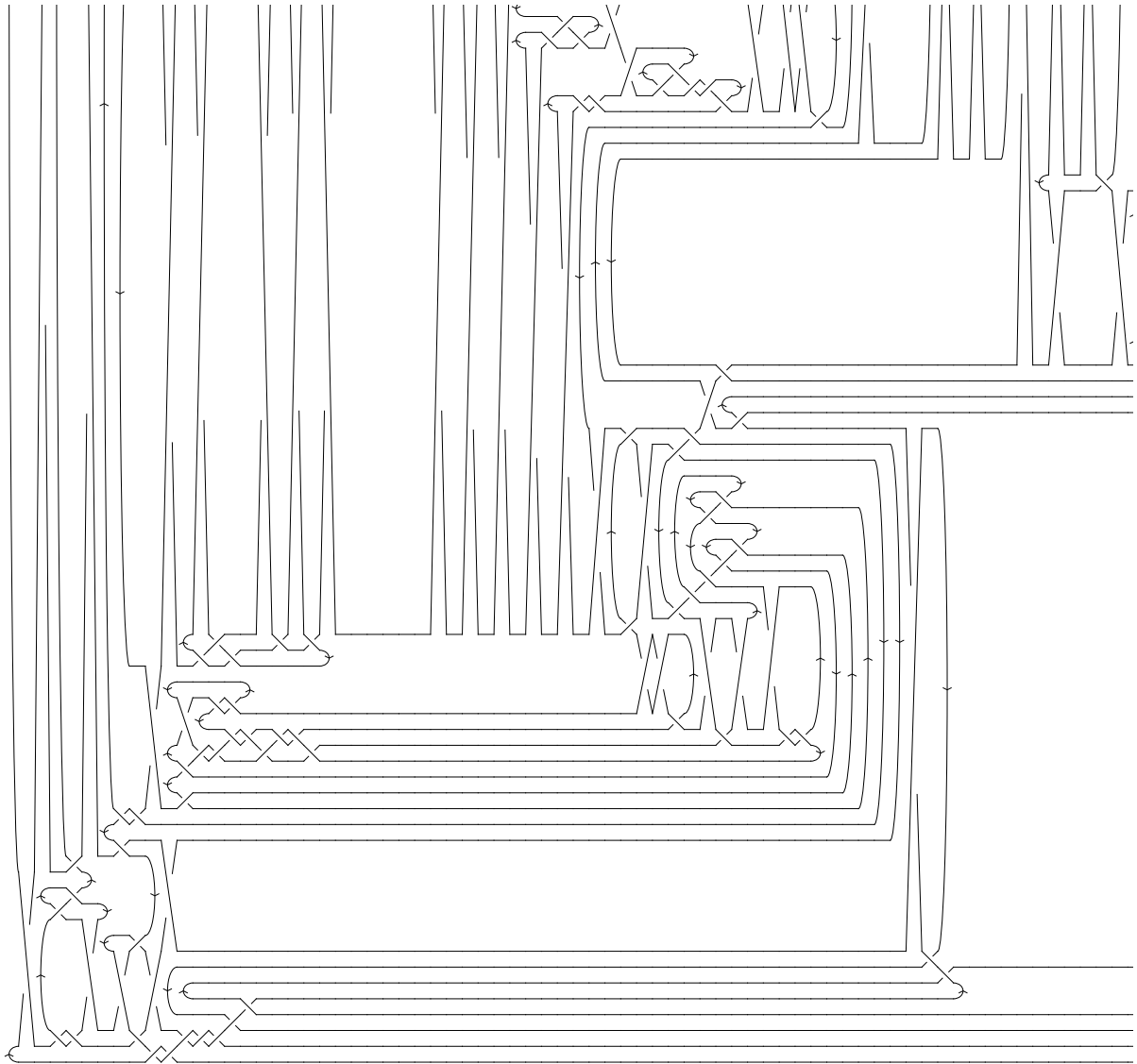
```

In[*]:= DrawMorseLink[DK[250]]

```

Out[*]=





`In[]:= Crossings [DK [576]]`

`Out[]:=`
576

`In[]:= AbsoluteTiming [θ [DK [3]]]`

`Out[]:=`

$$\{0.0061176, \left\{ \frac{1 - T + T^2}{T}, \frac{1 - T_1 + T_1^2 - T_2 - T_1^3 T_2 + T_2^2 + T_1^4 T_2^2 - T_1 T_2^3 - T_1^4 T_2^3 + T_1^2 T_2^4 - T_1^3 T_2^4 + T_1^4 T_2^4}{T_1^2 T_2^2} \right\} \}$$

`In[]:= AbsoluteTiming [θ [DK [30]] ;]`

`Out[]:=`
{2.91933, Null}

```
In[*]:= AbsoluteTiming[ $\theta$ [DK[60]]];
```

```
Out[*]=
{27.4555, Null}
```

```
In[*]:= AbsoluteTiming[ $\theta$ [DK[90]]];
```

```
Out[*]=
{227.389, Null}
```

```
In[*]:= AbsoluteTiming[ $\theta$ 120 =  $\theta$ [DK[120]]];
```

```
Out[*]=
{0.0003743, Null}
```

```
In[*]:= Put[ $\theta$ 120, "Theta4DK120.m"]
```

```
In[*]:= AbsoluteTiming[ $\theta$ [DK[150]]];
```

```
Out[*]=
{2357.39, Null}
```

(during the previous computation I biked home, so the AbsoluteTiming is too much)

```
In[*]:= AbsoluteTiming[ $\theta$ [DK[180]]];
```

```
Out[*]=
{5391.24, Null}
```

```
In[*]:= AbsoluteTiming[ $\theta$ [DK[210]]];
```

```
Out[*]=
{9613.68, Null}
```

```
In[*]:= AbsoluteTiming[ $\theta$ [DK[240]]];
```

```
Out[*]=
{22462.4, Null}
```

```
In[*]:= AbsoluteTiming[ $\theta$ [DK[270]]];
```

Mathematica crashed while trying the above computation.

```
In[*]:= AbsoluteTiming[ $\theta$ [DK[300]]];
```

```
In[*]:= Do[Echo /@ AbsoluteTiming[n  $\rightarrow$   $\theta$ 22/7,34/21[DK[n]]], {n, 100, 500, 100}]
```

```
» 1.34291
```

$$\gg 100 \rightarrow \left\{ \left\{ \frac{35\ 388\ 936\ 522\ 490\ 931\ 938\ 908\ 923\ 343\ 364\ 558\ 590\ 414\ 632\ 463\ 375\ 508\ 742\ 089}{264\ 554\ 736\ 545\ 069\ 605\ 885\ 631\ 471\ 128\ 764\ 401\ 339\ 301\ 535\ 744}, \right. \right. \\ \left. \frac{525\ 106\ 180\ 586\ 933\ 014\ 293\ 865\ 927\ 609\ 379\ 271\ 742\ 972\ 076\ 277\ 257\ 025\ 413\ 914\ 338\ 499}{37\ 324\ 734\ 431\ 368\ 634\ 257\ 516\ 595\ 221\ 111\ 791\ 096\ 751\ 570\ 183\ 668\ 795\ 296\ 664\ 772\ 608}, \right. \\ \left. \frac{50\ 463\ 574\ 955\ 913\ 231\ 815\ 385\ 186\ 261\ 134\ 862\ 814\ 456\ 979\ 779\ 055\ 953\ 806\ 229\ 018\ 368\ 595\ 827\ 102\ 502\ 222\ 063 - 117\ 299\ 430\ 053\ 887\ 387\ 799\ 738\ 329\ 099\ 644\ 807\ 147\ 011\ 110\ 057\ 363 / 78\ 995\ 482\ 272\ 843\ 339\ 527\ 758\ 555\ 299\ 340\ 636\ 345\ 228\ 530\ 305\ 737\ 655\ 210\ 586\ 135\ 944\ 082\ 585\ 735\ 874\ 960\ 483 - 459\ 404\ 024\ 632\ 393\ 880\ 311\ 552\ 802\ 816}{15\ 528\ 310\ 677\ 820\ 715\ 321\ 034\ 523\ 399\ 570\ 065\ 191\ 062\ 105\ 455\ 458\ 377\ 892\ 190\ 819\ 455\ 810\ 946\ 769\ 247\ 237\ 972\ 364 - 715\ 885\ 979\ 420\ 470\ 551\ 351\ 869\ 219\ 633\ 193\ 553\ 826\ 417\ 257\ 308\ 347\ 635\ 722\ 740\ 692\ 821\ 508\ 135\ 020\ 135\ 287\ 456 - 366\ 121\ 034\ 149\ 470\ 683\ 332\ 248\ 166\ 617\ 909\ 950\ 793\ 807\ 487\ 984\ 811\ 798\ 893\ 565\ 093\ 125\ 255\ 348\ 183\ 610\ 375\ 623 - 605\ 724\ 602\ 385 / 2\ 021\ 852\ 735\ 124\ 190\ 443\ 601\ 930\ 854\ 750\ 557\ 817\ 293\ 581\ 868\ 158\ 419\ 077\ 273\ 632\ 660\ 683\ 611\ 032\ 469\ 645\ 660 - 194\ 599\ 371\ 946\ 978\ 382\ 921\ 589\ 937\ 447\ 245\ 226\ 083\ 493\ 038\ 643\ 718\ 762\ 601\ 638\ 877\ 904\ 972\ 518\ 050\ 153\ 046 - 861\ 446\ 451\ 966\ 589\ 617\ 326\ 519\ 650\ 102\ 159\ 919\ 340\ 894\ 781\ 095\ 877\ 211\ 742\ 319\ 673\ 344} \right\},$$

» 6.88529

» 200 →

$$\{ \{ - (72\ 941\ 025\ 249\ 230\ 622\ 091\ 769\ 886\ 034\ 332\ 903\ 937\ 878\ 867\ 275\ 035\ 495\ 850\ 289\ 152\ 467\ 601\ 139\ 729\ 946\ 680 - 691\ 983\ 449\ 444\ 238\ 470\ 173\ 260\ 899\ 434\ 879\ 455\ 547\ 646\ 677 / 79\ 780\ 391\ 006\ 864\ 379\ 747\ 986\ 053\ 920\ 193\ 038\ 680\ 545\ 693\ 079\ 622\ 955\ 011\ 027\ 668\ 359\ 182\ 291\ 645\ 896\ 903 - 218\ 461\ 275\ 510\ 571\ 008), \} \\ 13\ 469\ 039\ 288\ 358\ 770\ 844\ 889\ 186\ 746\ 410\ 419\ 403\ 949\ 987\ 382\ 833\ 567\ 787\ 469\ 752\ 570\ 946\ 087\ 488\ 964\ 056\ 464 - 083\ 956\ 449\ 441\ 872\ 952\ 430\ 656\ 158\ 262\ 269\ 810\ 083\ 547\ 830\ 189\ 003\ 289\ 443\ 154\ 125 / 4\ 240\ 161\ 130\ 043\ 882\ 037\ 823\ 084\ 995\ 205\ 726\ 632\ 691\ 185\ 572\ 237\ 933\ 032\ 456\ 552\ 833\ 243\ 815\ 216\ 744\ 170\ 971 - 881\ 548\ 991\ 957\ 331\ 738\ 797\ 061\ 590\ 095\ 303\ 559\ 046\ 326\ 968\ 215\ 750\ 967\ 296, \\ 3\ 058\ 236\ 953\ 956\ 402\ 226\ 943\ 593\ 388\ 603\ 713\ 021\ 071\ 954\ 699\ 338\ 326\ 371\ 450\ 792\ 000\ 285\ 430\ 803\ 814\ 324\ 110 - 911\ 806\ 690\ 348\ 020\ 780\ 088\ 584\ 382\ 124\ 603\ 092\ 971\ 693\ 299\ 841\ 778\ 094\ 187\ 288\ 377\ 810\ 035\ 496\ 408\ 283\ 188 - 130\ 224\ 093\ 352\ 681\ 965\ 580\ 164\ 395\ 682\ 496\ 054\ 504\ 489\ 551\ 954\ 332\ 992\ 465\ 733\ 972\ 977\ 594\ 735\ 369\ 459\ 115 - 633\ 590\ 163\ 189\ 798\ 671\ 672\ 600\ 349\ 071\ 866\ 872\ 120\ 468\ 309\ 375 / 7\ 389\ 876\ 778\ 587\ 670\ 278\ 409\ 931\ 856\ 936\ 212\ 530\ 694\ 800\ 372\ 408\ 625\ 530\ 583\ 166\ 986\ 417\ 139\ 021\ 654\ 981\ 203 - 589\ 910\ 511\ 227\ 601\ 136\ 991\ 125\ 732\ 955\ 086\ 827\ 137\ 765\ 975\ 954\ 473\ 403\ 792\ 833\ 419\ 463\ 344\ 119\ 138\ 486\ 741 - 874\ 061\ 457\ 114\ 480\ 552\ 952\ 530\ 491\ 222\ 541\ 669\ 872\ 799\ 328\ 574\ 041\ 719\ 777\ 250\ 405\ 019\ 238\ 495\ 420\ 416 \}, \\ - (35\ 533\ 798\ 751\ 418\ 160\ 350\ 916\ 090\ 870\ 874\ 408\ 685\ 758\ 076\ 531\ 957\ 553\ 028\ 308\ 354\ 367\ 936\ 952\ 715\ 320\ 377\ 112 - 933\ 900\ 291\ 194\ 748\ 021\ 391\ 980\ 122\ 119\ 460\ 697\ 184\ 063\ 729\ 775\ 201\ 344\ 517\ 723\ 397\ 729\ 781\ 282\ 842\ 088\ 707 - 536\ 733\ 758\ 752\ 195\ 455\ 093\ 509\ 038\ 015\ 678\ 684\ 681\ 626\ 418\ 035\ 519\ 803\ 604\ 439\ 397\ 416\ 661\ 432\ 511\ 206\ 560 - 127\ 326\ 980\ 562\ 590\ 565\ 142\ 398\ 059\ 299\ 186\ 452\ 157\ 584\ 572\ 312\ 347\ 570\ 546\ 167\ 881\ 173\ 768\ 455\ 447\ 102\ 478 - 378\ 052\ 565\ 824\ 989\ 035\ 759\ 718\ 349\ 901\ 555\ 797\ 046\ 487\ 367\ 735\ 873\ 953\ 550\ 250\ 292\ 996\ 462\ 075\ 359\ 706\ 165 - 962\ 265\ 760\ 112\ 833\ 307\ 407\ 741\ 496\ 584\ 457\ 563\ 023\ 053\ 844\ 158\ 922\ 142\ 850\ 482\ 681\ 009\ 343\ 615\ 561\ 563\ 933 - 345\ 073\ 931\ 843\ 736\ 416\ 605\ 341\ 872\ 288\ 994\ 025\ 512\ 080\ 297\ 221\ 469\ 946\ 108\ 375\ 450\ 764\ 191\ 881\ 092\ 403\ 125 / 452\ 396\ 514\ 172\ 443\ 948\ 090\ 596\ 720\ 075\ 743\ 969\ 379\ 888\ 907\ 838\ 827\ 526\ 625\ 786\ 124\ 662\ 888\ 374\ 624\ 411\ 285 - 068\ 305\ 310\ 109\ 452\ 395\ 752\ 503\ 075\ 302\ 027\ 422\ 247\ 590\ 129\ 306\ 367\ 202\ 635\ 464\ 223\ 536\ 884\ 780\ 523\ 952\ 041 - 663\ 218\ 284\ 564\ 278\ 956\ 217\ 013\ 122\ 499\ 393\ 566\ 958\ 337\ 419\ 775\ 741\ 184\ 128\ 728\ 079\ 197\ 011\ 880\ 897\ 341\ 842 - 707\ 105\ 674\ 675\ 400\ 895\ 701\ 799\ 815\ 201\ 160\ 823\ 272\ 081\ 093\ 378\ 813\ 312\ 065\ 293\ 550\ 562\ 986\ 036\ 284\ 189\ 802 - 691\ 770\ 038\ 253\ 906\ 432\ 703\ 028\ 717\ 411\ 518\ 620\ 619\ 761\ 249\ 698\ 953\ 156\ 705\ 113\ 477\ 290\ 132\ 297\ 634\ 963\ 280 - 748\ 846\ 254\ 218\ 110\ 952\ 588\ 168\ 734\ 038\ 774\ 458\ 591\ 745\ 251\ 980\ 483\ 140\ 272\ 217\ 590\ 873\ 158\ 475\ 317\ 248) \}$$

» 81.2757

» 300 →

```
{ { 54 300 428 014 802 247 763 147 703 343 836 297 447 025 108 824 684 772 425 762 525 822 095 039 545 899 375 981 -
953 473 178 602 586 048 430 534 584 880 163 873 723 541 762 115 735 883 067 341 959 560 581 371 283 178 656 -
972 648 408 925 263 946 669 /
6 741 838 682 197 306 940 008 962 116 848 220 280 436 936 971 437 572 995 472 014 771 688 913 708 639 211 514 -
814 195 885 491 758 038 709 972 366 558 512 006 372 340 250 849 089 814 593 530 683 936 627 298 651 512 766 -
464,
1 084 128 382 249 743 436 824 663 986 171 685 150 273 646 351 713 912 937 150 171 700 202 730 323 922 010 700 -
294 161 035 743 289 238 368 194 879 507 950 682 627 574 784 328 439 797 605 967 434 628 113 238 619 877 448 -
933 104 349 915 804 145 167 106 117 098 828 582 214 168 974 179 /
458 816 114 715 914 322 691 410 371 538 510 819 835 906 604 695 828 488 701 592 446 861 566 683 983 329 916 -
364 046 021 667 534 630 113 436 786 891 827 119 466 479 256 930 424 597 743 983 452 685 367 746 981 696 618 -
500 346 273 956 034 473 567 578 882 100 679 941 606 973 898 752,
- (158 777 874 852 495 582 515 909 215 389 994 852 546 352 653 931 705 508 650 307 891 657 053 561 609 520 779 -
186 320 897 348 004 451 340 565 961 074 347 535 242 136 402 407 084 832 097 701 971 876 894 887 835 991 169 -
195 699 017 190 487 685 513 574 819 025 748 109 103 168 978 452 501 811 090 422 603 306 747 210 926 095 970 -
770 670 185 035 477 605 544 327 410 988 587 473 792 754 126 636 018 339 393 952 001 669 899 686 164 600 864 -
484 927 816 109 847 962 066 717 003 302 534 438 301 515 100 500 581 439 281 502 338 168 771 925 334 310 271 -
437 341 818 561 /
8 446 673 524 619 204 540 662 248 188 364 579 654 962 149 362 100 111 349 567 813 607 145 180 164 671 139 -
617 365 814 596 293 558 611 877 467 632 393 708 787 160 491 479 639 500 826 381 376 300 773 027 876 197 -
170 955 833 764 004 216 082 452 919 975 997 020 526 350 495 894 405 720 336 559 612 735 646 735 734 155 -
554 395 961 189 410 159 575 680 771 895 729 613 390 941 354 707 084 783 892 152 666 711 430 746 078 787 -
591 302 278 416 571 017 951 710 864 634 193 356 469 295 526 911 091 658 361 659 195 392) },
- (8 598 040 329 900 132 178 849 810 392 065 575 015 656 948 332 717 228 018 818 196 986 408 406 885 151 173 114 -
729 742 371 657 327 870 129 553 797 167 264 600 601 461 737 612 762 883 778 056 461 125 303 156 682 177 822 -
387 597 941 597 676 133 555 775 929 651 554 558 568 826 851 193 016 325 730 344 539 614 484 324 504 069 552 -
066 916 711 741 608 633 404 825 059 528 743 681 819 722 488 192 923 953 808 035 534 926 597 091 591 375 719 -
708 970 825 214 204 090 352 696 010 613 508 905 010 815 827 512 539 197 920 217 378 414 243 201 536 885 840 -
502 206 086 625 497 141 347 632 796 621 737 879 816 174 936 164 213 142 719 738 496 243 651 312 127 236 569 -
658 634 354 493 624 215 745 814 083 607 554 164 979 886 252 364 586 458 746 627 111 732 001 798 098 411 377 -
469 694 277 623 092 049 862 323 332 740 569 060 876 937 876 842 372 536 879 611 798 159 751 897 313 972 600 -
034 650 421 307 830 475 711 279 585 030 859 630 220 299 694 039 396 624 252 597 049 438 680 591 984 366 793 -
943 898 999 135 971 052 027 731 897 551 216 899 484 829 106 288 612 686 258 405 660 999 234 795 832 964 806 -
965 873 /
812 678 875 896 339 039 067 670 285 655 670 982 811 813 572 505 939 516 526 515 895 464 038 253 655 962 698 -
534 470 158 273 453 471 969 412 225 862 875 287 695 193 281 294 047 490 514 554 993 940 744 345 155 227 044 -
324 133 303 532 650 014 900 112 879 006 801 216 964 275 606 874 592 888 221 306 209 845 338 126 393 770 242 -
035 421 093 568 450 115 306 673 002 230 112 041 735 438 509 121 588 709 168 122 196 789 163 088 049 032 402 -
421 831 930 165 869 146 969 680 446 412 107 255 774 061 135 012 374 209 095 972 722 550 879 186 845 119 609 -
905 916 389 810 258 595 619 812 363 193 227 320 810 658 099 006 534 020 402 904 912 489 467 165 903 133 321 -
063 930 316 828 893 776 965 178 816 926 996 966 709 051 510 488 188 756 691 086 660 277 067 356 140 651 827 -
003 820 730 966 021 344 695 355 788 718 822 044 920 980 326 904 411 394 046 648 037 199 883 563 233 621 627 -
831 014 801 221 912 882 289 230 772 061 896 822 000 174 973 211 954 770 033 843 643 470 936 514 389 292 275 -
081 706 404 767 643 586 504 062 664 704) }
```

Out[*]=

\$Aborted

The following crashes at n=700:

```
In[*]:= Do[Echo /@ AbsoluteTiming[n →  $\Theta_{22/7,34/21}$ [DK[n]]], {n, 500, 1000, 100}]
```

Knot Genus

```
In[*]:= Import["KnotGenusFromKnotInfo.csv"][[2 ;;]] /.
  {K_String, pd_String, g_Integer} ⇒ (Genus[Knot[K]] = g);
```

```
In[*]:=  $\rho[K_] := \rho[K] = \text{Module}[\{R, Cs, \varphi, n, A, s, i, j, k, \Delta, G, \rho1\},
  R[s_, i_, j_] := s (g_{ji} (g_{j^+,j} + g_{j,j^+} - g_{ij}) - g_{ii} (g_{j,j^+} - 1) - 1/2);
  \{Cs, \varphi\} = \text{Rot}[K]; n = \text{Length}[Cs];
  A = \text{IdentityMatrix}[2 n + 1];
  \text{Cases}[Cs, \{s_, i_, j_\} ⇒ (A[[\{i, j\}, \{i + 1, j + 1\}]] += (

$$\begin{pmatrix} -T^s & T^s - 1 \\ 0 & -1 \end{pmatrix}$$

))];
  \Delta = T^{(-\text{Total}[\varphi] - \text{Total}[Cs[[All, 1]])]/2} \text{Det}[A];
  G = \text{Inverse}[A];
  \rho1 = \sum_{k=1}^n R@@Cs[[k]] - \sum_{k=1}^{2n} \varphi[[k]] (g_{kk} - 1/2);
  \text{Factor}@\{\Delta, \Delta^2 \rho1 /. \alpha_+ ⇒ \alpha + 1 /. g_{\alpha, \beta} ⇒ G[[\alpha, \beta]]\}];$ 
```

```
In[*]:= Table[
  {K, Genus[K], Exponent[ $\theta$ [K][[1]], T], Exponent[ $\rho$ [K][[2]], T] / 2, Exponent[ $\theta$ [K][[2]], T1] / 2,
  Exponent[ $\theta$ [K][[2]] /. {T1 → T, T2 → T2}, T] / 6}, {K, AllKnots[{3, 8]}}] // MatrixForm
```

```
Out[*]//MatrixForm=
(
  Knot[3, 1] 1 1 1 1 1
  Knot[4, 1] 1 1 -∞ -∞ -∞
  Knot[5, 1] 2 2 2 2 2
  Knot[5, 2] 1 1 1 1 1
  Knot[6, 1] 1 1 1 1 1
  Knot[6, 2] 2 2 2 2 2
  Knot[6, 3] 2 2 -∞ -∞ -∞
  Knot[7, 1] 3 3 3 3 3
  Knot[7, 2] 1 1 1 1 1
  Knot[7, 3] 2 2 2 2 2
  Knot[7, 4] 1 1 1 1 1
  Knot[7, 5] 2 2 2 2 2
  Knot[7, 6] 2 2 2 2 2
  Knot[7, 7] 2 2 1 1 1
  Knot[8, 1] 1 1 1 1 1
  Knot[8, 2] 3 3 3 3 3
  Knot[8, 3] 1 1 -∞ -∞ -∞
  Knot[8, 4] 2 2 2 2 2
  Knot[8, 5] 3 3 3 3 3
  Knot[8, 6] 2 2 2 2 2
  Knot[8, 7] 3 3 3 3 3
  Knot[8, 8] 2 2 2 2 2
  Knot[8, 9] 3 3 -∞ -∞ -∞
  Knot[8, 10] 3 3 3 3 3
  Knot[8, 11] 2 2 2 2 2
  Knot[8, 12] 2 2 -∞ -∞ -∞
  Knot[8, 13] 2 2 2 2 2
  Knot[8, 14] 2 2 2 2 2
  Knot[8, 15] 2 2 2 2 2
  Knot[8, 16] 3 3 3 3 3
  Knot[8, 17] 3 3 -∞ -∞ -∞
  Knot[8, 18] 3 3 -∞ -∞ -∞
  Knot[8, 19] 3 3 3 3 3
  Knot[8, 20] 2 2 1 1 1
  Knot[8, 21] 2 2 2 2 2
)
```

```
In[*]:= Select[AllKnots[{3, 12}], Genus[#] < Exponent[ $\theta$ [#][[1]], T] &]
```

 KnotTheory: Loading precomputed data in KnotTheory/12A.dts.

 KnotTheory: Loading precomputed data in KnotTheory/12N.dts.

```
Out[*]=
{}
```

```

In[*]:= Select[AllKnots[{3, 11}], Genus[#] > Exponent[Theta[#][1], T] &]
Out[*]=
{Knot[11, NonAlternating, 34], Knot[11, NonAlternating, 42],
 Knot[11, NonAlternating, 45], Knot[11, NonAlternating, 67], Knot[11, NonAlternating, 73],
 Knot[11, NonAlternating, 97], Knot[11, NonAlternating, 152]}

In[*]:= Select[AllKnots[{3, 12}], Genus[#] < Ceiling[Exponent[Theta[#][2], T1] / 2] &]
Out[*]=
{}

In[*]:= Select[AllKnots[{3, 10}], Genus[#] > Ceiling[Exponent[Theta[#][2], T1] / 2] &]
Out[*]=
{Knot[4, 1], Knot[6, 3], Knot[7, 7], Knot[8, 3], Knot[8, 9], Knot[8, 12],
 Knot[8, 17], Knot[8, 18], Knot[8, 20], Knot[9, 24], Knot[9, 27], Knot[9, 30],
 Knot[9, 33], Knot[9, 34], Knot[10, 17], Knot[10, 33], Knot[10, 37], Knot[10, 42],
 Knot[10, 43], Knot[10, 45], Knot[10, 48], Knot[10, 60], Knot[10, 71],
 Knot[10, 75], Knot[10, 79], Knot[10, 81], Knot[10, 88], Knot[10, 91], Knot[10, 96],
 Knot[10, 99], Knot[10, 104], Knot[10, 109], Knot[10, 115], Knot[10, 118],
 Knot[10, 123], Knot[10, 140], Knot[10, 141], Knot[10, 155], Knot[10, 158]}

In[*]:= Select[AllKnots[{3, 12}], Genus[#] < Ceiling[Exponent[Rho[#][2], T] / 2] &]
Out[*]=
{}

In[*]:= Select[AllKnots[{3, 10}], 2 Genus[#] > Exponent[Rho[#][2], T] &]
Out[*]=
{Knot[4, 1], Knot[6, 3], Knot[7, 7], Knot[8, 3], Knot[8, 9], Knot[8, 12], Knot[8, 17],
 Knot[8, 18], Knot[8, 20], Knot[9, 24], Knot[9, 27], Knot[9, 30], Knot[9, 33], Knot[9, 34],
 Knot[9, 44], Knot[10, 17], Knot[10, 31], Knot[10, 33], Knot[10, 37], Knot[10, 42],
 Knot[10, 43], Knot[10, 45], Knot[10, 48], Knot[10, 60], Knot[10, 71], Knot[10, 75],
 Knot[10, 79], Knot[10, 81], Knot[10, 88], Knot[10, 91], Knot[10, 96], Knot[10, 99],
 Knot[10, 104], Knot[10, 107], Knot[10, 109], Knot[10, 115], Knot[10, 118], Knot[10, 123],
 Knot[10, 132], Knot[10, 137], Knot[10, 140], Knot[10, 141], Knot[10, 155], Knot[10, 158]}

In[*]:= Select[AllKnots[{3, 10}], Exponent[Rho[#][2], T] > Exponent[Theta[#][2], T1] &]
Out[*]=
{}

In[*]:= Select[AllKnots[{3, 10}], Exponent[Rho[#][2], T] < Exponent[Theta[#][2], T1] &]
Out[*]=
{Knot[9, 24], Knot[10, 31], Knot[10, 107], Knot[10, 158]}

In[*]:= Factor@Alexander[Knot[10, 107]][T]
Out[*]=

$$\frac{1 - 8T + 22T^2 - 31T^3 + 22T^4 - 8T^5 + T^6}{T^3}$$


```

```
In[*]:= Select[AllKnots[{3, 12}], Genus[#] < Exponent[ $\theta$ [#][2] /. {T1 → T, T2 → T2}, T] / 6 &]
```

```
Out[*]=
{}

```

```
In[*]:= Select[AllKnots[{3, 10}], Genus[#] > Exponent[ $\theta$ [#][2] /. {T1 → T, T2 → T2}, T] / 6 &]
```

```
Out[*]=
{Knot[4, 1], Knot[6, 3], Knot[7, 7], Knot[8, 3], Knot[8, 9], Knot[8, 12], Knot[8, 17],
Knot[8, 18], Knot[8, 20], Knot[9, 24], Knot[9, 27], Knot[9, 30], Knot[9, 33], Knot[9, 34],
Knot[9, 44], Knot[10, 17], Knot[10, 31], Knot[10, 33], Knot[10, 37], Knot[10, 42],
Knot[10, 43], Knot[10, 45], Knot[10, 48], Knot[10, 60], Knot[10, 71], Knot[10, 75],
Knot[10, 79], Knot[10, 81], Knot[10, 88], Knot[10, 91], Knot[10, 96], Knot[10, 99],
Knot[10, 104], Knot[10, 107], Knot[10, 109], Knot[10, 115], Knot[10, 118], Knot[10, 123],
Knot[10, 132], Knot[10, 137], Knot[10, 140], Knot[10, 141], Knot[10, 155], Knot[10, 158]}
```

```
In[*]:= sel = Select[AllKnots[{3, 12}], Exponent[ $\rho$ [#][2], T] / 2 > Exponent[ $\theta$ [#][2], T1] / 2 &]
```

```
Out[*]=
{}

```

```
In[*]:= Select[AllKnots[{3, 11}], Ceiling[Exponent[ $\rho$ [#][2], T] / 2] < Exponent[ $\theta$ [#][2], T1] / 2 &]
```

```
Out[*]=
{Knot[10, 107], Knot[11, Alternating, 209], Knot[11, Alternating, 228],
Knot[11, NonAlternating, 34], Knot[11, NonAlternating, 45], Knot[11, NonAlternating, 152]}
```

```
In[*]:= Table[{K, Genus[K], Exponent[Theta[K][[1]], T], Exponent[Rho[K][[2]], T] / 2,
  Exponent[Theta[K][[2]], T1] / 2, Exponent[Theta[K][[2]] /. {T1 -> T, T2 -> T^2}, T] / 6},
  {K, AllKnots[{3, 7}] ~Join~ {Knot[9, 24], Knot[9, 30], Knot[10, 107],
  Knot["K11n34"], Knot["K11n42"], GST48}}] // MatrixForm
```

Out[*]//MatrixForm=

Knot[3, 1]	1	1	1	1	1
Knot[4, 1]	1	1	-∞	-∞	-∞
Knot[5, 1]	2	2	2	2	2
Knot[5, 2]	1	1	1	1	1
Knot[6, 1]	1	1	1	1	1
Knot[6, 2]	2	2	2	2	2
Knot[6, 3]	2	2	-∞	-∞	-∞
Knot[7, 1]	3	3	3	3	3
Knot[7, 2]	1	1	1	1	1
Knot[7, 3]	2	2	2	2	2
Knot[7, 4]	1	1	1	1	1
Knot[7, 5]	2	2	2	2	2
Knot[7, 6]	2	2	2	2	2
Knot[7, 7]	2	2	1	1	1
Knot[9, 24]	3	3	$\frac{3}{2}$	2	$\frac{11}{6}$
Knot[9, 30]	3	3	2	2	$\frac{11}{6}$
Knot[10, 107]	3	3	2	$\frac{5}{2}$	$\frac{7}{3}$
Knot[11, NonAlternating, 34]	3	0	$\frac{3}{2}$	3	$\frac{8}{3}$
Knot[11, NonAlternating, 42]	2	0	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{4}{3}$
GST48	Genus[GST48]	8	8	10	$\frac{55}{6}$

```
In[*]:= Union@Table[Simplify[Rho[K][[2]] == (-Theta[K][[2]] /. {T1 -> T, T2 -> 1})], {K, AllKnots[{3, 11}}]
```

Out[*]=
{True}

```
In[*]:= K = Knot[3, 1]; Factor@{Rho[K][[2]] /. T -> T^3, (-Theta[K][[2]] /. {T1 -> T, T2 -> T^2})}
```

Out[*]=

$$\left\{ \frac{(-1 + T)^2 (1 + T^2) (1 + T + T^2)^2 (1 - T^2 + T^4)}{T^6}, \frac{(-1 + T)^2 (1 + T^4) (1 + T + T^2 + T^3 + T^4 + T^5 + T^6)}{T^6} \right\}$$

```
In[ ]:= PolyPlotT1, T2[ $\Theta$ [Knot[10, 107]]][[2]]  
Out[ ]:=
```

