

Pensieve header: Finding the A_2 $\mathcal{S}d=1$ invariant using undetermined coefficients.

Searching for $Q + p_{xx} + \epsilon(ppx + 1 + px + ppx)$ solutions.

Initialization

```
In[*]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\HigherRank"];
Once[<< KnotTheory` ; << Rot.m];
<< FormalGaussianIntegration.m;
i_+ := i + 1;
```

Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.

Read more at <http://katlas.org/wiki/KnotTheory>.

Loading Rot.m from <http://drorbn.net/AP/Projects/HigherRank> to compute rotation numbers.

```
In[*]:= Features[Knot[8, 17]]
```

 KnotTheory: Loading precomputed data in PD4Knots`.

```
Out[*]=
```

```
Features[18,
C6[-1] C14[-1] X1,7[1] X3,9[-1] X5,13[-1] X8,16[1] X10,4[-1] X12,18[1] X15,2[-1] X17,11[1]]
```

```
In[*]:= T3 = T1 T2;
S = {x_, p_};
q[s_, i_, j_] := Sum[
  xv,i (pv,i+ - pv,i) + xv,j (pv,j+ - pv,j) + (Tv^S - 1) xv,i (pv,i+ - pv,j+),
  {v, 3}];
L[Xi_,j_[s_]] :=
  T3^S E[q[s, i, j] + B^-1 r0[s, i, j] + E B r1[s, i, j] + E r42[s, i, j] + O[epsilon]^2];
(* gamma1[phi_, k_] := phi (3/2 - X1,k p1,k - X2,k p2,k - X3,k p3,k) ; *)
L[Ck_[0]] := E[Sum[xv,k (pv,k+ - pv,k), {v, 3}] + O[epsilon]^2];
L[Ck_[phi_]] :=
  T3^phi E[Sum[xv,k (pv,k+ - pv,k), {v, 3}] + B^-1 gamma0[phi, k] + E B gamma1[phi, k] + E gamma42[phi, k] + O[epsilon]^2];
ps_i := Sequence[p1,i, p2,i, p3,i];
xs_i := Sequence[x1,i, x2,i, x3,i];
vs_i := Sequence[ps_i, xs_i];
F[is___] := E[Sum[pi,v,i pv,i, {i, {is}}, {v, 3}]];
L[K_] := (2 pi)^-Features[K][[1]] CF[L/@Features[K][[2]]];
vs[K_] := Union@@Table[{vs_i}, {i, Features[K][[1]]}]
```

```
In[*]:= vs_i
```

```
Out[*]=
```

```
Sequence[p1,i, p2,i, p3,i, x1,i, x2,i, x3,i]
```

The Various Terms (r_0)

The pxx Terms (r_0)

```
In[*]:=  $\kappa = 0$ ;
r0[1, i_, j_] := Evaluate[Sum[
  a++ $\kappa$  p3,k3 x1,k1 x2,k2,
  {k1, {i, j}}, {k2, {i, j}}, {k3, {i, j}}
]];
r0[1, i, j]
```

Out[*]=

$$a_1 p_{3,i} x_{1,i} x_{2,i} + a_2 p_{3,j} x_{1,i} x_{2,i} + a_5 p_{3,i} x_{1,j} x_{2,i} + a_6 p_{3,j} x_{1,j} x_{2,i} +$$

$$a_3 p_{3,i} x_{1,i} x_{2,j} + a_4 p_{3,j} x_{1,i} x_{2,j} + a_7 p_{3,i} x_{1,j} x_{2,j} + a_8 p_{3,j} x_{1,j} x_{2,j}$$

```
In[*]:=  $\kappa = 0$ ;
r0[-1, i_, j_] := Evaluate[Sum[
  d++ $\kappa$  p3,k3 x1,k1 x2,k2,
  {k1, {i, j}}, {k2, {i, j}}, {k3, {i, j}}
]];
r0[-1, i, j]
```

Out[*]=

$$d_1 p_{3,i} x_{1,i} x_{2,i} + d_2 p_{3,j} x_{1,i} x_{2,i} + d_5 p_{3,i} x_{1,j} x_{2,i} + d_6 p_{3,j} x_{1,j} x_{2,i} +$$

$$d_3 p_{3,i} x_{1,i} x_{2,j} + d_4 p_{3,j} x_{1,i} x_{2,j} + d_7 p_{3,i} x_{1,j} x_{2,j} + d_8 p_{3,j} x_{1,j} x_{2,j}$$

The pxx Terms (r_1)

```
In[*]:=  $\kappa = 0$ ;
r1[1, i_, j_] := Evaluate[Sum[
  b++ $\kappa$  x3,k3 p1,k1 p2,k2,
  {k1, {i, j}}, {k2, {i, j}}, {k3, {i, j}}
]];
r1[1, i, j]
```

Out[*]=

$$b_1 p_{1,i} p_{2,i} x_{3,i} + b_5 p_{1,j} p_{2,i} x_{3,i} + b_3 p_{1,i} p_{2,j} x_{3,i} + b_7 p_{1,j} p_{2,j} x_{3,i} +$$

$$b_2 p_{1,i} p_{2,i} x_{3,j} + b_6 p_{1,j} p_{2,i} x_{3,j} + b_4 p_{1,i} p_{2,j} x_{3,j} + b_8 p_{1,j} p_{2,j} x_{3,j}$$

```
In[*]:=  $\kappa = 0$ ;
r1[-1, i_, j_] := Evaluate[Sum[
  e++ $\kappa$  x3,k3 p1,k1 p2,k2,
  {k1, {i, j}}, {k2, {i, j}}, {k3, {i, j}}
]];
r1[-1, i, j]
```

Out[*]=

$$e_1 p_{1,i} p_{2,i} x_{3,i} + e_5 p_{1,j} p_{2,i} x_{3,i} + e_3 p_{1,i} p_{2,j} x_{3,i} + e_7 p_{1,j} p_{2,j} x_{3,i} +$$

$$e_2 p_{1,i} p_{2,i} x_{3,j} + e_6 p_{1,j} p_{2,i} x_{3,j} + e_4 p_{1,i} p_{2,j} x_{3,j} + e_8 p_{1,j} p_{2,j} x_{3,j}$$

The ppx Terms (r_{42})

```
In[*]:= x = 0;
Short[r42[1, i_, j_] = Evaluate[Plus[
  Sum[
    C+++ Xv1,k1 Pv1,k2 Xv2,k3 Pv2,k4,
    {k1, {i, j}}, {k2, {i, j}}, {k3, {i, j}}, {k4, {i, j}}, {v1, 2}, {v2, v1 + 1, 3}
  ],
  Sum[
    C+++ Xv,k1 Pv,k2,
    {k1, {i, j}}, {k2, {i, j}}, {v, 3}
  ]
]]]
```

```
Out[*]//Short=
C49 p1,i X1,i + C52 p1,j X1,i + <<57>> + C48 p2,j p3,j X2,j X3,j
```

```
In[*]:= x = 0;
Short[r42[-1, i_, j_] = Evaluate[Plus[
  Sum[
    f+++ Xv1,k1 Pv1,k2 Xv2,k3 Pv2,k4,
    {k1, {i, j}}, {k2, {i, j}}, {k3, {i, j}}, {k4, {i, j}}, {v1, 2}, {v2, v1 + 1, 3}
  ],
  Sum[
    f+++ Xv,k1 Pv,k2,
    {k1, {i, j}}, {k2, {i, j}}, {v, 3}
  ]
]]]
```

```
Out[*]//Short=
f49 p1,i X1,i + f52 p1,j X1,i + <<57>> + f48 p2,j p3,j X2,j X3,j
```

The γ Terms ($\gamma_0, \gamma_1, \gamma_{42}$)

```
In[*]:= x = 0;
 $\gamma_0[1, k_] := Evaluate[g+++ p3,k X1,k X2,k];$ 
 $\gamma_1[1, k_] := Evaluate[g+++ x3,k p1,k p2,k];$ 
 $\gamma_{42}[1, k_] := Evaluate[Plus[
  Sum[g+++ Xv,k Pv,k, {v, 3}],
  Sum[g+++ Xv1,k Pv1,k Xv2,k Pv2,k, {v1, 2}, {v2, v1 + 1, 3}]
]];
{ $\gamma_0[1, k]$ ,  $\gamma_0[1, k]$ ,  $\gamma_{42}[1, k]$ }$ 
```

```
Out[*]=
{g1 p3,k X1,k X2,k, g1 p3,k X1,k X2,k,
 g3 p1,k X1,k + g4 p2,k X2,k + g6 p1,k p2,k X1,k X2,k + g5 p3,k X3,k + g7 p1,k p3,k X1,k X3,k + g8 p2,k p3,k X2,k X3,k}
```

```
In[*]:= x = 0;
γ₀[-1, k_] := Evaluate[h+++ p3,k x1,k x2,k];
γ₁[-1, k_] := Evaluate[h+++ x3,k p1,k p2,k];
γ42[-1, k_] := Evaluate[Plus[
  Sum[h+++ xv,k pv,k, {v, 3}],
  Sum[h+++ xv1,k pv1,k xv2,k pv2,k, {v1, 2}, {v2, v1 + 1, 3}]
]];
{γ₀[-1, k], γ₀[-1, k], γ42[-1, k]}
```

```
Out[*]=
{h₁ p3,k x1,k x2,k, h₁ p3,k x1,k x2,k,
h₃ p1,k x1,k + h₄ p2,k x2,k + h₆ p1,k p2,k x1,k x2,k + h₅ p3,k x3,k + h₇ p1,k p3,k x1,k x3,k + h₈ p2,k p3,k x2,k x3,k}}
```

Reidemeister 3b

```
In[*]:= Timing[{LeftR3b} =
Cases[∫ ℱ[i, j, k] ℒ /@ (Xi,j[1] Xi+,k[1] Xj+,k+[1]) d{vsi, vsj, vsk, vsi+, vsj+, vsk+},
E[ε_] := ε, ∞]]
```

```
Out[*]=
{34.9844, {Series[
T₁² p1,2+i π1,i - (-1 + T₁) T₁ p1,2+j π1,i + (1 - T₁) p1,2+k π1,i + T₁ p1,2+j π1,j + (1 - T₁) p1,2+k π1,j + p1,2+k π1,k + ... 44 ... +
T₁² T₂² p3,2+i π3,i - T₁ T₂ (-1 + T₁ T₂) p3,2+j π3,i + (1 - T₁ T₂) p3,2+k π3,i + T₁ T₂ p3,2+j π3,j + (1 - T₁ T₂) p3,2+k π3,j + p3,2+k π3,k,
3 (a₁ b₁ + a₂ b₂ + a₃ b₃ + a₄ b₄ + a₅ b₅ + a₆ b₆ + a₇ b₇ + a₈ b₈ + c₁ + ... 8 ... + c₄₆ + c₄₇ + c₄₈ + c₄₉ + c₅₀ + c₅₁ + c₅₈ + c₅₉ + c₆₀) +
... 406 ... + ... 1 ... ]}}
```

Full expression not available (original memory size: 3.6 MB)

```
In[*]:= Timing[{RightR3b} =
Cases[∫ ℱ[i, j, k] ℒ /@ (Xj,k[1] Xi,k+[1] Xi+,j+[1]) d{vsi, vsj, vsk, vsi+, vsj+, vsk+},
E[ε_] := ε, ∞];
```

```
Out[*]=
{22.9688, Null}
```

```
In[*]:= Short[eqn = CF[LeftR3b[[1]] - RightR3b[[1]]]
cvs = Union@Cases[eqn, p_ | π_, ∞]
vars = Union@Cases[r_0[1, i, j], a_, ∞]
Short[eqns = CoefficientRules[eqn, cvs] /. (_ -> c_) :-> (c == 0), 3]
{sol} = Solve[eqns, vars]
```

Out[*]//Short=

$$\frac{T_1 T_2 \langle\langle 1 \rangle\rangle \langle\langle 1 \rangle\rangle \pi_{1,i} \pi_{2,i}}{B} - \frac{\langle\langle 1 \rangle\rangle}{B} + \langle\langle 32 \rangle\rangle + \frac{a_7 \langle\langle 5 \rangle\rangle \pi_{\langle\langle 1 \rangle\rangle}}{B}$$

Out[*]=

$$\{p_{3,2+i}, p_{3,2+j}, p_{3,2+k}, \pi_{1,i}, \pi_{1,j}, \pi_{1,k}, \pi_{2,i}, \pi_{2,j}, \pi_{2,k}\}$$

Out[*]=

$$\{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}$$

Out[*]//Short=

$$\left\{ -\frac{a_3 T_1^2 T_2^2}{B} + \frac{a_3 T_1^2 T_2^3}{B} == 0, \frac{a_3 T_1^2 T_2}{B} - \frac{a_3 T_1^2 T_2^2}{B} == 0, \right.$$

$$\left. \langle\langle 22 \rangle\rangle, -\frac{a_7}{B} - \frac{a_8}{B} + \frac{a_7 T_2}{B} + \frac{a_8 T_2}{B} + \frac{a_7 T_1 T_2}{B} - \frac{a_7 T_1 T_2^2}{B} == 0 \right\}$$

Solve: Equations may not give solutions for all "solve" variables.

Out[*]=

$$\left\{ \left\{ a_1 \rightarrow 0, a_3 \rightarrow 0, a_5 \rightarrow 0, a_6 \rightarrow -\frac{a_2}{T_1} - \frac{a_4 T_2}{T_1}, a_7 \rightarrow 0, a_8 \rightarrow 0 \right\} \right\}$$

```
In[*]:= sol /. (v_ -> val_) :-> (v = CF[val]);
r_0[1, i, j]
```

Out[*]=

$$a_2 p_{3,j} x_{1,i} x_{2,i} - \frac{(a_2 + a_4 T_2) p_{3,j} x_{1,j} x_{2,i}}{T_1} + a_4 p_{3,j} x_{1,i} x_{2,j}$$

```
In[*]:= Short[eqn = CF[Coefficient[
    LeftR3b[[2]] - RightR3b[[2]] /. v : (\pi | p) __ => \mu v,
    \mu^3
  ]], 5]
cvs = Union@Cases[eqn, p__ | \pi__, \infty]
vars = Union@Cases[r1[1, i, j], b_, \infty]
Short[eqns = CoefficientRules[eqn, cvs] /. (_ -> c_) => (c == 0), 3]
{sol} = Solve[eqns, vars]
```

Out[*]//Short=

$$B b_1 (-1 + T_1) T_1 T_2^2 p_{1,2+j} p_{2,2+i} \pi_{3,i} - B b_1 (-1 + T_1) T_1 T_2^2 p_{1,2+k} p_{2,2+i} \pi_{3,i} + \ll 31 \gg +$$

$$B (-1 + T_1) T_1 (-b_2 - b_4 + b_2 T_2) p_{1,2+i} p_{2,2+k} \pi_{3,k} - B (-1 + T_1) T_1 (-b_2 - b_4 + b_2 T_2) p_{1,2+j} p_{2,2+k} \pi_{3,k}$$

Out[*]=

$$\{p_{1,2+i}, p_{1,2+j}, p_{1,2+k}, p_{2,2+i}, p_{2,2+j}, p_{2,2+k}, \pi_{3,i}, \pi_{3,j}, \pi_{3,k}\}$$


Out[*]=

$$\{b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8\}$$

Out[*]//Short=

$$\{-B b_2 T_1^2 T_2^2 + B b_2 T_1^3 T_2^3 == 0, B b_2 T_1 T_2 - B b_2 T_1^2 T_2^2 == 0, \ll 1 \gg == 0, \ll 19 \gg, \ll 1 \gg == 0,$$

$$B b_1 + \ll 47 \gg == 0, -B b_6 T_1 - B b_8 T_1 - B b_4 T_2 - B b_8 T_2 + B b_2 T_1 T_2 + \ll 11 \gg + B b_2 T_1^2 T_2^2 == 0\}$$

 Solve: Equations may not give solutions for all "solve" variables. 

Out[*]=

$$\{\{b_1 \to 0, b_2 \to 0, b_4 \to 0, b_6 \to 0, b_7 \to -b_3 - b_5, b_8 \to 0\}\}$$

```
In[*]:= sol /. (v_ -> val_) => (v = CF[val]);
r1[1, i, j]
```

Out[*]=

$$b_5 p_{1,j} p_{2,i} x_{3,i} + b_3 p_{1,i} p_{2,j} x_{3,i} + (-b_3 - b_5) p_{1,j} p_{2,j} x_{3,i}$$

```
In[*]:= Short[eqn = CF[LeftR3b[[2]] - RightR3b[[2]], 5]
cvs = Union@Cases[eqn, p_ | π_, ∞]
vars = Union@Cases[r42[1, i, j], c_, ∞]
Short[eqns = CoefficientRules[eqn, cvs] /. (_ -> c_) :-> (c == 0), 3]
Short[{sol} = Solve[eqns, vars]]
```

Out[*]//Short=

$$- \left((C_{25} + C_{26} + C_{34} + C_{35} + C_{55}) (-1 + T_1) T_1^2 p_{1,2+j} \pi_{1,i} \right) -$$

$$(-1 + T_1) \left(C_1 + C_2 + C_{10} + C_{11} + C_{13} + C_{14} + C_{22} + C_{23} + C_{49} + C_{52} + \ll 5 \gg + C_{38} T_1 + \right.$$

$$C_{46} T_1 + C_{47} T_1 + C_{55} T_1 + C_{58} T_1 - C_{25} T_1^2 - C_{26} T_1^2 - C_{34} T_1^2 - C_{35} T_1^2 - C_{55} T_1^2 \left. \right) p_{1, \ll 1 \gg} \pi_{1,i} +$$

$$\ll 374 \gg + \ll 1 \gg - (-1 + T_2) T_2 (-C_{33} - C_{36} + C_{33} T_1 T_2) p_{2,2+j} p_{3,2+k} \pi_{2,k} \pi_{3,k}$$

Out[*]=

$$\{p_{1,2+i}, p_{1,2+j}, p_{1,2+k}, p_{2,2+i}, p_{2,2+j}, p_{2,2+k}, p_{3,2+i},$$

$$p_{3,2+j}, p_{3,2+k}, \pi_{1,i}, \pi_{1,j}, \pi_{1,k}, \pi_{2,i}, \pi_{2,j}, \pi_{2,k}, \pi_{3,i}, \pi_{3,j}, \pi_{3,k}\}$$

Out[*]=

$$\{C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8, C_9, C_{10}, C_{11}, C_{12}, C_{13}, C_{14}, C_{15}, C_{16}, C_{17}, C_{18}, C_{19}, C_{20}, C_{21}, C_{22},$$

$$C_{23}, C_{24}, C_{25}, C_{26}, C_{27}, C_{28}, C_{29}, C_{30}, C_{31}, C_{32}, C_{33}, C_{34}, C_{35}, C_{36}, C_{37}, C_{38}, C_{39}, C_{40}, C_{41},$$

$$C_{42}, C_{43}, C_{44}, C_{45}, C_{46}, C_{47}, C_{48}, C_{49}, C_{50}, C_{51}, C_{52}, C_{53}, C_{54}, C_{55}, C_{56}, C_{57}, C_{58}, C_{59}, C_{60}\}$$

Out[*]//Short=

$$\{-C_7 T_1^2 T_2^2 + C_7 T_1^2 T_2^3 == 0, \ll 250 \gg, C_8 T_1 T_2 + C_9 T_1 T_2 + C_{44} T_1 T_2 +$$

$$C_{45} T_1 T_2 + C_{57} T_1 T_2 - C_8 T_1^2 T_2^2 - C_9 T_1^2 T_2^2 - C_{44} T_1^2 T_2^2 - C_{45} T_1^2 T_2^2 - C_{57} T_1^2 T_2^2 == 0\}$$

 **Solve:** Equations may not give solutions for all "solve" variables. 

Out[*]//Short=

$$\left\{ \left\{ C_1 \rightarrow 0, \ll 46 \gg, C_{60} \rightarrow -\frac{C_{51}}{T_1 T_2} - \frac{C_{\ll 2 \gg}}{\ll 1 \gg \ll 1 \gg \ll 1 \gg} - \frac{\ll 8 \gg + \ll 1 \gg}{T_1^2 \ll 1 \gg (-1 + \ll 1 \gg)} \right\} \right\}$$

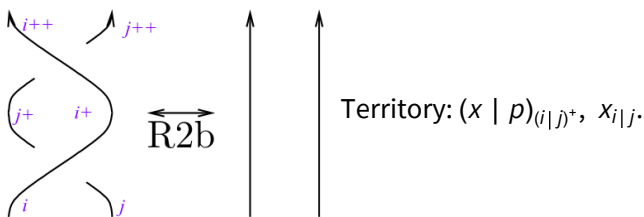
```
In[*]:= sol /. (v_ -> val_) :-> (v = CF[val]);
```

In[*]:= Short[CF[r₄₂[1, i, j]], 20]

Out[*]//Short=

$$\begin{aligned}
 & c_{49} p_{1,i} x_{1,i} + c_{52} p_{1,j} x_{1,i} - \frac{(c_{49} + c_{52}) p_{1,j} x_{1,j}}{T_1} + c_{50} p_{2,i} x_{2,i} + \\
 & c_{53} p_{2,j} x_{2,i} + c_{13} p_{1,j} p_{2,i} x_{1,i} x_{2,i} + c_4 p_{1,i} p_{2,j} x_{1,i} x_{2,i} + \ll 25 \gg + \\
 & \frac{1}{(-1 + T_1) (-1 + T_2)} \left(-a_4 b_5 - c_{15} + c_{15} T_1 + a_4 b_3 T_2 + a_4 b_5 T_2 - a_4 b_3 T_1 T_2 + c_{15} T_1 T_2 - c_{15} T_1^2 T_2 \right) \\
 & p_{2,j} p_{3,j} x_{2,j} x_{3,i} - \frac{1}{T_1^2 T_2 (-1 + T_1 T_2)} \\
 & \left(-a_2 b_5 - a_2 b_3 T_1 + a_4 b_3 T_1 - c_{51} T_1 - c_{54} T_1 - a_4 b_5 T_2 - a_4 b_3 T_1 T_2 + a_4 b_5 T_1 T_2 + c_{51} T_1^2 T_2 + c_{54} T_1^2 T_2 \right) \\
 & p_{3,j} x_{3,j} - \frac{(a_2 b_5 - a_4 b_5 - c_5 + c_{14} + c_5 T_1 + a_4 b_5 T_2 - c_{14} T_1 T_2) p_{1,i} p_{3,j} x_{1,i} x_{3,j}}{(-1 + T_1) (-1 + T_1 T_2)} - \\
 & \left((-a_2 b_3 + c_5 + a_2 b_3 T_1 - a_4 b_3 T_1 - c_5 T_1 - a_4 b_3 T_2 - c_5 T_2 + 2 a_4 b_3 T_1 T_2 + c_5 T_1 T_2) p_{1,j} p_{3,j} x_{1,i} x_{3,j} \right) / \\
 & \left((-1 + T_2) (-1 + T_1 T_2) \right) - \\
 & \left((a_2 b_3 - c_6 T_1 + c_{15} T_1 + a_4 b_3 T_2 - a_4 b_3 T_1 T_2 + c_6 T_1 T_2 - c_{15} T_1^2 T_2) p_{2,i} p_{3,j} x_{2,i} x_{3,j} \right) / \\
 & \left(T_1 (-1 + T_2) (-1 + T_1 T_2) \right) - \\
 & \left((c_6 T_1 - c_6 T_1^2 + a_2 b_5 T_2 - a_2 b_5 T_1 T_2 + a_4 b_5 T_1 T_2 - c_6 T_1 T_2 + c_6 T_1^2 T_2 + a_4 b_5 T_2^2 - 2 a_4 b_5 T_1 T_2^2) \right. \\
 & \left. p_{2,j} p_{3,j} x_{2,i} x_{3,j} \right) / \left((-1 + T_1) T_1 (-1 + T_1 T_2) \right)
 \end{aligned}$$

Reidemeister 2b



In[*]:= Timing[Short[LeftR2b = (∫ ℱ[i, j] ℒ / @ (X_{i,j}[1] X_{i+,j+}[-1]) d{vs_i, vs_j, vs_{i+}, vs_{j+}}) [[1]]]]

Out[*]=

$$\{2.01563, \in \text{Series}[p_{1,2+i} \pi_{1,i} + p_{1,2+j} \pi_{1,j} + \ll 12 \gg + p_{3,2+j} \pi_{3,j}, \ll 1 \gg]\}$$

In[*]:= RightR2b = eSeries[p_{1,2+i} π_{1,i} + p_{1,2+j} π_{1,j} + p_{2,2+i} π_{2,i} + p_{2,2+j} π_{2,j} + p_{3,2+i} π_{3,i} + p_{3,2+j} π_{3,j}, 0]

Out[*]=

$$\in \text{Series}[p_{1,2+i} \pi_{1,i} + p_{1,2+j} \pi_{1,j} + p_{2,2+i} \pi_{2,i} + p_{2,2+j} \pi_{2,j} + p_{3,2+i} \pi_{3,i} + p_{3,2+j} \pi_{3,j}, 0]$$


```
In[*]:= Short[eqn = CF[LeftR2b[[1]] - RightR2b[[1]]]
cvs = Union@Cases[eqn, p__ | π__, ∞]
vars = Union@Cases[r_[-1, i, j], d_, ∞]
Short[eqns = CoefficientRules[eqn, cvs] /. (_ -> c_) :-> (c == 0), 3]
{sol} = Solve[eqns, vars]
```

Out[*]//Short=

$$\frac{(d_7 + \ll 11 \gg + d_7 T_1 T_2) \ll 1 \gg \ll 1 \gg \ll 1 \gg \pi_{\ll 1 \gg}}{B T_1 T_2} + \ll 7 \gg + \frac{\ll 1 \gg}{B \ll 1 \gg T_2}$$

Out[*]=

$$\{p_{3,2+i}, p_{3,2+j}, \pi_{1,i}, \pi_{1,j}, \pi_{2,i}, \pi_{2,j}\}$$

Out[*]=

$$\{d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8\}$$

Out[*]//Short=

$$\left\{ \begin{aligned} \frac{d_1}{B} - \frac{d_3}{B} - \frac{d_5}{B} + \frac{d_7}{B} + \frac{d_5}{B T_1} - \frac{d_7}{B T_1} + \frac{d_3}{B T_2} - \frac{d_7}{B T_2} + \frac{d_7}{B T_1 T_2} &= 0, \\ \frac{d_3}{B T_2} - \frac{d_7}{B T_2} + \frac{d_7}{B T_1 T_2} &= 0, \ll 5 \gg, \frac{d_7}{B} + \frac{d_8}{B} - \frac{d_7}{B T_1 T_2} &= 0 \end{aligned} \right\}$$

Out[*]=

$$\left\{ \left\{ d_1 \rightarrow 0, d_2 \rightarrow -\frac{a_2 - a_4 T_1 + a_4 T_2}{T_1^2 T_2}, d_3 \rightarrow 0, d_4 \rightarrow -\frac{a_4}{T_1}, d_5 \rightarrow 0, d_6 \rightarrow -\frac{-a_2 - a_4 T_2}{T_1 T_2}, d_7 \rightarrow 0, d_8 \rightarrow 0 \right\} \right\}$$

```
In[*]:= sol /. (v_ -> val_) :-> (v = CF[val]);
r_[-1, i, j]
```

Out[*]=

$$\frac{(-a_2 + a_4 T_1 - a_4 T_2) p_{3,j} x_{1,i} x_{2,i}}{T_1^2 T_2} + \frac{(a_2 + a_4 T_2) p_{3,j} x_{1,j} x_{2,i}}{T_1 T_2} - \frac{a_4 p_{3,j} x_{1,i} x_{2,j}}{T_1}$$

```
In[*]:= Short[eqn = CF[LeftR2b[[2]] - RightR2b[[2]]]
cvs = Union@Cases[eqn, p_ | π_, ∞]
vars = Union@Cases[r1[-1, i, j] + r42[-1, i, j], e_ | f_, ∞]
Short[eqns = CoefficientRules[eqn, cvs] /. (_ -> c_) -> (c == 0), 3]
Short[{sol} = Solve[eqns, vars]]
```

Out[*]//Short=

$$\llcorner 85 \llcorner + \frac{\llcorner 1 \llcorner}{\llcorner 1 \llcorner} + \frac{\llcorner 1 \llcorner}{T_1 \llcorner 1 \llcorner}$$

Out[*]=

$$\{p_{1,2+i}, p_{1,2+j}, p_{2,2+i}, p_{2,2+j}, p_{3,2+i}, p_{3,2+j}, \pi_{1,i}, \pi_{1,j}, \pi_{2,i}, \pi_{2,j}, \pi_{3,i}, \pi_{3,j}\}$$

Out[*]=

$$\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}, f_{11}, f_{12}, f_{13}, f_{14}, f_{15}, f_{16}, f_{17}, f_{18}, f_{19}, f_{20}, f_{21}, f_{22}, f_{23}, f_{24}, f_{25}, f_{26}, f_{27}, f_{28}, f_{29}, f_{30}, f_{31}, f_{32}, f_{33}, f_{34}, f_{35}, f_{36}, f_{37}, f_{38}, f_{39}, f_{40}, f_{41}, f_{42}, f_{43}, f_{44}, f_{45}, f_{46}, f_{47}, f_{48}, f_{49}, f_{50}, f_{51}, f_{52}, f_{53}, f_{54}, f_{55}, f_{56}, f_{57}, f_{58}, f_{59}, f_{60}\}$$

Out[*]//Short=

$$\left\{ f_1 - f_7 - f_{25} + f_{31} + \frac{f_{25}}{T_1} - \frac{f_{31}}{T_1} + \frac{f_7}{T_2} - \frac{f_{31}}{T_2} + \frac{f_{31}}{T_1 T_2} = 0, \frac{f_7}{T_2} - \frac{\llcorner 1 \llcorner}{\llcorner 1 \llcorner} + \frac{f_{\llcorner 2 \llcorner}}{\llcorner 1 \llcorner \llcorner 1 \llcorner} = 0, \llcorner 66 \llcorner, \frac{2 a_4 b_3}{(1 - T_1)(1 - T_2)} + \frac{2 C_{49}}{(1 - T_1)(1 - T_2)} + \llcorner 204 \llcorner + \frac{f_{60} T_1 T_2}{(1 - T_1)(1 - T_2)} = 0 \right\}$$

Out[*]//Short=

$$\left\{ \left\{ e_1 \rightarrow 0, e_2 \rightarrow 0, \llcorner 64 \llcorner, f_{59} \rightarrow -\frac{\llcorner 1 \llcorner}{\llcorner 1 \llcorner}, f_{60} \rightarrow -\frac{a_2 b_5 + \llcorner 13 \llcorner}{T_1^2 T_2 (-1 + T_1 \llcorner 1 \llcorner)} \right\} \right\}$$

```
In[*]:= sol /. (v_ -> val_) -> (v = CF[val]);
```

```
In[*]:= r1[-1, i, j]
Short[CF[r42[-1, i, j]], 5]
```

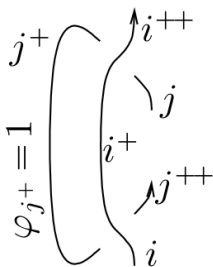
Out[*]=

$$-\frac{b_5 p_{1,j} p_{2,i} x_{3,i}}{T_1} - \frac{b_3 p_{1,i} p_{2,j} x_{3,i}}{T_2} + \frac{(b_3 T_1 + b_5 T_2) p_{1,j} p_{2,j} x_{3,i}}{T_1 T_2}$$

Out[*]//Short=

$$-C_{49} p_{1,i} x_{1,i} + \frac{(-C_{49} - C_{52} + C_{49} T_1^2) p_{1,j} x_{1,i}}{T_1^2} + \llcorner 42 \llcorner + \frac{\llcorner 1 \llcorner}{\llcorner 1 \llcorner} + \frac{(\llcorner 1 \llcorner) p_{2,i} p_{3, \llcorner 1 \llcorner} \llcorner 1 \llcorner x_{2,i} x_{3,j}}{T_1 (-1 + T_2) (-1 + T_1 T_2)} + \left((a_2 b_3 - a_2 b_3 T_1 + c_{15} T_1 - c_{15} T_1^2 + a_4 b_3 T_2 + \llcorner 6 \llcorner) + a_4 b_3 T_1^2 T_2 - c_{15} T_1^2 T_2 + c_{15} T_1^3 T_2 + a_4 b_5 T_2^2 - 2 a_4 b_5 T_1 T_2^2 \right) \llcorner 3 \llcorner x_{\llcorner 1 \llcorner} / ((-1 + T_1) T_1 T_2 (-1 + T_1 T_2))$$

Reidemeister 2c



```
In[*]:= Timing[ Short[ {LeftR2c} = Cases[
    ∫ ℱ[i, j] ℒ /@ (Xi+1,j[1] Xi,j+2[-1] Cj+1[1]) d{vsi, vsj, vsi+, vsj+, vsj+2}, E[ℰ_] := ℰ]
]]
```

```
Out[*]= {2.25, {∈Series[p1,2+i π1,i + p1,3+j π1,j + <<9>> + p3,3+j π3,j, g1 g2 + <<41>> + <<1>> ]}}
```

```
In[*]:= Timing[ Short[ {RightR2c} =
    Cases[ ∫ ℱ[i, j] ℒ /@ (Ci[0] Ci+1[0] Cj[0] Cj+1[1] Cj+2[0]) d{vsi, vsj, vsi+, vsj+, vsj+2},
    E[ℰ_] := ℰ]
]]
```

```
Out[*]= {0., {∈Series[p1,2+i π1,i + p1,3+j π1,j + <<4>> + p3,3+j π3,j, g1 g2 + <<12>> + <<1>> ]}}
```

```
In[*]:= Short[eqn = CF[LeftR2c[[1]] - RightR2c[[1]]]
cvs = Union@Cases[eqn, p__ | π__, ∞]
vars = Union@Cases[γ0[1, k], g_, ∞]
Short[eqns = CoefficientRules[eqn, cvs] /. (_ -> c_) := (c == 0), 3]
{sol} = Solve[eqns, vars]
```

```
Out[*]//Short=

$$\frac{g_1 (-1 + T_1) \ll 1 \gg \ll 1 \gg \ll 1 \gg \pi_{1, \ll 1 \gg} \ll 1 \gg \pi_{2, i}}{B T_1 T_2} - \frac{\ll 1 \gg}{B \ll 1 \gg \ll 1 \gg} - \frac{g_1 \ll 3 \gg \ll 1 \gg}{B T_1}$$

```

```
Out[*]= {p3,3+j, π1,i, π1,j, π2,i, π2,j}
```

```
Out[*]= {g1}
```

```
Out[*]//Short=

$$\left\{ \frac{g_1}{B} - \frac{g_1}{B T_1} - \frac{g_1}{B T_2} + \frac{g_1}{B T_1 T_2} == 0, -\frac{g_1}{B} + \frac{g_1}{B T_1} == 0, -\frac{g_1}{B} + \frac{g_1}{B T_2} == 0 \right\}$$

```

```
Out[*]= {{g1 -> 0}}
```

```
In[*]:= sol /. (v_ -> val_) :-> (v = CF[val]);
      \gamma_0[1, k]
```

```
Out[*]=
      0
```

```
In[*]:= Short[eqn = CF[LeftR2c[[2]] - RightR2c[[2]]]
      cvs = Union@Cases[eqn, p_ | \pi_, \infty]
      vars = Union@Cases[\gamma_1[1, k] + \gamma_{42}[1, k], g_, \infty]
      Short[eqns = CoefficientRules[eqn, cvs] /. (_ -> c_) :-> (c == 0), 3]
      Short[{sol} = Solve[eqns, vars]]
```

```
Out[*]//Short=
      - \frac{(g_3 + g_6 + g_7) (-1 + \ll 1 \gg) \ll 1 \gg \pi_{1,i}}{T_1} - \frac{\ll 1 \gg}{\ll 1 \gg} + \ll 18 \gg
```

```
Out[*]=
      {p_{1,3+j}, p_{2,3+j}, p_{3,3+j}, \pi_{1,i}, \pi_{1,j}, \pi_{2,i}, \pi_{2,j}, \pi_{3,i}, \pi_{3,j}}
```

```
Out[*]=
      {g_2, g_3, g_4, g_5, g_6, g_7, g_8}
```

```
Out[*]//Short=
      {g_6 - \frac{g_6}{T_1} - \frac{g_6}{T_2} + \frac{a_2 g_2}{T_1 T_2} + \frac{g_6}{T_1 T_2} == 0, -g_6 + \frac{a_4 g_2}{T_1} + \frac{g_6}{T_1} == 0, \ll 1 \gg == 0, \ll 7 \gg, -g_8 + \ll 1 \gg == 0,
      -g_4 - g_6 - g_8 + \frac{g_4}{T_2} + \frac{g_6}{T_2} + \frac{g_8}{T_2} == 0, \frac{2 a_4 b_3}{(1 - T_1) (1 - T_2)} + \frac{a_4 b_3}{(1 - T_1) T_1^2 (1 - T_2)} + \ll 50 \gg == 0}
```

```
Out[*]//Short=
      {{g_2 -> 0, g_3 -> 0, g_4 -> 0, g_5 -> -\frac{\ll 1 \gg}{\ll 1 \gg}, g_6 -> 0, g_7 -> 0, g_8 -> 0}}
```

```
In[*]:= sol /. (v_ -> val_) :-> (v = CF[val]);
```

```
In[*]:= \gamma_1[1, k]
      Short[CF[\gamma_{42}[1, k]], 5]
```

```
Out[*]=
      0
```

```
Out[*]//Short=
      \frac{(-b_3 + b_5 + b_3 T_1 - b_5 T_2) (-a_2 + a_2 T_1 - a_4 T_1 - a_4 T_2 + 2 a_4 T_1 T_2) p_{3,k} x_{3,k}}{(-1 + T_1) T_1 (-1 + T_2) (-1 + T_1 T_2)}
```

C_k[1] and C_k[-1] are inverses

```
In[*]:= Timing[Short[{LeftCC} = Cases[{{\int \mathcal{F}[k] \mathcal{L} / @ (C_k[1] C_{k+1}[-1]) d\{vs_k, vs_{k'}\}}, \mathbb{E}[\mathcal{E}_-] :-> \mathcal{E}}]]]
```

```
Out[*]=
      {0.015625, {Series[p_{1,2+k} \pi_{1,k} + p_{2,2+k} \pi_{2,k} + \frac{\ll 1 \gg}{B} + p_{3,2+k} \pi_{3,k}, \frac{\ll 1 \gg}{\ll 1 \gg} + \ll 6 \gg + \ll 1 \gg ]}}}
```

```
In[*]:= Timing [ Short [ { RightCC } = Cases [ { {  $\int \mathcal{F}[k] \mathcal{L} / @ (C_k[\theta] C_{k+1}[\theta]) d\{v_{s_k}, v_{s_k^*}\}$  },  $\mathbb{E}[\mathcal{E}_-] \Rightarrow \mathcal{E}$  } ] ] ]
```

```
Out[*]= { 0., {Series [ p_{1,2+k} \pi_{1,k} + p_{2,2+k} \pi_{2,k} + p_{3,2+k} \pi_{3,k}, \theta ] } }
```

```
In[*]:= Short [ eqn = CF [ LeftCC [ 1 ] - RightCC [ 1 ] ] ]
cvs = Union @ Cases [ eqn, p_ | \pi_, \infty ]
vars = Union @ Cases [ \gamma_0 [ -1, k ], h_, \infty ]
Short [ eqns = CoefficientRules [ eqn, cvs ] /. ( _ -> c_ ) => ( c == 0 ), 3 ]
{ sol } = Solve [ eqns, vars ]
```

```
Out[*]//Short= 
$$\frac{h_1 p_3 \ll 1 \gg 2 \ll 1 \gg \ll 1 \gg \pi \ll 1 \gg \pi_{2,k}}{B}$$

```

```
Out[*]= { p_{3,2+k}, \pi_{1,k}, \pi_{2,k} }
```

```
Out[*]= { h_1 }
```

```
Out[*]//Short= {  $\frac{h_1}{B} == 0$  }
```

```
Out[*]= { { h_1 -> 0 } }
```

```
In[*]:= sol /. ( v_ -> val_ ) => ( v = CF [ val ] );
\gamma_0 [ -1, k ]
```

```
Out[*]= 0
```

```
In[*]:= Short[eqn = CF[LeftCC[[2]] - RightCC[[2]]]
cvs = Union@Cases[eqn, p__ |  $\pi$ __,  $\infty$ ]
vars = Union@Cases[ $\gamma_1[-1, k] + \gamma_{42}[-1, k], h_, \infty$ ]
Short[eqns = CoefficientRules[eqn, cvs] /. (_ -> c_) :-> (c == 0), 3]
Short[{sol} = Solve[eqns, vars]]
```

```
Out[*]//Short=
<<1>>
```

```
Out[*]=
{p1,2+k, p2,2+k, p3,2+k,  $\pi_{1,k}$ ,  $\pi_{2,k}$ ,  $\pi_{3,k}$ }
```

```
Out[*]=
{h2, h3, h4, h5, h6, h7, h8}
```

```
Out[*]//Short=
{h6 == 0, B h2 == 0, h7 == 0, h3 + h6 + h7 == 0, h8 == 0, h4 + h6 + h8 == 0,
   $\frac{2 a_2 b_3}{(1 - T_1) (1 - T_2) (1 - T_1 T_2)} - \frac{a_4 b_3}{(1 - T_1) (1 - \langle\langle 1 \rangle\rangle) (1 - T_1 T_2)} - \frac{\langle\langle 1 \rangle\rangle \langle\langle 1 \rangle\rangle \langle\langle 1 \rangle\rangle}{\langle\langle 1 \rangle\rangle} + \langle\langle 48 \rangle\rangle == 0,$ 
   $\langle\langle 1 \rangle\rangle == 0$ }
```

```
Out[*]//Short=
{{h2 -> 0, h3 -> 0, h4 -> 0, h5 -> - $\frac{\langle\langle 1 \rangle\rangle}{\langle\langle 1 \rangle\rangle}$ , h6 -> 0, h7 -> 0, h8 -> 0}}
```

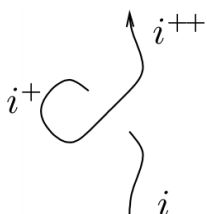
```
In[*]:= sol /. (v_ -> val_) :-> (v = CF[val]);
```

```
In[*]:=  $\gamma_1[-1, k]$ 
Short[CF[ $\gamma_{42}[-1, k]$ ], 5]
```

```
Out[*]=
0
```

```
Out[*]//Short=
 $\frac{(-b_3 + b_5 + b_3 T_1 - b_5 T_2) (-a_2 + a_2 T_1 - a_4 T_1 - a_4 T_2 + 2 a_4 T_1 T_2) p_{3,k} x_{3,k}}{(-1 + T_1) T_1 (-1 + T_2) (-1 + T_1 T_2)}$ 
```

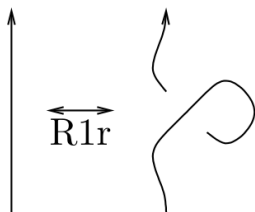
Invariance Under R1l



```
In[*]:= {LeftR1l} = Cases[{{ $\int \mathcal{F}[i] \mathcal{L} / @ (X_{i+2,i}[1] C_{i+1}[1]) d\{v_{s_i}, v_{s_{i+}}, v_{s_{i+2}}\}$ ,  $\mathbb{E}[\mathcal{E}_-] :-> \mathcal{E}, \infty$ }}
```

```
Out[*]=
{ $\in$ Series[p1,3+i  $\pi_{1,i}$  + p2,3+i  $\pi_{2,i}$  + p3,3+i  $\pi_{3,i}$ , 0]}
```

Invariance Under R1r



```
In[*]:= {LeftR1r} = Cases [ { { \int \mathcal{F}[i] \mathcal{L} / @ (X_{i,i+2} [1] C_{i+1} [-1]) d {v_{s_i}, v_{s_{i+}}, v_{s_{i+2}}} } , \mathbb{E} [\mathcal{E}_-] \Rightarrow \mathcal{E}, \infty ]
```

```
Out[*]:=
{ \in Series [ p_{1,3+i} \pi_{1,i} + p_{2,3+i} \pi_{2,i} + p_{3,3+i} \pi_{3,i},
  \frac{1}{T_1^2 T_2 (-1 + T_1 T_2)} ( a_2 b_5 + a_2 b_3 T_1 - a_4 b_3 T_1 + c_{51} T_1 + c_{54} T_1 + c_{50} T_1^2 + c_{53} T_1^2 + a_4 b_5 T_2 +
  a_4 b_3 T_1 T_2 - a_4 b_5 T_1 T_2 + c_{49} T_1 T_2 + c_{52} T_1 T_2 - c_{49} T_1^2 T_2 - c_{50} T_1^2 T_2 - 2 c_{51} T_1^2 T_2 -
  c_{54} T_1^2 T_2 - c_{50} T_1^3 T_2 - c_{53} T_1^3 T_2 - c_{49} T_1^2 T_2^2 - c_{52} T_1^2 T_2^2 + c_{49} T_1^3 T_2^2 + c_{50} T_1^3 T_2^2 + c_{51} T_1^3 T_2^2 ) ] }
```

```
In[*]:= {RightR1r} = Cases [ { { \int \mathcal{F}[i] \mathcal{L} / @ (C_i [0] C_{i+1} [0] C_{i+2} [0]) d {v_{s_i}, v_{s_{i+}}, v_{s_{i+2}}} } , \mathbb{E} [\mathcal{E}_-] \Rightarrow \mathcal{E}, \infty ]
```

```
Out[*]:=
{ \in Series [ p_{1,3+i} \pi_{1,i} + p_{2,3+i} \pi_{2,i} + p_{3,3+i} \pi_{3,i}, 0 ] }
```

```
In[*]:= LeftR1r[[1]] == RightR1r[[1]]
```

```
Out[*]:=
True
```

```
In[*]:= Short [ eqn = CF [ LeftR1r [[2]] - RightR1r [[2]] ]
  cvs = Union @ Cases [ eqn, p_ | \pi_, \infty ]
  vars = Union @ Cases [ eqn, (c | d | e | f | g | h)_ , \infty ]
  Short [ eqns = CoefficientRules [ eqn, cvs ] /. (_ -> c_) \Rightarrow (c == 0), 3 ]
  Short [ {sol} = Solve [ eqns, vars ] ]
```

```
Out[*]//Short=
\frac{a_2 b_5 + \langle\langle 31 \rangle\rangle + c_{51} T_1^3 T_2^2}{T_1^2 T_2 (-1 + T_1 T_2)}
```

```
Out[*]=
{ }
```

```
Out[*]=
{ C_{49}, C_{50}, C_{51}, C_{52}, C_{53}, C_{54} }
```

```
Out[*]//Short=
{ \frac{a_2 b_5 + a_2 b_3 T_1 - a_4 b_3 T_1 + \langle\langle 26 \rangle\rangle + c_{49} T_1^3 T_2^2 + c_{50} T_1^3 T_2^2 + c_{51} T_1^3 T_2^2}{T_1^2 T_2 (-1 + T_1 T_2)} == 0 }
```

Solve: Equations may not give solutions for all "solve" variables.

```
Out[*]//Short=
{ { C_{54} \to \langle\langle 1 \rangle\rangle } }
```

```
In[*]:= sol /. (v_ -> val_) \Rightarrow (v = CF [val]);
```