Pensieve header: Finding the A2 $d=1$ invariant using undetermined coefficients.

Searching for $Q + pxx + \epsilon(ppx + 1 + px + ppxx)$ solutions.

Initialization

```
In[◦]:=  SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\HigherRank"];
         Once[<< KnotTheory`; << Rot.m];
         << FormalGaussianIntegration.m;
         i_⁺ := i + 1;
```

Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.

Loading Rot.m from http://drorbn.net/AP/Projects/HigherRank to compute rotation numbers.

```
In[◦]:=  Features[Knot[8, 17]]
```

⊙⊙ KnotTheory: Loading precomputed data in PD4Knots`.

```
Out[◦]=
         Features[18,
           C_6[-1] C_14[-1] X_{1,7}[1] X_{3,9}[-1] X_{5,13}[-1] X_{8,16}[1] X_{10,4}[-1] X_{12,18}[1] X_{15,2}[-1] X_{17,11}[1]]
```

```
In[◦]:=  T_3 = T_1 T_2;
         S = {x__, p__};
         q[s_, i_, j_] := Sum[
             x_{v,i} (p_{v,i⁺} - p_{v,i}) + x_{v,j} (p_{v,j⁺} - p_{v,j}) + (T_v^s - 1) x_{v,i} (p_{v,i⁺} - p_{v,j⁺}),
             {v, 3}];
         L[X_{i_,j_}[s_]] :=
           T_3^s 𝔼[q[s, i, j] + B^{-1} r_0[s, i, j] + ϵ B r_1[s, i, j] + ϵ r_42[s, i, j] + O[ϵ]²];
         (* γ_1[φ_, k_] := φ (3/2 - x_{1,k} p_{1,k} - x_{2,k} p_{2,k} - x_{3,k} p_{3,k}); *)
         L[C_k_[0]] := 𝔼[Sum[x_{v,k} (p_{v,k⁺} - p_{v,k}), {v, 3}] + O[ϵ]²];
         L[C_k_[φ_]] :=
           T_3^φ 𝔼[Sum[x_{v,k} (p_{v,k⁺} - p_{v,k}), {v, 3}] + B^{-1} γ_0[φ, k] + ϵ B γ_1[φ, k] + ϵ γ_42[φ, k] + O[ϵ]²];
         ps_{i_} := Sequence[p_{1,i}, p_{2,i}, p_{3,i}];
         xs_{i_} := Sequence[x_{1,i}, x_{2,i}, x_{3,i}];
         vs_{i_} := Sequence[ps_i, xs_i];
         F[is___] := 𝔼[Sum[π_{v,i} p_{v,i}, {i, {is}}, {v, 3}]];
         L[K_] := (2 π)^{-Features[K][[1]]} CF[L /@ Features[K][[2]]];
         vs[K_] := Union @@ Table[{vs_i}, {i, Features[K][[1]]}]
```

```
In[◦]:=  vs_i
```

```
Out[◦]=
         Sequence[p_{1,i}, p_{2,i}, p_{3,i}, x_{1,i}, x_{2,i}, x_{3,i}]
```

The Various Terms (r_0)

The pxx Terms (r_0)

```
In[•]:= κ = 0;
      r₀[1, i_, j_] := Evaluate[Sum[
          a₊₊κ p₃,ₖ₃ x₁,ₖ₁ x₂,ₖ₂,
          {k1, {i, j}}, {k2, {i, j}}, {k3, {i, j}}
          ]];
      r₀[1, i, j]
```

Out[•]=

$a_1\,p_{3,i}\,x_{1,i}\,x_{2,i} + a_2\,p_{3,j}\,x_{1,i}\,x_{2,i} + a_5\,p_{3,i}\,x_{1,j}\,x_{2,i} + a_6\,p_{3,j}\,x_{1,j}\,x_{2,i} +$
$a_3\,p_{3,i}\,x_{1,i}\,x_{2,j} + a_4\,p_{3,j}\,x_{1,i}\,x_{2,j} + a_7\,p_{3,i}\,x_{1,j}\,x_{2,j} + a_8\,p_{3,j}\,x_{1,j}\,x_{2,j}$

```
In[•]:= κ = 0;
      r₀[-1, i_, j_] := Evaluate[Sum[
          d₊₊κ p₃,ₖ₃ x₁,ₖ₁ x₂,ₖ₂,
          {k1, {i, j}}, {k2, {i, j}}, {k3, {i, j}}
          ]];
      r₀[-1, i, j]
```

Out[•]=

$d_1\,p_{3,i}\,x_{1,i}\,x_{2,i} + d_2\,p_{3,j}\,x_{1,i}\,x_{2,i} + d_5\,p_{3,i}\,x_{1,j}\,x_{2,i} + d_6\,p_{3,j}\,x_{1,j}\,x_{2,i} +$
$d_3\,p_{3,i}\,x_{1,i}\,x_{2,j} + d_4\,p_{3,j}\,x_{1,i}\,x_{2,j} + d_7\,p_{3,i}\,x_{1,j}\,x_{2,j} + d_8\,p_{3,j}\,x_{1,j}\,x_{2,j}$

The ppx Terms (r_1)

```
In[•]:= κ = 0;
      r₁[1, i_, j_] := Evaluate[Sum[
          b₊₊κ x₃,ₖ₃ p₁,ₖ₁ p₂,ₖ₂,
          {k1, {i, j}}, {k2, {i, j}}, {k3, {i, j}}
          ]];
      r₁[1, i, j]
```

Out[•]=

$b_1\,p_{1,i}\,p_{2,i}\,x_{3,i} + b_5\,p_{1,j}\,p_{2,i}\,x_{3,i} + b_3\,p_{1,i}\,p_{2,j}\,x_{3,i} + b_7\,p_{1,j}\,p_{2,j}\,x_{3,i} +$
$b_2\,p_{1,i}\,p_{2,i}\,x_{3,j} + b_6\,p_{1,j}\,p_{2,i}\,x_{3,j} + b_4\,p_{1,i}\,p_{2,j}\,x_{3,j} + b_8\,p_{1,j}\,p_{2,j}\,x_{3,j}$

```
In[•]:= κ = 0;
      r₁[-1, i_, j_] := Evaluate[Sum[
          e₊₊κ x₃,ₖ₃ p₁,ₖ₁ p₂,ₖ₂,
          {k1, {i, j}}, {k2, {i, j}}, {k3, {i, j}}
          ]];
      r₁[-1, i, j]
```

Out[•]=

$e_1\,p_{1,i}\,p_{2,i}\,x_{3,i} + e_5\,p_{1,j}\,p_{2,i}\,x_{3,i} + e_3\,p_{1,i}\,p_{2,j}\,x_{3,i} + e_7\,p_{1,j}\,p_{2,j}\,x_{3,i} +$
$e_2\,p_{1,i}\,p_{2,i}\,x_{3,j} + e_6\,p_{1,j}\,p_{2,i}\,x_{3,j} + e_4\,p_{1,i}\,p_{2,j}\,x_{3,j} + e_8\,p_{1,j}\,p_{2,j}\,x_{3,j}$

The ppxx Terms (r_{42})

```
In[•]:= κ = 0;
       Short[r₄₂[1, i_, j_] = Evaluate[Plus[
           Sum[
             c₊₊ₖ x_{v1,k1} p_{v1,k2} x_{v2,k3} p_{v2,k4},
             {k1, {i, j}}, {k2, {i, j}}, {k3, {i, j}}, {k4, {i, j}}, {v1, 2}, {v2, v1 + 1, 3}
             ],
           Sum[
             c₊₊ₖ x_{v,k1} p_{v,k2},
             {k1, {i, j}}, {k2, {i, j}}, {v, 3}
             ]
           ]]]
```

```
Out[•]//Short=
       c₄₉ p_{1,i} x_{1,i} + c₅₂ p_{1,j} x_{1,i} + c₅₅ p_{1,i} x_{1,j} + ≪54≫ +
         c₄₅ p_{2,j} p_{3,i} x_{2,j} x_{3,j} + c₃₆ p_{2,i} p_{3,j} x_{2,j} x_{3,j} + c₄₈ p_{2,j} p_{3,j} x_{2,j} x_{3,j}
```

```
In[•]:= κ = 0;
       Short[r₄₂[-1, i_, j_] = Evaluate[Plus[
           Sum[
             f₊₊ₖ x_{v1,k1} p_{v1,k2} x_{v2,k3} p_{v2,k4},
             {k1, {i, j}}, {k2, {i, j}}, {k3, {i, j}}, {k4, {i, j}}, {v1, 2}, {v2, v1 + 1, 3}
             ],
           Sum[
             f₊₊ₖ x_{v,k1} p_{v,k2},
             {k1, {i, j}}, {k2, {i, j}}, {v, 3}
             ]
           ]]]
```

```
Out[•]//Short=
       f₄₉ p_{1,i} x_{1,i} + f₅₂ p_{1,j} x_{1,i} + f₅₅ p_{1,i} x_{1,j} + ≪54≫ +
         f₄₅ p_{2,j} p_{3,i} x_{2,j} x_{3,j} + f₃₆ p_{2,i} p_{3,j} x_{2,j} x_{3,j} + f₄₈ p_{2,j} p_{3,j} x_{2,j} x_{3,j}
```

The γ Terms (γ_0, γ_1, γ_{42})

```
In[•]:= κ = 0;
       γ₀[1, k_] := Evaluate[g₊₊ₖ p_{3,k} x_{1,k} x_{2,k}];
       γ₁[1, k_] := Evaluate[g₊₊ₖ x_{3,k} p_{1,k} p_{2,k}];
       γ₄₂[1, k_] := Evaluate[Plus[
           Sum[g₊₊ₖ x_{v,k} p_{v,k}, {v, 3}],
           Sum[g₊₊ₖ x_{v1,k} p_{v1,k} x_{v2,k} p_{v2,k}, {v1, 2}, {v2, v1 + 1, 3}]
           ]];
       {γ₀[1, k], γ₀[1, k], γ₄₂[1, k]}
```

```
Out[•]=
       {g₁ p_{3,k} x_{1,k} x_{2,k}, g₁ p_{3,k} x_{1,k} x_{2,k},
        g₃ p_{1,k} x_{1,k} + g₄ p_{2,k} x_{2,k} + g₆ p_{1,k} p_{2,k} x_{1,k} x_{2,k} + g₅ p_{3,k} x_{3,k} + g₇ p_{1,k} p_{3,k} x_{1,k} x_{3,k} + g₈ p_{2,k} p_{3,k} x_{2,k} x_{3,k}}
```

```
In[ ]:= κ = 0;
        γ₀[-1, k_] := Evaluate[h₊₊ₖ p₃,ₖ x₁,ₖ x₂,ₖ];
        γ₁[-1, k_] := Evaluate[h₊₊ₖ x₃,ₖ p₁,ₖ p₂,ₖ];
        γ₄₂[-1, k_] := Evaluate[Plus[
            Sum[h₊₊ₖ xᵥ,ₖ pᵥ,ₖ, {ν, 3}],
            Sum[h₊₊ₖ xᵥ₁,ₖ pᵥ₁,ₖ xᵥ₂,ₖ pᵥ₂,ₖ, {ν1, 2}, {ν2, ν1 + 1, 3}]
        ]];
        {γ₀[-1, k], γ₀[-1, k], γ₄₂[-1, k]}
```

```
Out[ ]=
        {h₁ p₃,ₖ x₁,ₖ x₂,ₖ, h₁ p₃,ₖ x₁,ₖ x₂,ₖ,
         h₃ p₁,ₖ x₁,ₖ + h₄ p₂,ₖ x₂,ₖ + h₆ p₁,ₖ p₂,ₖ x₁,ₖ x₂,ₖ + h₅ p₃,ₖ x₃,ₖ + h₇ p₁,ₖ p₃,ₖ x₁,ₖ x₃,ₖ + h₈ p₂,ₖ p₃,ₖ x₂,ₖ x₃,ₖ}
```

Reidemeister 3b

```
In[ ]:= Timing[{LeftR3b} =
          Cases[∫𝓕[i, j, k] 𝓛 /@ (Xᵢ,ⱼ[1] Xᵢ⁺,ₖ[1] Xⱼ⁺,ₖ⁺[1]) ⅆ{vsᵢ, vsⱼ, vsₖ, vsᵢ⁺, vsⱼ⁺, vsₖ⁺},
            𝔼[𝓔_] ⇝ 𝓔, ∞]]
```

```
Out[ ]=
```



```
{3.23438,
 {∈Series[T₁² p₁,₂₊ᵢ π₁,ᵢ − (−1 + T₁) T₁ p₁,₂₊ⱼ π₁,ᵢ + (1 − T₁) p₁,₂₊ₖ π₁,ᵢ + T₁ p₁,₂₊ⱼ π₁,ⱼ + (1 − T₁) p₁,₂₊ₖ π₁,ⱼ + p₁,₂₊ₖ π₁,ₖ +
   T₂² p₂,₂₊ᵢ π₂,ᵢ − (−1 + T₂) T₂ p₂,₂₊ⱼ π₂,ᵢ + (1 − T₂) p₂,₂₊ₖ π₂,ᵢ + ⋯ 33 ⋯ + a₃ T₁ T₂ p₃,₂₊ⱼ π₁,ⱼ π₂,ₖ / B −
   (−a₃−a₄+a₃ T₁ T₂) p₃,₂₊ₖ π₁,ⱼ π₂,ₖ / B + a₇ T₁ T₂ p₃,₂₊ᵢ π₁,ₖ π₂,ₖ / B + a₇ T₁ T₂ p₃,₂₊ⱼ π₁,ₖ π₂,ₖ / B − 2 (−a₇−a₈+a₇ T₁ T₂) p₃,₂₊ₖ π₁,ₖ π₂,ₖ / B + T₁² T₂² p₃,₂₊ᵢ π₃,ᵢ −
   T₁ T₂ (−1 + T₁ T₂) p₃,₂₊ⱼ π₃,ᵢ + (1 − T₁ T₂) p₃,₂₊ₖ π₃,ᵢ + T₁ T₂ p₃,₂₊ⱼ π₃,ⱼ + (1 − T₁ T₂) p₃,₂₊ₖ π₃,ⱼ + p₃,₂₊ₖ π₃,ₖ,
   3 (a₁ b₁ + a₂ b₂ + a₃ b₃ + a₄ b₄ + a₅ b₅ + a₆ b₆ + a₇ b₇ + a₈ b₈ + c₁ + c₂ + c₃ + c₁₀ + c₁₁ + c₁₂ +
      c₃₇ + c₃₈ + c₃₉ + c₄₆ + c₄₇ + c₄₈ + c₄₉ + c₅₀ + c₅₁ + c₅₈ + c₅₉ + c₆₀) + ⋯ 406 ⋯ +
   (2 a₃ b₂ + 2 a₄ b₂ + a₇ b₂ + a₈ b₂ + 2 a₃ b₄ + 2 a₄ b₄ + a₇ b₄ + a₈ b₄ + 3 a₇ b₆ + 3 a₈ b₆ + 3 a₇ b₈ + 3 a₈ b₈ + 2 c₃₃ + ⋯ 42 ⋯ +
      a₇ b₂ T₁² T₂ + a₇ b₄ T₁² T₂ + 2 a₃ b₂ T₁ T₂² + a₇ b₂ T₁ T₂² + 3 a₇ b₆ T₁ T₂² + 2 c₃₃ T₁ T₂² − a₇ b₂ T₁² T₂²) p₂,₂₊ₖ p₃,₂₊ₖ π₂,ₖ π₃,ₖ]}}
```

Full expression not available (original memory size: 3.6 MB)   ⚙

```
In[ ]:= Timing[{RightR3b} =
          Cases[∫𝓕[i, j, k] 𝓛 /@ (Xⱼ,ₖ[1] Xᵢ,ₖ⁺[1] Xᵢ⁺,ⱼ⁺[1]) ⅆ{vsᵢ, vsⱼ, vsₖ, vsᵢ⁺, vsⱼ⁺, vsₖ⁺},
            𝔼[𝓔_] ⇝ 𝓔, ∞];]
```

```
Out[ ]=
        {1.98438, Null}
```

$In[\bullet]:=$ **Short[eqn = CF[LeftR3b[1] − RightR3b[1]]]**
cvs = Union@Cases[eqn, p__ | π__, ∞]
vars = Union@Cases[r₀[1, i, j], a_, ∞]
Short[eqns = CoefficientRules[eqn, cvs] /. (_ → c_) :> (c == 0), 3]
{sol} = Solve[eqns, vars]

$Out[\bullet]//Short=$

$$\frac{T_1 \, T_2 \left(-a_1 \, T_1 + \ll 20 \gg + a_7 \, T_1^2 \, T_2^2 \right) p_{3,2+j} \, \pi_{1,i} \, \pi_{2,i}}{B} - \frac{\ll 1 \gg}{B} + \frac{a_5 \, \ll 5 \gg \pi_{\ll 1 \gg}}{B} - \frac{\ll 1 \gg}{B} + \ll 29 \gg + \frac{a_7 \, T_1 \, T_2 \left(-1 + T_1 \, T_2 \right) p_{3,2+j} \, \pi_{1,k} \, \pi_{2,k}}{B}$$

$Out[\bullet]=$

$$\{ p_{3,2+i}, \, p_{3,2+j}, \, p_{3,2+k}, \, \pi_{1,i}, \, \pi_{1,j}, \, \pi_{1,k}, \, \pi_{2,i}, \, \pi_{2,j}, \, \pi_{2,k} \}$$

$Out[\bullet]=$

$$\{ a_1, \, a_2, \, a_3, \, a_4, \, a_5, \, a_6, \, a_7, \, a_8 \}$$

$Out[\bullet]//Short=$

$$\left\{ -\frac{a_3 \, T_1^2 \, T_2^2}{B} + \frac{a_3 \, T_1^2 \, T_2^3}{B} == 0, \; \frac{a_3 \, T_1^2 \, T_2}{B} - \frac{a_3 \, T_1^2 \, T_2^2}{B} == 0, \right.$$

$$-\frac{a_5 \, T_1^2 \, T_2^2}{B} + \frac{a_5 \, T_1^3 \, T_2^2}{B} == 0, \; -\frac{a_7 \, T_1^2 \, T_2^2}{B} + \frac{a_7 \, T_1^3 \, T_2^2}{B} + \frac{a_7 \, T_1^2 \, T_2^3}{B} - \frac{a_7 \, T_1^3 \, T_2^3}{B} == 0,$$

$$\ll 18 \gg, \; -\frac{a_7}{B} - \frac{a_8}{B} + \frac{a_7 \, T_1}{B} + \frac{a_8 \, T_1}{B} + \frac{a_7 \, T_1 \, T_2}{B} - \frac{a_7 \, T_1^2 \, T_2}{B} == 0,$$

$$\frac{a_7 \, T_2}{B} + \frac{a_8 \, T_2}{B} - \frac{a_7 \, T_2^2}{B} - \frac{a_8 \, T_2^2}{B} - \frac{a_7 \, T_1 \, T_2^2}{B} + \frac{a_7 \, T_1 \, T_2^3}{B} == 0,$$

$$\left. -\frac{a_7}{B} - \frac{a_8}{B} + \frac{a_7 \, T_2}{B} + \frac{a_8 \, T_2}{B} + \frac{a_7 \, T_1 \, T_2}{B} - \frac{a_7 \, T_1 \, T_2^2}{B} == 0 \right\}$$

⋯ Solve: Equations may not give solutions for all "solve" variables. ⓘ

$Out[\bullet]=$

$$\left\{ \left\{ a_1 \to 0, \, a_3 \to 0, \, a_5 \to 0, \, a_6 \to -\frac{a_2}{T_1} - \frac{a_4 \, T_2}{T_1}, \, a_7 \to 0, \, a_8 \to 0 \right\} \right\}$$

$In[\bullet]:=$ **sol /. (v_ → val_) :> (v = CF[val]);**
r₀[1, i, j]

$Out[\bullet]=$

$$a_2 \, p_{3,j} \, x_{1,i} \, x_{2,i} - \frac{(a_2 + a_4 \, T_2) \, p_{3,j} \, x_{1,j} \, x_{2,i}}{T_1} + a_4 \, p_{3,j} \, x_{1,i} \, x_{2,j}$$

```
In[ ]:= Short[eqn = CF[Coefficient[
          LeftR3b[[2]] - RightR3b[[2]] /. v : (π | p)__ ⧴ μ v,
          μ³
        ]], 5]
     cvs = Union@Cases[eqn, p__ | π__, ∞]
     vars = Union@Cases[r₁[1, i, j], b_, ∞]
     Short[eqns = CoefficientRules[eqn, cvs] /. (_ → c_) ⧴ (c == 0), 3]
     {sol} = Solve[eqns, vars]
```

$Out[ ]//Short=$

$B\ b_1\ (-1 + T_1)\ T_1\ T_2^2\ p_{1,2+j}\ p_{2,2+i}\ \pi_{3,i} -$
$B\ b_1\ (-1 + T_1)\ T_1\ T_2^2\ p_{1,2+k}\ p_{2,2+i}\ \pi_{3,i} + B\ b_1\ T_1^2\ (-1 + T_2)\ T_2\ p_{1,2+i}\ p_{2,2+j}\ \pi_{3,i} -$
$B\ T_1\ T_2\ \left(-b_1\ T_1 - b_1\ T_2 + 2\ b_1\ T_1\ T_2 - b_2\ T_1\ T_2 + b_2\ T_1^2\ T_2^2\right)\ p_{1,2+j}\ p_{2,2+j}\ \pi_{3,i} + \ll 23 \gg +$
$B\ b_2\ T_1^2\ (-1 + T_2)\ T_2\ p_{1,2+i}\ p_{2,2+j}\ \pi_{3,k} - B\ b_2\ (-1 + T_1)\ T_1\ (-1 + T_2)\ T_2\ p_{1,2+j}\ p_{2,2+j}\ \pi_{3,k} -$
$B\ (-b_2 - b_6 + b_2\ T_1)\ (-1 + T_2)\ T_2\ p_{1,2+k}\ p_{2,2+j}\ \pi_{3,k} +$
$B\ (-1 + T_1)\ T_1\ (-b_2 - b_4 + b_2\ T_2)\ p_{1,2+i}\ p_{2,2+k}\ \pi_{3,k} - B\ (-1 + T_1)\ T_1\ (-b_2 - b_4 + b_2\ T_2)\ p_{1,2+j}\ p_{2,2+k}\ \pi_{3,k}$

$Out[ ]=$

$\{p_{1,2+i},\ p_{1,2+j},\ p_{1,2+k},\ p_{2,2+i},\ p_{2,2+j},\ p_{2,2+k},\ \pi_{3,i},\ \pi_{3,j},\ \pi_{3,k}\}$

$Out[ ]=$

$\{b_1,\ b_2,\ b_3,\ b_4,\ b_5,\ b_6,\ b_7,\ b_8\}$

$Out[ ]//Short=$

$\left\{-B\ b_2\ T_1^2\ T_2^2 + B\ b_2\ T_1^3\ T_2^3 == 0,\ B\ b_2\ T_1\ T_2 - B\ b_2\ T_1^2\ T_2^2 == 0,\ \ll 22 \gg,\right.$
$-B\ b_6\ T_1 - B\ b_8\ T_1 - B\ b_4\ T_2 - B\ b_8\ T_2 + B\ b_2\ T_1\ T_2 + 2\ B\ b_4\ T_1\ T_2 + 2\ B\ b_6\ T_1\ T_2 +$
$\left. 2\ B\ b_8\ T_1\ T_2 - B\ b_2\ T_1^2\ T_2 - B\ b_4\ T_1^2\ T_2 - B\ b_2\ T_1\ T_2^2 - B\ b_6\ T_1\ T_2^2 + B\ b_2\ T_1^2\ T_2^2 == 0\right\}$

     ⚬⚬⚬ Solve: Equations may not give solutions for all "solve" variables. ⓘ

$Out[ ]=$

$\{\{b_1 \to 0,\ b_2 \to 0,\ b_4 \to 0,\ b_6 \to 0,\ b_7 \to -b_3 - b_5,\ b_8 \to 0\}\}$

```
In[ ]:= sol /. (v_ → val_) ⧴ (v = CF[val]);
     r₁[1, i, j]
```

$Out[ ]=$

$b_5\ p_{1,j}\ p_{2,i}\ x_{3,i} + b_3\ p_{1,i}\ p_{2,j}\ x_{3,i} + (-b_3 - b_5)\ p_{1,j}\ p_{2,j}\ x_{3,i}$

```
In[ ]:= Short[eqn = CF[LeftR3b〚2〛 - RightR3b〚2〛], 5]
        cvs = Union@Cases[eqn, p__ | π__, ∞]
        vars = Union@Cases[r₄₂[1, i, j], c_, ∞]
        Short[eqns = CoefficientRules[eqn, cvs] /. (_ → c_) ⧴ (c == 0), 3]
        Short[{sol} = Solve[eqns, vars]]
```

Out[ ]//Short=

$$- \Big((c_{25} + c_{26} + c_{34} + c_{35} + c_{55}) \ (-1 + T_1) \ T_1^2 \ p_{1,2+j} \ \pi_{1,i}\Big) -$$
$$(-1 + T_1) \ \Big(c_1 + c_2 + c_{10} + c_{11} + c_{13} + c_{14} + c_{22} + c_{23} + c_{49} + c_{52} + c_{25} T_1 + c_{26} T_1 + c_{34} T_1 + c_{35} T_1 + c_{37} T_1 +$$
$$c_{38} T_1 + c_{46} T_1 + c_{47} T_1 + c_{55} T_1 + c_{58} T_1 - c_{25} T_1^2 - c_{26} T_1^2 - c_{34} T_1^2 - c_{35} T_1^2 - c_{55} T_1^2\Big) \ p_{1,2+k} \ \pi_{1,i} +$$
$$\ll 374 \gg + (-1 + T_2) \ T_2 \ (-c_{33} - c_{36} + c_{33} T_1 T_2) \ p_{2,2+i} \ p_{3,2+k} \ \pi_{2,k} \ \pi_{3,k} -$$
$$(-1 + T_2) \ T_2 \ (-c_{33} - c_{36} + c_{33} T_1 T_2) \ p_{2,2+j} \ p_{3,2+k} \ \pi_{2,k} \ \pi_{3,k}$$

Out[ ]=

$$\{p_{1,2+i}, p_{1,2+j}, p_{1,2+k}, p_{2,2+i}, p_{2,2+j}, p_{2,2+k}, p_{3,2+i},$$
$$p_{3,2+j}, p_{3,2+k}, \pi_{1,i}, \pi_{1,j}, \pi_{1,k}, \pi_{2,i}, \pi_{2,j}, \pi_{2,k}, \pi_{3,i}, \pi_{3,j}, \pi_{3,k}\}$$

Out[ ]=

$$\{c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}, c_{11}, c_{12}, c_{13}, c_{14}, c_{15}, c_{16}, c_{17}, c_{18}, c_{19}, c_{20}, c_{21}, c_{22},$$
$$c_{23}, c_{24}, c_{25}, c_{26}, c_{27}, c_{28}, c_{29}, c_{30}, c_{31}, c_{32}, c_{33}, c_{34}, c_{35}, c_{36}, c_{37}, c_{38}, c_{39}, c_{40}, c_{41},$$
$$c_{42}, c_{43}, c_{44}, c_{45}, c_{46}, c_{47}, c_{48}, c_{49}, c_{50}, c_{51}, c_{52}, c_{53}, c_{54}, c_{55}, c_{56}, c_{57}, c_{58}, c_{59}, c_{60}\}$$

Out[ ]//Short=

$$\Big\{- c_7 \ T_1^2 \ T_2^2 + c_7 \ T_1^2 \ T_2^3 == 0, \ c_7 \ T_1^2 \ T_2 - c_7 \ T_1^2 \ T_2^2 == 0,$$
$$\ll 249 \gg, \ c_8 \ T_1 \ T_2 + c_9 \ T_1 \ T_2 + c_{44} \ T_1 \ T_2 + c_{45} \ T_1 \ T_2 + c_{57} \ T_1 \ T_2 -$$
$$c_8 \ T_1^2 \ T_2^2 - c_9 \ T_1^2 \ T_2^2 - c_{44} \ T_1^2 \ T_2^2 - c_{45} \ T_1^2 \ T_2^2 - c_{57} \ T_1^2 \ T_2^2 == 0\Big\}$$

> ⋯ Solve: Equations may not give solutions for all "solve" variables. ⓘ

Out[ ]//Short=

$$\Big\{\Big\{c_1 \to 0, \ c_2 \to 0, \ c_3 \to 0, \ \ll 42 \gg, \ c_{58} \to - \ll 1 \gg - \ll 1 \gg, \ c_{59} \to - \frac{c_{50}}{T_2} - \frac{c_{53}}{T_2},$$
$$c_{60} \to - \frac{c_{51}}{T_1 \ T_2} - \frac{c_{54}}{T_1 \ T_2} - \frac{- a_2 \ b_5 - a_2 \ b_3 \ T_1 + a_4 \ b_3 \ T_1 - a_4 \ \ll 1 \gg \ \ll 1 \gg \ \ll 1 \gg - a_4 \ b_3 \ T_1 \ T_2 + a_4 \ b_5 \ T_1 \ T_2}{T_1^2 \ T_2 \ (-1 + T_1 \ T_2)}\Big\}\Big\}$$

```
In[ ]:= sol /. (v_ → val_) ⧴ (v = CF[val]);
```

```
In[ ]:= Short[CF[r₄₂[1, i, j]], 20]
```

Out[ ]//Short=

$$c_{49} \ p_{1,i} \ x_{1,i} + c_{52} \ p_{1,j} \ x_{1,j} - \frac{(c_{49} + c_{52}) \ p_{1,j} \ x_{1,j}}{T_1} + c_{50} \ p_{2,i} \ x_{2,i} +$$
$$c_{53} \ p_{2,j} \ x_{2,i} + c_{13} \ p_{1,j} \ p_{2,i} \ x_{1,i} \ x_{2,i} + c_4 \ p_{1,i} \ p_{2,j} \ x_{1,i} \ x_{2,i} + \frac{1}{-1 + T_1 \ T_2}$$
$$\Big(a_2 \ b_3 + c_4 + c_{13} - c_4 \ T_1 + a_4 \ b_3 \ T_2 + a_2 \ b_5 \ T_2 - a_4 \ b_5 \ T_2 - c_{13} \ T_2 - a_4 \ b_3 \ T_1 \ T_2 - c_4 \ T_1 \ T_2 - c_{13} \ T_1 \ T_2 +$$
$$c_4 \ T_1^2 \ T_2 + a_4 \ b_5 \ T_2^2 + c_{13} \ T_1 \ T_2^2\Big) \ p_{1,j} \ p_{2,j} \ x_{1,i} \ x_{2,i} + \frac{1}{(-1 + T_1) \ T_1 \ (-1 + T_2) \ (-1 + T_1 \ T_2)}$$
$$\Big(- a_2 \ b_3 + a_2 \ b_5 + a_2 \ b_3 \ T_1 + c_4 \ T_1 - c_{13} \ T_1 - c_4 \ T_1^2 - a_4 \ b_3 \ T_2 - a_2 \ b_5 \ T_2 + a_4 \ b_5 \ T_2 + 2 \ a_4 \ b_3 \ T_1 \ T_2 -$$
$$a_4 \ b_5 \ T_1 \ T_2 + c_{13} \ T_1 \ T_2 - a_4 \ b_3 \ T_1^2 \ T_2 - c_4 \ T_1^2 \ T_2 + c_{13} \ T_1^2 \ T_2 + c_4 \ T_1^3 \ T_2 - a_4 \ b_5 \ T_2^2 + a_4 \ b_5 \ T_1 \ T_2^2 - c_{13} \ T_1^2 \ T_2^2\Big)$$
$$p_{1,j} \ p_{2,i} \ x_{1,j} \ x_{2,i} - \frac{(a_2 \ b_5 - c_{13} \ T_1 + a_4 \ b_5 \ T_2 + c_{13} \ T_1 \ T_2) \ p_{1,j} \ p_{2,j} \ x_{1,j} \ x_{2,i}}{(-1 + T_1) \ T_1} -$$

$$\frac{(c_{50} + c_{53})\, p_{2,j}\, x_{2,j}}{T_2} - \frac{1}{(-1 + T_1)\,(-1 + T_2)\,(-1 + T_1 T_2)}$$

$$\big(-a_2 b_3 + a_4 b_3 + a_2 b_5 - a_4 b_5 + c_4 - c_{13} + a_2 b_3 T_1 - a_4 b_3 T_1 - c_4 T_1 - a_4 b_3 T_2 - a_2 b_5 T_2 + 2 a_4 b_5 T_2 +$$
$$c_{13} T_2 + a_4 b_3 T_1 T_2 - c_4 T_1 T_2 + c_{13} T_1 T_2 + c_4 T_1^2 T_2 - a_4 b_5 T_2^2 - c_{13} T_1 T_2^2\big)\, p_{1,i}\, p_{2,j}\, x_{1,i}\, x_{2,j} +$$

$$\frac{(a_4 b_3 + c_4 - c_4 T_1)\, p_{1,j}\, p_{2,j}\, x_{1,i}\, x_{2,j}}{-1 + T_2} + c_{51}\, p_{3,i}\, x_{3,i} + \ll 5 \gg + \frac{b_3\,(a_2 + a_4 T_2)\, p_{1,i}\, p_{3,j}\, x_{1,j}\, x_{3,i}}{T_1\,(-1 + T_2)} -$$

$$\frac{1}{(-1 + T_1)\, T_1\,(-1 + T_2)}$$

$$\big(-a_2 b_3 + a_2 b_3 T_1 + a_2 b_5 T_1 + c_{14} T_1 - a_4 b_3 T_2 + a_4 b_3 T_1 T_2 - a_2 b_5 T_1 T_2 +$$
$$a_4 b_5 T_1 T_2 - c_{14} T_1 T_2 - c_{14} T_1^2 T_2 - a_4 b_5 T_1 T_2^2 + c_{14} T_1^2 T_2^2\big)\, p_{1,j}\, p_{3,j}\, x_{1,j}\, x_{3,i} +$$

$$c_{15}\, p_{2,j}\, p_{3,i}\, x_{2,i}\, x_{3,i} + c_6\, p_{2,i}\, p_{3,j}\, x_{2,i}\, x_{3,i} - \frac{1}{(-1 + T_1)\, T_1}$$

$$\big(-a_2 b_3 + a_2 b_3 T_1 - c_6 T_1 - c_{15} T_1 + c_6 T_1^2 + c_{15} T_1^2 - a_4 b_3 T_2 - a_2 b_5 T_2 + 2 a_4 b_3 T_1 T_2 + a_2 b_5 T_1 T_2 -$$
$$a_4 b_5 T_1 T_2 + c_6 T_1 T_2 - a_4 b_3 T_1^2 T_2 - c_6 T_1^2 T_2 + c_{15} T_1^2 T_2 - c_{15} T_1^3 T_2 - a_4 b_5 T_2^2 + 2 a_4 b_5 T_1 T_2^2\big)$$

$$p_{2,j}\, p_{3,j}\, x_{2,i}\, x_{3,i} + \frac{1}{(-1 + T_1)\, T_1\,(-1 + T_2)\,(-1 + T_1 T_2)}$$

$$\big(-a_2 b_5 - a_2 b_3 T_1 + a_4 b_3 T_1 + a_2 b_5 T_1 - a_4 b_5 T_1 + c_6 T_1 - c_{15} T_1 + a_2 b_3 T_1^2 - a_4 b_3 T_1^2 - c_6 T_1^2 + c_{15} T_1^2 +$$
$$a_2 b_5 T_2 - a_4 b_5 T_2 - a_4 b_3 T_1 T_2 - a_2 b_5 T_1 T_2 + 3 a_4 b_5 T_1 T_2 - c_6 T_1 T_2 + a_4 b_3 T_1^2 T_2 + c_6 T_1^2 T_2 +$$
$$c_{15} T_1^2 T_2 - c_{15} T_1^3 T_2 + a_4 b_5 T_2^2 - 2 a_4 b_5 T_1 T_2^2\big)\, p_{2,j}\, p_{3,i}\, x_{2,j}\, x_{3,i} - \frac{a_4 b_5\, p_{2,i}\, p_{3,j}\, x_{2,j}\, x_{3,i}}{-1 + T_1} +$$

$$\frac{\big(-a_4 b_5 - c_{15} + c_{15} T_1 + a_4 b_3 T_2 + a_4 b_5 T_2 - a_4 b_3 T_1 T_2 + c_{15} T_1 T_2 - c_{15} T_1^2 T_2\big)\, p_{2,j}\, p_{3,j}\, x_{2,j}\, x_{3,i}}{(-1 + T_1)\,(-1 + T_2)} -$$

$$\frac{1}{T_1^2 T_2\,(-1 + T_1 T_2)}$$

$$\big(-a_2 b_5 - a_2 b_3 T_1 + a_4 b_3 T_1 - c_{51} T_1 - c_{54} T_1 - a_4 b_5 T_2 - a_4 b_3 T_1 T_2 + a_4 b_5 T_1 T_2 + c_{51} T_1^2 T_2 + c_{54} T_1^2 T_2\big)$$

$$p_{3,j}\, x_{3,j} - \frac{(a_2 b_5 - a_4 b_5 - c_5 + c_{14} + c_5 T_1 + a_4 b_5 T_2 - c_{14} T_1 T_2)\, p_{1,i}\, p_{3,j}\, x_{1,i}\, x_{3,j}}{(-1 + T_1)\,(-1 + T_1 T_2)} -$$

$$\frac{(-a_2 b_3 + c_5 + a_2 b_3 T_1 - a_4 b_3 T_1 - c_5 T_1 - a_4 b_3 T_2 - c_5 T_2 + 2 a_4 b_3 T_1 T_2 + c_5 T_1 T_2)\, p_{1,j}\, p_{3,j}\, x_{1,i}\, x_{3,j}}{(-1 + T_2)\,(-1 + T_1 T_2)} -$$

$$\frac{\big(a_2 b_3 - c_6 T_1 + c_{15} T_1 + a_4 b_3 T_2 - a_4 b_3 T_1 T_2 + c_6 T_1 T_2 - c_{15} T_1^2 T_2\big)\, p_{2,i}\, p_{3,j}\, x_{2,i}\, x_{3,j}}{T_1\,(-1 + T_2)\,(-1 + T_1 T_2)} -$$

$$\frac{1}{(-1 + T_1)\, T_1\,(-1 + T_1 T_2)}$$

$$\big(c_6 T_1 - c_6 T_1^2 + a_2 b_5 T_2 - a_2 b_5 T_1 T_2 + a_4 b_5 T_1 T_2 - c_6 T_1 T_2 + c_6 T_1^2 T_2 + a_4 b_5 T_2^2 - 2 a_4 b_5 T_1 T_2^2\big)$$

$$p_{2,j}\, p_{3,j}\, x_{2,i}\, x_{3,j}$$

Reidemeister 2b

Territory: $(x \mid p)_{(i \mid j)^+}, x_{i \mid j}.$
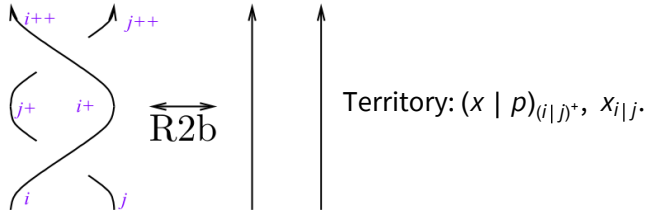
$In[\circ]:=$ **Timing$\Big[$ Short$\Big[$ LeftR2b** $= \left(\int \mathcal{F}[\mathtt{i}, \mathtt{j}] \mathcal{L} \text{ /@ } (X_{\mathtt{i},\mathtt{j}}[\mathtt{1}] X_{\mathtt{i}^+,\mathtt{j}^+}[-\mathtt{1}]) \, \mathbb{d}\{vs_{\mathtt{i}}, vs_{\mathtt{j}}, vs_{\mathtt{i}^+}, vs_{\mathtt{j}^+}\}\right)$ $[\![\mathtt{1}]\!]$

$\Big]\Big]$

$Out[\circ]=$

$\Big\{0.234375, \epsilon\text{Series}\Big[p_{1,2+i}\,\pi_{1,i} + p_{1,2+j}\,\pi_{1,j} + p_{2,2+i}\,\pi_{2,i} + \ll 10 \gg + p_{3,2+i}\,\pi_{3,i} + p_{3,2+j}\,\pi_{3,j},$

$\dfrac{\ll 220 \gg + f_{60}\,T_1^3\,T_2^2}{(-1+T_1)\,T_1^2\,(-1+T_2)\,T_2} + \dfrac{\ll 1 \gg}{\ll 1 \gg} + \ll 74 \gg + \dfrac{\ll 1 \gg}{\ll 1 \gg} + \dfrac{(\ll 1 \gg)\,\ll 3 \gg\,\pi_{\ll 1 \gg}}{T_1\,T_2^2}\Big]\Big\}$

$In[\circ]:=$ **RightR2b** $= \epsilon\text{Series}[p_{1,2+i}\,\pi_{1,i} + p_{1,2+j}\,\pi_{1,j} + p_{2,2+i}\,\pi_{2,i} + p_{2,2+j}\,\pi_{2,j} + p_{3,2+i}\,\pi_{3,i} + p_{3,2+j}\,\pi_{3,j}, \mathtt{0}]$

$Out[\circ]=$

$\epsilon\text{Series}[p_{1,2+i}\,\pi_{1,i} + p_{1,2+j}\,\pi_{1,j} + p_{2,2+i}\,\pi_{2,i} + p_{2,2+j}\,\pi_{2,j} + p_{3,2+i}\,\pi_{3,i} + p_{3,2+j}\,\pi_{3,j}, \mathtt{0}]$

$In[\bullet]:=$ `Short[eqn = CF[LeftR2b⟦1⟧ - RightR2b⟦1⟧]]`
`cvs = Union@Cases[eqn, p__ | π__, ∞]`
`vars = Union@Cases[r₀[-1, i, j], d_, ∞]`
`Short[eqns = CoefficientRules[eqn, cvs] /. (_ → c_) :> (c == 0), 3]`
`{sol} = Solve[eqns, vars]`

$Out[\bullet]//Short=$

$$\frac{(d_7 + d_3 T_1 - d_7 T_1 + d_5 T_2 - d_7 T_2 + d_1 T_1 T_2 - d_3 T_1 T_2 - d_5 T_1 T_2 + d_7 T_1 T_2) \; \ll 2 \gg \pi_{\ll 1 \gg}}{B \, T_1 \, T_2} +$$

$$\frac{(\ll 1 \gg) \; \ll 2 \gg \; \ll 1 \gg}{B \, T_1 \, T_2} + \ll 5 \gg + \frac{\ll 1 \gg}{\ll 1 \gg} + \frac{(-d_7 + d_7 T_1 T_2 + d_8 T_1 T_2) \; \ll 2 \gg \; \pi_{\ll 1 \gg}}{B \, T_1 \, T_2}$$

$Out[\bullet]=$

$$\{p_{3,2+i}, \; p_{3,2+j}, \; \pi_{1,i}, \; \pi_{1,j}, \; \pi_{2,i}, \; \pi_{2,j}\}$$

$Out[\bullet]=$

$$\{d_1, \; d_2, \; d_3, \; d_4, \; d_5, \; d_6, \; d_7, \; d_8\}$$

$Out[\bullet]//Short=$

$$\left\{ \frac{d_1}{B} - \frac{d_3}{B} - \frac{d_5}{B} + \frac{d_7}{B} + \frac{d_5}{B T_1} - \frac{d_7}{B T_1} + \frac{d_3}{B T_2} - \frac{d_7}{B T_2} + \frac{d_7}{B T_1 T_2} == 0, \; \frac{d_3}{B T_2} - \frac{d_7}{B T_2} + \frac{d_7}{B T_1 T_2} == 0, \right.$$

$$\frac{d_5}{B T_1} - \frac{d_7}{B T_1} + \frac{d_7}{B T_1 T_2} == 0, \; \frac{d_7}{B T_1 T_2} == 0, \; \frac{a_2}{B} - \frac{d_1}{B} + \frac{d_3}{B} + \ll 32 \gg + \frac{d_7 T_1 T_2}{B} + \frac{d_8 T_1 T_2}{B} == 0,$$

$$\frac{a_4}{B} + \frac{d_7}{B} + \frac{d_8}{B} + \frac{d_3 T_1}{B} + \frac{d_4 T_1}{B} - \frac{d_7 T_1}{B} - \frac{d_8 T_1}{B} - \frac{d_3}{B T_2} + \frac{d_7}{B T_2} - \frac{d_7}{B T_1 T_2} == 0,$$

$$\frac{d_7}{B} + \frac{d_8}{B} - \frac{a_2}{B T_1} - \frac{d_5}{B T_1} + \frac{d_7}{B T_1} - \frac{d_7}{B T_1 T_2} + \frac{d_5 T_2}{B} + \frac{d_6 T_2}{B} - \frac{d_7 T_2}{B} - \frac{d_8 T_2}{B} - \frac{a_4 T_2}{B T_1} == 0,$$

$$\left. \frac{d_7}{B} + \frac{d_8}{B} - \frac{d_7}{B T_1 T_2} == 0 \right\}$$

$Out[\bullet]=$

$$\left\{ \left\{ d_1 \to 0, \; d_2 \to -\frac{a_2 - a_4 T_1 + a_4 T_2}{T_1^2 T_2}, \; d_3 \to 0, \; d_4 \to -\frac{a_4}{T_1}, \; d_5 \to 0, \; d_6 \to -\frac{-a_2 - a_4 T_2}{T_1 T_2}, \; d_7 \to 0, \; d_8 \to 0 \right\} \right\}$$

$In[\bullet]:=$ `sol /. (v_ → val_) :> (v = CF[val]);`
`r₀[-1, i, j]`

$Out[\bullet]=$

$$\frac{(-a_2 + a_4 T_1 - a_4 T_2) \, p_{3,j} \, x_{1,i} \, x_{2,i}}{T_1^2 T_2} + \frac{(a_2 + a_4 T_2) \, p_{3,j} \, x_{1,j} \, x_{2,i}}{T_1 T_2} - \frac{a_4 \, p_{3,j} \, x_{1,i} \, x_{2,j}}{T_1}$$

In[•]:= **Short[eqn = CF[LeftR2b⟦2⟧ - RightR2b⟦2⟧]]**
cvs = Union@Cases[eqn, p__ | π__, ∞]
vars = Union@Cases[r₁[-1, i, j] + r₄₂[-1, i, j], e_ | f_, ∞]
Short[eqns = CoefficientRules[eqn, cvs] /. (_ → c_) :> (c == 0), 3]
Short[{sol} = Solve[eqns, vars]]

Out[•]//Short=

$$\ll 85 \gg + \frac{\ll 1 \gg}{\ll 1 \gg} + \frac{\left(f_{33} + \ll 16 \gg + f_{48}\, T_1\, T_2^2\right) \ll 3 \gg \pi_{\ll 1 \gg}}{T_1\, T_2^2}$$

Out[•]=

$$\{p_{1,2+i},\ p_{1,2+j},\ p_{2,2+i},\ p_{2,2+j},\ p_{3,2+i},\ p_{3,2+j},\ \pi_{1,i},\ \pi_{1,j},\ \pi_{2,i},\ \pi_{2,j},\ \pi_{3,i},\ \pi_{3,j}\}$$

Out[•]=

$$\{e_1,\ e_2,\ e_3,\ e_4,\ e_5,\ e_6,\ e_7,\ e_8,\ f_1,\ f_2,\ f_3,\ f_4,\ f_5,\ f_6,\ f_7,\ f_8,\ f_9,\ f_{10},$$
$$f_{11},\ f_{12},\ f_{13},\ f_{14},\ f_{15},\ f_{16},\ f_{17},\ f_{18},\ f_{19},\ f_{20},\ f_{21},\ f_{22},\ f_{23},\ f_{24},\ f_{25},\ f_{26},$$
$$f_{27},\ f_{28},\ f_{29},\ f_{30},\ f_{31},\ f_{32},\ f_{33},\ f_{34},\ f_{35},\ f_{36},\ f_{37},\ f_{38},\ f_{39},\ f_{40},\ f_{41},\ f_{42},\ f_{43},$$
$$f_{44},\ f_{45},\ f_{46},\ f_{47},\ f_{48},\ f_{49},\ f_{50},\ f_{51},\ f_{52},\ f_{53},\ f_{54},\ f_{55},\ f_{56},\ f_{57},\ f_{58},\ f_{59},\ f_{60}\}$$

Out[•]//Short=

$$\left\{ f_1 - f_7 - f_{25} + f_{31} + \frac{f_{25}}{T_1} - \frac{f_{31}}{T_1} + \frac{f_7}{T_2} - \frac{f_{31}}{T_2} + \frac{f_{31}}{T_1\, T_2} == 0, \right.$$

$$\frac{f_7}{T_2} - \frac{f_{31}}{T_2} + \frac{f_{31}}{T_1\, T_2} == 0,\ \ll 1 \gg\ == 0,\ \ll 63 \gg,\ \ll 1 \gg == 0,\ \ll 1 \gg == 0,$$

$$\frac{2\, a_4\, b_3}{(1 - T_1)\,(1 - T_2)} + \frac{2\, c_{49}}{(1 - T_1)\,(1 - T_2)} + \frac{2\, c_{50}}{(1 - T_1)\,(1 - T_2)} + \frac{c_{52}}{(1 - T_1)\,(1 - T_2)} + \frac{c_{53}}{(1 - T_1)\,(1 - T_2)} -$$

$$\frac{c_{54}}{(1 - T_1)\,(1 - T_2)} + \ll 194 \gg + \frac{f_{49}\, T_1\, T_2}{(1 - T_1)\,(1 - T_2)} + \frac{f_{50}\, T_1\, T_2}{(1 - T_1)\,(1 - T_2)} +$$

$$\left. \frac{f_{51}\, T_1\, T_2}{(1 - T_1)\,(1 - T_2)} + \frac{f_{58}\, T_1\, T_2}{(1 - T_1)\,(1 - T_2)} + \frac{f_{59}\, T_1\, T_2}{(1 - T_1)\,(1 - T_2)} + \frac{f_{60}\, T_1\, T_2}{(1 - T_1)\,(1 - T_2)} == 0 \right\}$$

Out[•]//Short=

$$\left\{ \left\{ e_1 \to 0,\ e_2 \to 0,\ e_3 \to -\frac{b_3}{T_2},\ e_4 \to 0,\ e_5 \to -\frac{b_5}{T_1},\ e_6 \to 0,\ \ll 56 \gg,\ f_{55} \to 0,\ f_{56} \to 0,\ f_{57} \to 0, \right.\right.$$

$$\left.\left. f_{58} \to -\frac{-c_{49} - c_{52}}{T_1},\ f_{59} \to -\frac{-c_{50} - c_{53}}{T_2},\ f_{60} \to -\frac{a_2\, b_5 + a_2\, b_3\, T_1 - a_4\, b_3\, T_1 + \ll 10 \gg}{T_1^2\, T_2\,(-1 + T_1\, T_2)} \right\}\right\}$$

In[•]:= **sol /. (v_ → val_) :> (v = CF[val]);**

In[ ]:= **r₁[-1, i, j]**
Short[CF[r₄₂[-1, i, j]], 5]

Out[ ]=

$$-\frac{b_5\,p_{1,j}\,p_{2,i}\,x_{3,i}}{T_1} - \frac{b_3\,p_{1,i}\,p_{2,j}\,x_{3,i}}{T_2} + \frac{(b_3\,T_1 + b_5\,T_2)\,p_{1,j}\,p_{2,j}\,x_{3,i}}{T_1\,T_2}$$

Out[ ]//Short=

$$-c_{49}\,p_{1,i}\,x_{1,i} + \frac{\left(-c_{49} - c_{52} + c_{49}\,T_1^2\right)\,p_{1,j}\,x_{1,i}}{T_1^2} + \ll 43 \gg +$$

$$\frac{\left(a_2\,b_3 - c_6\,T_1 + c_{15}\,T_1 + a_4\,b_3\,T_2 - a_4\,b_3\,T_1\,T_2 + c_6\,T_1\,T_2 - c_{15}\,T_1^2\,T_2\right)\,p_{2,i}\,p_{3,j}\,x_{2,i}\,x_{3,j}}{T_1\,(-1 + T_2)\,(-1 + T_1\,T_2)} +$$

$$\left(\left(a_2\,b_3 - a_2\,b_3\,T_1 + c_{15}\,T_1 - c_{15}\,T_1^2 + a_4\,b_3\,T_2 + a_2\,b_5\,T_2 - 2\,a_4\,b_3\,T_1\,T_2 - a_2\,b_5\,T_1\,T_2 + a_4\,b_5\,T_1\,T_2 + a_4\,b_3\,T_1^2\,T_2 - \right.\right.$$
$$\left.\left. c_{15}\,T_1^2\,T_2 + c_{15}\,T_1^3\,T_2 + a_4\,b_5\,T_2^2 - 2\,a_4\,b_5\,T_1\,T_2^2\right)\,p_{2,j}\,p_{3,j}\,x_{2,i}\,x_{3,j}\right) \Big/ \left((-1 + T_1)\,T_1\,T_2\,(-1 + T_1\,T_2)\right)$$

Reidemeister 2c
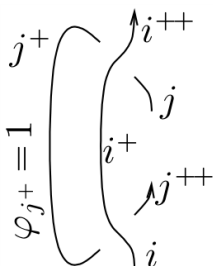
In[ ]:= **Timing[Short[{LeftR2c} = Cases[**
∫𝓕[i, j] 𝓛 /@ (X_{i+1,j}[1] X_{i,j+2}[-1] C_{j+1}[1]) d⫲{vs_i, vs_j, vs_{i⁺}, vs_{j⁺}, vs_{j+2}}, 𝔼[ℰ_] ⤇ ℰ]
]]

Out[ ]=
{0.171875, {⟨⟨1⟩⟩}}

In[ ]:= **Timing[Short[{RightR2c} =**
Cases[∫𝓕[i, j] 𝓛 /@ (C_i[0] C_{i+1}[0] C_j[0] C_{j+1}[1] C_{j+2}[0]) d⫲{vs_i, vs_j, vs_{i⁺}, vs_{j⁺}, vs_{j+2}},
𝔼[ℰ_] ⤇ ℰ]
]]

Out[ ]=
$$\Big\{0.015625,$$
$$\Big\{\epsilon\text{Series}\Big[p_{1,2+i}\,\pi_{1,i} + p_{1,3+j}\,\pi_{1,j} + p_{2,2+i}\,\pi_{2,i} + p_{2,3+j}\,\pi_{2,j} + \frac{g_1\,p_{3,3+j}\,\pi_{1,j}\,\pi_{2,j}}{B} + p_{3,2+i}\,\pi_{3,i} + p_{3,3+j}\,\pi_{3,j},$$
$$g_1\,g_2 + \ll 12 \gg + (g_1\,g_2 + g_8)\,p_{2,3+j}\,p_3 \ll 1 \gg \ll 1 \gg \pi_{2,j}\,\pi_{3,j}\Big]\Big\}\Big\}$$

```
In[ ]:= Short[eqn = CF[LeftR2c〚1〛 - RightR2c〚1〛]]
        cvs = Union@Cases[eqn, p__ | π__, ∞]
        vars = Union@Cases[γ₀[1, k], g_, ∞]
        Short[eqns = CoefficientRules[eqn, cvs] /. (_ → c_) ⧴ (c == 0), 3]
        {sol} = Solve[eqns, vars]
```

Out[ ]//Short=

$$\frac{g_1 \ (-1 + T_1) \ (-1 + T_2) \ p_{3,3+j} \ \pi_{1,i} \ \pi_{2,i}}{B \ T_1 \ T_2} - \frac{g_1 \ (-1 + T_2) \ p_{3, \ll 1 \gg} \ \pi_{1,j} \ \pi_{2,i}}{B \ T_2} - \frac{g_1 \ (-1 + T_1) \ p_{3,3+j} \ \pi_{1,i} \ \pi_{2,j}}{B \ T_1}$$

Out[ ]=

$$\{p_{3,3+j}, \ \pi_{1,i}, \ \pi_{1,j}, \ \pi_{2,i}, \ \pi_{2,j}\}$$

Out[ ]=

$$\{g_1\}$$

Out[ ]//Short=

$$\left\{\frac{g_1}{B} - \frac{g_1}{B \ T_1} - \frac{g_1}{B \ T_2} + \frac{g_1}{B \ T_1 \ T_2} == 0, \ -\frac{g_1}{B} + \frac{g_1}{B \ T_1} == 0, \ -\frac{g_1}{B} + \frac{g_1}{B \ T_2} == 0\right\}$$

Out[ ]=

$$\{\{g_1 \to 0\}\}$$

```
In[ ]:= sol /. (v_ → val_) ⧴ (v = CF[val]);
        γ₀[1, k]
```

Out[ ]=

        0

```
In[ ]:= Short[eqn = CF[LeftR2c[[2]] - RightR2c[[2]]]]
        cvs = Union@Cases[eqn, p__ | π__, ∞]
        vars = Union@Cases[γ₁[1, k] + γ₄₂[1, k], g_, ∞]
        Short[eqns = CoefficientRules[eqn, cvs] /. (_ → c_) :> (c == 0), 3]
        Short[{sol} = Solve[eqns, vars]]
```

Out[ ]//Short=

$$-\frac{(g_3 + g_6 + g_7)\ (-1 + T_1)\ p_{1,3+j}\ \pi_{1,i}}{T_1} - \frac{(g_4 + g_6 + g_8)\ (\ll 1 \gg)\ \ll 1 \gg\ \pi_{2,i}}{T_2} + \ll 18 \gg$$

Out[ ]=

$$\{p_{1,3+j},\ p_{2,3+j},\ p_{3,3+j},\ \pi_{1,i},\ \pi_{1,j},\ \pi_{2,i},\ \pi_{2,j},\ \pi_{3,i},\ \pi_{3,j}\}$$

Out[ ]=

$$\{g_2,\ g_3,\ g_4,\ g_5,\ g_6,\ g_7,\ g_8\}$$

Out[ ]//Short=

$$\left\{g_6 - \frac{g_6}{T_1} - \frac{g_6}{T_2} + \frac{a_2\ g_2}{T_1\ T_2} + \frac{g_6}{T_1\ T_2} == 0,\ -g_6 + \frac{\ll 1 \gg\ \ll 1 \gg}{T_1} + \frac{g_6}{T_1} == 0,\ \ll 9 \gg,\ \ll 1 \gg == 0,\right.$$

$$\frac{2\ a_4\ b_3}{(1 - T_1)\ (1 - T_2)} + \frac{a_4\ b_3}{(1 - T_1)\ T_1^2\ (1 - T_2)} + \frac{a_2\ b_5}{(1 - T_1)\ T_1^2\ (1 - T_2)} - \frac{a_4\ b_5}{(1 - T_1)\ T_1^2\ (1 - T_2)} -$$

$$\frac{3\ a_4\ b_3}{(1 - T_1)\ T_1\ (1 - T_2)} - \frac{a_2\ b_5}{(1 - T_1)\ T_1\ (1 - T_2)} + \ll 33 \gg + \frac{g_8\ T_2}{(1 - T_1)\ (1 - T_2)} + \frac{a_4\ b_5\ T_2}{(1 - T_1)\ T_1^2\ (1 - T_2)} -$$

$$\left.\frac{2\ a_4\ b_5\ T_2}{(1 - T_1)\ T_1\ (1 - T_2)} - \frac{g_5\ T_1\ T_2}{(1 - T_1)\ (1 - T_2)} - \frac{g_7\ T_1\ T_2}{(1 - T_1)\ (1 - T_2)} - \frac{g_8\ T_1\ T_2}{(1 - T_1)\ (1 - T_2)} == 0\right\}$$

Out[ ]//Short=

$$\left\{\left\{g_2 \to 0,\ g_3 \to 0,\ g_4 \to 0,\ g_5 \to -\frac{-a_2\ b_3 + a_2\ b_5 + \ll 22 \gg + 2\ a_4\ b_5\ T_1\ T_2^2}{(-1 + T_1)\ T_1\ (-1 + T_2)\ (-1 + T_1\ T_2)},\ g_6 \to 0,\ g_7 \to 0,\ g_8 \to 0\right\}\right\}$$

```
In[ ]:= sol /. (v_ → val_) :> (v = CF[val]);
```

```
In[ ]:= γ₁[1, k]
        Short[CF[γ₄₂[1, k]], 5]
```

Out[ ]=

0

Out[ ]//Short=

$$\frac{(-b_3 + b_5 + b_3\ T_1 - b_5\ T_2)\ (-a_2 + a_2\ T_1 - a_4\ T_1 - a_4\ T_2 + 2\ a_4\ T_1\ T_2)\ p_{3,k}\ x_{3,k}}{(-1 + T_1)\ T_1\ (-1 + T_2)\ (-1 + T_1\ T_2)}$$

$C_k[1]$ and $C_k[-1]$ are inverses

```
In[ ]:= Timing[ Short[{LeftCC} = Cases[{∫ℱ[k] ℒ /@ (C_k[1] C_{k+1}[-1]) d{vs_k, vs_{k⁺}}}, 𝔼[ℰ_] :> ℰ]
        ]]
```

Out[ ]=

$$\left\{0.,\ \left\{\in\text{Series}\left[p_{1,2+k}\ \pi_{1,k} + p_{2,2+k}\ \pi_{2,k} + \frac{h_1\ p_{3,2+k}\ \pi_{1,k}\ \pi_{2,k}}{B} + p_{3,2+k}\ \pi_{3,k},\ \ll 1 \gg\right]\right\}\right\}$$

In[◦]:= **Timing**[**Short**[{**RightCC**} = **Cases**[{∫ $\mathcal{F}$[k] $\mathcal{L}$ /@ (C_k[0] C_{k+1}[0]) ⅆ{vs_k, vs_{k^+}}}, $\mathbb{E}$[$\mathcal{E}$_] :> $\mathcal{E}$]

]]

Out[◦]=

{0., {∈Series[$p_{1,2+k}$ $\pi_{1,k}$ + $p_{2,2+k}$ $\pi_{2,k}$ + $p_{3,2+k}$ $\pi_{3,k}$, 0]}}

In[◦]:= **Short**[eqn = **CF**[**LeftCC**〚1〛 - **RightCC**〚1〛]]
cvs = **Union**@**Cases**[eqn, p__ | π__, ∞]
vars = **Union**@**Cases**[γ_0[-1, k], h_, ∞]
Short[eqns = **CoefficientRules**[eqn, cvs] /. (_ → *c*_) :> (*c* == 0), 3]
{sol} = **Solve**[eqns, vars]

Out[◦]//Short=

$$\frac{h_1 \, p_{3,2+k} \, \pi_{1,k} \, \pi_{2,k}}{B}$$

Out[◦]=

{$p_{3,2+k}$, $\pi_{1,k}$, $\pi_{2,k}$}

Out[◦]=

{h_1}

Out[◦]//Short=

$$\left\{ \frac{h_1}{B} == 0 \right\}$$

Out[◦]=

{{h_1 → 0}}

In[◦]:= **sol** /. (*v*_ → *val*_) :> (*v* = **CF**[*val*]);
γ_0[-1, k]

Out[◦]=

0

In[∘]:= **Short[eqn = CF[LeftCC〚2〛 - RightCC〚2〛]]**
cvs = Union@Cases[eqn, p__ | π__, ∞]
vars = Union@Cases[γ₁[-1, k] + γ₄₂[-1, k], h_, ∞]
Short[eqns = CoefficientRules[eqn, cvs] /. (_ → c_) :> (c == 0), 3]
Short[{sol} = Solve[eqns, vars]]

Out[∘]//Short=

$$\frac{a_2\,b_3 - a_2\,b_5 - 2\,a_2\,b_3\,T_1 + a_4\,b_3\,T_1 + \ll 70 \gg + h_6\,T_1^3\,T_2^2 + h_7\,T_1^3\,T_2^2 + h_8\,T_1^3\,T_2^2}{(-1+T_1)\,T_1\,(-1+T_2)\,(-1+T_1\,T_2)} +$$

$$\ll 6 \gg + h_8\,p_{2,2+k}\,p_{3,2+k}\,\pi_{2,k}\,\pi_{3,k}$$

Out[∘]=

$$\{p_{1,2+k},\ p_{2,2+k},\ p_{3,2+k},\ \pi_{1,k},\ \pi_{2,k},\ \pi_{3,k}\}$$

Out[∘]=

$$\{h_2,\ h_3,\ h_4,\ h_5,\ h_6,\ h_7,\ h_8\}$$

Out[∘]//Short=

$$\Big\{ h_6 == 0,\ \ll 6 \gg,$$

$$\frac{2\,a_2\,b_3}{(1-T_1)\,(1-T_2)\,(1-T_1\,T_2)} - \frac{a_4\,b_3}{(1-T_1)\,(1-T_2)\,(1-T_1\,T_2)} - \frac{a_2\,b_5}{(1-T_1)\,(1-T_2)\,(1-T_1\,T_2)} +$$

$$\frac{a_4\,b_5}{(1-T_1)\,(1-T_2)\,(1-T_1\,T_2)} + \frac{h_3}{(1-T_1)\,(1-T_2)\,(1-T_1\,T_2)} + \frac{h_4}{\ll 1 \gg} + \ll 72 \gg\ == 0 \Big\}$$

Out[∘]//Short=

$$\Big\{\Big\{ h_2 \to 0,\ h_3 \to 0,\ h_4 \to 0,$$

$$h_5 \to -\frac{\ll 1 \gg}{\frac{1}{(1-T_1)\,(\ll 1 \gg)\,(1-T_1\,\ll 1 \gg)} - \frac{T_1}{\ll 1 \gg} - \ll 1 \gg + \ll 1 \gg + \frac{\ll 1 \gg}{\ll 1 \gg} - \frac{T_1^2\,\ll 1 \gg}{\ll 1 \gg}},\ h_6 \to 0,\ h_7 \to 0,\ h_8 \to 0 \Big\}\Big\}$$

In[∘]:= **sol /. (v_ → val_) :> (v = CF[val]);**

In[∘]:= **γ₁[-1, k]**
Short[CF[γ₄₂[-1, k]], 5]

Out[∘]=
0

Out[∘]//Short=

$$-\frac{(-b_3 + b_5 + b_3\,T_1 - b_5\,T_2)\,(-a_2 + a_2\,T_1 - a_4\,T_1 - a_4\,T_2 + 2\,a_4\,T_1\,T_2)\,p_{3,k}\,x_{3,k}}{(-1+T_1)\,T_1\,(-1+T_2)\,(-1+T_1\,T_2)}$$

Invariance Under R1l
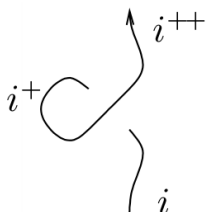
In[◦]:= {LeftR1l} = Cases[{∫ℱ[i] ℒ /@ (X$_{i+2,i}$[1] C$_{i+1}$[1]) ⅆ{vs$_i$, vs$_{i^+}$, vs$_{i+2}$}}, 𝔼[ℰ_] ⤇ ℰ, ∞]

Out[◦]=

{∈Series[p$_{1,3+i}$ π$_{1,i}$ + p$_{2,3+i}$ π$_{2,i}$ + p$_{3,3+i}$ π$_{3,i}$, 0]}

Invariance Under R1r
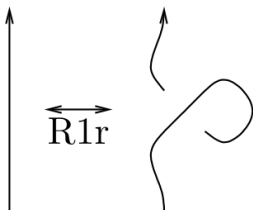
In[◦]:= {LeftR1r} = Cases[{∫ℱ[i] ℒ /@ (X$_{i,i+2}$[1] C$_{i+1}$[-1]) ⅆ{vs$_i$, vs$_{i^+}$, vs$_{i+2}$}}, 𝔼[ℰ_] ⤇ ℰ, ∞]

Out[◦]=

{∈Series[p$_{1,3+i}$ π$_{1,i}$ + p$_{2,3+i}$ π$_{2,i}$ + p$_{3,3+i}$ π$_{3,i}$, 0]}

In[◦]:= {RightR1r} = Cases[{∫ℱ[i] ℒ /@ (C$_i$[0] C$_{i+1}$[0] C$_{i+2}$[0]) ⅆ{vs$_i$, vs$_{i^+}$, vs$_{i+2}$}}, 𝔼[ℰ_] ⤇ ℰ, ∞]

Out[◦]=

{∈Series[p$_{1,3+i}$ π$_{1,i}$ + p$_{2,3+i}$ π$_{2,i}$ + p$_{3,3+i}$ π$_{3,i}$, 0]}

In[◦]:= LeftR1r == RightR1r

Out[◦]=

True

Invariance Under Sw
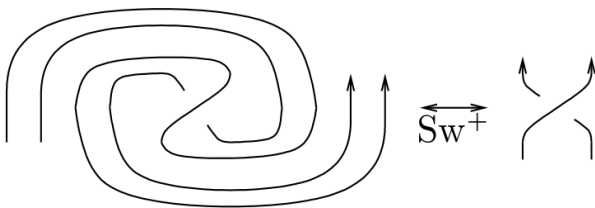
In[◦]:= Timing[Short[{LeftSw} = Cases[

{∫ℱ[i, j] ℒ /@ (X$_{i+1,j+1}$[1] C$_i$[-1] C$_j$[-1] C$_{i+2}$[1] C$_{j+2}$[1]) ⅆ{vs$_i$, vs$_j$, vs$_{i^+}$, vs$_{j^+}$, vs$_{j+2}$}},

𝔼[ℰ_] ⤇ ℰ, ∞]

]]

Out[◦]=

{0.03125, {∈Series[T$_1$ p$_{1,2+i}$ π$_{1,i}$ + ≪20≫ + p$_{3,3+i}$ x$_{3,2+i}$, $\frac{≪1≫}{≪1≫}$ + ≪48≫ + $\frac{≪1≫}{≪1≫}$]}}

$In[\circ]:=$ `Timing[ Short[{RightSw} = Cases[`

$$\left\{ \int \mathcal{F}[i, j] \, \mathcal{L} \, /@ \, (X_{i+1,j+1}[1] \, C_i[0] \, C_j[0] \, C_{i+2}[0] \, C_{j+2}[0]) \, \mathbb{d}\{vs_i, vs_j, vs_{i^+}, vs_{j^+}, vs_{j+2}\}\right\},$$

$$\mathbb{E}[\mathcal{E}\_] :\mapsto \mathcal{E}, \infty]$$

`]]`

$Out[\circ]=$

$$\left\{0.078125, \left\{\epsilon \text{Series}\left[T_1 \, p_{1,2+i} \, \pi_{1,i} + \ll20\gg + p_{3,3+i} \, x_{3,2+i}, \frac{\ll1\gg \ll1\gg \ll1\gg}{\ll1\gg} + \ll48\gg + \ll1\gg\right]\right\}\right\}$$

$In[\circ]:=$ `LeftSw[1] == RightSw[1]`

$Out[\circ]=$

True

$In[\circ]:=$ `Short[eqn = CF[LeftSw[2] - RightSw[2]]]`
`cvs = Union@Cases[eqn, p__ | π__, ∞]`
`vars = Union@Cases[eqn, (c | d | e | f | g | h)_, ∞]`
`Short[eqns = CoefficientRules[eqn, cvs] /. (_ → c_) :> (c == 0), 3]`
`Short[{sol} = Solve[eqns, vars]]`

$Out[\circ]//Short=$

$$-\frac{(-b_3 + b_5 + b_3 \ll1\gg - b_5 \, T_2) \, (\ll1\gg)}{(-1 + T_1) \, T_1 \, (\ll1\gg) \, (-1 + T_1 \, T_2)} - \ll1\gg + \frac{\ll1\gg}{\ll1\gg}$$

$Out[\circ]=$

$$\{p_{3,2+i}, \pi_{3,i}\}$$

$Out[\circ]=$

$$\{\}$$

$Out[\circ]//Short=$

$$\{ \ll1\gg \}$$

⬤ Set: Lists {sol} and $\left\{\left\{a_2 \to \frac{a_4 \, T_1 + a_4 \, T_2 - 2 \, a_4 \, T_1 \, T_2}{-1 + T_1}\right\}, \left\{T_2 \to \frac{-b_3 + b_5 + b_3 \, T_1}{b_5}\right\}, \{b_3 \to 0, b_5 \to 0\}\right\}$ are not the same shape. ⓘ

$Out[\circ]//Short=$

$$\left\{\left\{a_2 \to \frac{a_4 \, T_1 + a_4 \, \ll1\gg - 2 \, a_4 \, T_1 \, T_2}{-1 + T_1}\right\}, \left\{T_2 \to \frac{\ll1\gg}{\ll1\gg}\right\}, \{b_3 \to 0, b_5 \to 0\}\right\}$$

$In[\circ]:=$ `Factor[eqn]`

$Out[\circ]=$

$$- (((-b_3 + b_5 + b_3 \, T_1 - b_5 \, T_2) \, (-a_2 + a_2 \, T_1 - a_4 \, T_1 - a_4 \, T_2 + 2 \, a_4 \, T_1 \, T_2)$$
$$(1 + T_1 \, T_2 \, p_{3,2+i} \, \pi_{3,i} - p_{3,2+i} \, x_{3,2+i})) \, / \, ((-1 + T_1) \, T_1 \, (-1 + T_2) \, (-1 + T_1 \, T_2)))$$