Pensieve header: Proof of invariance of ρ_2 using integration techniques.

Initialization

In[∘]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\HigherRank"];
Once[<< KnotTheory`; << Rot.m];
<< FormalGaussianIntegration.m;
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
Loading Rot.m from http://drorbn.net/AP/Projects/HigherRank to compute rotation numbers.
```

In[∘]:=
```
T2z[p_] := Module[{q = Expand[p], n, c},
   If[q === 0, 0, c = Coefficient[q, T, n = Exponent[q, T]];
    c z^(2 n) + T2z[q - c (T^(1/2) - T^(-1/2))^(2 n)]]];
```

In[∘]:=
```
Features[Knot[8, 17]]
```

KnotTheory: Loading precomputed data in PD4Knots`.

Out[∘]=

Features[18,
$C_7[-1]\ C_{15}[-1]\ X_{1,7}[1]\ X_{3,9}[-1]\ X_{5,13}[-1]\ X_{8,16}[1]\ X_{10,4}[-1]\ X_{12,18}[1]\ X_{15,2}[-1]\ X_{17,11}[1]$]

The ρ_2 Integrand

Adopted from pensieve://Talks//Oaxaca-2210/Rho.nb.

In[∘]:=

```
𝒮 = {x_, p_};
q[s_, i_, j_] := x_i (p_{i+1} - p_i) + x_j (p_{j+1} - p_j) + (T^s - 1) x_i (p_{i+1} - p_{j+1});
r1[s_, i_, j_] := s/2 (x_i (p_i - p_j) ((T^s - 1) x_i p_j + 2 (1 - p_j x_j)) - 1);
r2[1, i_, j_] :=
  (-6 p_i x_i + 6 p_j x_i - 3 (-1 + 3 T) p_i p_j x_i^2 + 3 (-1 + 3 T) p_j^2 x_i^2 + 4 (-1 + T) p_i^2 p_j x_i^3 -
     2 (-1 + T) (5 + T) p_i p_j^2 x_i^3 + 2 (-1 + T) (3 + T) p_j^3 x_i^3 + 18 p_i p_j x_i x_j - 18 p_j^2 x_i x_j -
     6 p_i^2 p_j x_i^2 x_j + 6 (2 + T) p_i p_j^2 x_i^2 x_j - 6 (1 + T) p_j^3 x_i^2 x_j - 6 p_i p_j^2 x_i x_j^2 + 6 p_j^3 x_i x_j^2) / 12;
r2[-1, i_, j_] :=
  (-6 T^2 p_i x_i + 6 T^2 p_j x_i + 3 (-3 + T) T p_i p_j x_i^2 - 3 (-3 + T) T p_j^2 x_i^2 - 4 (-1 + T) T p_i^2 p_j x_i^3 +
     2 (-1 + T) (1 + 5 T) p_i p_j^2 x_i^3 - 2 (-1 + T) (1 + 3 T) p_j^3 x_i^3 + 18 T^2 p_i p_j x_i x_j -
     18 T^2 p_j^2 x_i x_j - 6 T^2 p_i^2 p_j x_i^2 x_j + 6 T (1 + 2 T) p_i p_j^2 x_i^2 x_j -
     6 T (1 + T) p_j^3 x_i^2 x_j - 6 T^2 p_i p_j^2 x_i x_j^2 + 6 T^2 p_j^3 x_i x_j^2) / (12 T^2);
γ1[φ_, k_] := φ (1 / 2 - x_k p_k);
γ2[φ_, k_] := -φ^2 p_k x_k / 2;
ℒ[X_{i_,j_}[s_]] := T^{s/2} 𝔼[q[s, i, j] + ϵ r1[s, i, j] + ϵ^2 r2[s, i, j] + O[ϵ]^3];
ℒ[C_{k_}[φ_]] := T^{φ/2} 𝔼[-x_k (p_k - p_{k+1}) + ϵ γ1[φ, k] + ϵ^2 γ2[φ, k] + O[ϵ]^3];
ℒ[K_] := (2 π)^(-Features[K][[1]]) CF[ℒ /@ Features[K][[2]]];
vs[K_] := Union @@ Table[{p_i, x_i}, {i, Features[K][[1]]}]
```

In[∘]:=

```
Features[Knot[3, 1]]
```

Out[∘]=

```
Features[7, C_4[-1] X_{2,6}[-1] X_{5,1}[-1] X_{7,3}[-1]]
```

In[∘]:= `ℒ[Knot[3, 1]]`

Out[∘]=

$$\frac{1}{128\,\pi^7\,T^2}\,\mathbb{E}\Big[\epsilon\text{Series}\Big[-p_1 x_1+p_2 x_1-p_2 x_2+\frac{p_3 x_2}{T}+\frac{(-1+T)\,p_7 x_2}{T}-p_3 x_3+p_4 x_3-p_4 x_4+p_5 x_4+\frac{(-1+T)\,p_2 x_5}{T}-p_5 x_5+\frac{p_6 x_5}{T}-p_6 x_6+p_7 x_6+\frac{(-1+T)\,p_4 x_7}{T}-p_7 x_7+\frac{p_8 x_7}{T},\ 1-p_2 x_2+p_6 x_2+\frac{(-1+T)\,p_2 p_6 x_2^2}{2T}-\frac{(-1+T)\,p_6^2 x_2^2}{2T}+p_4 x_4+p_1 x_5-p_5 x_5-p_1^2 x_1 x_5+p_1 p_5 x_1 x_5-\frac{(-1+T)\,p_1^2 x_5^2}{2T}+\frac{(-1+T)\,p_1 p_5 x_5^2}{2T}+p_2 p_6 x_2 x_6-p_6^2 x_2 x_6+p_3 x_7-p_7 x_7-p_3^2 x_3 x_7+p_3 p_7 x_3 x_7-\frac{(-1+T)\,p_3^2 x_7^2}{2T}+\frac{(-1+T)\,p_3 p_7 x_7^2}{2T},$$
$$-\frac{1}{2}p_2 x_2+\frac{p_6 x_2}{2}+\frac{(-3+T)\,p_2 p_6 x_2^2}{4T}-\frac{(-3+T)\,p_6^2 x_2^2}{4T}-\frac{(-1+T)\,p_2^2 p_6 x_2^3}{3T}+\frac{(-1+T)\,(1+5T)\,p_2 p_6^2 x_2^3}{6T^2}-\frac{(-1+T)\,(1+3T)\,p_6^3 x_2^3}{6T^2}-\frac{p_4 x_4}{2}+\frac{p_1 x_5}{2}-\frac{p_5 x_5}{2}-\frac{3}{2}p_1^2 x_1 x_5+\frac{3}{2}p_1 p_5 x_1 x_5+\frac{1}{2}p_1^3 x_1^2 x_5-\frac{1}{2}p_1^2 p_5 x_1^2 x_5-\frac{(-3+T)\,p_1^2 x_5^2}{4T}+\frac{(-3+T)\,p_1 p_5 x_5^2}{4T}-\frac{(1+T)\,p_1^3 x_1 x_5^2}{2T}+\frac{(1+2T)\,p_1^2 p_5 x_1 x_5^2}{2T}-\frac{1}{2}p_1 p_5^2 x_1 x_5^2-\frac{(-1+T)\,(1+3T)\,p_1^3 x_5^3}{6T^2}+\frac{(-1+T)\,(1+5T)\,p_1^2 p_5 x_5^3}{6T^2}-\frac{(-1+T)\,p_1 p_5^2 x_5^3}{3T}+\frac{3}{2}p_2 p_6 x_2 x_6-\frac{3}{2}p_6^2 x_2 x_6-\frac{1}{2}p_2^2 p_6 x_2^2 x_6+\frac{(1+2T)\,p_2 p_6^2 x_2^2 x_6}{2T}-\frac{(1+T)\,p_6^3 x_2^2 x_6}{2T}-\frac{1}{2}p_2 p_6^2 x_2 x_6^2+\frac{1}{2}p_6^3 x_2 x_6^2+\frac{p_3 x_7}{2}-\frac{p_7 x_7}{2}-\frac{3}{2}p_3^2 x_3 x_7+\frac{3}{2}p_3 p_7 x_3 x_7+\frac{1}{2}p_3^3 x_3^2 x_7-\frac{1}{2}p_3^2 p_7 x_3^2 x_7-\frac{(-3+T)\,p_3^2 x_7^2}{4T}+\frac{(-3+T)\,p_3 p_7 x_7^2}{4T}-\frac{(1+T)\,p_3^3 x_3 x_7^2}{2T}+\frac{(1+2T)\,p_3^2 p_7 x_3 x_7^2}{2T}-\frac{1}{2}p_3 p_7^2 x_3 x_7^2-\frac{(-1+T)\,(1+3T)\,p_3^3 x_7^3}{6T^2}+\frac{(-1+T)\,(1+5T)\,p_3^2 p_7 x_7^3}{6T^2}-\frac{(-1+T)\,p_3 p_7^2 x_7^3}{3T}\Big]\Big]$$

In[∘]:= `vs[Knot[3, 1]]`

Out[∘]=

$\{p_1, p_2, p_3, p_4, p_5, p_6, p_7, x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$

In[∘]:= `K = Knot[3, 1]; ∫ℒ[K] ⅆ(vs@K)`

Out[∘]=

$$-\frac{\mathbb{i}\,T\,\mathbb{E}\Big[\epsilon\text{Series}\Big[0,\ \frac{(-1+T)^2\,(1+T^2)}{(1-T+T^2)^2},\ -\frac{T^2\,(1-4T^2+T^4)}{2\,(1-T+T^2)^4}\Big]\Big]}{1-T+T^2}$$

In[∘]:= `T2z[T^-2 (1 - 4 T^2 + T^4)]`

Out[∘]=

$-2+4z^2+z^4$

In[∘]:=
```
K = Knot[5, 2]; ∫ℒ[K] ⅆ(vs@K)
```

KnotTheory: Loading precomputed data in PD4Knots`.

Out[∘]=

$$-\frac{\mathrm{i}\, T\, \mathbb{E}\left[\epsilon\text{Series}\left[0, \frac{(-1+T)^2 (5-4T+5T^2)}{(2-3T+2T^2)^2}, \frac{1-4T+11T^2-44T^3+76T^4-44T^5+11T^6-4T^7+T^8}{2(2-3T+2T^2)^4}\right]\right]}{2-3T+2T^2}$$

In[∘]:=
```
T2z[(1 - 4 T + 11 T^2 - 44 T^3 + 76 T^4 - 44 T^5 + 11 T^6 - 4 T^7 + T^8) / T^4]
```

Out[∘]=

$$4 - 20 z^2 + 7 z^4 + 4 z^6 + z^8$$

In[∘]:=
```
K = Knot[8, 19]; ∫ℒ[K] ⅆ(vs@K)
```

Out[∘]=

$$\frac{1}{1-T+T^3-T^5+T^6} T^3\, \mathbb{E}\left[\epsilon\text{Series}\left[0, -\frac{(-1+T)^2 (1+T^4)(3+4T^3+3T^6)}{(1-T+T^2)^2 (1-T^2+T^4)^2}, -\left(\left(T^3 (4-7T-12T^2+62T^3-100T^4+95T^5-60T^6-42T^7+208T^8-306T^9+208T^{10}-42T^{11}-60T^{12}+95T^{13}-100T^{14}+62T^{15}-12T^{16}-7T^{17}+4T^{18})\right) / \left(2 (1-T+T^2)^4 (1-T^2+T^4)^4\right)\right)\right]\right]$$

In[∘]:=
```
K = Knot[3, 1]; Features[K]
```

Out[∘]=

$$\text{Features}[7, C_4[-1]\, X_{2,6}[-1]\, X_{5,1}[-1]\, X_{7,3}[-1]]$$

In[∘]:=
```
{L} = Cases[ℒ[K], ϵSeries[Q_, ___] :→ Q, ∞]
```

Out[∘]=

$$\left\{-p_1 x_1 + p_2 x_1 - p_2 x_2 + \frac{p_3 x_2}{T} + \frac{(-1+T)\, p_7 x_2}{T} - p_3 x_3 + p_4 x_3 - p_4 x_4 + p_5 x_4 + \frac{(-1+T)\, p_2 x_5}{T} - p_5 x_5 + \frac{p_6 x_5}{T} - p_6 x_6 + p_7 x_6 + \frac{(-1+T)\, p_4 x_7}{T} - p_7 x_7 + \frac{p_8 x_7}{T}\right\}$$

In[∘]:=
```
∫𝔼[L + ϵ x4 p1 + O[ϵ]^2] ⅆ(vs@K)
```

Out[∘]=

$$-\frac{128\, \mathrm{i}\, \pi^7 T^3\, \mathbb{E}[\epsilon\text{Series}[0, T]]}{1-T+T^2}$$

In[∘]:=
```
∫𝔼[L + ϵ x4 p2 + O[ϵ]^2] ⅆ(vs@K)
```

Out[∘]=

$$-\frac{128\, \mathrm{i}\, \pi^7 T^3\, \mathbb{E}[\epsilon\text{Series}[0, T]]}{1-T+T^2}$$

In[∘]:= $\int \mathbb{E}\left[L + \epsilon\, x_4\, p_6 + O[\epsilon]^2\right] d(vs@K)$

Out[∘]=

$$-\frac{128\, i\, \pi^7\, T^3\, \mathbb{E}\left[\epsilon\text{Series}\left[0, \frac{(-1+T)\, T^2}{1-T+T^2}\right]\right]}{1 - T + T^2}$$

In[∘]:= $\int \mathbb{E}\left[L + \epsilon\, x_4\, p_7 + O[\epsilon]^2\right] d(vs@K)$

Out[∘]=

$$-\frac{128\, i\, \pi^7\, T^3\, \mathbb{E}\left[\epsilon\text{Series}\left[0, \frac{(-1+T)\, T^2}{1-T+T^2}\right]\right]}{1 - T + T^2}$$

In[∘]:= $\int \mathbb{E}\left[L + \epsilon\, x_4\, p_8 + O[\epsilon]^2\right] d(vs@K)$

Out[∘]=

$$-\frac{128\, i\, \pi^7\, T^3\, \mathbb{E}[\epsilon\text{Series}[0, 0]]}{1 - T + T^2}$$

In[∘]:= $\int \mathbb{E}\left[L + \epsilon\, x_1\, p_1 + O[\epsilon]^2\right] d(vs@K)$

Out[∘]=

$$-\frac{128\, i\, \pi^7\, T^3\, \mathbb{E}[\epsilon\text{Series}[0, 1]]}{1 - T + T^2}$$

In[∘]:= $\int \mathbb{E}\left[L + \epsilon\, x_1\, p_2 + O[\epsilon]^2\right] d(vs@K)$

Out[∘]=

$$-\frac{128\, i\, \pi^7\, T^3\, \mathbb{E}[\epsilon\text{Series}[0, 0]]}{1 - T + T^2}$$

Invariance Under Reidemeister 3b

In[∘]:=

$$\text{lhs} = \int (\mathbb{E}[\pi_i\, p_i + \pi_j\, p_j + \pi_k\, p_k]\ \mathcal{L}\ /@\ (X_{i,j}[1]\ X_{i+1,k}[1]\ X_{j+1,k+1}[1]))\ d\{p_i, p_j, p_k, p_{i+1}, p_{j+1}, p_{k+1}, x_i, x_j, x_k, x_{i+1}, x_{j+1}, x_{k+1}\}$$

$$\text{rhs} = \int (\mathbb{E}[\pi_i\, p_i + \pi_j\, p_j + \pi_k\, p_k]\ \mathcal{L}\ /@\ (X_{j,k}[1]\ X_{i,k+1}[1]\ X_{i+1,j+1}[1]))\ d\{p_i, p_j, p_k, p_{i+1}, p_{j+1}, p_{k+1}, x_i, x_j, x_k, x_{i+1}, x_{j+1}, x_{k+1}\};$$

lhs == rhs

Out[∘]=

$$64\,\pi^6\,T^{3/2}\,\mathbb{E}\Big[\epsilon\text{Series}\Big[T^2 p_{2+i}\pi_i - T p_{2+j}(-\pi_i + T\pi_i - \pi_j) + p_{2+k}(\pi_i - T\pi_i + \pi_j - T\pi_j + \pi_k),$$
$$-\frac{3}{2} + \frac{1}{2}T^3 p_{2+i} p_{2+j}\pi_i(-\pi_i + T\pi_i - 2\pi_j) - \frac{1}{2}T^3 p_{2+j}^2\pi_i(-\pi_i + T\pi_i - 2\pi_j) +$$
$$T p_{2+j}(T\pi_i - \pi_j) - \frac{1}{2}T p_{2+k}^2(\pi_i + \pi_j)(-\pi_i + T\pi_i - \pi_j + T\pi_j - 2\pi_k) +$$
$$\frac{1}{2}T^2 p_{2+i} p_{2+k}\pi_i(-\pi_i + T\pi_i - 2\pi_j + 2T\pi_j - 2\pi_k) + p_{2+k}(T\pi_i - \pi_j + 2T\pi_j - 2\pi_k) - \frac{1}{2}T p_{2+j} p_{2+k}$$
$$\left(\pi_i^2 - 2T\pi_i^2 + T^2\pi_i^2 + 2\pi_i\pi_j - 4T\pi_i\pi_j + 2T^2\pi_i\pi_j + \pi_j^2 - T\pi_j^2 + 2\pi_i\pi_k - 2T\pi_i\pi_k + 2\pi_j\pi_k\right),$$
$$\frac{1}{4}T^3 p_{2+j}^2\pi_i(-3\pi_i + 5T\pi_i - 10\pi_j) - \frac{1}{4}T^3 p_{2+i} p_{2+j}\pi_i(-\pi_i + 3T\pi_i - 6\pi_j) -$$
$$\frac{1}{6}T^5 p_{2+i}^2 p_{2+j}\pi_i^2(-\pi_i + T\pi_i - 3\pi_j) - \frac{1}{2}T p_{2+j}(T\pi_i - \pi_j) +$$
$$\frac{1}{6}T^4 p_{2+i} p_{2+j}^2\pi_i\left(\pi_i^2 - 5T\pi_i^2 + 4T^2\pi_i^2 + 3\pi_i\pi_j - 12T\pi_i\pi_j + 3\pi_j^2\right) -$$
$$\frac{1}{6}T^4 p_{2+j}^3\pi_i\left(\pi_i^2 - 4T\pi_i^2 + 3T^2\pi_i^2 + 3\pi_i\pi_j - 9T\pi_i\pi_j + 3\pi_j^2\right) -$$
$$\frac{1}{4}T^2 p_{2+i} p_{2+k}\pi_i(-\pi_i + 3T\pi_i - 6\pi_j + 10T\pi_j - 10\pi_k) -$$
$$\frac{1}{6}T^4 p_{2+i}^2 p_{2+k}\pi_i^2(-\pi_i + T\pi_i - 3\pi_j + 3T\pi_j - 3\pi_k) + \frac{1}{2}p_{2+k}(-T\pi_i + \pi_j - 4T\pi_j + 4\pi_k) +$$
$$\frac{1}{4}T p_{2+k}^2\left(-3\pi_i^2 + 5T\pi_i^2 - 10\pi_i\pi_j + 14T\pi_i\pi_j - 7\pi_j^2 + 9T\pi_j^2 - 14\pi_i\pi_k - 18\pi_j\pi_k\right) +$$
$$\frac{1}{4}T p_{2+j} p_{2+k}\left(\pi_i^2 - 6T\pi_i^2 + 5T^2\pi_i^2 + 6\pi_i\pi_j - 20T\pi_i\pi_j + 14T^2\pi_i\pi_j + 5\pi_j^2 - 7T\pi_j^2 + 10\pi_i\pi_k -\right.$$
$$\left.14T\pi_i\pi_k + 14\pi_j\pi_k\right) + \frac{1}{6}T^3 p_{2+i} p_{2+j} p_{2+k}\pi_i\left(5\pi_i^2 - 10T\pi_i^2 + 5T^2\pi_i^2 + 15\pi_i\pi_j -\right.$$
$$\left.27T\pi_i\pi_j + 12T^2\pi_i\pi_j + 12\pi_j^2 - 12T\pi_j^2 + 12\pi_i\pi_k - 12T\pi_i\pi_k + 18\pi_j\pi_k\right) - \frac{1}{6}T^2 p_{2+j}^2 p_{2+k}$$
$$\left(-\pi_i^3 + 6T\pi_i^3 - 9T^2\pi_i^3 + 4T^3\pi_i^3 - 3\pi_i^2\pi_j + 18T\pi_i^2\pi_j - 24T^2\pi_i^2\pi_j + 9T^3\pi_i^2\pi_j - 3\pi_i\pi_j^2 + 15T\pi_i\pi_j^2 -\right.$$
$$\left.12T^2\pi_i\pi_j^2 - \pi_j^3 + T\pi_j^3 - 3\pi_i^2\pi_k + 12T\pi_i^2\pi_k - 9T^2\pi_i^2\pi_k - 6\pi_i\pi_j\pi_k + 18T\pi_i\pi_j\pi_k - 3\pi_j^2\pi_k\right) +$$
$$\frac{1}{6}T^2 p_{2+i} p_{2+k}^2\pi_i\left(\pi_i^2 - 5T\pi_i^2 + 4T^2\pi_i^2 + 3\pi_i\pi_j - 15T\pi_i\pi_j + 12T^2\pi_i\pi_j + 3\pi_j^2 -\right.$$
$$\left.12T\pi_j^2 + 9T^2\pi_j^2 + 3\pi_i\pi_k - 12T\pi_i\pi_k + 6\pi_j\pi_k - 18T\pi_j\pi_k + 3\pi_k^2\right) -$$
$$\frac{1}{6}T p_{2+k}^3(\pi_i + \pi_j)\left(\pi_i^2 - 4T\pi_i^2 + 3T^2\pi_i^2 + 2\pi_i\pi_j - 8T\pi_i\pi_j + 6T^2\pi_i\pi_j + \pi_j^2 -\right.$$
$$\left.4T\pi_j^2 + 3T^2\pi_j^2 + 3\pi_i\pi_k - 9T\pi_i\pi_k + 3\pi_j\pi_k - 9T\pi_j\pi_k + 3\pi_k^2\right) -$$
$$\frac{1}{6}T p_{2+j} p_{2+k}^2\left(-\pi_i^3 + 6T\pi_i^3 - 9T^2\pi_i^3 + 4T^3\pi_i^3 - 3\pi_i^2\pi_j + 18T\pi_i^2\pi_j - 27T^2\pi_i^2\pi_j + 12T^3\pi_i^2\pi_j - 3\pi_i\pi_j^2 +\right.$$
$$18T\pi_i\pi_j^2 - 24T^2\pi_i\pi_j^2 + 9T^3\pi_i\pi_j^2 - \pi_j^3 + 5T\pi_j^3 - 4T^2\pi_j^3 - 3\pi_i^2\pi_k + 15T\pi_i^2\pi_k - 12T^2\pi_i^2\pi_k -$$
$$\left.6\pi_i\pi_j\pi_k + 30T\pi_i\pi_j\pi_k - 18T^2\pi_i\pi_j\pi_k - 3\pi_j^2\pi_k + 12T\pi_j^2\pi_k - 3\pi_i\pi_k^2 + 3T\pi_i\pi_k^2 - 3\pi_j\pi_k^2\right)\Big]\Big]$$

Out[∘]=

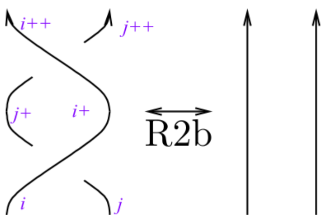
True

Scattering by the crossings

In[∘]:=
$$\left\{\frac{1}{4\pi^2}\int \mathbb{E}[\pi_i p_i + \pi_j p_j + O[\epsilon]]\ \mathcal{L}[X_{i,j}[1]]\ \mathrm{d}\{x_i, x_j, p_i, p_j\},\ \frac{1}{4\pi^2}\int \mathbb{E}[\pi_i p_i + \pi_j p_j + O[\epsilon]]\ \mathcal{L}[X_{i,j}[-1]]\ \mathrm{d}\{x_i, x_j, p_i, p_j\}\right\}$$

Out[∘]=
$$\left\{\sqrt{T}\ \mathbb{E}[\epsilon\text{Series}[T\, p_{1+i}\,\pi_i + (1-T)\, p_{1+j}\,\pi_i + p_{1+j}\,\pi_j]],\ \frac{\mathbb{E}\left[\epsilon\text{Series}\left[\frac{p_{1+i}\,\pi_i}{T} + \frac{(-1+T)\, p_{1+j}\,\pi_i}{T} + p_{1+j}\,\pi_j\right]\right]}{\sqrt{T}}\right\}$$

Invariance Under Reidemeister 2b



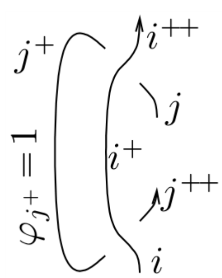
Territory: $(x \mid p)_{(i\,|\,j)^+},\ x_{i\,|\,j}$.

In[∘]:=
$$\text{lhs} = \int \mathbb{E}[\pi_i p_i + \pi_j p_j]\ \mathcal{L}\ /@\ (X_{i,j}[1]\ X_{i+1,j+1}[-1])\ \mathrm{d}\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}\}$$
$$\text{rhs} = \int \mathbb{E}[\pi_i p_i + \pi_j p_j]\ \mathcal{L}\ /@\ (C_i[0]\ C_{i+1}[0]\ C_j[0]\ C_{j+1}[0])\ \mathrm{d}\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}\};$$
$$\text{lhs} == \text{rhs}$$

Out[∘]=
$$16\,\pi^4\ \mathbb{E}[\epsilon\text{Series}[p_{2+i}\,\pi_i + p_{2+j}\,\pi_j, 0, 0]]$$

Out[∘]=
True

Invariance Under R2c



In[∘]:=
```
lhs = ∫𝔼[π_i p_i + π_j p_j] ℒ /@ (X_{i+1,j}[1] X_{i,j+2}[-1] C_{j+1}[1])
    ⅆ{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{j+2}, p_{j+2}}
rhs = ∫𝔼[π_i p_i + π_j p_j] ℒ /@ (C_i[0] C_{i+1}[0] C_j[0] C_{j+1}[1] C_{j+2}[0])
    ⅆ{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{j+2}, p_{j+2}};
lhs == rhs
```

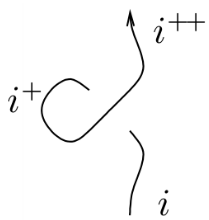
Out[∘]=

$$-32\, i\, \pi^5 \sqrt{T}\ \mathbb{E}\left[\epsilon\text{Series}\left[p_{2+i}\,\pi_i + p_{3+j}\,\pi_j,\ -\frac{1}{2} - p_{3+j}\,\pi_j,\ \frac{1}{2}\,p_{3+j}\,\pi_j\right]\right]$$

Out[∘]=

True

Invariance Under R1l



In[∘]:=
```
lhs = ∫𝔼[π_i p_i] ℒ /@ (X_{i+2,i}[1] C_{i+1}[1]) ⅆ{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}}
rhs = ∫𝔼[π_i p_i] ℒ /@ (C_i[0] C_{i+1}[0] C_{i+2}[0]) ⅆ{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}};
lhs == rhs
```

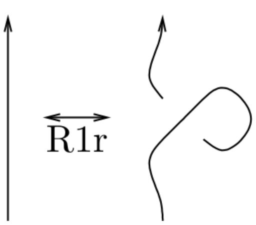
Out[∘]=

$$-8\, i\, \pi^3\ \mathbb{E}[\epsilon\text{Series}[p_{3+i}\,\pi_i, 0, 0]]$$

Out[∘]=

True

Invariance Under R1r



In[∘]:=

$$\text{lhs} = \int \mathbb{E}[\pi_i\, p_i]\ \mathcal{L}\ \text{/@}\ (X_{i,i+2}[1]\ C_{i+1}[-1])\ d\{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}\}$$

$$\text{rhs} = \int \mathbb{E}[\pi_i\, p_i]\ \mathcal{L}\ \text{/@}\ (C_i[0]\ C_{i+1}[0]\ C_{i+2}[0])\ d\{x_i, p_i, x_{i+1}, p_{i+1}, x_{i+2}, p_{i+2}\};$$

$$\text{lhs} == \text{rhs}$$

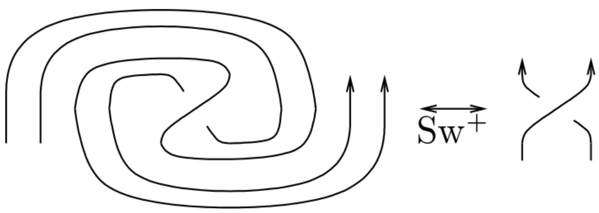
Out[∘]=

$$-8\, i\, \pi^3\, \mathbb{E}[\epsilon\text{Series}[p_{3+i}\, \pi_i, 0, 0]]$$

Out[∘]=

True

Invariance Under Sw



In[∘]:=

$$\text{lhs} = \int \mathbb{E}[\pi_i\, p_i + \pi_j\, p_j]\ \mathcal{L}\ \text{/@}\ (X_{i+1,j+1}[1]\ C_i[-1]\ C_j[-1]\ C_{i+2}[1]\ C_{j+2}[1])\ d\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{i+2}, p_{i+2}, x_{j+2}, p_{j+2}\}$$

$$\text{rhs} = \int \mathbb{E}[\pi_i\, p_i + \pi_j\, p_j]\ \mathcal{L}\ \text{/@}\ (X_{i+1,j+1}[1]\ C_i[0]\ C_j[0]\ C_{i+2}[0]\ C_{j+2}[0])\ d\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}, x_{i+2}, p_{i+2}, x_{j+2}, p_{j+2}\};$$

$$\text{lhs} == \text{rhs}$$

Out[∘]=

$$64\, \pi^6\, \sqrt{T}\ \mathbb{E}\Big[\epsilon\text{Series}\Big[T\, p_{3+i}\, \pi_i + p_{3+j}\, (\pi_i - T\, \pi_i + \pi_j),$$
$$-\frac{1}{2} + \frac{1}{2}\, T\, p_{3+i}\, p_{3+j}\, \pi_i\, (-\pi_i + T\, \pi_i - 2\, \pi_j) - \frac{1}{2}\, T\, p_{3+j}^2\, \pi_i\, (-\pi_i + T\, \pi_i - 2\, \pi_j) + p_{3+j}\, (T\, \pi_i - \pi_j),$$
$$\frac{1}{4}\, T\, p_{3+j}^2\, \pi_i\, (-3\, \pi_i + 5\, T\, \pi_i - 10\, \pi_j) - \frac{1}{4}\, T\, p_{3+i}\, p_{3+j}\, \pi_i\, (-\pi_i + 3\, T\, \pi_i - 6\, \pi_j) -$$
$$\frac{1}{6}\, T^2\, p_{3+i}^2\, p_{3+j}\, \pi_i^2\, (-\pi_i + T\, \pi_i - 3\, \pi_j) + \frac{1}{2}\, p_{3+j}\, (-T\, \pi_i + \pi_j) +$$
$$\frac{1}{6}\, T\, p_{3+i}\, p_{3+j}^2\, \pi_i\, \left(\pi_i^2 - 5\, T\, \pi_i^2 + 4\, T^2\, \pi_i^2 + 3\, \pi_i\, \pi_j - 12\, T\, \pi_i\, \pi_j + 3\, \pi_j^2\right) -$$
$$\frac{1}{6}\, T\, p_{3+j}^3\, \pi_i\, \left(\pi_i^2 - 4\, T\, \pi_i^2 + 3\, T^2\, \pi_i^2 + 3\, \pi_i\, \pi_j - 9\, T\, \pi_i\, \pi_j + 3\, \pi_j^2\right)\Big]\Big]$$

Out[∘]=

True