

Pensieve header: Formal Gaussian integration over an arbitrary “Feynman Ring”.

What must a Feynman Ring F have? (Over some set of labels S with elements x, y, \dots)

- * A vector space over \mathbb{Q} .
- * Has a symmetric linear $Z \mapsto \partial_{x,y} Z$ and a symmetric bilinear $(Z_1, Z_2) \mapsto (\partial_x Z_1)(\partial_y Z_2)$ that satisfy the axioms of (roughly) a connected circuit algebra.
- * Has $q_{x,y} : F \rightarrow \mathbb{Q}$ in some sense dual to some $\theta_{x,y} \in F$.
- * Has $\text{Ev}_{vs \rightarrow 0} : F \rightarrow F$.

Further axioms must be worked out.

Goals.

- * Define \int .
- * Prove a Fubini theorem.
- * Prove a theorem about the injectivity of the Laplace transform.

Initialization

```
In[*]:= CCF[ $\mathcal{E}$ _] := ExpandDenominator@ExpandNumerator@Together[ $\mathcal{E}$ ];
CCF[ $\mathcal{E}$ _] := Factor[ $\mathcal{E}$ ];
CF[ $\omega$  .  $\mathcal{E}$ _E] := CF[ $\omega$ ] CF /@  $\mathcal{E}$ ;
CF[ $\mathcal{E}$ _List] := CF /@  $\mathcal{E}$ ;
CF[ $\mathcal{E}$ _] := Module[{vs = Cases[ $\mathcal{E}$ , {x | p |  $\pi$ }_,  $\infty$ ]  $\cup$  {x, p,  $\epsilon$ }, ps, c},
  Total[CoefficientRules[Expand[ $\mathcal{E}$ ], vs] /. (ps_ -> c_) := CCF[c] (Times @@ vsps) ]];
```

The Basic Feynman Ring

```
In[*]:= S = {x, x_, y, z};
qx,y[f_] := (∂x,y f) /. Thread[S -> 0];
 $\theta_{x,y}$  := x y;
f_ == 0 := f === 0;
Evvs_List -> 0[f_] := CF[f /. Thread[vs -> 0]]
```

The ϵ Series Feynman Ring

With a hack to remove more- x -than- p terms at the highest power of ϵ .

The variable B carries the p vs x balance: $p \rightarrow B, x \rightarrow B^{-1}$.

```
In[*]:=
S = {y, z, ϕ, x_, p_, p, x};
qx,y[ser_eSeries] := (∂x,yser[[1]]) /. Thread[S → 0];
θx,y := x y;
eSeries /: D[ser_eSeries, vs___] := D[#, vs] & /@ ser;
eSeries /: Plus[ss___eSeries] /; Length[{ss}] > 1 := Module[{l = Min[Length /@ {ss}]},
  eSeries @@ Total[Take[List @@ #, l] & /@ {ss}]]
eSeries /: t_ + ser_eSeries := MapAt[(# + t) &, ser, 1];
eSeries /: s1_eSeries * s2_eSeries := eSeries @@ MapAt[(# /. Bbc /. ; bc < 0 → 0) &, -1] @
  Table[Collect[Sum[s1[[ii + 1]] s2[[kk - ii + 1]], {ii, 0, kk}], B],
  {kk, 0, Min[Length@s1, Length@s2] - 1}];
eSeries /: c_ * ser_eSeries := (c #) & /@ ser;
ser_eSeries ≡ 0 := And @@ (# === 0) & /@ ser;
eSeries /: Integrate[ser_eSeries, pars_] := eSeries @@ (Integrate[#, pars] & /@ ser);
eSeries /: EVvs_List→0[ser_eSeries] := ser /. Thread[vs → 0];
CF[ser_eSeries] := CF /@ ser;
```

Integration

Using Picard Iteration!

```
In[*]:=
E /: E[A_] E[B_] := E[A + B]
```

```
In[*]:=
E[sd_SeriesData] /; (List @@ sd) [{1, 2, 4, 6}] === {e, 0, 0, 1} :=
  E[eSeries @@ PadRight[sd[[3]], sd[[5]], 0]]
```

pdf

Following a program in Projects/FullDoPeGDO/Engine.nb, we write $Z_\lambda = \sum Z[m] \lambda^m$.

```

In[*]:= Unprotect[Integrate];
Integrate::sing = "How dare you ask me to integrate a singular Gaussian!";
∫ ω_ . E[L_] d(vs_List) := Module[{n, Q, Δ, G, a, b, m, m1, $m}, Clear[Z];
n = Length@vs;
Q = Table[q_{vs[[a]], vs[[b]] [L], {a, n}, {b, n}];
If[(Δ = CF@Det[-Q]) == 0, Message[Integrate::sing]; Return[]];
G = CF[-Inverse[Q] / 2];
Z[] = Z[0] = CF[L - Sum[Q[[a, b]] θ_{vs[[a]], vs[[b]]}, {a, n}, {b, n}] / 2];
Z[m_, a_] := Z[m, a] = CF@D[Z[m], vs[[a]];
Z[m_, a_, b_] /; a ≤ b := Z[m, a, b] = CF@D[Z[m, a], vs[[b]];
Z[m_, a_, b_] /; a > b := Z[m, b, a];
For[$m = m = 0, m ≤ 2 $m, ++m,
Z[m + 1] = CF@Sum[Sum[If[G[[a, b]] == 0, 0,
G[[a, b]] / (m + 1) (Z[m, a, b] + Sum[Z[m1, a] Z[m - m1, b], {m1, 0, m})]],
{a, n}], {b, n}];
If[!(Z[m + 1] == 0), $m = m + 1; Z[] += Z[m + 1]];
];
PowerExpand@Factor[ω Δ^{-1/2}] E[CF[E_{vs→0}[Z[]]]];
Protect[Integrate];

```

```

In[*]:= Assuming[λ > 0, ∫_{-∞}^∞ Exp[-λ x_1^2 / 2] d{x_1}

```

Out[*]=

$$\frac{\sqrt{2\pi}}{\sqrt{\lambda}}$$

```

In[*]:= ∫ E[-λ x_1^2 / 2] d{x_1}

```

Out[*]=

$$\frac{E[0]}{\sqrt{\lambda}}$$

```

In[*]:= ∫ E[i λ x_1^2 / 2] d{x_1}

```

Out[*]=

$$\frac{(-1)^{1/4} E[0]}{\sqrt{\lambda}}$$

$$\text{In[*]} := \int \mathbb{E} \left[-i \lambda \mathbf{x}_1^2 / 2 \right] \mathbb{d} \{ \mathbf{x}_1 \}$$

$$\text{Out[*]} = \frac{(-1)^{3/4} \mathbb{E} [\theta]}{\sqrt{\lambda}}$$

$$\text{In[*]} := \int \mathbb{E} \left[\frac{i}{2} \{ \mathbf{x}_1, \mathbf{x}_2 \} \cdot \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{b} & \mathbf{c} \end{pmatrix} \cdot \{ \mathbf{x}_1, \mathbf{x}_2 \} \right] \mathbb{d} \{ \mathbf{x}_1, \mathbf{x}_2 \}$$

$$\text{Out[*]} = \frac{\mathbb{E} [\theta]}{\sqrt{\mathbf{b}^2 - \mathbf{a} \mathbf{c}}}$$

$$\text{In[*]} := \int \mathbb{E} \left[-\lambda \mathbf{x}_1^2 / 2 + \xi \mathbf{x}_1 \right] \mathbb{d} \{ \mathbf{x}_1 \}$$

$$\text{Out[*]} = \frac{\mathbb{E} \left[\frac{\xi^2}{2 \lambda} \right]}{\sqrt{\lambda}}$$

$$\text{In[*]} := \int \mathbb{E} \left[-\frac{1}{2} \{ \mathbf{x}_1, \mathbf{x}_2 \} \cdot \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{b} & \mathbf{c} \end{pmatrix} \cdot \{ \mathbf{x}_1, \mathbf{x}_2 \} + \{ \xi_1, \xi_2 \} \cdot \{ \mathbf{x}_1, \mathbf{x}_2 \} \right] \mathbb{d} \{ \mathbf{x}_1, \mathbf{x}_2 \}$$

$$\text{Out[*]} = \frac{\mathbb{E} \left[\frac{\mathbf{c} \xi_1^2 - 2 \mathbf{b} \xi_1 \xi_2 + \mathbf{a} \xi_2^2}{2 (-\mathbf{b}^2 + \mathbf{a} \mathbf{c})} \right]}{\sqrt{-\mathbf{b}^2 + \mathbf{a} \mathbf{c}}}$$

$$\text{In[*]} := \mathbf{I1} = \int \mathbb{E} \left[-\frac{1}{2} \{ \mathbf{x}_1, \mathbf{x}_2 \} \cdot \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{b} & \mathbf{c} \end{pmatrix} \cdot \{ \mathbf{x}_1, \mathbf{x}_2 \} + \{ \xi_1, \xi_2 \} \cdot \{ \mathbf{x}_1, \mathbf{x}_2 \} \right] \mathbb{d} \{ \mathbf{x}_1 \}$$

$$\text{Out[*]} = \frac{\mathbb{E} \left[-\frac{(-\mathbf{b}^2 + \mathbf{a} \mathbf{c}) \mathbf{x}_2^2}{2 \mathbf{a}} + \frac{\xi_1^2}{2 \mathbf{a}} + \frac{\mathbf{x}_2 (-\mathbf{b} \xi_1 + \mathbf{a} \xi_2)}{\mathbf{a}} \right]}{\sqrt{\mathbf{a}}}$$

$$\text{In[*]} := \int \mathbf{I1} \mathbb{d} \{ \mathbf{x}_2 \}$$

$$\text{Out[*]} = \frac{\mathbb{E} \left[\frac{\mathbf{c} \xi_1^2 - 2 \mathbf{b} \xi_1 \xi_2 + \mathbf{a} \xi_2^2}{2 (-\mathbf{b}^2 + \mathbf{a} \mathbf{c})} \right]}{\sqrt{-\mathbf{b}^2 + \mathbf{a} \mathbf{c}}}$$

$$\text{In[*]} := \int \mathbb{E} \left[\xi \mathbf{x} + \eta \mathbf{y} + \mathbf{z} (\mathbf{x} - \mathbf{y}) + \mathbf{x}^2 \right] \mathbb{d} \{ \mathbf{x}, \mathbf{z} \}$$

$$\text{Out[*]} = -2 i \pi \mathbb{E} [\mathbf{y} (\mathbf{y} + \eta + \xi)]$$

Integration of ϵ Series

$$\text{In[*]} := \int \mathbb{E} \left[-x^2 / 2 + \epsilon x^3 / 6 + 0[\epsilon]^7 \right] d\{x\}$$

Out[*]=

$$\mathbb{E} \left[\epsilon \text{Series} \left[0, 0, \frac{5}{24}, 0, \frac{5}{16}, 0, \frac{1105}{1152} \right] \right]$$

$$\text{In[*]} := \int \mathbb{E} \left[-\phi^2 / 2 + \epsilon \phi^4 / 24 + 0[\epsilon]^7 \right] d\{\phi\}$$

Out[*]=

$$\mathbb{E} \left[\epsilon \text{Series} \left[0, \frac{1}{8}, \frac{1}{12}, \frac{11}{96}, \frac{17}{72}, \frac{619}{960}, \frac{709}{324} \right] \right]$$

$$\text{In[*]} := \int \mathbb{E} \left[p x + \epsilon p^2 x + 0[\epsilon]^5 \right] d\{p, x\}$$

Out[*]=

$$-i \mathbb{E} \left[\epsilon \text{Series} \left[0, 0, 0, 0, 0 \right] \right]$$

$$\text{In[*]} := \int \mathbb{E} \left[p x + B p x^2 + \epsilon B^{-1} p^2 x + 0[\epsilon]^2 \right] d\{p, x\}$$

$$\gg \{ \{0, 1\}, \{1, 0\} \}$$

$$\gg \epsilon \text{Series} \left[-2 B x + B p x^2 - 2 B^2 p x^3, -\frac{2 p}{B} + \frac{p^2 x}{B} - 5 p^2 x^2 \right]$$

$$\gg \epsilon \text{Series} \left[-2 B x + 4 B^2 x^2 + B p x^2 - 2 B^2 p x^3 + 5 B^3 p x^4, -\frac{2 p}{B} + 14 p x + \frac{p^2 x}{B} - 5 p^2 x^2 \right]$$

$$\gg \epsilon \text{Series} \left[-2 B x + 4 B^2 x^2 + B p x^2 - \frac{32 B^3 x^3}{3} - 2 B^2 p x^3 + 5 B^3 p x^4 - 14 B^4 p x^5, -6 - \frac{2 p}{B} + 14 p x + \frac{p^2 x}{B} - 5 p^2 x^2 \right]$$

$$\gg \epsilon \text{Series} \left[-2 B x + 4 B^2 x^2 + B p x^2 - \frac{32 B^3 x^3}{3} - 2 B^2 p x^3 + 32 B^4 x^4 + 5 B^3 p x^4 - 14 B^4 p x^5 + 42 B^5 p x^6, \right. \\ \left. -6 - \frac{2 p}{B} + 14 p x + \frac{p^2 x}{B} - 5 p^2 x^2 \right]$$

$$\gg \epsilon \text{Series} \left[-2 B x + 4 B^2 x^2 + B p x^2 - \frac{32 B^3 x^3}{3} - 2 B^2 p x^3 + 32 B^4 x^4 + 5 B^3 p x^4 - \right. \\ \left. \frac{512 B^5 x^5}{5} - 14 B^4 p x^5 + 42 B^5 p x^6 - 132 B^6 p x^7, -6 - \frac{2 p}{B} + 14 p x + \frac{p^2 x}{B} - 5 p^2 x^2 \right]$$

Out[*]=

\$Aborted

```
In[*]:= Block[{$\pi = Total@Select[MonomialList[#, {\epsilon, x, p}],
    mon \mapsto And[
        Exponent[mon, \epsilon] \le 2,
        Exponent[mon, x] == Exponent[mon, p]
    ] &},
    \int \mathbb{E}[p x + a x^2 p + \epsilon b x^3 p^3] d\{p, x\}
```

Out[*]=

$$-\frac{i \mathbb{E}[-6 b \epsilon + 342 b^2 \epsilon^2]}{2 \pi}$$

```
In[*]:= MatrixForm@Table[
    \int \mathbb{E}[x_1 p_2 + x_2 p_3 + x_3 p_1 + \xi_i x_i + \pi_j p_j] d\{x_1, x_2, x_3, p_1, p_2, p_3\},
    {i, 3}, {j, 3}]
```

Out[*]//MatrixForm=

$$\begin{pmatrix} -8 i \pi^3 \mathbb{E}[\emptyset] & -8 i \pi^3 \mathbb{E}[-\pi_2 \xi_1] & -8 i \pi^3 \mathbb{E}[\emptyset] \\ -8 i \pi^3 \mathbb{E}[\emptyset] & -8 i \pi^3 \mathbb{E}[\emptyset] & -8 i \pi^3 \mathbb{E}[-\pi_3 \xi_2] \\ -8 i \pi^3 \mathbb{E}[-\pi_1 \xi_3] & -8 i \pi^3 \mathbb{E}[\emptyset] & -8 i \pi^3 \mathbb{E}[\emptyset] \end{pmatrix}$$