

Pensieve header: Mathematica notebook for the 2-variable perturbed Alexander invariant.

## Initialization

```
In[*]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\HigherRank"];
Once[<< KnotTheory` ; << Rot.m];
T3 = T1 T2;
( $\alpha$ _+)^+ :=  $\alpha$ ^{++}; (* this is for cosmetic reasons only *)
 $\delta_{i,j}$  := If[i === j, 1, 0];
```

## The Target Program

```
R1[{s_, i_, j_}] := TBD;
 $\theta$ [{s0_, i0_, j0_}, {s1_, i1_, j1_}] := TBD;
CF[ $\mathcal{E}$ ] := Factor@Together[ $\mathcal{E}$ ];
 $\lambda$ [K_] := Module[{Cs,  $\varphi$ , n, A, s, i, j, k,  $\Delta$ , G, gEval, c,  $\lambda$ 1},
  {Cs,  $\varphi$ } = Rot[K]; n = Length[Cs];
  A = IdentityMatrix[2 n + 1];
  Cases[Cs, {s_, i_, j_}  $\Rightarrow$  (A[[{i, j}, {i + 1, j + 1}]] += ( $\begin{matrix} -T^s & T^s & -1 \\ \mathbf{0} & & -1 \end{matrix}$ ))];
   $\Delta$  = T^{(-Total[ $\varphi$ ]-Total[Cs[[All,1]])/2} Det[A];
  G = Inverse[A];
  gEval[ $\mathcal{E}$ ] := CF[ $\mathcal{E}$  /. {g $\nu$ _,  $\alpha$ _,  $\beta$ _  $\Rightarrow$  (G[[ $\alpha$ ,  $\beta$ ]] /. T  $\rightarrow$  T $\nu$ )}];
   $\lambda$ 1 = Plus[
    Sum[R1[Cs[[k]], {k, n}],
    Sum[ $\theta$ [Cs[[k0]], Cs[[k1]], {k0, n}, {k1, n}],
    Sum[3 / 2 - g1,k,k - g2,k,k - g3,k,k, {k, 2 n}];
  { $\Delta$ , ( $\Delta^2$  /. T  $\rightarrow$  T1) ( $\Delta^2$  /. T  $\rightarrow$  T2) ( $\Delta^2$  /. T  $\rightarrow$  T3)  $\lambda$ 1} // gEval];
```

## Step-by-step Run-Through

```
In[*]:= CF[ $\mathcal{E}$ ] := Factor@Together[ $\mathcal{E}$ ];
```

```
In[*]:= K = {
  PD[X[4, 2, 5, 1], X[2, 6, 3, 5], X[6, 4, 7, 3]],
  Knot[6, 2]
}[[1]]
```

```
Out[*]= PD[X[4, 2, 5, 1], X[2, 6, 3, 5], X[6, 4, 7, 3]]
```

```
In[*]:= {Cs, φ} = Rot[K]
Out[*]=
{{{1, 1, 4}, {1, 5, 2}, {1, 3, 6}}, {0, 0, 0, -1, 0, 0}}

In[*]:= n = Length[Cs];
A = IdentityMatrix[2 n + 1];
Cases[Cs, {s_, i_, j_} => (A[[{i, j}, {i + 1, j + 1}]] += (

$$\begin{pmatrix} -T^s & T^s - 1 \\ 0 & -1 \end{pmatrix}$$

))];

In[*]:= A // MatrixForm
```

```
Out[*]//MatrixForm=

$$\begin{pmatrix} 1 & -T & 0 & 0 & -1+T & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -T & 0 & 0 & -1+T \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1+T & 0 & 1 & -T & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

```

```
In[*]:= Δ = T^(-Total[φ]-Total[Cs[[All,1]]])/2 Det[A]
Out[*]=

$$\frac{1 - T + T^2}{T}$$

```

```
In[*]:= G = Inverse[A];
G // MatrixForm
```

```
Out[*]//MatrixForm=

$$\begin{pmatrix} 1 & \frac{T-T^2+T^3}{1-T+T^2} & 1 & \frac{T-T^2+T^3}{1-T+T^2} & 1 & \frac{T-T^2+T^3}{1-T+T^2} & 1 \\ 0 & 1 & \frac{1}{1-T+T^2} & \frac{T}{1-T+T^2} & \frac{T}{1-T+T^2} & \frac{T^2}{1-T+T^2} & 1 \\ 0 & 0 & \frac{1}{1-T+T^2} & \frac{T}{1-T+T^2} & \frac{T}{1-T+T^2} & \frac{T^2}{1-T+T^2} & 1 \\ 0 & 0 & \frac{1-T}{1-T+T^2} & \frac{1}{1-T+T^2} & \frac{1}{1-T+T^2} & \frac{T}{1-T+T^2} & 1 \\ 0 & 0 & \frac{1-T}{1-T+T^2} & \frac{T-T^2}{1-T+T^2} & \frac{1}{1-T+T^2} & \frac{T}{1-T+T^2} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

```

```
In[*]:= gEval[ε_] := CF[ε /. {gv,α,β => (G[[α, β]] /. T -> Tv})}];
```

## The g-Rules

```
In[*]:= gRuless,i,j := {
  gv,i,β => δi,β + Tvs gv,i+1,β + (1 - Tvs) gv,j+1,β, gv,j,β => δj,β + gv,j+1,β,
  gv,α,i => Tv-s (gv,α,i+1 - δα,i+1), gv,α,j => gv,α,j+1 - (1 - Tvs) gv,α,i - δα,j+1
}
```

In[\*]:= (gs = Table[{g<sub>1,α,β</sub>, g<sub>2,α,β</sub>, g<sub>3,α,β</sub>}, {α, 2 n + 1}, {β, 2 n + 1}][[1 ;; 2, 1 ;; 2, 1 ;; 2]])

Out[\*]= {{{g<sub>1,1,1</sub>, g<sub>2,1,1</sub>}, {g<sub>1,1,2</sub>, g<sub>2,1,2</sub>}}, {{g<sub>1,2,1</sub>, g<sub>2,2,1</sub>}, {g<sub>1,2,2</sub>, g<sub>2,2,2</sub>}}}

In[\*]:= Cs

Out[\*]= {{1, 1, 4}, {1, 5, 2}, {1, 3, 6}}

In[\*]:= (gs /. gRules<sub>Sequence@@Cs[[1]]</sub>)[[1 ;; 2, 1 ;; 2, 1 ;; 2]]

Out[\*]= { {{1 + T<sub>1</sub> g<sub>1,2,1</sub> + (1 - T<sub>1</sub>) g<sub>1,5,1</sub>, 1 + T<sub>2</sub> g<sub>2,2,1</sub> + (1 - T<sub>2</sub>) g<sub>2,5,1</sub>},  
 {T<sub>1</sub> g<sub>1,2,2</sub> + (1 - T<sub>1</sub>) g<sub>1,5,2</sub>, T<sub>2</sub> g<sub>2,2,2</sub> + (1 - T<sub>2</sub>) g<sub>2,5,2</sub>}},  
 {{ $\frac{-1 + g_{1,2,2}}{T_1}$ ,  $\frac{-1 + g_{2,2,2}}{T_2}$ }, {g<sub>1,2,2</sub>, g<sub>2,2,2</sub>}}}

In[\*]:= gEval[gs - (gs /. Flatten@Table[gRules<sub>Sequence@@c</sub>, {c, Cs}])] ]

Out[\*]= {{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}},  
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}},  
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}},  
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}},  
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}},  
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}},  
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}

## Fixing $\theta$ .

Sum[g<sub>3,</sub>]

---

## Invariance of $y_{\alpha\beta\gamma}$ under remote R2s

```
In[ ]:= Clear[i, j];
Cs = {{1, i, j}, {-1, i+, j+}}
Z = Module[{s, i, j}, Sum[{s, i, j} = c;
  g3,α,i (S^s (T^s - 1) g1,j+,β g2,i+,γ + (S^s - 1) T^s g1,i+,β g2,j+,γ + (S^s + T^s - 2 T S^s) g1,j+,β g2,j+,γ),
  {c, Cs}]]]
Simplify[Z //. gRules1,i,j ∪ gRules-1,i+,j+ /. _If → 0]

Out[ ]:=
{{1, i, j}, {-1, i+, j+}}

Out[ ]:=
(S (-1 + T) g1,j+,β g2,i+,γ + (-1 + S) T g1,i+,β g2,j+,γ + (S + T - 2 S T) g1,j+,β g2,j+,γ) g3,α,i +
  ⎛
  ⎜
  ⎝
  (-1 +  $\frac{1}{T}$ ) g1,j+,β g2,i+,γ +  $\frac{(-1 + \frac{1}{S}) g1,i+,β g2,j+,γ}{T}$  +  $(\frac{1}{S} + \frac{1}{T} - \frac{2}{S T}) g1,j+,β g2,j+,γ$ 
  ⎞
  ⏟
  g3,α,i

Out[ ]:=
0
```

## Invariance of $y_{\alpha\beta\gamma}$ under remote R3s

```
In[ ]:= Clear[i, j, k];
Cs = {{1, i, j}, {1, i+, k}, {1, j+, k+}}
Z = Module[{s, i, j}, Sum[{s, i, j} = c;
  g3,α,i (S^s (T^s - 1) g1,j+,β g2,i+,γ + (S^s - 1) T^s g1,i+,β g2,j+,γ + (S^s + T^s - 2 T S^s) g1,j+,β g2,j+,γ),
  {c, Cs}]]]
lhs = Simplify[Z //. gRules1,i,j ∪ gRules1,i+,k ∪ gRules1,j+,k+ /. _If → 0]

Out[ ]:=
{{1, i, j}, {1, i+, k}, {1, j+, k+}}

Out[ ]:=
(S (-1 + T) g1,j+,β g2,i+,γ + (-1 + S) T g1,i+,β g2,j+,γ + (S + T - 2 S T) g1,j+,β g2,j+,γ) g3,α,i +
(S (-1 + T) g1,k+,β g2,i+,γ + (-1 + S) T g1,i+,β g2,k+,γ + (S + T - 2 S T) g1,k+,β g2,k+,γ) g3,α,i+ +
(S (-1 + T) g1,k+,β g2,j+,γ + (-1 + S) T g1,j+,β g2,k+,γ + (S + T - 2 S T) g1,k+,β g2,k+,γ) g3,α,j+

Out[ ]:=

$$\frac{1}{S^2 T^2} \left( (S (-1 + T) (T g1,j+,β - (-1 + T) g1,k+,β) (S g2,i+,γ - (-1 + S) g2,k+,γ) + \right.$$


$$(-1 + S) T (T g1,i+,β - (-1 + T) g1,k+,β) (S g2,j+,γ - (-1 + S) g2,k+,γ) +$$


$$(S + T - 2 S T) (T g1,j+,β - (-1 + T) g1,k+,β) (S g2,j+,γ - (-1 + S) g2,k+,γ) \Big) g3,α,i+ +$$


$$S T ((-1 + S) T g1,i+,β g2,k+,γ + g1,k+,β (S (-1 + T) g2,i+,γ + (S + T - 2 S T) g2,k+,γ)) g3,α,i+ +$$


$$S T ((-1 + S) T g1,j+,β g2,k+,γ + g1,k+,β (S (-1 + T) g2,j+,γ + (S + T - 2 S T) g2,k+,γ)) g3,α,j+ \Big)$$

```

```

In[ ]:= Clear[i, j, k];
Cs = {{1, j, k}, {1, i, k}, {1, i, j}}
Z = Module[{s, i, j}, Sum[{s, i, j} = c;
  g3,α,i (S^s (T^s - 1) g1,j+,β g2,i+,γ + (S^s - 1) T^s g1,i+,β g2,j+,γ + (S^s + T^s - 2 T S^s) g1,j+,β g2,j+,γ),
  {c, Cs}]]
rhs = Simplify[Z //. gRules1,j,k ∪ gRules1,i,k ∪ gRules1,i,j /. _If → 0]

Out[ ]:=
{{1, j, k}, {1, i, k}, {1, i, j}}

Out[ ]:=
((S + T - 2 S T) g1,k+,β g2,k+,γ + (-1 + S) T g1,i+,β g2,k+,γ + S (-1 + T) g1,k+,β g2,i+,γ) g3,α,i +
(S (-1 + T) g1,k+,β g2,j+,γ + (-1 + S) T g1,j+,β g2,k+,γ + (S + T - 2 S T) g1,k+,β g2,k+,γ) g3,α,j +
(S (-1 + T) g1,j+,β g2,i+,γ + (-1 + S) T g1,i+,β g2,j+,γ + (S + T - 2 S T) g1,j+,β g2,j+,γ) g3,α,i

Out[ ]:=
1
S^2 T^2 (S T ((-1 + S) T g1,i+,β g2,j+,γ + g1,j+,β (S (-1 + T) g2,i+,γ + (S + T - 2 S T) g2,j+,γ)) g3,α,i++ +
(S (-1 + T) g1,k+,β (S g2,i+,γ - (-1 + S) g2,j+,γ) +
(-1 + S) T (T g1,i+,β - (-1 + T) g1,j+,β) g2,k+,γ + (S + T - 2 S T) g1,k+,β g2,k+,γ) g3,α,i++ +
((-1 + S) T g1,j+,β g2,k+,γ + g1,k+,β (S (-1 + T) g2,j+,γ + (S + T - 2 S T) g2,k+,γ))
((-1 + S T) g3,α,i++ + S T g3,α,j++))

In[ ]:= Simplify[lhs == rhs]

Out[ ]:=
True

```

## Setting up $R_1$

```

In[ ]:= gs = Flatten@Table[{gp,i,i, gp,i,j, gp,j,i, gp,j,j}, {p, 3}];
bas =
  {1} ∪ gs ∪ Flatten@Table[gs[[p]] × gs[[q]], {p, Length[gs]}, {q, p, Length[gs]}] ∪ Flatten@
  Table[gs[[p]] × gs[[q]] × gs[[r]], {p, Length[gs]}, {q, p, Length[gs]}, {r, q, Length[gs]}] ∪
  {Yi,i,i, Yi,i,j, Yi,j,i, Yi,j,j, Yj,i,i, Yj,i,j, Yj,j,i, Yj,j,j};
len = Length[bas]
vars = Table[Cp = 0; Cp = .; Cp, {p, 2 len}];
R1[1, i_, j_] := Evaluate[Sum[Cp bas[[p]], {p, len}]];
R1[-1, i_, j_] := Evaluate[Sum[C1len+p bas[[p]], {p, Length[bas]}]];

Out[ ]:=
463

```

## Solving R2b

```
In[*]:= Clear[i, j];
eqn = Expand[R1[1, i, j] + R1[-1, i+, j+] /.
  gRules_{1,i,j} U gRules_{-1,i+,j+} U yRules_{1,i,j} U yRules_{-1,i+,j+} /. _If -> 0]
```

Out[\*]=

$$C_1 + C_{464} + C_2 g_{1,i^{++},i^{++}} - C_{457} g_{1,i^{++},i^{++}} + 2 S C_{457} g_{1,i^{++},i^{++}} + \frac{S C_{457} g_{1,i^{++},i^{++}}}{T^2} + \frac{C_{457} g_{1,i^{++},i^{++}}}{T} - \frac{3 S C_{457} g_{1,i^{++},i^{++}}}{T} + \dots + \frac{C_{922} y_{j^{++},j^{++},i^{++}}}{T} - \frac{C_{922} y_{j^{++},j^{++},i^{++}}}{S T} + S C_{925} y_{j^{++},j^{++},i^{++}} + C_{926} y_{j^{++},j^{++},i^{++}} - S C_{926} y_{j^{++},j^{++},i^{++}} + C_{463} y_{j^{++},j^{++},j^{++}} + C_{922} y_{j^{++},j^{++},j^{++}} - \frac{C_{922} y_{j^{++},j^{++},j^{++}}}{S T} + C_{926} y_{j^{++},j^{++},j^{++}}$$

Full expression not available (original memory size: 14 MB)

```
In[*]:= gys = Cases[eqn, g_., _., _] U Cases[eqn, y_., _]
```

Out[\*]=

- {g<sub>1,i<sup>++</sup>,i<sup>++</sup></sub>, g<sub>1,i<sup>++</sup>,j<sup>++</sup></sub>, g<sub>1,j<sup>++</sup>,i<sup>++</sup></sub>, g<sub>1,j<sup>++</sup>,j<sup>++</sup></sub>, g<sub>2,i<sup>++</sup>,i<sup>++</sup></sub>, g<sub>2,i<sup>++</sup>,j<sup>++</sup></sub>,
- g<sub>2,j<sup>++</sup>,i<sup>++</sup></sub>, g<sub>2,j<sup>++</sup>,j<sup>++</sup></sub>, g<sub>3,i<sup>++</sup>,i<sup>++</sup></sub>, g<sub>3,i<sup>++</sup>,j<sup>++</sup></sub>, g<sub>3,j<sup>++</sup>,i<sup>++</sup></sub>, g<sub>3,j<sup>++</sup>,j<sup>++</sup></sub>, y<sub>i<sup>++</sup>,i<sup>++</sup>,i<sup>++</sup></sub>,
- y<sub>i<sup>++</sup>,i<sup>++</sup>,j<sup>++</sup></sub>, y<sub>i<sup>++</sup>,j<sup>++</sup>,i<sup>++</sup></sub>, y<sub>i<sup>++</sup>,j<sup>++</sup>,j<sup>++</sup></sub>, y<sub>j<sup>++</sup>,i<sup>++</sup>,i<sup>++</sup></sub>, y<sub>j<sup>++</sup>,i<sup>++</sup>,j<sup>++</sup></sub>, y<sub>j<sup>++</sup>,j<sup>++</sup>,i<sup>++</sup></sub>, y<sub>j<sup>++</sup>,j<sup>++</sup>,j<sup>++</sup></sub>}

```
In[*]:= eqns = CoefficientRules[eqn, gys] /. (_ -> c_) :-> (c == 0)
```

Out[\*]=

$$\left\{ \begin{aligned} C_4 + C_{467} - C_{470} + \frac{C_{470}}{T} + C_{472} + \frac{C_{472}}{T^2} - \frac{2 C_{472}}{T} - C_{473} + \frac{C_{473}}{T^3} - \frac{3 C_{473}}{T^2} + \frac{3 C_{473}}{T} &= 0, C_7 + \frac{C_{470}}{T} + \frac{2 C_{472}}{T^2} - \frac{2 C_{472}}{T} + \frac{3 C_{473}}{T^3} - \frac{6 C_{473}}{T^2} + \frac{3 C_{473}}{T} = 0, \\ C_{13} - 3 C_{467} + 3 T C_{467} + 6 C_{470} - \frac{3 C_{470}}{T} - 3 T C_{470} - 9 C_{472} - \frac{3 C_{472}}{T^2} + \frac{9 C_{472}}{T} + 3 T C_{472} + 12 C_{473} - \frac{3 C_{473}}{T^3} + \frac{12 C_{473}}{T^2} - \frac{18 C_{473}}{T} - 3 T C_{473} + T C_{476} + \\ C_{478} - T C_{478} - 2 C_{479} + \frac{C_{479}}{T} + T C_{479} + C_{486} - T C_{486} - 2 C_{488} + \frac{C_{488}}{T} + T C_{488} + 3 C_{489} + \frac{C_{489}}{T^2} - \frac{3 C_{489}}{T} - T C_{489} &= 0, \dots + 457 \dots, \\ C_{462} + S C_{921} - \frac{C_{921}}{T} + C_{922} - S C_{922} + \frac{C_{922}}{T} - \frac{C_{922}}{S T} + S C_{925} + C_{926} - S C_{926} &= 0, C_{463} + C_{922} - \frac{C_{922}}{S T} + C_{926} = 0, C_1 + C_{464} = 0 \end{aligned} \right\}$$

Full expression not available (original memory size: 3.2 MB)

```
In[*]:= Short[{sol} = Solve[eqns, vars], 5]
```

Solve: Equations may not give solutions for all "solve" variables.

Out[\*]//Short=

$$\left\{ \begin{aligned} C_2 &\rightarrow -C_{465} - \left(-1 + \frac{1}{T}\right) C_{468} - \frac{(-S - T + 3 S T + T^2 - 2 S T^2) C_{920}}{S T^2}, \\ C_3 &\rightarrow -C_{466} - \left(-1 + \frac{1}{T}\right) C_{469} - \frac{(1 - 2 T + T^2) C_{471}}{T^2}, \\ C_4 &\rightarrow -C_{467} - \left(-1 + \frac{1}{T}\right) C_{470} - \frac{(1 - 2 T + T^2) C_{472}}{T^2} - \left(-1 + \frac{1}{T^3} - \frac{3}{T^2} + \frac{3}{T}\right) C_{473}, \\ C_5 &\rightarrow -\frac{C_{468}}{T} - \frac{(-S - T + 2 S T) C_{920}}{S T^2}, \ll 456 \gg, \\ C_{462} &\rightarrow -\frac{(-1 + S T) C_{921}}{T} - \frac{(-1 + S + S T - S^2 T) C_{922}}{S T} - S C_{925} - (1 - S) C_{926}, \\ C_{463} &\rightarrow -\frac{(-1 + S T) C_{922}}{S T} - C_{926}, C_{464} \rightarrow -C_1 \end{aligned} \right\}$$

```
In[*]:= Short[sol /. (v_ -> val_) :-> (v = CF[val])]
Out[*]//Short=
{ - $\frac{S T^2 c_{465} + \langle\langle 11 \rangle\rangle}{S T^2}$ ,  $-\frac{T^2 c_{466} + \langle\langle 6 \rangle\rangle + T^2 c_{471}}{T^2}$ ,  $-\frac{\langle\langle 1 \rangle\rangle}{T^3}$ ,
   $\langle\langle 457 \rangle\rangle$ ,  $-\frac{\langle\langle 1 \rangle\rangle}{S \langle\langle 1 \rangle\rangle \langle\langle 1 \rangle\rangle}$ ,  $-\frac{-c_{922} + \langle\langle 1 \rangle\rangle + S T c_{\langle\langle 3 \rangle\rangle}}{S T}$ ,  $-c_1$  }
```

```
In[*]:= Last@R1[1, i, j]
Out[*]=

$$\frac{(-c_{922} + S T c_{922} + S T c_{926}) y_{j,j,j}}{S T}$$

```

```
In[*]:= Short[R1[1, i, j], 10]
Out[*]//Short=
<<1>>
```

### Solving R3

```
In[*]:= Clear[i, j, k];
lhs = Expand[R1[1, i, j] + R1[1, i+, k] + R1[1, j+, k+] // . gRules1,i,j U
  gRules1,i+,k U gRules1,j+,k+ U yRules1,i,j U yRules1,i+,k U yRules1,j+,k+ /. _If -> 0];
rhs = Expand[R1[1, j, k] + R1[1, i, k+] + R1[1, i+, j+] // . gRules1,j,k U
  gRules1,i,k+ U gRules1,i+,j+ U yRules1,j,k U yRules1,i,k+ U yRules1,i+,j+ /. _If -> 0];
eqn = lhs - rhs
```

```
Out[*]=

$$-2 c_{922} + \frac{c_{922}}{S} + \frac{c_{922}}{T} - c_{468} g_{1,i+,j++} + \frac{c_{468} g_{1,i+,j++}}{T} - \frac{c_{920} g_{1,i+,j++}}{T^2} + \frac{2 c_{920} g_{1,i+,j++}}{T} -$$


$$\frac{c_{920} g_{1,i+,j++}}{S T} - c_{469} g_{1,i+,i++} g_{1,i+,j++} + \dots 29\ 005 \dots + \frac{c_{926} y_{k+,j++,k++}}{S} + \frac{c_{926} y_{k+,j++,k++}}{T} - \frac{2 c_{926} y_{k+,j++,k++}}{S T} -$$


$$c_{926} y_{k+,j++,k++} + \frac{c_{926} y_{k+,j++,k++}}{T} - \frac{c_{926} y_{k+,k++,i++}}{S^2} + \frac{c_{926} y_{k+,k++,i++}}{S} - c_{926} y_{k+,k++,j++} + \frac{c_{926} y_{k+,k++,j++}}{S}$$

Full expression not available (original memory size: 18.7 MB)
```

```
In[*]:= Short[vars = Union@Cases[eqn, c_, ∞]]
Out[*]//Short=
{c465, c466, c467, c468, c469, c470, c471, c472, <<446>>, c919, c920, c921, c922, c923, c924, c925, c926}
```

```
In[*]:= gys = Cases[eqn, g_., ∞] U Cases[eqn, y_., ∞]
Out[*]=
{g1,i+,i++, g1,i+,j++, g1,i+,k++, g1,j+,i++, g1,j+,j++, g1,j+,k++, g1,k+,i++, g1,k+,j++,
  g1,k+,k++, g2,i+,i++, g2,i+,j++, g2,i+,k++, g2,j+,i++, g2,j+,j++, g2,j+,k++, g2,k+,i++,
  g2,k+,j++, g2,k+,k++, g3,i+,i++, g3,i+,j++, g3,i+,k++, g3,j+,i++, g3,j+,j++, g3,j+,k++, g3,k+,i++,
  g3,k+,j++, g3,k+,k++, yi+,i+,j++, yi+,i+,k++, yi+,j+,i++, yi+,j+,j++, yi+,j+,k++, yi+,k+,i++,
  yi+,k+,j++, yi+,k+,k++, yj+,i+,i++, yj+,i+,j++, yj+,i+,k++, yj+,j+,i++, yj+,j+,k++, yj+,k+,i++, yj+,k+,k++,
  yk+,i+,i++, yk+,i+,j++, yk+,i+,k++, yk+,j+,i++, yk+,j+,j++, yk+,j+,k++, yk+,k+,i++, yk+,k+,j++}
```

In[\*]:= Short[eqns = CoefficientRules[eqn, gys] /. (\_ -> c\_) :-> (c == 0)]

Out[\*]//Short=

$$\left\{ -c_{470} + \frac{c_{470}}{T} == 0, -c_{470} + T c_{470} == 0, \ll 2799 \gg, -c_{926} + \frac{c_{926}}{S} == 0, -2 c_{922} + \frac{c_{922}}{S} + \frac{c_{922}}{T} == 0 \right\}$$

In[\*]:= Short[{sol} = Solve[eqns, vars], 5]

 Solve: Equations may not give solutions for all "solve" variables. 

Out[\*]//Short=

$$\left\{ \left\{ c_{466} \rightarrow 0, c_{467} \rightarrow 0, c_{468} \rightarrow 0, c_{469} \rightarrow 0, c_{470} \rightarrow 0, c_{471} \rightarrow 0, c_{472} \rightarrow 0, c_{473} \rightarrow 0, \right. \right.$$

$$c_{477} \rightarrow -\frac{T c_{475}}{1+T} - \frac{T c_{490}}{1+T}, c_{478} \rightarrow 0, c_{479} \rightarrow 0, c_{480} \rightarrow -\frac{(-1+T) c_{475}}{1+T} - \frac{(1-T) c_{490}}{T(1+T)},$$

$$c_{481} \rightarrow -\frac{(-1+T) c_{476}}{T}, c_{482} \rightarrow 0, c_{483} \rightarrow 0, c_{484} \rightarrow -T c_{465} - T c_{474}, \ll 402 \gg, c_{911} \rightarrow 0,$$

$$c_{912} \rightarrow 0, c_{913} \rightarrow -\frac{(-1+S) T c_{800}}{2(-1+ST)}, c_{914} \rightarrow 0, c_{915} \rightarrow 0, c_{916} \rightarrow 0, c_{917} \rightarrow -c_{804}, c_{918} \rightarrow 0,$$

$$\left. c_{919} \rightarrow 0, c_{920} \rightarrow 0, c_{921} \rightarrow 0, c_{922} \rightarrow 0, c_{923} \rightarrow 0, c_{924} \rightarrow 0, c_{925} \rightarrow 0, c_{926} \rightarrow 0 \right\}$$

In[\*]:= Short[sol /. (v\_ -> val\_) :-> (v = CF[val])]

Out[\*]//Short=

$$\left\{ 0, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{T(c_{475} + c_{490})}{1+T}, 0, 0, \ll 412 \gg, 0, -c_{804}, 0, 0, 0, 0, 0, 0, 0, 0, 0 \right\}$$

In[\*]:= Short[R1[1, i, j], 5]

Out[\*]//Short=

$$c_1 - c_{465} g_{1,i,i} - \frac{(-S T^2 c_{465} + S T^3 c_{465} + S T^3 c_{474} - S T^3 (c_{465} + c_{474}) + S T^4 (c_{465} + c_{474})) g_{1,j,i}}{S T^2}$$

$$\frac{\left( T^3 c_{475} - \frac{2 \ll 1 \gg (c_{475} + c_{490})}{1+T} + \frac{2 T^4 (c_{475} + c_{490})}{1+T} \right) g_{1,i,i} g_{1,j,i}}{T^2} -$$

$$T c_{476} g_{1,i,i}^2 g_{1,j,i} + \ll 190 \gg + S T c_{800} g_{2,j,j} g_{3,j,i} g_{3,j,j} - \frac{\ll 1 \gg}{S^3 T^3} +$$

$$\frac{S(-1+T) T c_{796} g_{1,j,i} g_{3,j,j}^2}{2(-1+ST)} + \frac{(-1+S) S T c_{800} g_{2,j,i} g_{3,j,j}^2}{2(-1+ST)} + S T c_{804} g_{3,j,i} g_{3,j,j}^2$$

In[\*]:= R1[1, i, j] // Simplify

Out[\*]=

$$c_1 - c_{465} g_{1,i,i} - \left( (-1+T^2) c_{465} + T^2 c_{474} \right) g_{1,j,i} - \frac{T((-1+3T) c_{475} + 2(-1+T) c_{490}) g_{1,i,i} g_{1,j,i}}{1+T}$$

$$T c_{476} g_{1,i,i}^2 g_{1,j,i} + \frac{T(c_{475} + c_{490}) g_{1,i,j} g_{1,j,i}}{1+T} - \frac{(-1+T) T((-1+2T) c_{475} + (-2+T) c_{490}) g_{1,j,i}^2}{1+T}$$

$$(-1+T) T c_{476} g_{1,i,i} g_{1,j,i}^2 + T(c_{465} + c_{474}) g_{1,j,j} + \frac{T(c_{475} + c_{490}) g_{1,i,i} g_{1,j,j}}{1+T} +$$

$$\frac{T(2(-1+T) c_{475} + (-3+T) c_{490}) g_{1,j,i} g_{1,j,j}}{1+T} - (-1+T) T c_{476} g_{1,j,i}^2 g_{1,j,j} +$$



$$\begin{aligned}
 & T C_{476} g_{1,j,i} g_{1,j,j}^2 - C_{499} g_{2,i,i} - \frac{S(-1+T) C_{549} g_{1,j,i} g_{2,i,i}}{-1+S} - T C_{506} g_{1,i,i} g_{1,j,i} g_{2,i,i} - \\
 & \frac{1}{2} (-1+T) T C_{506} g_{1,j,i}^2 g_{2,i,i} + \left( -\frac{T C_{505}}{-1+T} + \frac{S C_{549}}{-1+S} \right) g_{1,j,j} g_{2,i,i} - \frac{S(-1+T) C_{564} g_{1,j,i} g_{2,i,i}^2}{2(-1+S)} - \\
 & \left( (-1+S^2) C_{499} + S^2 C_{548} \right) g_{2,j,i} - \frac{(-1+S) T C_{505} g_{1,i,i} g_{2,j,i}}{-1+T} - \frac{(-1+S) T C_{506} g_{1,i,i}^2 g_{2,j,i}}{2(-1+T)} - \\
 & ((-1+S) T C_{505} + S(-1+T) C_{549}) g_{1,j,i} g_{2,j,i} - (-1+S) T C_{506} g_{1,i,i} g_{1,j,i} g_{2,j,i} + \\
 & S C_{549} g_{1,j,j} g_{2,j,i} - (-1+S) T C_{506} g_{1,j,i} g_{1,j,j} g_{2,j,i} + \frac{(-1+S) T C_{506} g_{1,j,j}^2 g_{2,j,i}}{2(-1+T)} - \\
 & \frac{S((-1+3S) C_{563} + 2(-1+S) C_{612}) g_{2,i,i} g_{2,j,i}}{1+S} - S C_{564} g_{1,i,i} g_{2,i,i} g_{2,j,i} - \\
 & S(-1+T) C_{564} g_{1,j,i} g_{2,i,i} g_{2,j,i} - S C_{568} g_{2,i,i}^2 g_{2,j,i} + \frac{S(C_{563} + C_{612}) g_{2,i,j} g_{2,j,i}}{1+S} - \\
 & \frac{(-1+S) S((-1+2S) C_{563} + (-2+S) C_{612}) g_{2,j,i}^2}{1+S} - \frac{1}{2} (-1+S) S C_{564} g_{1,i,i} g_{2,j,i}^2 - \\
 & \frac{1}{2} (-1+S) S C_{564} g_{1,j,j} g_{2,j,i}^2 - (-1+S) S C_{568} g_{2,i,i} g_{2,j,i}^2 + S(C_{499} + C_{548}) g_{2,j,j} + \\
 & \frac{(-1+S) T C_{505} - S(-1+T) C_{549}}{(-1+S)(-1+T)} g_{1,i,i} g_{2,j,j} + T C_{505} g_{1,j,i} g_{2,j,j} - \frac{1}{2} (-1+T) T C_{506} g_{1,j,i}^2 g_{2,j,j} + \\
 & T C_{506} g_{1,j,i} g_{1,j,j} g_{2,j,j} + \frac{S(C_{563} + C_{612}) g_{2,i,i} g_{2,j,j}}{1+S} + \frac{S(2(-1+S) C_{563} + (-3+S) C_{612}) g_{2,j,i} g_{2,j,j}}{1+S} - \\
 & S(-1+T) C_{564} g_{1,j,i} g_{2,j,i} g_{2,j,j} + S C_{564} g_{1,j,j} g_{2,j,i} g_{2,j,j} - (-1+S) S C_{568} g_{2,j,i}^2 g_{2,j,j} + \\
 & \frac{S(-1+T) C_{564} g_{1,j,i} g_{2,j,j}^2}{2(-1+S)} + S C_{568} g_{2,j,i} g_{2,j,j}^2 - C_{629} g_{3,i,i} - \frac{S(-1+T) T C_{751} g_{1,j,i} g_{3,i,i}}{-1+ST} - \\
 & T C_{636} g_{1,i,i} g_{1,j,i} g_{3,i,i} - \frac{1}{2} (-1+T) T C_{636} g_{1,j,i}^2 g_{3,i,i} + T \left( -\frac{C_{635}}{-1+T} + \frac{S C_{751}}{-1+ST} \right) g_{1,j,j} g_{3,i,i} - \\
 & T C_{647} g_{1,j,i} g_{2,i,i} g_{3,i,i} - \frac{(-1+S) S T C_{765} g_{2,j,i} g_{3,i,i}}{-1+ST} - \frac{(-1+S) T C_{647} g_{1,i,i} g_{2,j,i} g_{3,i,i}}{-1+T} - \\
 & (-1+S) T C_{647} g_{1,j,i} g_{2,j,i} g_{3,i,i} - S C_{662} g_{2,i,i} g_{2,j,i} g_{3,i,i} - \frac{1}{2} (-1+S) S C_{662} g_{2,j,i}^2 g_{3,i,i} + \\
 & S \left( -\frac{C_{657}}{-1+S} + \frac{T C_{765}}{-1+ST} \right) g_{2,j,j} g_{3,i,i} + \frac{S(-1+T) T C_{796} g_{1,j,i} g_{3,i,i}^2}{2-2ST} + \frac{(-1+S) S T C_{800} g_{2,j,i} g_{3,i,i}^2}{2-2ST} - \\
 & \left( (-1+S^2 T^2) C_{629} + S^2 T^2 C_{750} \right) g_{3,j,i} - \frac{T(-1+ST) C_{635} g_{1,i,i} g_{3,j,i}}{-1+T} - \\
 & \frac{T(-1+ST) C_{636} g_{1,i,i}^2 g_{3,j,i}}{2(-1+T)} - T((-1+ST) C_{635} + S(-1+T) C_{751}) g_{1,j,i} g_{3,j,i} - \\
 & T(-1+ST) C_{636} g_{1,i,i} g_{1,j,i} g_{3,j,i} + S T C_{751} g_{1,j,j} g_{3,j,i} - T(-1+ST) C_{636} g_{1,j,i} g_{1,j,j} g_{3,j,i} + \\
 & \frac{T(-1+ST) C_{636} g_{1,j,j}^2 g_{3,j,i}}{2(-1+T)} - \frac{S(-1+ST) C_{657} g_{2,i,i} g_{3,j,i}}{-1+S} - \\
 & \frac{T(-1+ST) C_{647} g_{1,i,i} g_{2,i,i} g_{3,j,i}}{-1+T} - T(-1+ST) C_{647} g_{1,j,i} g_{2,i,i} g_{3,j,i} -
 \end{aligned}$$

$$\begin{aligned}
 & \frac{S(-1+ST) c_{662} g_{2,i,i}^2 g_{3,j,i}}{2(-1+S)} - S((-1+ST) c_{657} + (-1+S) T c_{765}) g_{2,j,i} g_{3,j,i} - \\
 & \frac{(-1+S) T (-1+ST) c_{647} g_{1,i,i} g_{2,j,i} g_{3,j,i}}{-1+T} - \frac{(-1+S) T (-1+ST) c_{647} g_{1,j,j} g_{2,j,i} g_{3,j,i}}{-1+T} - \\
 & \frac{S(-1+ST) c_{662} g_{2,i,i} g_{2,j,i} g_{3,j,i} + ST c_{765} g_{2,j,j} g_{3,j,i} - T(-1+ST) c_{647} g_{1,j,i} g_{2,j,j} g_{3,j,i} +}{-1+T} \\
 & \frac{T(-1+ST) c_{647} g_{1,j,j} g_{2,j,j} g_{3,j,i}}{-1+T} - S(-1+ST) c_{662} g_{2,j,i} g_{2,j,j} g_{3,j,i} + \\
 & \frac{S(-1+ST) c_{662} g_{2,j,j}^2 g_{3,j,i}}{2(-1+S)} - \frac{ST((-1+3ST) c_{795} + 2(-1+ST) c_{894}) g_{3,i,i} g_{3,j,i}}{1+ST} - \\
 & ST c_{796} g_{1,i,i} g_{3,i,i} g_{3,j,i} - S(-1+T) T c_{796} g_{1,j,i} g_{3,i,i} g_{3,j,i} - ST c_{800} g_{2,i,i} g_{3,i,i} g_{3,j,i} - \\
 & (-1+S) ST c_{800} g_{2,j,i} g_{3,i,i} g_{3,j,i} - ST c_{804} g_{3,i,i}^2 g_{3,j,i} + \frac{ST(c_{795} + c_{894}) g_{3,i,j} g_{3,j,i}}{1+ST} - \\
 & \frac{ST(-1+ST)((-1+2ST) c_{795} + (-2+ST) c_{894}) g_{3,j,i}^2}{1+ST} - \frac{1}{2} ST(-1+ST) c_{796} g_{1,i,i} g_{3,j,i}^2 - \\
 & \frac{1}{2} ST(-1+ST) c_{796} g_{1,j,j} g_{3,j,i}^2 - \frac{1}{2} ST(-1+ST) c_{800} g_{2,i,i} g_{3,j,i}^2 - \\
 & \frac{1}{2} ST(-1+ST) c_{800} g_{2,j,j} g_{3,j,i}^2 - ST(-1+ST) c_{804} g_{3,i,i} g_{3,j,i}^2 + ST(c_{629} + c_{750}) g_{3,j,j} + \\
 & \frac{T((-1+ST) c_{635} - S(-1+T) c_{751}) g_{1,i,i} g_{3,j,j}}{(-1+T)(-1+ST)} + T c_{635} g_{1,j,i} g_{3,j,j} - \frac{1}{2} (-1+T) T c_{636} g_{1,j,i}^2 g_{3,j,j} + \\
 & T c_{636} g_{1,j,i} g_{1,j,j} g_{3,j,j} + \frac{S((-1+ST) c_{657} - (-1+S) T c_{765}) g_{2,i,i} g_{3,j,j}}{(-1+S)(-1+ST)} + \\
 & S c_{657} g_{2,j,i} g_{3,j,j} - (-1+S) T c_{647} g_{1,j,i} g_{2,j,i} g_{3,j,j} + \frac{(-1+S) T c_{647} g_{1,j,j} g_{2,j,i} g_{3,j,j}}{-1+T} - \\
 & \frac{1}{2} (-1+S) S c_{662} g_{2,j,i}^2 g_{3,j,j} + T c_{647} g_{1,j,i} g_{2,j,j} g_{3,j,j} + S c_{662} g_{2,j,i} g_{2,j,j} g_{3,j,j} + \\
 & \frac{ST(c_{795} + c_{894}) g_{3,i,i} g_{3,j,j}}{1+ST} + \frac{ST(2(-1+ST) c_{795} + (-3+ST) c_{894}) g_{3,j,i} g_{3,j,j}}{1+ST} - \\
 & S(-1+T) T c_{796} g_{1,j,i} g_{3,j,i} g_{3,j,j} + ST c_{796} g_{1,j,j} g_{3,j,i} g_{3,j,j} - \\
 & (-1+S) ST c_{800} g_{2,j,i} g_{3,j,i} g_{3,j,j} + ST c_{800} g_{2,j,j} g_{3,j,i} g_{3,j,j} - ST(-1+ST) c_{804} g_{3,j,i}^2 g_{3,j,j} + \\
 & \frac{S(-1+T) T c_{796} g_{1,j,i} g_{3,j,j}^2}{-2+2ST} + \frac{(-1+S) ST c_{800} g_{2,j,i} g_{3,j,j}^2}{-2+2ST} + ST c_{804} g_{3,j,i} g_{3,j,j}^2
 \end{aligned}$$