Pensieve header: Mathematica notebook for the 2-variable perturbed Alexander invariant.

Initialization

```
In[•]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\HigherRank"];
Once[<< KnotTheory`; << ../APAI/Rot.m];
TS = T S;
(α_⁺)⁺ := α^"++"; (* this is for cosmetic reasons only *)
δ_i_,j_ := If[i == j, 1, 0];
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.

Loading Rot.m from http://drorbn.net/APAI to compute rotation numbers.
```

Lower Rank 2 Gassner: (Modified from Rank2Gassner.nb)

```
In[•]:= LR2G_i_,j_[ℰ_] := Expand[ℰ /. {e_i ↦ T e_i + (1 - T) e_j, f_i ↦ S f_i + (1 - S) f_j,
      g_i ↦ T S g_i + (1 - T S) g_j
        + S (T - 1) e_j f_i + (S - 1) T e_i f_j + (S + T - 2 S T) e_j f_j}];
LR2G̅_i_,j_[ℰ_] := Expand[ℰ /. {e_i ↦ T^-1 e_i + (1 - T^-1) e_j, f_i ↦ S^-1 f_i + (1 - S^-1) f_j,
      g_i ↦ T^-1 S^-1 g_i + (1 - T^-1 S^-1) g_j
        - S^-1 (1 - T^-1) e_j f_i - T^-1 (1 - S^-1) e_i f_j + (S^-1 + T^-1 - 2 S^-1 T^-1) e_j f_j}];
bas =
  {e_1, e_2, e_3, f_1, f_2, f_3, e_1 f_1, e_1 f_2, e_1 f_3, e_2 f_1, e_2 f_2, e_2 f_3, e_3 f_1, e_3 f_2, e_3 f_3, g_1, g_2, g_3};
(bas // LR2G_1,2 // LR2G̅_1,2) == bas
(bas // LR2G_3,2 // LR2G̅_3,2) == bas
(lhs = bas // LR2G_1,2 // LR2G_1,3 // LR2G_2,3) == (bas // LR2G_2,3 // LR2G_1,3 // LR2G_1,2)
```

```
Out[•]=
True

Out[•]=
True

Out[•]=
True
```

The Target Program

```
In[•]:= R₁[s_, i_, j_] := TBD; (* a sign(s)-dependent quadratic polynomial in g_{1αβ}, g_{2αβ}, g_{3αβ},
        and y_{αβγ}, where deg y_{αβγ}=2 and α,β,γ∈{i,j}, with coefficients in ℤ[T^±1,S^±1]. *)
        CF[ℰ_] := Factor@Together[ℰ];
        λ[K_] := Module[{Cs, φ, n, A, s, i, j, k, Δ, G, gEval, Y, yEval, c, λ1},
          {Cs, φ} = Rot[K]; n = Length[Cs];
          A = IdentityMatrix[2 n + 1];
          Cases[Cs, {s_, i_, j_} ⧴ (A⟦{i, j}, {i + 1, j + 1}⟧ += ({{-T^s, T^s - 1}, {0, -1}}))];
          Δ = T^((-Total[φ] - Total[Cs⟦All, 1⟧])/2) Det[A];
          G = Inverse[A];
          gEval[ℰ_] := CF[ℰ /. α_^+ ⧴ α + 1 /.
            {g_{1,α_,β_} ⧴ G⟦α, β⟧, g_{2,α_,β_} ⧴ (G⟦α, β⟧ /. T → S), g_{3,α_,β_} ⧴ (G⟦α, β⟧ /. T → TS)}];
          Y[α_, β_, γ_] := Sum[{s, i, j} = c;
            g_{3,α,i} (S^s (T^s - 1) g_{1,j^+,β} g_{2,i^+,γ} + (S^s - 1) T^s g_{1,i^+,β} g_{2,j^+,γ} + (S^s + T^s - 2 TS^s) g_{1,j^+,β} g_{2,j^+,γ}),
            {c, Cs}];
          yEval[ℰ_] := CF[ℰ /. y_{α_,β_,γ_} ⧴ Y[α, β, γ]];
          λ1 = Sum_{k=1}^{n} R₁ @@ Cs⟦k⟧ - Sum_{k=1}^{2 n} φ⟦k⟧ (g_{1,k,k} + g_{2,k,k} + g_{3,k,k});
          {Δ, Δ (Δ /. T → S) (Δ /. T → TS) λ1} // yEval // gEval
        ];
```

Step-by-step Run-Through

```
In[•]:= CF[ℰ_] := Factor@Together[ℰ];
```

```
In[•]:= K = {
          PD[X[4, 2, 5, 1], X[2, 6, 3, 5], X[6, 4, 7, 3]],
          Knot[6, 2]
        }⟦1⟧
```

```
Out[•]=
PD[X[4, 2, 5, 1], X[2, 6, 3, 5], X[6, 4, 7, 3]]
```

```
In[•]:= {Cs, φ} = Rot[K]
```

```
Out[•]=
{{{1, 1, 4}, {1, 5, 2}, {1, 3, 6}}, {0, 0, 0, -1, 0, 0}}
```

```
In[•]:= n = Length[Cs];
        A = IdentityMatrix[2 n + 1];
        Cases[Cs, {s_, i_, j_} ⧴ (A⟦{i, j}, {i + 1, j + 1}⟧ += ({{-T^s, T^s - 1}, {0, -1}}))];
```

```
In[•]:= A // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 1 & -T & 0 & 0 & -1+T & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -T & 0 & 0 & -1+T \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1+T & 0 & 1 & -T & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

```
In[•]:= Δ = T^(-Total[φ]-Total[Cs[[All,1]]])/2 Det[A]
```

Out[•]=

$$\frac{1-T+T^2}{T}$$

```
In[•]:= G = Inverse[A];
        G // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 1 & \frac{T-T^2+T^3}{1-T+T^2} & 1 & \frac{T-T^2+T^3}{1-T+T^2} & 1 & \frac{T-T^2+T^3}{1-T+T^2} & 1 \\ 0 & 1 & \frac{1}{1-T+T^2} & \frac{T}{1-T+T^2} & \frac{T}{1-T+T^2} & \frac{T^2}{1-T+T^2} & 1 \\ 0 & 0 & \frac{1}{1-T+T^2} & \frac{T}{1-T+T^2} & \frac{T}{1-T+T^2} & \frac{T^2}{1-T+T^2} & 1 \\ 0 & 0 & \frac{1-T}{1-T+T^2} & \frac{1}{1-T+T^2} & \frac{1}{1-T+T^2} & \frac{T}{1-T+T^2} & 1 \\ 0 & 0 & \frac{1-T}{1-T+T^2} & \frac{T-T^2}{1-T+T^2} & \frac{1}{1-T+T^2} & \frac{T}{1-T+T^2} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

```
In[•]:= gEval[ℰ_] := CF[ℰ /. α_^+ :> α + 1 /.
          {g_{1,α_,β_} :> G[[α, β]], g_{2,α_,β_} :> (G[[α, β]] /. T → S), g_{3,α_,β_} :> (G[[α, β]] /. T → TS)}];
```

```
In[•]:= Clear[Y]
        Y[α_, β_, γ_] := Y[α, β, γ] = Sum[{s, i, j} = c;
          g_{3,α,i} (S^s (T^s - 1) g_{1,j^+,β} g_{2,i^+,γ} + (S^s - 1) T^s g_{1,i^+,β} g_{2,j^+,γ} + (S^s + T^s - 2 TS^s) g_{1,j^+,β} g_{2,j^+,γ}),
          {c, Cs}]

        Column[MatrixForm /@ Table[Y[α, β, γ], {α, 3}, {β, 3}, {γ, 3}]]
```

Out[•]=

$$\begin{pmatrix} (S(-1+T)\, g_{1,4^+,1}\, g_{2,1^+,1} + (-1+S)\, T\, g_{1,1^+,1}\, g_{2,4^+,1} + (S+T-2ST)\, g_{1,4^+,1}\, g_{2,4^+,1})\, g_{3,1,1} + (S(-1+T)\, g_1 \\ (S(-1+T)\, g_{1,4^+,2}\, g_{2,1^+,1} + (-1+S)\, T\, g_{1,1^+,2}\, g_{2,4^+,1} + (S+T-2ST)\, g_{1,4^+,2}\, g_{2,4^+,1})\, g_{3,1,1} + (S(-1+T)\, g_1 \\ (S(-1+T)\, g_{1,4^+,3}\, g_{2,1^+,1} + (-1+S)\, T\, g_{1,1^+,3}\, g_{2,4^+,1} + (S+T-2ST)\, g_{1,4^+,3}\, g_{2,4^+,1})\, g_{3,1,1} + (S(-1+T)\, g_1 \end{pmatrix}$$

$$\begin{pmatrix} (S(-1+T)\, g_{1,4^+,1}\, g_{2,1^+,1} + (-1+S)\, T\, g_{1,1^+,1}\, g_{2,4^+,1} + (S+T-2ST)\, g_{1,4^+,1}\, g_{2,4^+,1})\, g_{3,2,1} + (S(-1+T)\, g_1 \\ (S(-1+T)\, g_{1,4^+,2}\, g_{2,1^+,1} + (-1+S)\, T\, g_{1,1^+,2}\, g_{2,4^+,1} + (S+T-2ST)\, g_{1,4^+,2}\, g_{2,4^+,1})\, g_{3,2,1} + (S(-1+T)\, g_1 \\ (S(-1+T)\, g_{1,4^+,3}\, g_{2,1^+,1} + (-1+S)\, T\, g_{1,1^+,3}\, g_{2,4^+,1} + (S+T-2ST)\, g_{1,4^+,3}\, g_{2,4^+,1})\, g_{3,2,1} + (S(-1+T)\, g_1 \end{pmatrix}$$

$$\begin{pmatrix} (S(-1+T)\, g_{1,4^+,1}\, g_{2,1^+,1} + (-1+S)\, T\, g_{1,1^+,1}\, g_{2,4^+,1} + (S+T-2ST)\, g_{1,4^+,1}\, g_{2,4^+,1})\, g_{3,3,1} + (S(-1+T)\, g_1 \\ (S(-1+T)\, g_{1,4^+,2}\, g_{2,1^+,1} + (-1+S)\, T\, g_{1,1^+,2}\, g_{2,4^+,1} + (S+T-2ST)\, g_{1,4^+,2}\, g_{2,4^+,1})\, g_{3,3,1} + (S(-1+T)\, g_1 \\ (S(-1+T)\, g_{1,4^+,3}\, g_{2,1^+,1} + (-1+S)\, T\, g_{1,1^+,3}\, g_{2,4^+,1} + (S+T-2ST)\, g_{1,4^+,3}\, g_{2,4^+,1})\, g_{3,3,1} + (S(-1+T)\, g_1 \end{pmatrix}$$

```
In[•]:= Column[MatrixForm /@ Table[Y[α, β, γ] // gEval, {α, 3}, {β, 3}, {γ, 3}]]
```

Out[•]=

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{(-1+S)^2\,T}{1-S+S^2} \\ 0 & -\frac{S\,(-1+T)^2}{1-T+T^2} & -\frac{-S+S^2-T+2\,S\,T-2\,S^2\,T+T^2-2\,S\,T^2+2\,S^2\,T^2}{(1-S+S^2)\,(1-T+T^2)} \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{S\,T\,(-S-T+2\,S\,T)}{(1-S+S^2)\,(1-T+T^2)\,(1-S\,T+S^2\,T^2)} \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{S\,T\,(-S-T+2\,S\,T)}{(1-S+S^2)\,(1-T+T^2)\,(1-S\,T+S^2\,T^2)} \end{pmatrix}$$

```
In[•]:= yEval[ℰ_] := CF[ℰ /. y_α_,β_,γ_ :> Y[α, β, γ]]
```

The g-Rules

```
In[•]:= gRules_s_,i_,j_ := {
  g1,i,β_ :> δi,β + T^s g1,i⁺,β + (1 - T^s) g1,j⁺,β, g1,j,β_ :> δj,β + g1,j⁺,β,
  g1,α_,i :> T^-s (g1,α,i⁺ - δα,i⁺), g1,α_,j :> g1,α,j⁺ - (1 - T^s) g1,α,i - δα,j⁺,
  g2,i,β_ :> δi,β + S^s g2,i⁺,β + (1 - S^s) g2,j⁺,β, g2,j,β_ :> δj,β + g2,j⁺,β,
  g2,α_,i :> S^-s (g2,α,i⁺ - δα,i⁺), g2,α_,j :> g2,α,j⁺ - (1 - S^s) g2,α,i - δα,j⁺,
  g3,i,β_ :> δi,β + TS^s g3,i⁺,β + (1 - TS^s) g3,j⁺,β, g3,j,β_ :> δj,β + g3,j⁺,β,
  g3,α_,i :> TS^-s (g3,α,i⁺ - δα,i⁺), g3,α_,j :> g3,α,j⁺ - (1 - TS^s) g3,α,i - δα,j⁺
 }
```

```
In[•]:= (gs = Table[{g1,α,β, g2,α,β, g3,α,β}, {α, 2 n + 1}, {β, 2 n + 1}])⟦1 ;; 2, 1 ;; 2, 1 ;; 2⟧
```

Out[•]=

$$\{\{\{g_{1,1,1}, g_{2,1,1}\}, \{g_{1,1,2}, g_{2,1,2}\}\}, \{\{g_{1,2,1}, g_{2,2,1}\}, \{g_{1,2,2}, g_{2,2,2}\}\}\}$$

```
In[•]:= Cs
```

Out[•]=

$$\{\{1, 1, 4\}, \{1, 5, 2\}, \{1, 3, 6\}\}$$

```
In[•]:= (gs /. gRules_Sequence@@Cs⟦1⟧)⟦1 ;; 2, 1 ;; 2, 1 ;; 2⟧
```

Out[•]=

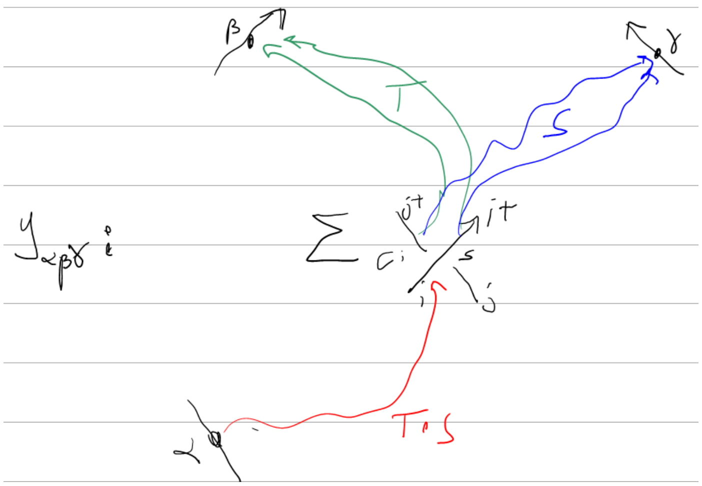
$$\left\{\{\{1 + T\,g_{1,1^+,1} + (1 - T)\,g_{1,4^+,1},\ 1 + S\,g_{2,1^+,1} + (1 - S)\,g_{2,4^+,1}\},\ \{T\,g_{1,1^+,2} + (1 - T)\,g_{1,4^+,2},\ S\,g_{2,1^+,2} + (1 - S)\,g_{2,4^+,2}\}\},\ \left\{\left\{\frac{-\text{If}[2 == 1^+, 1, 0] + g_{1,2,1^+}}{T},\ \frac{-\text{If}[2 == 1^+, 1, 0] + g_{2,2,1^+}}{S}\right\}, \{g_{1,2,2}, g_{2,2,2}\}\right\}\right\}$$

In[∘]:= $\texttt{gEval}\left[\texttt{gs} - \left(\texttt{gs} \;/.\; \texttt{Flatten@Table}\left[\texttt{gRules}_{\texttt{Sequence@@c}}, \{\texttt{c}, \texttt{Cs}\}\right]\right)\right]$

Out[∘]=

```
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

The y-Rules



In[∘]:=

$$\texttt{yRules}_{s\_,i\_,j\_} := \Big\{$$
$$y_{i,\beta\_,\gamma\_} \mapsto T S^s\, y_{i^+,\beta,\gamma} + \left(1 - T S^s\right) y_{j^+,\beta,\gamma} + \left(S^s\left(T^s - 1\right) g_{1,j^+,\beta}\, g_{2,i^+,\gamma} + \left(S^s - 1\right) T^s\, g_{1,i^+,\beta}\, g_{2,j^+,\gamma} + \left(S^s + T^s - 2\, T S^s\right) g_{1,j^+,\beta}\, g_{2,j^+,\gamma}\right),$$
$$y_{j,\beta\_,\gamma\_} \mapsto y_{j^+,\beta,\gamma},$$
$$y_{\alpha\_,i,\gamma\_} \mapsto T^{-s}\, y_{\alpha,i^+,\gamma} + \left(1 - S^s\right) g_{3,\alpha,i}\, g_{2,j^+,\gamma},$$
$$y_{\alpha\_,j,\gamma\_} \mapsto y_{\alpha,j^+,\gamma} - \left(1 - T^s\right) y_{\alpha,i,\gamma} - g_{3,\alpha,i}\left(S^s\left(T^s - 1\right) g_{2,i^+,\gamma} + \left(S^s + T^s - 2\, T S^s\right) g_{2,j^+,\gamma}\right),$$
$$y_{\alpha\_,\beta\_,i} \mapsto S^{-s}\, y_{\alpha,\beta,i^+} + \left(1 - T^s\right) g_{3,\alpha,i}\, g_{1,j^+,\beta},$$
$$y_{\alpha\_,\beta\_,j} \mapsto y_{\alpha,\beta,j^+} - \left(1 - S^s\right) y_{\alpha,\beta,i} - g_{3,\alpha,i}\left(T^s\left(S^s - 1\right) g_{1,i^+,\beta} + \left(S^s + T^s - 2\, T S^s\right) g_{1,\beta,j^+}\right)$$
$$\Big\}$$

In[∘]:= $\left(\texttt{ys} = \texttt{Table}\left[y_{\alpha,\beta,\gamma}, \{\alpha, 2n+1\}, \{\beta, 2n+1\}, \{\gamma, 2n+1\}\right]\right)[\![1\;;;\;2, 1\;;;\;2, 1\;;;\;2]\!]$

Out[∘]=

$\{\{\{y_{1,1,1}, y_{1,1,2}\}, \{y_{1,2,1}, y_{1,2,2}\}\}, \{\{y_{2,1,1}, y_{2,1,2}\}, \{y_{2,2,1}, y_{2,2,2}\}\}\}$

In[∘]:= **Cs**

Out[∘]=

```
{{1, 1, 4}, {1, 5, 2}, {1, 3, 6}}
```

In[•]:=
```
(ys /. yRules_{Sequence@@Cs[[1]]})[[1 ;; 2, 1 ;; 2, 1 ;; 2]]
```

Out[•]=

$$\Big\{\{\{S\,(-1+T)\,g_{1,4^+,1}\,g_{2,1^+,1}+(-1+S)\,T\,g_{1,1^+,1}\,g_{2,4^+,1}+(S+T-2\,S\,T)\,g_{1,4^+,1}\,g_{2,4^+,1}+S\,T\,y_{1^+,1,1}+(1-S\,T)\,y_{4^+,1,1},\ S\,(-1+T)\,g_{1,4^+,1}\,g_{2,1^+,2}+(-1+S)\,T\,g_{1,1^+,1}\,g_{2,4^+,2}+(S+T-2\,S\,T)\,g_{1,4^+,1}\,g_{2,4^+,2}+S\,T\,y_{1^+,1,2}+(1-S\,T)\,y_{4^+,1,2}\},\ \{S\,(-1+T)\,g_{1,4^+,2}\,g_{2,1^+,1}+(-1+S)\,T\,g_{1,1^+,2}\,g_{2,4^+,1}+(S+T-2\,S\,T)\,g_{1,4^+,2}\,g_{2,4^+,1}+S\,T\,y_{1^+,2,1}+(1-S\,T)\,y_{4^+,2,1},\ S\,(-1+T)\,g_{1,4^+,2}\,g_{2,1^+,2}+(-1+S)\,T\,g_{1,1^+,2}\,g_{2,4^+,2}+(S+T-2\,S\,T)\,g_{1,4^+,2}\,g_{2,4^+,2}+S\,T\,y_{1^+,2,2}+(1-S\,T)\,y_{4^+,2,2}\}\},\ \Big\{\Big\{(1-S)\,g_{2,4^+,1}\,g_{3,2,1}+\frac{y_{2,1^+,1}}{T},\ (1-S)\,g_{2,4^+,2}\,g_{3,2,1}+\frac{y_{2,1^+,2}}{T}\Big\},\ \Big\{(1-T)\,g_{1,4^+,2}\,g_{3,2,1}+\frac{y_{2,2,1^+}}{S},\ y_{2,2,2}\Big\}\Big\}\Big\}$$

In[•]:=
```
yEval[ys - (ys /. yRules_{Sequence@@Cs[[1]]})][[1, 1, 1]]
```

Out[•]=

$$\begin{aligned}
&S\,g_{1,4^+,1}\,g_{2,1^+,1}-S\,T\,g_{1,4^+,1}\,g_{2,1^+,1}+T\,g_{1,1^+,1}\,g_{2,4^+,1}-S\,T\,g_{1,1^+,1}\,g_{2,4^+,1}-S\,g_{1,4^+,1}\,g_{2,4^+,1}-\\
&T\,g_{1,4^+,1}\,g_{2,4^+,1}+2\,S\,T\,g_{1,4^+,1}\,g_{2,4^+,1}-S\,g_{1,4^+,1}\,g_{2,1^+,1}\,g_{3,1,1}+S\,T\,g_{1,4^+,1}\,g_{2,1^+,1}\,g_{3,1,1}-\\
&T\,g_{1,1^+,1}\,g_{2,4^+,1}\,g_{3,1,1}+S\,T\,g_{1,1^+,1}\,g_{2,4^+,1}\,g_{3,1,1}+S\,g_{1,4^+,1}\,g_{2,4^+,1}\,g_{3,1,1}+T\,g_{1,4^+,1}\,g_{2,4^+,1}\,g_{3,1,1}-\\
&2\,S\,T\,g_{1,4^+,1}\,g_{2,4^+,1}\,g_{3,1,1}-S\,g_{1,6^+,1}\,g_{2,3^+,1}\,g_{3,1,3}+S\,T\,g_{1,6^+,1}\,g_{2,3^+,1}\,g_{3,1,3}-T\,g_{1,3^+,1}\,g_{2,6^+,1}\,g_{3,1,3}+\\
&S\,T\,g_{1,3^+,1}\,g_{2,6^+,1}\,g_{3,1,3}+S\,g_{1,6^+,1}\,g_{2,6^+,1}\,g_{3,1,3}+T\,g_{1,6^+,1}\,g_{2,6^+,1}\,g_{3,1,3}-2\,S\,T\,g_{1,6^+,1}\,g_{2,6^+,1}\,g_{3,1,3}+\\
&S\,g_{1,2^+,1}\,g_{2,2^+,1}\,g_{3,1,5}+T\,g_{1,2^+,1}\,g_{2,2^+,1}\,g_{3,1,5}-2\,S\,T\,g_{1,2^+,1}\,g_{2,2^+,1}\,g_{3,1,5}-T\,g_{1,5^+,1}\,g_{2,2^+,1}\,g_{3,1,5}+\\
&S\,T\,g_{1,5^+,1}\,g_{2,2^+,1}\,g_{3,1,5}-S\,g_{1,2^+,1}\,g_{2,5^+,1}\,g_{3,1,5}+S\,T\,g_{1,2^+,1}\,g_{2,5^+,1}\,g_{3,1,5}+S^2\,T\,g_{1,4^+,1}\,g_{2,1^+,1}\,g_{3,1^+,1}-\\
&S^2\,T^2\,g_{1,4^+,1}\,g_{2,1^+,1}\,g_{3,1^+,1}+S\,T^2\,g_{1,1^+,1}\,g_{2,4^+,1}\,g_{3,1^+,1}-S^2\,T^2\,g_{1,1^+,1}\,g_{2,4^+,1}\,g_{3,1^+,1}-\\
&S^2\,T\,g_{1,4^+,1}\,g_{2,4^+,1}\,g_{3,1^+,1}-S\,T^2\,g_{1,4^+,1}\,g_{2,4^+,1}\,g_{3,1^+,1}+2\,S^2\,T^2\,g_{1,4^+,1}\,g_{2,4^+,1}\,g_{3,1^+,1}+\\
&S^2\,T\,g_{1,6^+,1}\,g_{2,3^+,1}\,g_{3,1^+,3}-S^2\,T^2\,g_{1,6^+,1}\,g_{2,3^+,1}\,g_{3,1^+,3}+S\,T^2\,g_{1,3^+,1}\,g_{2,6^+,1}\,g_{3,1^+,3}-\\
&S^2\,T^2\,g_{1,3^+,1}\,g_{2,6^+,1}\,g_{3,1^+,3}-S^2\,T\,g_{1,6^+,1}\,g_{2,6^+,1}\,g_{3,1^+,3}-S\,T^2\,g_{1,6^+,1}\,g_{2,6^+,1}\,g_{3,1^+,3}+\\
&2\,S^2\,T^2\,g_{1,6^+,1}\,g_{2,6^+,1}\,g_{3,1^+,3}-S^2\,T\,g_{1,2^+,1}\,g_{2,2^+,1}\,g_{3,1^+,5}-S\,T^2\,g_{1,2^+,1}\,g_{2,2^+,1}\,g_{3,1^+,5}+\\
&2\,S^2\,T^2\,g_{1,2^+,1}\,g_{2,2^+,1}\,g_{3,1^+,5}+S\,T^2\,g_{1,5^+,1}\,g_{2,2^+,1}\,g_{3,1^+,5}-S^2\,T^2\,g_{1,5^+,1}\,g_{2,2^+,1}\,g_{3,1^+,5}+\\
&S^2\,T\,g_{1,2^+,1}\,g_{2,5^+,1}\,g_{3,1^+,5}-S^2\,T^2\,g_{1,2^+,1}\,g_{2,5^+,1}\,g_{3,1^+,5}+S\,g_{1,4^+,1}\,g_{2,1^+,1}\,g_{3,4^+,1}-S\,T\,g_{1,4^+,1}\,g_{2,1^+,1}\,g_{3,4^+,1}-\\
&S^2\,T\,g_{1,4^+,1}\,g_{2,1^+,1}\,g_{3,4^+,1}+S^2\,T^2\,g_{1,4^+,1}\,g_{2,1^+,1}\,g_{3,4^+,1}+T\,g_{1,1^+,1}\,g_{2,4^+,1}\,g_{3,4^+,1}-S\,T\,g_{1,1^+,1}\,g_{2,4^+,1}\,g_{3,4^+,1}-\\
&S\,T^2\,g_{1,1^+,1}\,g_{2,4^+,1}\,g_{3,4^+,1}+S^2\,T^2\,g_{1,1^+,1}\,g_{2,4^+,1}\,g_{3,4^+,1}-S\,g_{1,4^+,1}\,g_{2,4^+,1}\,g_{3,4^+,1}-T\,g_{1,4^+,1}\,g_{2,4^+,1}\,g_{3,4^+,1}+\\
&2\,S\,T\,g_{1,4^+,1}\,g_{2,4^+,1}\,g_{3,4^+,1}+S^2\,T\,g_{1,4^+,1}\,g_{2,4^+,1}\,g_{3,4^+,1}+S\,T^2\,g_{1,4^+,1}\,g_{2,4^+,1}\,g_{3,4^+,1}-\\
&2\,S^2\,T^2\,g_{1,4^+,1}\,g_{2,4^+,1}\,g_{3,4^+,1}+S\,g_{1,6^+,1}\,g_{2,3^+,1}\,g_{3,4^+,3}-S\,T\,g_{1,6^+,1}\,g_{2,3^+,1}\,g_{3,4^+,3}-S^2\,T\,g_{1,6^+,1}\,g_{2,3^+,1}\,g_{3,4^+,3}+\\
&S^2\,T^2\,g_{1,6^+,1}\,g_{2,3^+,1}\,g_{3,4^+,3}+T\,g_{1,3^+,1}\,g_{2,6^+,1}\,g_{3,4^+,3}-S\,T\,g_{1,3^+,1}\,g_{2,6^+,1}\,g_{3,4^+,3}-S\,T^2\,g_{1,3^+,1}\,g_{2,6^+,1}\,g_{3,4^+,3}+\\
&S^2\,T^2\,g_{1,3^+,1}\,g_{2,6^+,1}\,g_{3,4^+,3}-S\,g_{1,6^+,1}\,g_{2,6^+,1}\,g_{3,4^+,3}-T\,g_{1,6^+,1}\,g_{2,6^+,1}\,g_{3,4^+,3}+2\,S\,T\,g_{1,6^+,1}\,g_{2,6^+,1}\,g_{3,4^+,3}+\\
&S^2\,T\,g_{1,6^+,1}\,g_{2,6^+,1}\,g_{3,4^+,3}+S\,T^2\,g_{1,6^+,1}\,g_{2,6^+,1}\,g_{3,4^+,3}-2\,S^2\,T^2\,g_{1,6^+,1}\,g_{2,6^+,1}\,g_{3,4^+,3}-\\
&S\,g_{1,2^+,1}\,g_{2,2^+,1}\,g_{3,4^+,5}-T\,g_{1,2^+,1}\,g_{2,2^+,1}\,g_{3,4^+,5}+2\,S\,T\,g_{1,2^+,1}\,g_{2,2^+,1}\,g_{3,4^+,5}+S^2\,T\,g_{1,2^+,1}\,g_{2,2^+,1}\,g_{3,4^+,5}+\\
&S\,T^2\,g_{1,2^+,1}\,g_{2,2^+,1}\,g_{3,4^+,5}-2\,S^2\,T^2\,g_{1,2^+,1}\,g_{2,2^+,1}\,g_{3,4^+,5}+T\,g_{1,5^+,1}\,g_{2,2^+,1}\,g_{3,4^+,5}-\\
&S\,T\,g_{1,5^+,1}\,g_{2,2^+,1}\,g_{3,4^+,5}-S\,T^2\,g_{1,5^+,1}\,g_{2,2^+,1}\,g_{3,4^+,5}+S^2\,T^2\,g_{1,5^+,1}\,g_{2,2^+,1}\,g_{3,4^+,5}+\\
&S\,g_{1,2^+,1}\,g_{2,5^+,1}\,g_{3,4^+,5}-S\,T\,g_{1,2^+,1}\,g_{2,5^+,1}\,g_{3,4^+,5}-S^2\,T\,g_{1,2^+,1}\,g_{2,5^+,1}\,g_{3,4^+,5}+S^2\,T^2\,g_{1,2^+,1}\,g_{2,5^+,1}\,g_{3,4^+,5}
\end{aligned}$$

In[∘]:=
```
gEval@yEval[ys - (ys /. Flatten@Table[gRules_{Sequence@@c}, {c, Cs}])]
```

Out[∘]=
```
{{{0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}},
  {{0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}},
  {{0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}},
  {{0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}},
  {{0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}},
  {{0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}},
  {{0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}}}
```

Invariance of $y_{\alpha\beta\gamma}$ under remote R2s

In[∘]:=
```
Clear[i, j];
Cs = {{1, i, j}, {-1, i⁺, j⁺}}
Z = Module[{s, i, j}, Sum[{s, i, j} = c;
    g_{3,α,i} (S^s (T^s - 1) g_{1,j⁺,β} g_{2,i⁺,γ} + (S^s - 1) T^s g_{1,i⁺,β} g_{2,j⁺,γ} + (S^s + T^s - 2 TS^s) g_{1,j⁺,β} g_{2,j⁺,γ}),
    {c, Cs}]]
Simplify[Z //. gRules_{1,i,j} ⋃ gRules_{-1,i⁺,j⁺} /. _If → 0]
```

Out[∘]=

{{1, i, j}, {-1, i⁺, j⁺}}

Out[∘]=

$$(S\,(-1+T)\,g_{1,j^+,\beta}\,g_{2,i^+,\gamma} + (-1+S)\,T\,g_{1,i^+,\beta}\,g_{2,j^+,\gamma} + (S+T-2\,S\,T)\,g_{1,j^+,\beta}\,g_{2,j^+,\gamma})\,g_{3,\alpha,i} + \left(\frac{\left(-1+\frac{1}{T}\right)g_{1,j^{++},\beta}\,g_{2,i^{++},\gamma}}{S} + \frac{\left(-1+\frac{1}{S}\right)g_{1,i^{++},\beta}\,g_{2,j^{++},\gamma}}{T} + \left(\frac{1}{S}+\frac{1}{T}-\frac{2}{S\,T}\right)g_{1,j^{++},\beta}\,g_{2,j^{++},\gamma}\right)g_{3,\alpha,i^+}$$

Out[∘]=

0

Invariance of $y_{\alpha\beta\gamma}$ under remote R3s

```
In[•]:= Clear[i, j, k];
Cs = {{1, i, j}, {1, i⁺, k}, {1, j⁺, k⁺}}
Z = Module[{s, i, j}, Sum[{s, i, j} = c;
    g3,α,i (S^s (T^s - 1) g1,j⁺,β g2,i⁺,γ + (S^s - 1) T^s g1,i⁺,β g2,j⁺,γ + (S^s + T^s - 2 TS^s) g1,j⁺,β g2,j⁺,γ),
    {c, Cs}]]
lhs = Simplify[Z //. gRules1,i,j ⋃ gRules1,i⁺,k ⋃ gRules1,j⁺,k⁺ /. _If → 0]
```

```
Out[•]= {{1, i, j}, {1, i⁺, k}, {1, j⁺, k⁺}}
```

```
Out[•]= (S (-1 + T) g1,j⁺,β g2,i⁺,γ + (-1 + S) T g1,i⁺,β g2,j⁺,γ + (S + T - 2 S T) g1,j⁺,β g2,j⁺,γ) g3,α,i +
  (S (-1 + T) g1,k⁺,β g2,i⁺⁺,γ + (-1 + S) T g1,i⁺⁺,β g2,k⁺,γ + (S + T - 2 S T) g1,k⁺,β g2,k⁺,γ) g3,α,i⁺ +
  (S (-1 + T) g1,k⁺⁺,β g2,j⁺⁺,γ + (-1 + S) T g1,j⁺⁺,β g2,k⁺⁺,γ + (S + T - 2 S T) g1,k⁺⁺,β g2,k⁺⁺,γ) g3,α,j⁺
```

```
Out[•]= 1/(S^2 T^2) ((S (-1 + T) (T g1,j⁺⁺,β - (-1 + T) g1,k⁺⁺,β) (S g2,i⁺⁺,γ - (-1 + S) g2,k⁺⁺,γ) +
        (-1 + S) T (T g1,i⁺⁺,β - (-1 + T) g1,k⁺⁺,β) (S g2,j⁺⁺,γ - (-1 + S) g2,k⁺⁺,γ) +
        (S + T - 2 S T) (T g1,j⁺⁺,β - (-1 + T) g1,k⁺⁺,β) (S g2,j⁺⁺,γ - (-1 + S) g2,k⁺⁺,γ)) g3,α,i⁺⁺ +
    S T ((-1 + S) T g1,i⁺⁺,β g2,k⁺⁺,γ + g1,k⁺⁺,β (S (-1 + T) g2,i⁺⁺,γ + (S + T - 2 S T) g2,k⁺⁺,γ)) g3,α,i⁺⁺ +
    S T ((-1 + S) T g1,j⁺⁺,β g2,k⁺⁺,γ + g1,k⁺⁺,β (S (-1 + T) g2,j⁺⁺,γ + (S + T - 2 S T) g2,k⁺⁺,γ)) g3,α,j⁺⁺)
```

```
In[•]:= Clear[i, j, k];
Cs = {{1, j, k}, {1, i, k⁺}, {1, i⁺, j⁺}}
Z = Module[{s, i, j}, Sum[{s, i, j} = c;
    g3,α,i (S^s (T^s - 1) g1,j⁺,β g2,i⁺,γ + (S^s - 1) T^s g1,i⁺,β g2,j⁺,γ + (S^s + T^s - 2 TS^s) g1,j⁺,β g2,j⁺,γ),
    {c, Cs}]]
rhs = Simplify[Z //. gRules1,j,k ⋃ gRules1,i,k⁺ ⋃ gRules1,i⁺,j⁺ /. _If → 0]
```

```
Out[•]= {{1, j, k}, {1, i, k⁺}, {1, i⁺, j⁺}}
```

```
Out[•]= ((S + T - 2 S T) g1,k⁺⁺,β g2,k⁺⁺,γ + (-1 + S) T g1,i⁺,β g2,k⁺⁺,γ + S (-1 + T) g1,k⁺⁺,β g2,i⁺,γ) g3,α,i +
  (S (-1 + T) g1,k⁺,β g2,j⁺,γ + (-1 + S) T g1,j⁺,β g2,k⁺,γ + (S + T - 2 S T) g1,k⁺,β g2,k⁺,γ) g3,α,j +
  (S (-1 + T) g1,j⁺⁺,β g2,i⁺⁺,γ + (-1 + S) T g1,i⁺⁺,β g2,j⁺⁺,γ + (S + T - 2 S T) g1,j⁺⁺,β g2,j⁺⁺,γ) g3,α,i⁺
```

```
Out[•]= 1/(S^2 T^2) (S T ((-1 + S) T g1,i⁺⁺,β g2,j⁺⁺,γ + g1,j⁺⁺,β (S (-1 + T) g2,i⁺⁺,γ + (S + T - 2 S T) g2,j⁺⁺,γ)) g3,α,i⁺⁺ +
    (S (-1 + T) g1,k⁺⁺,β (S g2,i⁺⁺,γ - (-1 + S) g2,j⁺⁺,γ) +
      (-1 + S) T (T g1,i⁺⁺,β - (-1 + T) g1,j⁺⁺,β) g2,k⁺⁺,γ + (S + T - 2 S T) g1,k⁺⁺,β g2,k⁺⁺,γ) g3,α,i⁺⁺ +
    ((-1 + S) T g1,j⁺⁺,β g2,k⁺⁺,γ + g1,k⁺⁺,β (S (-1 + T) g2,j⁺⁺,γ + (S + T - 2 S T) g2,k⁺⁺,γ))
     ((-1 + S T) g3,α,i⁺⁺ + S T g3,α,j⁺⁺))
```

```
In[•]:= Simplify[lhs == rhs]
```

```
Out[•]=
True
```

Setting up R_1

```
In[•]:= gs = Flatten@Table[{g_{p,i,i}, g_{p,i,j}, g_{p,j,i}, g_{p,j,j}}, {p, 3}];
bas =
  {1} ⋃ gs ⋃ Flatten@Table[gs[[p]] gs[[q]], {p, Length[gs]}, {q, p, Length[gs]}] ⋃ Flatten@
    Table[gs[[p]] gs[[q]] gs[[r]], {p, Length[gs]}, {q, p, Length[gs]}, {r, q, Length[gs]}] ⋃
   {y_{i,i,i}, y_{i,i,j}, y_{i,j,i}, y_{i,j,j}, y_{j,i,i}, y_{j,i,j}, y_{j,j,i}, y_{j,j,j}};
len = Length[bas]
vars = Table[c_p = 0; c_p =.; c_p, {p, 2 len}];
R_1[1, i_, j_] := Evaluate[Sum[c_p bas[[p]], {p, len}]];
R_1[-1, i_, j_] := Evaluate[Sum[c_{len+p} bas[[p]], {p, Length[bas]}]];
```

```
Out[•]=
463
```

Solving R2b

```
In[•]:= Clear[i, j];
eqn = Expand[R_1[1, i, j] + R_1[-1, i^+, j^+] //.
    gRules_{1,i,j} ⋃ gRules_{-1,i^+,j^+} ⋃ yRules_{1,i,j} ⋃ yRules_{-1,i^+,j^+} /. _If → 0]
```

```
Out[•]=
c_1 + c_464 + c_2 g_{1,i^{++},i^{++}} − c_457 g_{1,i^{++},i^{++}} + 2 S c_457 g_{1,i^{++},i^{++}} + (S c_457 g_{1,i^{++},i^{++}})/T^2 + (c_457 g_{1,i^{++},i^{++}})/T − (3 S c_457 g_{1,i^{++},i^{++}})/T + ⋯ 22 533 ⋯ + (c_922 y_{j^{++},j^{++},i^{++}})/T −
(c_922 y_{j^{++},j^{++},i^{++}})/(S T) + S c_925 y_{j^{++},j^{++},i^{++}} + c_926 y_{j^{++},j^{++},i^{++}} − S c_926 y_{j^{++},j^{++},i^{++}} + c_463 y_{j^{++},j^{++},j^{++}} + c_922 y_{j^{++},j^{++},j^{++}} − (c_922 y_{j^{++},j^{++},j^{++}})/(S T) + c_926 y_{j^{++},j^{++},j^{++}}
```

Full expression not available (original memory size: 14 MB)

```
In[•]:= gys = Cases[eqn, g_,_,_, ∞] ⋃ Cases[eqn, y__, ∞]
```

```
Out[•]=
{g_{1,i^{++},i^{++}}, g_{1,i^{++},j^{++}}, g_{1,j^{++},i^{++}}, g_{1,j^{++},j^{++}}, g_{2,i^{++},i^{++}}, g_{2,i^{++},j^{++}},
 g_{2,j^{++},i^{++}}, g_{2,j^{++},j^{++}}, g_{3,i^{++},i^{++}}, g_{3,i^{++},j^{++}}, g_{3,j^{++},i^{++}}, g_{3,j^{++},j^{++}}, y_{i^{++},i^{++},i^{++}},
 y_{i^{++},i^{++},j^{++}}, y_{i^{++},j^{++},i^{++}}, y_{i^{++},j^{++},j^{++}}, y_{j^{++},i^{++},i^{++}}, y_{j^{++},i^{++},j^{++}}, y_{j^{++},j^{++},i^{++}}, y_{j^{++},j^{++},j^{++}}}
```

```
In[•]:= eqns = CoefficientRules[eqn, gys] /. (_ → c_) :> (c == 0)
```

```
Out[•]=
{c_4 + c_467 − c_470 + (c_470)/T + c_472 + (c_472)/T^2 − (2 c_472)/T − c_473 + (c_473)/T^3 − (3 c_473)/T^2 + (3 c_473)/T == 0, c_7 + (c_470)/T + (2 c_472)/T^2 − (2 c_472)/T + (3 c_473)/T^3 − (6 c_473)/T^2 + (3 c_473)/T == 0,
 c_13 − 3 c_467 + 3 T c_467 + 6 c_470 − (3 c_470)/T − 3 T c_470 − 9 c_472 − (3 c_472)/T^2 + (9 c_472)/T + 3 T c_472 + 12 c_473 − (3 c_473)/T^3 + (12 c_473)/T^2 − (18 c_473)/T − 3 T c_473 + T c_476 +
   c_478 − T c_478 − 2 c_479 + (c_479)/T + T c_479 + c_486 − T c_486 − 2 c_488 + (c_488)/T + T c_488 + 3 c_489 + (c_489)/T^2 − (3 c_489)/T − T c_489 == 0, ⋯ 457 ⋯,
 c_462 + S c_921 − (c_921)/T + c_922 − S c_922 + (c_922)/T − (c_922)/(S T) + S c_925 + c_926 − S c_926 == 0, c_463 + c_922 − (c_922)/(S T) + c_926 == 0, c_1 + c_464 == 0}
```

Full expression not available (original memory size: 3.2 MB)

In[•]:=
```
Short[{sol} = Solve[eqns, vars], 5]
```

Solve: Equations may not give solutions for all "solve" variables.

Out[•]//Short=

$$\left\{\left\{c_2 \to -c_{465} - \left(-1+\frac{1}{T}\right) c_{468} - \frac{\left(-S-T+3\,S\,T+T^2-2\,S\,T^2\right) c_{920}}{S\,T^2},\right.\right.$$
$$c_3 \to -c_{466} - \left(-1+\frac{1}{T}\right) c_{469} - \frac{\left(1-2\,T+T^2\right) c_{471}}{T^2},$$
$$c_4 \to -c_{467} - \left(-1+\frac{1}{T}\right) c_{470} - \frac{\left(1-2\,T+T^2\right) c_{472}}{T^2} - \left(-1+\frac{1}{T^3}-\frac{3}{T^2}+\frac{3}{T}\right) c_{473},$$
$$c_5 \to -\frac{c_{468}}{T} - \frac{(-S-T+2\,S\,T)\,c_{920}}{S\,T^2},\ \text{<<456>>},$$
$$c_{462} \to -\frac{(-1+S\,T)\,c_{921}}{T} - \frac{\left(-1+S+S\,T-S^2\,T\right) c_{922}}{S\,T} - S\,c_{925} - (1-S)\,c_{926},$$
$$\left.\left.c_{463} \to -\frac{(-1+S\,T)\,c_{922}}{S\,T} - c_{926},\ c_{464} \to -c_1\right\}\right\}$$

In[•]:=
```
Short[sol /. (v_ → val_) :> (v = CF[val])]
```

Out[•]//Short=

$$\left\{-\frac{S\,T^2\,c_{465}+\text{<<11>>}}{S\,T^2},\ -\frac{T^2\,c_{466}+\text{<<6>>}+T^2\,c_{471}}{T^2},\ -\frac{\text{<<1>>}}{T^3},\right.$$
$$\text{<<457>>},\ -\frac{\text{<<1>>}}{S\ \text{<<1>>}\ \text{<<1>>}},\ -\frac{-c_{922}+\text{<<1>>}+S\,T\,c_{\text{<<3>>}}}{S\,T},\ \left.-c_1\right\}$$

In[•]:=
```
Last@R1[1, i, j]
```

Out[•]=

$$-\frac{(-c_{922}+S\,T\,c_{922}+S\,T\,c_{926})\,y_{j,j,j}}{S\,T}$$

In[•]:=
```
Short[R1[1, i, j], 10]
```

Out[•]//Short=

<<1>>

Solving R3

In[•]:=
```
Clear[i, j, k];
lhs = Expand[R1[1, i, j] + R1[1, i⁺, k] + R1[1, j⁺, k⁺] //. gRules1,i,j ⋃
      gRules1,i⁺,k ⋃ gRules1,j⁺,k⁺ ⋃ yRules1,i,j ⋃ yRules1,i⁺,k ⋃ yRules1,j⁺,k⁺ /. _If → 0];
rhs = Expand[R1[1, j, k] + R1[1, i, k⁺] + R1[1, i⁺, j⁺] //. gRules1,j,k ⋃
      gRules1,i,k⁺ ⋃ gRules1,i⁺,j⁺ ⋃ yRules1,j,k ⋃ yRules1,i,k⁺ ⋃ yRules1,i⁺,j⁺ /. _If → 0];
eqn = lhs - rhs
```

Out[•]=

$$-2\,c_{922} + \frac{c_{922}}{S} + \frac{c_{922}}{T} - c_{468}\,g_{1,i^{++},j^{++}} + \frac{c_{468}\,g_{1,i^{++},j^{++}}}{T} - \frac{c_{920}\,g_{1,i^{++},j^{++}}}{T^2} + \frac{2\,c_{920}\,g_{1,i^{++},j^{++}}}{T} -$$
$$\frac{c_{920}\,g_{1,i^{++},j^{++}}}{S\,T} - c_{469}\,g_{1,i^{++},i^{++}}\,g_{1,i^{++},j^{++}} + \langle\!\langle 29005 \rangle\!\rangle + \frac{c_{926}\,y_{k^{++},j^{++},j^{++}}}{S} + \frac{c_{926}\,y_{k^{++},j^{++},j^{++}}}{T} - \frac{2\,c_{926}\,y_{k^{++},j^{++},j^{++}}}{S\,T} -$$
$$c_{926}\,y_{k^{++},j^{++},k^{++}} + \frac{c_{926}\,y_{k^{++},j^{++},k^{++}}}{T} - \frac{c_{926}\,y_{k^{++},k^{++},i^{++}}}{S^2} + \frac{c_{926}\,y_{k^{++},k^{++},i^{++}}}{S} - c_{926}\,y_{k^{++},k^{++},j^{++}} + \frac{c_{926}\,y_{k^{++},k^{++},j^{++}}}{S}$$

Full expression not available (original memory size: 18.7 MB)

In[•]:=
```
Short[vars = Union@Cases[eqn, c_, ∞]]
```

Out[•]//Short=

$$\{c_{465}, c_{466}, c_{467}, c_{468}, c_{469}, c_{470}, c_{471}, c_{472}, \langle\!\langle 446 \rangle\!\rangle, c_{919}, c_{920}, c_{921}, c_{922}, c_{923}, c_{924}, c_{925}, c_{926}\}$$

In[•]:=
```
gys = Cases[eqn, g_,_,_, ∞] ⋃ Cases[eqn, y__, ∞]
```

Out[•]=

$$\{g_{1,i^{++},i^{++}}, g_{1,i^{++},j^{++}}, g_{1,i^{++},k^{++}}, g_{1,j^{++},i^{++}}, g_{1,j^{++},j^{++}}, g_{1,j^{++},k^{++}}, g_{1,k^{++},i^{++}}, g_{1,k^{++},j^{++}},$$
$$g_{1,k^{++},k^{++}}, g_{2,i^{++},i^{++}}, g_{2,i^{++},j^{++}}, g_{2,i^{++},k^{++}}, g_{2,j^{++},i^{++}}, g_{2,j^{++},j^{++}}, g_{2,j^{++},k^{++}}, g_{2,k^{++},i^{++}},$$
$$g_{2,k^{++},j^{++}}, g_{2,k^{++},k^{++}}, g_{3,i^{++},i^{++}}, g_{3,i^{++},j^{++}}, g_{3,i^{++},k^{++}}, g_{3,j^{++},i^{++}}, g_{3,j^{++},j^{++}}, g_{3,j^{++},k^{++}}, g_{3,k^{++},i^{++}},$$
$$g_{3,k^{++},j^{++}}, g_{3,k^{++},k^{++}}, y_{i^{++},i^{++},j^{++}}, y_{i^{++},i^{++},k^{++}}, y_{i^{++},j^{++},i^{++}}, y_{i^{++},j^{++},j^{++}}, y_{i^{++},j^{++},k^{++}}, y_{i^{++},k^{++},i^{++}},$$
$$y_{i^{++},k^{++},j^{++}}, y_{i^{++},k^{++},k^{++}}, y_{j^{++},i^{++},i^{++}}, y_{j^{++},i^{++},j^{++}}, y_{j^{++},i^{++},k^{++}}, y_{j^{++},j^{++},i^{++}}, y_{j^{++},k^{++},i^{++}}, y_{j^{++},k^{++},k^{++}},$$
$$y_{k^{++},i^{++},i^{++}}, y_{k^{++},i^{++},j^{++}}, y_{k^{++},i^{++},k^{++}}, y_{k^{++},j^{++},i^{++}}, y_{k^{++},j^{++},j^{++}}, y_{k^{++},j^{++},k^{++}}, y_{k^{++},k^{++},i^{++}}, y_{k^{++},k^{++},j^{++}}\}$$

In[•]:=
```
Short[eqns = CoefficientRules[eqn, gys] /. (_ → c_) :> (c == 0)]
```

Out[•]//Short=

$$\left\{-c_{470} + \frac{c_{470}}{T} == 0,\ -c_{470} + T\,c_{470} == 0,\ \langle\!\langle 2799 \rangle\!\rangle,\ -c_{926} + \frac{c_{926}}{S} == 0,\ -2\,c_{922} + \frac{c_{922}}{S} + \frac{c_{922}}{T} == 0\right\}$$

In[•]:=
```
Short[{sol} = Solve[eqns, vars], 5]
```

Solve: Equations may not give solutions for all "solve" variables.

Out[•]//Short=

$$\left\{\left\{c_{466}\to 0,\ c_{467}\to 0,\ c_{468}\to 0,\ c_{469}\to 0,\ c_{470}\to 0,\ c_{471}\to 0,\ c_{472}\to 0,\ c_{473}\to 0,\right.\right.$$
$$c_{477}\to -\frac{T\,c_{475}}{1+T}-\frac{T\,c_{490}}{1+T},\ c_{478}\to 0,\ c_{479}\to 0,\ c_{480}\to -\frac{(-1+T)\,c_{475}}{1+T}-\frac{(1-T)\,c_{490}}{T\,(1+T)},$$
$$c_{481}\to -\frac{(-1+T)\,c_{476}}{T},\ c_{482}\to 0,\ c_{483}\to 0,\ c_{484}\to -T\,c_{465}-T\,c_{474},\ \text{<<}402\text{>>},\ c_{911}\to 0,$$
$$c_{912}\to 0,\ c_{913}\to -\frac{(-1+S)\,T\,c_{800}}{2\,(-1+S\,T)},\ c_{914}\to 0,\ c_{915}\to 0,\ c_{916}\to 0,\ c_{917}\to -c_{804},\ c_{918}\to 0,$$
$$\left.\left.c_{919}\to 0,\ c_{920}\to 0,\ c_{921}\to 0,\ c_{922}\to 0,\ c_{923}\to 0,\ c_{924}\to 0,\ c_{925}\to 0,\ c_{926}\to 0\right\}\right\}$$

In[•]:=
```
Short[sol /. (v_ → val_) :> (v = CF[val])]
```

Out[•]//Short=

$$\left\{0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ -\frac{T\,(c_{475}+c_{490})}{1+T},\ 0,\ 0,\ \text{<<}412\text{>>},\ 0,\ -c_{804},\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\right\}$$

In[•]:=
```
Short[R1[1, i, j], 5]
```

Out[•]//Short=

$$c_1-c_{465}\,g_{1,i,i}-\frac{\left(-S\,T^2\,c_{465}+S\,T^3\,c_{465}+S\,T^3\,c_{474}-S\,T^3\,(c_{465}+c_{474})+S\,T^4\,(c_{465}+c_{474})\right)\,g_{1,j,i}}{S\,T^2}-$$
$$\frac{\left(T^3\,c_{475}-\frac{2\ \text{<<}1\text{>>}\ (c_{475}+c_{\text{<<}3\text{>>}})}{1+T}+\frac{2\,T^4\,(c_{475}+c_{490})}{1+T}\right)\,g_{1,i,i}\,g_{1,j,i}}{T^2}-$$
$$T\,c_{476}\,g_{1,i,i}^2\,g_{1,j,i}+\text{<<}190\text{>>}+S\,T\,c_{800}\,g_{2,j,j}\,g_{3,j,i}\,g_{3,j,j}-\frac{\text{<<}1\text{>>}}{S^3\,T^3}+$$
$$\frac{S\,(-1+T)\,T\,c_{796}\,g_{1,j,i}\,g_{3,j,j}^2}{2\,(-1+S\,T)}+\frac{(-1+S)\,S\,T\,c_{800}\,g_{2,j,i}\,g_{3,j,j}^2}{2\,(-1+S\,T)}+S\,T\,c_{804}\,g_{3,j,i}\,g_{3,j,j}^2$$

In[•]:=
```
R1[1, i, j] // Simplify
```

Out[•]=

$$c_1-c_{465}\,g_{1,i,i}-\left(\left(-1+T^2\right)\,c_{465}+T^2\,c_{474}\right)\,g_{1,j,i}-\frac{T\,((-1+3\,T)\,c_{475}+2\,(-1+T)\,c_{490})\,g_{1,i,i}\,g_{1,j,i}}{1+T}-$$
$$T\,c_{476}\,g_{1,i,i}^2\,g_{1,j,i}+\frac{T\,(c_{475}+c_{490})\,g_{1,i,j}\,g_{1,j,i}}{1+T}-\frac{(-1+T)\,T\,((-1+2\,T)\,c_{475}+(-2+T)\,c_{490})\,g_{1,j,i}^2}{1+T}-$$
$$(-1+T)\,T\,c_{476}\,g_{1,i,i}\,g_{1,j,i}^2+T\,(c_{465}+c_{474})\,g_{1,j,j}+\frac{T\,(c_{475}+c_{490})\,g_{1,i,i}\,g_{1,j,j}}{1+T}+$$
$$\frac{T\,(2\,(-1+T)\,c_{475}+(-3+T)\,c_{490})\,g_{1,j,i}\,g_{1,j,j}}{1+T}-(-1+T)\,T\,c_{476}\,g_{1,j,i}^2\,g_{1,j,j}+$$
$$T\,c_{476}\,g_{1,j,i}\,g_{1,j,j}^2-c_{499}\,g_{2,i,i}-\frac{S\,(-1+T)\,c_{549}\,g_{1,j,i}\,g_{2,i,i}}{-1+S}-T\,c_{506}\,g_{1,i,i}\,g_{1,j,i}\,g_{2,i,i}-$$
$$\frac{1}{2}\,(-1+T)\,T\,c_{506}\,g_{1,j,i}^2\,g_{2,i,i}+\left(-\frac{T\,c_{505}}{-1+T}+\frac{S\,c_{549}}{-1+S}\right)\,g_{1,j,j}\,g_{2,i,i}-\frac{S\,(-1+T)\,c_{564}\,g_{1,j,i}\,g_{2,i,i}^2}{2\,(-1+S)}-$$

$$\left(\left(-1+S^2\right)\,c_{499}+S^2\,c_{548}\right)\,g_{2,j,i}-\frac{(-1+S)\,T\,c_{505}\,g_{1,i,i}\,g_{2,j,i}}{-1+T}-\frac{(-1+S)\,T\,c_{506}\,g_{1,i,i}^2\,g_{2,j,i}}{2\,(-1+T)}-$$

$$((-1+S)\,T\,c_{505}+S\,(-1+T)\,c_{549})\,g_{1,j,i}\,g_{2,j,i}-(-1+S)\,T\,c_{506}\,g_{1,i,i}\,g_{1,j,i}\,g_{2,j,i}+$$

$$S\,c_{549}\,g_{1,j,j}\,g_{2,j,i}-(-1+S)\,T\,c_{506}\,g_{1,j,i}\,g_{1,j,j}\,g_{2,j,i}+\frac{(-1+S)\,T\,c_{506}\,g_{1,j,j}^2\,g_{2,j,i}}{2\,(-1+T)}-$$

$$\frac{S\,((-1+3\,S)\,c_{563}+2\,(-1+S)\,c_{612})\,g_{2,i,i}\,g_{2,j,i}}{1+S}-S\,c_{564}\,g_{1,i,i}\,g_{2,i,i}\,g_{2,j,i}-$$

$$S\,(-1+T)\,c_{564}\,g_{1,j,i}\,g_{2,i,i}\,g_{2,j,i}-S\,c_{568}\,g_{2,i,i}^2\,g_{2,j,i}+\frac{S\,(c_{563}+c_{612})\,g_{2,i,j}\,g_{2,j,i}}{1+S}-$$

$$\frac{(-1+S)\,S\,((-1+2\,S)\,c_{563}+(-2+S)\,c_{612})\,g_{2,j,i}^2}{1+S}-\frac{1}{2}\,(-1+S)\,S\,c_{564}\,g_{1,i,i}\,g_{2,j,i}^2-$$

$$\frac{1}{2}\,(-1+S)\,S\,c_{564}\,g_{1,j,j}\,g_{2,j,i}^2-(-1+S)\,S\,c_{568}\,g_{2,i,i}\,g_{2,j,i}^2+S\,(c_{499}+c_{548})\,g_{2,j,j}+$$

$$\frac{((-1+S)\,T\,c_{505}-S\,(-1+T)\,c_{549})\,g_{1,i,i}\,g_{2,j,j}}{(-1+S)\,(-1+T)}+T\,c_{505}\,g_{1,j,i}\,g_{2,j,j}-\frac{1}{2}\,(-1+T)\,T\,c_{506}\,g_{1,j,i}^2\,g_{2,j,j}+$$

$$T\,c_{506}\,g_{1,j,i}\,g_{1,j,j}\,g_{2,j,j}+\frac{S\,(c_{563}+c_{612})\,g_{2,i,i}\,g_{2,j,j}}{1+S}+\frac{S\,(2\,(-1+S)\,c_{563}+(-3+S)\,c_{612})\,g_{2,j,i}\,g_{2,j,j}}{1+S}-$$

$$S\,(-1+T)\,c_{564}\,g_{1,j,i}\,g_{2,j,i}\,g_{2,j,j}+S\,c_{564}\,g_{1,j,j}\,g_{2,j,i}\,g_{2,j,j}-(-1+S)\,S\,c_{568}\,g_{2,j,i}^2\,g_{2,j,j}+$$

$$\frac{S\,(-1+T)\,c_{564}\,g_{1,j,i}\,g_{2,j,j}^2}{2\,(-1+S)}+S\,c_{568}\,g_{2,j,i}\,g_{2,j,j}^2-c_{629}\,g_{3,i,i}-\frac{S\,(-1+T)\,T\,c_{751}\,g_{1,j,i}\,g_{3,i,i}}{-1+S\,T}-$$

$$T\,c_{636}\,g_{1,i,i}\,g_{1,j,i}\,g_{3,i,i}-\frac{1}{2}\,(-1+T)\,T\,c_{636}\,g_{1,j,i}^2\,g_{3,i,i}+T\left(-\frac{c_{635}}{-1+T}+\frac{S\,c_{751}}{-1+S\,T}\right)\,g_{1,j,j}\,g_{3,i,i}-$$

$$T\,c_{647}\,g_{1,j,i}\,g_{2,i,i}\,g_{3,i,i}-\frac{(-1+S)\,S\,T\,c_{765}\,g_{2,j,i}\,g_{3,i,i}}{-1+S\,T}-\frac{(-1+S)\,T\,c_{647}\,g_{1,i,i}\,g_{2,j,i}\,g_{3,i,i}}{-1+T}-$$

$$(-1+S)\,T\,c_{647}\,g_{1,j,i}\,g_{2,j,i}\,g_{3,i,i}-S\,c_{662}\,g_{2,i,i}\,g_{2,j,i}\,g_{3,i,i}-\frac{1}{2}\,(-1+S)\,S\,c_{662}\,g_{2,j,i}^2\,g_{3,i,i}+$$

$$S\left(-\frac{c_{657}}{-1+S}+\frac{T\,c_{765}}{-1+S\,T}\right)\,g_{2,j,j}\,g_{3,i,i}+\frac{S\,(-1+T)\,T\,c_{796}\,g_{1,j,i}\,g_{3,i,i}^2}{2-2\,S\,T}+\frac{(-1+S)\,S\,T\,c_{800}\,g_{2,j,i}\,g_{3,i,i}^2}{2-2\,S\,T}-$$

$$\left(\left(-1+S^2\,T^2\right)\,c_{629}+S^2\,T^2\,c_{750}\right)\,g_{3,j,i}-\frac{T\,(-1+S\,T)\,c_{635}\,g_{1,i,i}\,g_{3,j,i}}{-1+T}-$$

$$\frac{T\,(-1+S\,T)\,c_{636}\,g_{1,i,i}^2\,g_{3,j,i}}{2\,(-1+T)}-T\,((-1+S\,T)\,c_{635}+S\,(-1+T)\,c_{751})\,g_{1,j,i}\,g_{3,j,i}-$$

$$T\,(-1+S\,T)\,c_{636}\,g_{1,i,i}\,g_{1,j,i}\,g_{3,j,i}+S\,T\,c_{751}\,g_{1,j,j}\,g_{3,j,i}-T\,(-1+S\,T)\,c_{636}\,g_{1,j,i}\,g_{1,j,j}\,g_{3,j,i}+$$

$$\frac{T\,(-1+S\,T)\,c_{636}\,g_{1,j,j}^2\,g_{3,j,i}}{2\,(-1+T)}-\frac{S\,(-1+S\,T)\,c_{657}\,g_{2,i,i}\,g_{3,j,i}}{-1+S}-$$

$$\frac{T\,(-1+S\,T)\,c_{647}\,g_{1,i,i}\,g_{2,i,i}\,g_{3,j,i}}{-1+T}-T\,(-1+S\,T)\,c_{647}\,g_{1,j,i}\,g_{2,i,i}\,g_{3,j,i}-$$

$$\frac{S\,(-1+S\,T)\,c_{662}\,g_{2,i,i}^2\,g_{3,j,i}}{2\,(-1+S)}-S\,((-1+S\,T)\,c_{657}+(-1+S)\,T\,c_{765})\,g_{2,j,i}\,g_{3,j,i}-$$

$$\frac{(-1+S)\,T\,(-1+S\,T)\,c_{647}\,g_{1,i,i}\,g_{2,j,i}\,g_{3,j,i}}{-1+T}-\frac{(-1+S)\,T\,(-1+S\,T)\,c_{647}\,g_{1,j,j}\,g_{2,j,i}\,g_{3,j,i}}{-1+T}-$$

$$S\,(-1+S\,T)\,c_{662}\,g_{2,i,i}\,g_{2,j,i}\,g_{3,j,i}+S\,T\,c_{765}\,g_{2,j,j}\,g_{3,j,i}-T\,(-1+S\,T)\,c_{647}\,g_{1,j,i}\,g_{2,j,j}\,g_{3,j,i}+\frac{T\,(-1+S\,T)\,c_{647}\,g_{1,j,j}\,g_{2,j,j}\,g_{3,j,i}}{-1+T}-S\,(-1+S\,T)\,c_{662}\,g_{2,j,i}\,g_{2,j,j}\,g_{3,j,i}+\frac{S\,(-1+S\,T)\,c_{662}\,g_{2,j,j}^2\,g_{3,j,i}}{2\,(-1+S)}-\frac{S\,T\,((-1+3\,S\,T)\,c_{795}+2\,(-1+S\,T)\,c_{894})\,g_{3,i,i}\,g_{3,j,i}}{1+S\,T}-S\,T\,c_{796}\,g_{1,i,i}\,g_{3,i,i}\,g_{3,j,i}-S\,(-1+T)\,T\,c_{796}\,g_{1,j,i}\,g_{3,i,i}\,g_{3,j,i}-S\,T\,c_{800}\,g_{2,i,i}\,g_{3,i,i}\,g_{3,j,i}-(-1+S)\,S\,T\,c_{800}\,g_{2,j,i}\,g_{3,i,i}\,g_{3,j,i}-S\,T\,c_{804}\,g_{3,i,i}^2\,g_{3,j,i}+\frac{S\,T\,(c_{795}+c_{894})\,g_{3,i,j}\,g_{3,j,i}}{1+S\,T}-\frac{S\,T\,(-1+S\,T)\,((-1+2\,S\,T)\,c_{795}+(-2+S\,T)\,c_{894})\,g_{3,j,i}^2}{1+S\,T}-\frac{1}{2}\,S\,T\,(-1+S\,T)\,c_{796}\,g_{1,i,i}\,g_{3,j,i}^2-\frac{1}{2}\,S\,T\,(-1+S\,T)\,c_{796}\,g_{1,j,j}\,g_{3,j,i}^2-\frac{1}{2}\,S\,T\,(-1+S\,T)\,c_{800}\,g_{2,i,i}\,g_{3,j,i}^2-\frac{1}{2}\,S\,T\,(-1+S\,T)\,c_{800}\,g_{2,j,j}\,g_{3,j,i}^2-S\,T\,(-1+S\,T)\,c_{804}\,g_{3,i,i}\,g_{3,j,i}^2+S\,T\,(c_{629}+c_{750})\,g_{3,j,j}+\frac{T\,((-1+S\,T)\,c_{635}-S\,(-1+T)\,c_{751})\,g_{1,i,i}\,g_{3,j,j}}{(-1+T)\,(-1+S\,T)}+T\,c_{635}\,g_{1,j,i}\,g_{3,j,j}-\frac{1}{2}\,(-1+T)\,T\,c_{636}\,g_{1,j,i}^2\,g_{3,j,j}+T\,c_{636}\,g_{1,j,i}\,g_{1,j,j}\,g_{3,j,j}+\frac{S\,((-1+S\,T)\,c_{657}-(-1+S)\,T\,c_{765})\,g_{2,i,i}\,g_{3,j,j}}{(-1+S)\,(-1+S\,T)}+S\,c_{657}\,g_{2,j,i}\,g_{3,j,j}-(-1+S)\,T\,c_{647}\,g_{1,j,i}\,g_{2,j,i}\,g_{3,j,j}+\frac{(-1+S)\,T\,c_{647}\,g_{1,j,j}\,g_{2,j,i}\,g_{3,j,j}}{-1+T}-\frac{1}{2}\,(-1+S)\,S\,c_{662}\,g_{2,j,i}^2\,g_{3,j,j}+T\,c_{647}\,g_{1,j,i}\,g_{2,j,j}\,g_{3,j,j}+S\,c_{662}\,g_{2,j,i}\,g_{2,j,j}\,g_{3,j,j}+\frac{S\,T\,(c_{795}+c_{894})\,g_{3,i,i}\,g_{3,j,j}}{1+S\,T}+\frac{S\,T\,(2\,(-1+S\,T)\,c_{795}+(-3+S\,T)\,c_{894})\,g_{3,j,i}\,g_{3,j,j}}{1+S\,T}-S\,(-1+T)\,T\,c_{796}\,g_{1,j,i}\,g_{3,j,i}\,g_{3,j,j}+S\,T\,c_{796}\,g_{1,j,j}\,g_{3,j,i}\,g_{3,j,j}-(-1+S)\,S\,T\,c_{800}\,g_{2,j,i}\,g_{3,j,i}\,g_{3,j,j}+S\,T\,c_{800}\,g_{2,j,j}\,g_{3,j,i}\,g_{3,j,j}-S\,T\,(-1+S\,T)\,c_{804}\,g_{3,j,i}^2\,g_{3,j,j}+\frac{S\,(-1+T)\,T\,c_{796}\,g_{1,j,i}\,g_{3,j,j}^2}{-2+2\,S\,T}+\frac{(-1+S)\,S\,T\,c_{800}\,g_{2,j,i}\,g_{3,j,j}^2}{-2+2\,S\,T}+S\,T\,c_{804}\,g_{3,j,i}\,g_{3,j,j}^2$$