In[∘]:=
```
SetDirectory["C:\\Users\\T15Roland\\Wiskunde\\Bn\\HigherRank"];
Once[<< KnotTheory`];
 << Rot.m
(α_⁺)⁺ := α"++";
(* this is for cosmetic reasons only *)
```

ParentDirectory: Argument File should be a positive machine-size integer, a nonempty string, or a File specification.

ParentDirectory: Argument File should be a positive machine-size integer, a nonempty string, or a File specification.

ToFileName: String or list of strings expected at position 1 in ToFileName[{File, WikiLink, mathematica}].

ToFileName: String or list of strings expected at position 1 in ToFileName[{File, QuantumGroups}].

```
Loading KnotTheory` version of September 6, 2014, 13:37:37.2841.
Read more at http://katlas.org/wiki/KnotTheory.
Loading Rot.m from http://drorbn.net/AP/Projects/HigherRank to compute rotation numbers.
```

In[∘]:=

$$r_0[1, i\_, j\_] := p_{3,j}\, x_{1,i}\, x_{2,i} - \frac{p_{3,j}\, x_{1,j}\, x_{2,i}}{T_1} \quad \text{(*from r0p*)}$$

$$r_0[-1, i\_, j\_] := -\frac{p_{3,j}\, x_{1,i}\, x_{2,i}}{T_1^2\, T_2} + \frac{p_{3,j}\, x_{1,j}\, x_{2,i}}{T_1\, T_2}$$

$$\begin{aligned} r_1[1, i\_, j\_] := {} & \frac{T_2\, p_{1,j}\, p_{2,j}\, x_{1,i}\, x_{2,i}}{-1 + T_1 T_2} - \frac{p_{1,j}\, p_{2,i}\, x_{1,j}\, x_{2,i}}{(-1 + T_1)\, T_1\, (-1 + T_1 T_2)} - \\ & \frac{p_{1,j}\, p_{2,j}\, x_{1,j}\, x_{2,i}}{(-1 + T_1)\, T_1} + \frac{p_{1,i}\, p_{2,j}\, x_{1,i}\, x_{2,j}}{(-1 + T_1)\, (-1 + T_1 T_2)} + p_{1,j}\, p_{2,i}\, x_{3,i} - p_{1,j}\, p_{2,j}\, x_{3,i} + \\ & \frac{p_{3,j}\, x_{3,i}}{T_1\, (-1 + T_1 T_2)} - p_{1,j}\, p_{3,j}\, x_{1,i}\, x_{3,i} + \frac{p_{1,j}\, p_{3,i}\, x_{1,j}\, x_{3,i}}{(-1 + T_1)\, T_1\, (-1 + T_1 T_2)} + \frac{p_{1,j}\, p_{3,j}\, x_{1,j}\, x_{3,i}}{-1 + T_1} - \\ & \frac{T_2\, p_{2,j}\, p_{3,j}\, x_{2,i}\, x_{3,i}}{T_1} - \frac{p_{2,j}\, p_{3,i}\, x_{2,j}\, x_{3,i}}{T_1\, (-1 + T_1 T_2)} - \frac{p_{1,i}\, p_{3,j}\, x_{1,i}\, x_{3,j}}{(-1 + T_1)\, (-1 + T_1 T_2)} + \frac{T_2\, p_{2,j}\, p_{3,j}\, x_{2,i}\, x_{3,j}}{T_1\, (-1 + T_1 T_2)} \end{aligned}$$

$$\begin{aligned} r_1[-1, i\_, j\_] := {} & \\ & \frac{p_{1,j}\, p_{2,i}\, x_{1,i}\, x_{2,i}}{T_1^2\, (-1 + T_1 T_2)} - \frac{(-1 + T_2)\, p_{1,i}\, p_{2,j}\, x_{1,i}\, x_{2,i}}{(-1 + T_1)\, T_2\, (-1 + T_1 T_2)} + \frac{(-T_1 - T_2 + T_1 T_2)\, p_{1,j}\, p_{2,j}\, x_{1,i}\, x_{2,i}}{T_1^2\, T_2\, (-1 + T_1 T_2)} + \\ & \frac{p_{1,j}\, p_{2,i}\, x_{1,j}\, x_{2,i}}{(-1 + T_1)\, T_1\, (-1 + T_1 T_2)} + \frac{p_{1,j}\, p_{2,j}\, x_{1,j}\, x_{2,i}}{T_1\, (-1 + T_1 T_2)} - \frac{p_{1,i}\, p_{2,j}\, x_{1,i}\, x_{2,j}}{(-1 + T_1)\, (-1 + T_1 T_2)} + \frac{p_{1,j}\, p_{2,j}\, x_{1,i}\, x_{2,j}}{T_1\, (-1 + T_1 T_2)} - \\ & \frac{p_{1,j}\, p_{2,i}\, x_{3,i}}{T_1} + \frac{p_{1,j}\, p_{2,j}\, x_{3,i}}{T_1} - \frac{p_{3,j}\, x_{3,i}}{T_1\, (-1 + T_1 T_2)} - \frac{p_{1,j}\, p_{3,i}\, x_{1,i}\, x_{3,i}}{T_1^2\, (-1 + T_1 T_2)} + \frac{p_{1,i}\, p_{3,j}\, x_{1,i}\, x_{3,i}}{(-1 + T_1)\, T_1\, T_2} - \\ & \frac{p_{1,j}\, p_{3,j}\, x_{1,i}\, x_{3,i}}{T_1^2\, T_2} - \frac{p_{1,j}\, p_{3,i}\, x_{1,j}\, x_{3,i}}{(-1 + T_1)\, T_1\, (-1 + T_1 T_2)} + \frac{(-1 + T_2)\, p_{2,j}\, p_{3,i}\, x_{2,i}\, x_{3,i}}{T_1\, T_2\, (-1 + T_1 T_2)} + \\ & \frac{p_{2,i}\, p_{3,j}\, x_{2,i}\, x_{3,i}}{T_1^2\, T_2} - \frac{(-1 + 2\, T_2)\, p_{2,j}\, p_{3,j}\, x_{2,i}\, x_{3,i}}{T_1^2\, T_2^2} + \frac{p_{2,j}\, p_{3,i}\, x_{2,j}\, x_{3,i}}{T_1\, (-1 + T_1 T_2)} - \\ & \frac{p_{2,j}\, p_{3,j}\, x_{2,j}\, x_{3,i}}{T_1^2\, T_2} + \frac{p_{1,i}\, p_{3,j}\, x_{1,i}\, x_{3,j}}{(-1 + T_1)\, (-1 + T_1 T_2)} - \frac{p_{1,j}\, p_{3,j}\, x_{1,i}\, x_{3,j}}{T_1\, (-1 + T_1 T_2)} - \frac{p_{2,j}\, p_{3,j}\, x_{2,i}\, x_{3,j}}{T_1\, (-1 + T_1 T_2)} \end{aligned}$$

In[∘]:=
```
g2px[ℰ_] := Module[{λ}, Expand[ℰ /. g_{α_,i_,j_} :→ λ p_{α,i} x_{α,j}] /. λ^{k_.} :→ 1 / k!]
```

In[∘]:=
```
{p^*, x^*, π^*, ξ^*} = {π, ξ, p, x};  (u_{i__})^* := (u^*)_i;
```

In[∘]:=
```
Zip_{{}}[ℰ_] := ℰ;
Zip_{{ζ_,ζs___}}[ℰ_] := (Collect[ℰ // Zip_{{ζs}}, ζ] /. f_. ζ^{d_.} :→ (D[f, {ζ^*, d}])) /. ζ^* → 0
```

In[∘]:=
```
px2g[ℰ_] := Module[{ps, xs, Q},
  ps = Union[Cases[ℰ, p__, ∞]];
  xs = Union[Cases[ℰ, x__, ∞]];
  Q = Sum[p0^* x0^* g_{p0[[2]],x0[[2]],p0[[3]],x0[[3]]}, {p0, ps}, {x0, xs}];
  Expand[Zip_{ps⋃xs}[ℰ e^Q] /. g_{α_,β_,i_,j_} :→ If[α === β, g_{α,i,j}, 0]]
 ]
```

In[∘]:=
```
R_1[1, i_, j_] := Evaluate[px2g[r_1[1, i, j]] +
    (Coefficient[r_1[1, i, j] /. t : (x | p)__ :→ λ t, λ^3] /. x_{3,α_} p_{1,β_} p_{2,γ_} :→ y_{α,β,γ})]
R_1[-1, i_, j_] := Evaluate[px2g[r_1[-1, i, j]] +
    (Coefficient[r_1[-1, i, j] /. t : (x | p)__ :→ λ t, λ^3] /. x_{3,α_} p_{1,β_} p_{2,γ_} :→ y_{α,β,γ})]
Piv_{i_} := -1/(T_1 (-1 + T_1 T_2)) g_{3,i,i} (* -((-2+T_1+T_2) (-T_1-T_2+2 T_1 T_2) g_{3,i,i})/((-1+T_1) (-1+T_2) (-1+T_1 T_2)) *)
```

In[∘]:=
```
θ[1, i_, j_, α_, β_, γ_] :=
  Evaluate[r_0[1, i, j] /. {p_{3,j_} :→ g_{3,j,α}, x_{1,i_} :→ g_{1,β,i}, x_{2,i_} :→ g_{2,γ,i}}] ;
 (* The θ graph with light (pxx) vertex  at (1,i,j) and
 unspecified heavy (xpp) vertex *)
θ[-1, i_, j_, α_, β_, γ_] :=
  Evaluate[r_0[-1, i, j] /. {p_{3,j_} :→ g_{3,j,α}, x_{1,i_} :→ g_{1,β,i}, x_{2,i_} :→ g_{2,γ,i}}] ;
 (* The θ graph with light (pxx) vertex  at (-1,i,j)
 and unspecified heavy (xpp) vertex *)
θ[1, 5, 8, 21, 22, 23]
```

Out[∘]=

$$g_{1,22,5}\, g_{2,23,5}\, g_{3,8,21} - \frac{g_{1,22,8}\, g_{2,23,5}\, g_{3,8,21}}{T_1}$$

In[ ]:=
```
T₃ = T₁ T₂;
CF[ℰ_] := Factor@Together[ℰ];
λ[K_] := Module[{Cs, φ, n, A, s, i, j, k, Δ, G, gEval, Y, yEval, c, λ1},
   {Cs, φ} = Rot[K]; n = Length[Cs];
   A = IdentityMatrix[2 n + 1];
   Cases[Cs, {s_, i_, j_} ⧴ (A[[{i, j}, {i + 1, j + 1}]] += ( -T^s  T^s - 1 ; 0  -1 ))];
   Δ = T^((-Total[φ] - Total[Cs[[All,1]]])/2) Det[A];
   G = Inverse[A];
   gEval[ℰ_] := CF[ℰ /.
      {g_{1,α_,β_} ⧴ (G[[α, β]] /. T → T₁),
       g_{2,α_,β_} ⧴ (G[[α, β]] /. T → T₂), g_{3,α_,β_} ⧴ (G[[α, β]] /. T → T₃)}];
   Y[α_, β_, γ_] :=
    Y[α, β, γ] = Sum[{s, i, j} = c; (* The expectation value of x_{3,α} p_{1,β} p_{2,γ} *)
      Θ[s, i, j, α, β, γ],
      {c, Cs}];
   yEval[ℰ_] := ℰ /. y_{α_,β_,γ_} ⧴ Y[α, β, γ];
   λ1 = Sum_{k=1}^{n} R₁ @@ Cs[[k]] + Sum_{k=1}^{2n} φ[[k]] Piv_k;
   {Δ, (1 - T₃) (Δ /. T → T₁) (Δ /. T → T₂) (Δ /. T → T₃) λ1} // yEval // gEval // Expand
  ];
Θ[K_] := Module[{L = λ[K]},
  {L[[1]], T₁ L[[2]] + (T D[L[[1]], T] /. T → T₃) (L[[1]] /. T → T₁) (L[[1]] /. T → T₂)} // Expand]
```

In[•]:=
```
CF[ℰ_] := Factor@Together[ℰ];
Nλ_{p1_,p2_}[K_] := Module[{G1, G2, G3, Δ1, Δ2, Δ3,
    A1, A2, A3, Cs, φ, n, A, s, i, j, k, Δ, G, gEval, Y, yEval, c, λ1},
   {Cs, φ} = Rot[K]; n = Length[Cs];
   A = IdentityMatrix[2 n + 1];
   Cases[Cs, {s_, i_, j_} ⧴ (A⟦{i, j}, {i + 1, j + 1}⟧ += ( -T^s  T^s - 1
                                                              0     -1  ))];
   A1 = A /. T → p1; A2 = A /. T → p2; A3 = A /. T → p1 p2;
   Δ1 = p1^((-Total[φ]-Total[Cs⟦All,1⟧])/2) Det[A1];
   Δ2 = p2^((-Total[φ]-Total[Cs⟦All,1⟧])/2) Det[A2];
   Δ3 = (p1 p2)^((-Total[φ]-Total[Cs⟦All,1⟧])/2) Det[A3];
   G1 = Inverse[A1]; G2 = Inverse[A2]; G3 = Inverse[A3];
   gEval[ℰ_] := CF[ℰ /.
      {g_{1,α_,β_} ⧴ G1⟦α, β⟧, g_{2,α_,β_} ⧴ G2⟦α, β⟧, g_{3,α_,β_} ⧴ G3⟦α, β⟧}];
   Y[α_, β_, γ_] :=
    Y[α, β, γ] = Sum[{s, i, j} = c; (* The expectation value of x_{3,α}p_{1,β}p_{2,γ} *)
       θ[s, i, j, α, β, γ],
       {c, Cs}] /. {T_1 → p1, T_2 → p2};
   yEval[ℰ_] := ℰ /. y_{α_,β_,γ_} ⧴ Y[α, β, γ];
   λ1 = Σ_{k=1}^{n} R_1 @@ Cs⟦k⟧ + Σ_{k=1}^{2 n} φ⟦k⟧ Piv_k /. {T_1 → p1, T_2 → p2};
   {Δ1, (1 - p1 p2) Δ1 Δ2 Δ3 λ1} // yEval // gEval // Expand
  ];
```

In[•]:=
```
Rrho_1[s_, i_, j_] := s (g_{ji} (g_{j+1,j} + g_{j,j+1} - g_{ij}) - g_{ii} (g_{j,j+1} - 1) - 1 / 2);
ρ[K_] := ρ[K] = Module[{Cs, φ, n, A, s, i, j, k, Δ, G, ρ1},
    {Cs, φ} = Rot[K]; n = Length[Cs];
    A = IdentityMatrix[2 n + 1];
    Cases[Cs, {s_, i_, j_} ⧴ (A⟦{i, j}, {i + 1, j + 1}⟧ += ( -T^s  T^s - 1
                                                               0     -1  ))];
    Δ = T^((-Total[φ]-Total[Cs⟦All,1⟧])/2) Det[A];
    G = Inverse[A];
    ρ1 = Σ_{k=1}^{n} Rrho_1 @@ Cs⟦k⟧ - Σ_{k=1}^{2 n} φ⟦k⟧ (g_{kk} - 1 / 2);
    Expand@Together@{Δ, Δ^2 ρ1 /. g_{α_,β_} ⧴ G⟦α, β⟧}

   ];
```

```
In[∘]:= ColFun[t_] := If[t > 0, {t, 0, 0}, {0, 0, t}]
        Renorm[t_] := If[t == 0, 0, Sign[t] × Log[Abs[t] + 10]]
        Poly2Pic[P_] := Module[{e1 = Exponent[P, T1^-1], e2 = Exponent[P, T2^-1], Mat},
          If[P === 0, P, Mat =
            Map[Renorm, Normal@SparseArray[CoefficientRules[T1^(e1+1) T2^(e2+1) P, {T1, T2}]], {2}];
           MatrixPlot[Mat (*,ColorFunction→(RGBColor[If[#==0,0,1],0,0]&)*)]]
         ]
```

Relation to ρ_1:

```
In[∘]:= CheckRelationTorho1[K_] := Module[{th = θ[K][[2]], rh = ρ[K][[2]]},
          ({th /. {T1 → 1}, th /. {T2 → 1}} + rh) /. T_ → T // Together]

In[∘]:= CheckRelationTorho1 /@ AllKnots[{3, 8}]

Out[∘]=
{{0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0},
 {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0},
 {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}}
```

Symmetries

```
In[∘]:= CheckT12swapsym[K_] := Module[{th = θ[K][[2]]}, {th - (th /. {T1 → T2, T2 → T1})}]

In[∘]:= CheckT12swapsym /@ AllKnots[{3, 8}] // Union

Out[∘]=
{{0}}

In[∘]:= CheckT12swapsym[Knot[11, NonAlternating, 34]]

Out[∘]=
{0}

In[∘]:= CheckMirr[K_] := Module[{th = θ[K][[2]], thm = θ[Mirror@K][[2]]}, {th + thm}]
        CheckMirr /@ AllKnots[{3, 7}] // Union

Out[∘]=
{{0}}

In[∘]:= CheckMirr[Knot[11, NonAlternating, 34]]

Out[∘]=
{0}

In[∘]:= CheckT1T2palin[K_] := Module[{th = θ[K][[2]]}, {th - (th /. {T1 → T1^-1, T2 → T2^-1})}]

In[∘]:= CheckT1T2palin /@ AllKnots[{3, 8}] // Union

Out[∘]=
{{0}}
```

In[∘]:=
```
CheckT1T2palin[Knot[11, NonAlternating, 34]]
```
Out[∘]=
```
{0}
```

In[∘]:=
```
CheckT1T2mystsym[K_] := Module[{th = θ[K][[2]]}, {th - (th /. {T1 → T1, T2 → T3^-1})}]
CheckT1T2mystsym /@ AllKnots[{3, 8}]
```
Out[∘]=
```
{{0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0},
 {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}}
```

In[∘]:=
```
CheckT1T2mystsym[Knot[11, NonAlternating, 34]]
```
Out[∘]=
```
{0}
```

Moving to better variables, very similar to Garoufalidis-Kashaev:

$$u = T_1 + T_1^{-1} + T_2 + T_2^{-1} + T_3 + T_3^{-1} - 2$$

$$v = T_1^2\, T_2 + T_1^{-2}\, T_2^{-1} + T_2^2\, T_1 + T_2^{-2}\, T_1^{-1} + T_1\, T_2^{-1} - T_1^{-1}\, T_2 - 2$$

In[∘]:=
```
{u - (u /. {s → t, t → s}), v - (v /. {s → t, t → s})}
{u - (u /. {s → s^-1, t → t^-1}), v - (v /. {s → s^-1, t → t^-1})}
{u - (u /. {t → 1 / (s t)}), v - (v /. {t → 1 / (s t)})} // Together
```
Out[∘]=
```
{0, 0}
```
Out[∘]=
```
{0, 0}
```
Out[∘]=
```
{0, 0}
```

In[∘]:=
```
pp[x_] := x + x^-1
u = pp[s] + pp[t] + pp[s t] + 1;
v = pp[s^2 t] + pp[s t^2] + pp[s t^-1] + 1;
Monomials_k_ [a_, b_] := Flatten@Table[a^m b^n, {m, 0, k}, {n, 0, k - m}]
```

In[•]:=

```
(*This code is not optimal and runs too slowly!*)
ToUV[Q_] :=
 Module[{P = Q /. {T1 → s, T2 → t}, deg, degs, degt, ShiftP, UVMons, Coefs, sol, eqs, cr},
  If[P == 0, Return[0]];
  deg = Exponent[P /. {t → s}, s];
  UVMons = Expand[Monomials_deg[u, v]];
  degs = Exponent[P /. s → 1 / s, s];
  degt = Exponent[P /. t → 1 / t, t];

  degs = Max@Append[Table[Exponent[μ /. s → 1 / s, s], {μ, UVMons}], degs];
  degt = Max@Append[Table[Exponent[μ /. t → 1 / t, t], {μ, UVMons}], degt];
  UVMons = s^degs t^degt UVMons // Expand;
  ShiftP  =  Expand[P s^degs t^degt];


  Coefs = Table[f_i, {i, 1, Length[UVMons]}];
  cr = CoefficientRules[(UVMons.Coefs - ShiftP), {s, t}];
  eqs = cr /. {(r_ → w_) :→ w == 0};
  {sol} =
   Solve[eqs, Coefs];

  Monomials_deg[U, V].Coefs /. sol
 ]
ToUV[-1/T1^2 - T1^2 - 1/T2^2 - 1/(T1^2 T2^2) + 1/(T1 T2^2) + 1/(T1^2 T2) + T1/T2 + T2/T1 + T1^2 T2 - T2^2 + T1 T2^2 - T1^2 T2^2]
Renorm[t_] := If[t == 0, 0, Sign[t] × Log[Abs[t] + 10]]
DrawUVPol[P_] := Module[{Mat},
  If[P === 0, Return[P]],
   Mat = Map[Renorm, Normal@SparseArray[CoefficientRules[U V P, {U, V}]], {2}]];
  MatrixPlot[Mat]
 ]
```

Out[•]=

$4U - U^2 + 3V$

Rolfsen table

```
In[•]:= UVTable = {#, ToUV[θ[#][[2]]]} & /@ AllKnots[{3, 7}];
        Column[% // Factor]
```

Out[•]=

$\{\text{Knot}[3, 1], 4U - U^2 + 3V\}$
$\{\text{Knot}[4, 1], 0\}$
$\{\text{Knot}[5, 1], -22U - 11U^2 + 12U^3 - 2U^4 - 13V - 30UV + 10U^2V - 10V^2\}$
$\{\text{Knot}[5, 2], 14 + 30U - 9U^2 + 31V\}$
$\{\text{Knot}[6, 1], -28 + 2U + U^2 - 5V\}$
$\{\text{Knot}[6, 2], 73U - 18U^2 - 4U^3 + U^4 + 39V + 19UV - 7U^2V + 11V^2\}$
$\{\text{Knot}[6, 3], 0\}$
$\{\text{Knot}[7, 1], -21 + 29U + 109U^2 - 45U^3 - 44U^4 + 24U^5 - 3U^6 + 2V + 141UV + 58U^2V - 105U^3V + 21U^4V + 47V^2 + 84UV^2 - 42U^2V^2 + 21V^3\}$
$\{\text{Knot}[7, 2], -2(-49 - 54U + 18U^2 - 65V)\}$
$\{\text{Knot}[7, 3], -14 + 267U + 88U^2 - 106U^3 + 17U^4 + 127V + 307UV - 93U^2V + 109V^2\}$
$\{\text{Knot}[7, 4], 8(-35 - 28U + 10U^2 - 37V)\}$
$\{\text{Knot}[7, 5], 70 - 207U - 153U^2 + 118U^3 - 17U^4 - 76V - 367UV + 101U^2V - 141V^2\}$
$\{\text{Knot}[7, 6], 56 + 157U - 67U^2 + 2U^3 + U^4 + 164V + UV - 9U^2V + 19V^2\}$
$\{\text{Knot}[7, 7], 56 - 8U - U^2 + 7V\}$

```
In[•]:= UVTable // Column
```

```
In[•]:= {#[[1]], DrawUVPol[#[[2]]]} & /@ UVTable // MatrixForm
```

Ribbon Knot table:

Genus bound:

It appears that $\deg_V \leq g$. Or perhaps $2\deg_V + \deg_U \leq 2g$ is sharper.
See also in the Conway and KT cases below. Conway has genus 3, KT genus 2.

Specific knots

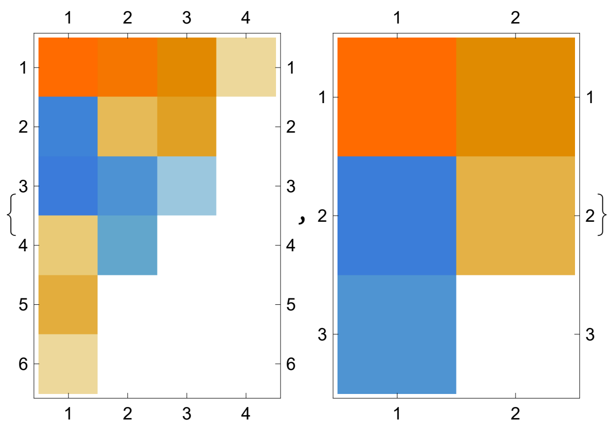
Conway and Kinoshita-Terasaka

In[∘]:=
```
{UVConway = ToUV[θ[Knot[11, NonAlternating, 34]]⟦2⟧],
 UVKT = ToUV[θ[Knot[11, NonAlternating, 42]]⟦2⟧]}
DrawUVPol /@ %
```

Out[∘]=

$\{2856 - 518\,U - 612\,U^2 + 20\,U^3 + 40\,U^4 + 4\,U^5 + 1544\,V + 33\,U\,V - 196\,U^2\,V - 28\,U^3\,V + 224\,V^2 + 44\,U\,V^2 - U^2\,V^2 + 4\,V^3,\ 40 - 6\,U - 4\,U^2 + 8\,V + U\,V\}$

Out[∘]=



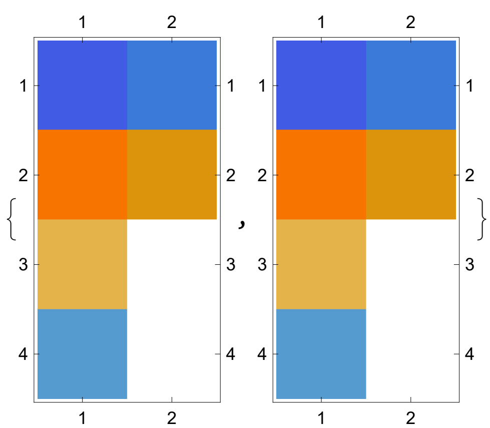
Mutant ninja turtles

In[∘]:=
```
{UVConway = ToUV[θ[Knot[11, NonAlternating, 73]]⟦2⟧],
 UVKT = ToUV[θ[Knot[11, NonAlternating, 74]]⟦2⟧]}
DrawUVPol /@ %
```

Out[∘]=

$\{-88 + 38\,U + 4\,U^2 - 2\,U^3 - 24\,V + 6\,U\,V,\ -88 + 38\,U + 4\,U^2 - 2\,U^3 - 24\,V + 6\,U\,V\}$

Out[∘]=



GST knot.

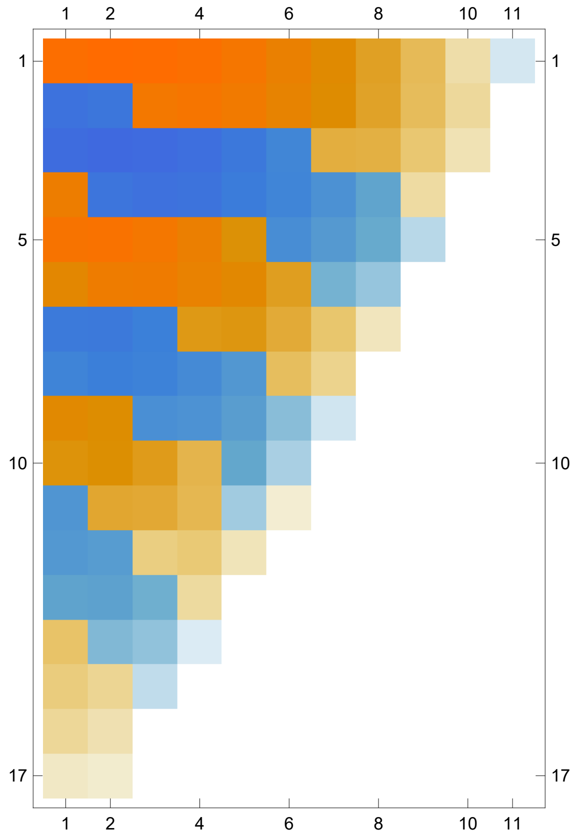
```
In[•]:= PD[GST48] = PD[X[1, 15, 2, 14], X[29, 2, 30, 3], X[40, 4, 41, 3],
    X[4, 44, 5, 43], X[5, 26, 6, 27], X[95, 7, 96, 6], X[7, 1, 8, 96], X[8, 14, 9, 13],
    X[28, 9, 29, 10], X[41, 11, 42, 10], X[11, 43, 12, 42], X[12, 27, 13, 28],
    X[15, 31, 16, 30], X[61, 16, 62, 17], X[72, 17, 73, 18], X[83, 18, 84, 19],
    X[34, 20, 35, 19], X[20, 89, 21, 90], X[92, 21, 93, 22], X[22, 79, 23, 80],
    X[23, 68, 24, 69], X[24, 57, 25, 58], X[56, 25, 57, 26], X[31, 63, 32, 62],
    X[32, 74, 33, 73], X[33, 85, 34, 84], X[35, 50, 36, 51], X[81, 37, 82, 36],
    X[70, 38, 71, 37], X[59, 39, 60, 38], X[54, 39, 55, 40], X[55, 45, 56, 44],
    X[45, 59, 46, 58], X[46, 70, 47, 69], X[47, 81, 48, 80], X[91, 49, 92, 48],
    X[49, 91, 50, 90], X[82, 52, 83, 51], X[71, 53, 72, 52], X[60, 54, 61, 53],
    X[74, 63, 75, 64], X[85, 64, 86, 65], X[65, 76, 66, 77], X[66, 87, 67, 88],
    X[94, 67, 95, 68], X[86, 75, 87, 76], X[77, 88, 78, 89], X[93, 78, 94, 79]];

In[•]:= KGST48 = θ[PD@GST48];

In[•]:= UVGST48 = ToUV[KGST48[[2]]]
```

Out[•]=

$$6\,230\,829\,076 - 1\,649\,181\,286\,U - 5\,550\,362\,737\,U^2 + 633\,563\,170\,U^3 + 2\,149\,291\,095\,U^4 + 57\,738\,350\,U^5 - 442\,863\,600\,U^6 - 68\,037\,954\,U^7 + 47\,087\,638\,U^8 + 13\,742\,818\,U^9 - 1\,713\,126\,U^{10} - 1\,133\,034\,U^{11} - 93\,673\,U^{12} + 27\,628\,U^{13} + 7084\,U^{14} + 634\,U^{15} + 21\,U^{16} + 13\,167\,733\,457\,V - 742\,113\,426\,U\,V - 10\,317\,864\,060\,U^2\,V - 780\,044\,732\,U^3\,V + 3\,238\,407\,625\,U^4\,V + 638\,880\,245\,U^5\,V - 474\,970\,634\,U^6\,V - 158\,493\,853\,U^7\,V + 24\,648\,280\,U^8\,V + 16\,630\,248\,U^9\,V + 1\,117\,975\,U^{10}\,V - 597\,951\,U^{11}\,V - 131\,649\,U^{12}\,V - 6085\,U^{13}\,V + 927\,U^{14}\,V + 120\,U^{15}\,V + 4\,U^{16}\,V + 11\,869\,957\,279\,V^2 + 1\,596\,094\,282\,U\,V^2 - 7\,694\,098\,809\,U^2\,V^2 - 1\,915\,654\,735\,U^3\,V^2 + 1\,772\,355\,983\,U^4\,V^2 + 673\,776\,096\,U^5\,V^2 - 139\,570\,447\,U^6\,V^2 - 95\,990\,994\,U^7\,V^2 - 4\,878\,592\,U^8\,V^2 + 4\,956\,644\,U^9\,V^2 + 1\,012\,288\,U^{10}\,V^2 + 5355\,U^{11}\,V^2 - 18\,588\,U^{12}\,V^2 - 2124\,U^{13}\,V^2 - 76\,U^{14}\,V^2 + 5\,974\,726\,186\,V^3 + 1\,846\,197\,822\,U\,V^3 - 2\,937\,035\,760\,U^2\,V^3 - 1\,250\,175\,184\,U^3\,V^3 + 401\,371\,993\,U^4\,V^3 + 272\,656\,716\,U^5\,V^3 + 6\,202\,565\,U^6\,V^3 - 20\,912\,710\,U^7\,V^3 - 3\,998\,030\,U^8\,V^3 + 181\,761\,U^9\,V^3 + 132\,950\,U^{10}\,V^3 + 14\,623\,U^{11}\,V^3 + 480\,U^{12}\,V^3 - 5\,U^{13}\,V^3 + 1\,838\,914\,446\,V^4 + 858\,092\,040\,U\,V^4 - 591\,691\,979\,U^2\,V^4 - 383\,311\,959\,U^3\,V^4 + 15\,686\,538\,U^4\,V^4 + 48\,517\,081\,U^5\,V^4 + 8\,278\,217\,U^6\,V^4 - 1\,141\,018\,U^7\,V^4 - 488\,295\,U^8\,V^4 - 48\,732\,U^9\,V^4 - 807\,U^{10}\,V^4 + 80\,U^{11}\,V^4 + 354\,683\,158\,V^5 + 214\,618\,897\,U\,V^5 - 52\,915\,707\,U^2\,V^5 - 59\,477\,229\,U^3\,V^5 - 7\,719\,781\,U^4\,V^5 + 3\,142\,057\,U^5\,V^5 + 991\,283\,U^6\,V^5 + 74\,251\,U^7\,V^5 - 3605\,U^8\,V^5 - 492\,U^9\,V^5 + U^{10}\,V^5 + 41\,939\,725\,V^6 + 30\,223\,366\,U\,V^6 + 486\,587\,U^2\,V^6 - 4\,238\,868\,U^3\,V^6 - 1\,043\,085\,U^4\,V^6 - 15\,128\,U^5\,V^6 + 18\,462\,U^6\,V^6 + 1428\,U^7\,V^6 - 13\,U^8\,V^6 + 2\,800\,418\,V^7 + 2\,267\,506\,U\,V^7 + 390\,623\,U^2\,V^7 - 87\,915\,U^3\,V^7 - 30\,306\,U^4\,V^7 - 1835\,U^5\,V^7 + 63\,U^6\,V^7 + 84\,191\,V^8 + 74\,924\,U\,V^8 + 17\,376\,U^2\,V^8 + 474\,U^3\,V^8 - 136\,U^4\,V^8 + 272\,V^9 + 596\,U\,V^9 + 115\,U^2\,V^9 - 12\,V^{10}$$

In[∘]:=
```
DrawUVPol[UVGST48]
```

Out[∘]=



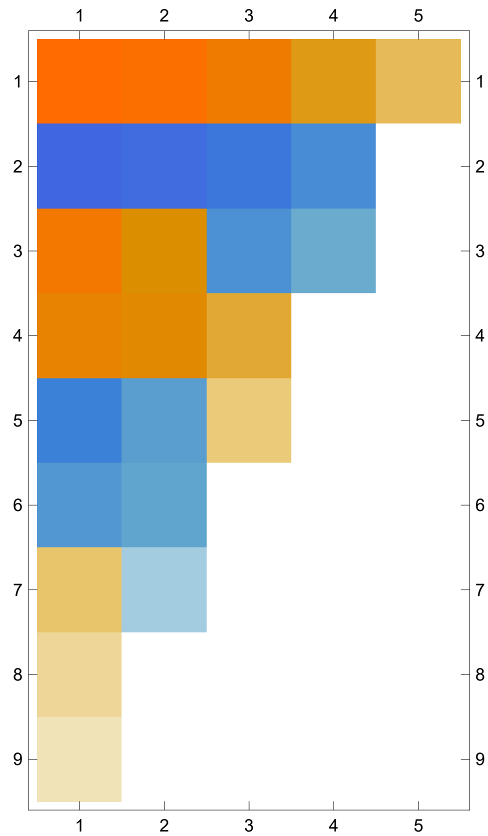
In[∘]:=
```
DunfieldKnotList =
  ReadList["C:\\Users\\T15Roland\\Wiskunde\\Bn\\HigherRank\\nmd_random_knots.txt"] /.
   {i_Integer :> i + 1};
```

In[ ]:=
```
ToUV[θ[DunfieldKnotList[[10]]][[2]]]
DrawUVPol@%
```

Out[ ]=

$$99168 - 131978\,U + 31970\,U^2 + 16662\,U^3 - 5055\,U^4 - 1038\,U^5 + 172\,U^6 + 40\,U^7 + 2\,U^8 + 90274\,V - 89599\,U\,V + 7613\,U^2\,V + 10324\,U^3\,V - 648\,U^4\,V - 438\,U^5\,V - 30\,U^6\,V + 30861\,V^2 - 20290\,U\,V^2 - 1512\,U^2\,V^2 + 1496\,U^3\,V^2 + 162\,U^4\,V^2 + 4720\,V^3 - 1542\,U\,V^3 - 364\,U^2\,V^3 + 274\,V^4$$

Out[ ]=

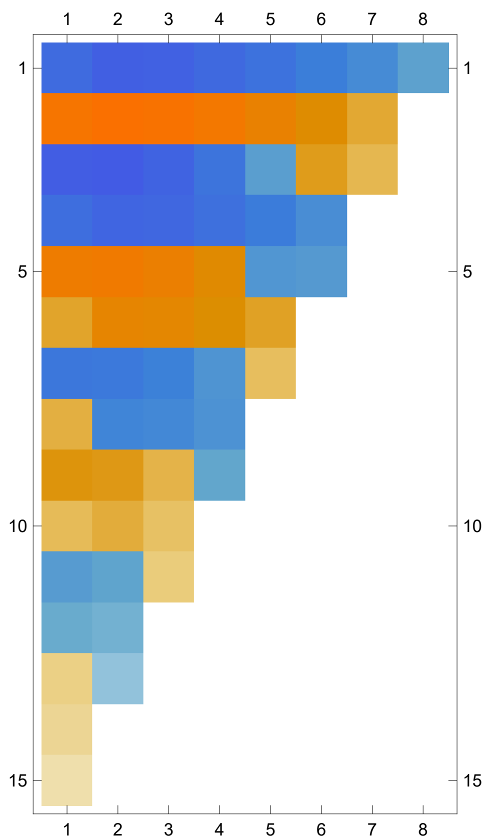


In[ ]:=
```
ToUV[θ[DunfieldKnotList[[30]]][[2]]]
DrawUVPol@%
```

Out[ ]=

$$-20\,959\,356\,192 + 82\,648\,870\,670\,U - 61\,420\,204\,654\,U^2 - 12\,889\,058\,040\,U^3 + 21\,952\,491\,586\,U^4 +$$
$$75\,909\,790\,U^5 - 3\,467\,252\,696\,U^6 + 32\,343\,128\,U^7 + 314\,994\,260\,U^8 + 11\,593\,600\,U^9 - 15\,968\,084\,U^{10} -$$
$$1\,697\,514\,U^{11} + 310\,109\,U^{12} + 64\,702\,U^{13} + 3195\,U^{14} - 49\,508\,478\,050\,V + 147\,417\,992\,421\,U\,V -$$
$$79\,306\,207\,340\,U^2\,V - 31\,830\,212\,699\,U^3\,V + 25\,568\,303\,784\,U^4\,V + 3\,664\,498\,263\,U^5\,V -$$
$$3\,278\,502\,945\,U^6\,V - 405\,420\,878\,U^7\,V + 203\,012\,405\,U^8\,V + 34\,035\,364\,U^9\,V - 4\,288\,830\,U^{10}\,V -$$
$$1\,229\,093\,U^{11}\,V - 70\,217\,U^{12}\,V - 48\,238\,331\,920\,V^2 + 108\,765\,255\,504\,U\,V^2 - 37\,844\,869\,967\,U^2\,V^2 -$$
$$25\,625\,045\,308\,U^3\,V^2 + 10\,333\,553\,045\,U^4\,V^2 + 3\,195\,589\,246\,U^5\,V^2 - 902\,413\,026\,U^6\,V^2 -$$
$$257\,790\,994\,U^7\,V^2 + 21\,112\,027\,U^8\,V^2 + 9\,629\,496\,U^9\,V^2 + 653\,692\,U^{10}\,V^2 - 25\,424\,737\,904\,V^3 +$$
$$42\,535\,474\,929\,U\,V^3 - 7\,222\,442\,748\,U^2\,V^3 - 9\,373\,889\,543\,U^3\,V^3 + 1\,477\,014\,251\,U^4\,V^3 +$$
$$933\,076\,873\,U^5\,V^3 - 36\,018\,686\,U^6\,V^3 - 39\,726\,750\,U^7\,V^3 - 3\,330\,993\,U^8\,V^3 - 7\,883\,961\,088\,V^4 +$$
$$9\,307\,650\,913\,U\,V^4 - 10\,398\,780\,U^2\,V^4 - 1\,621\,282\,746\,U^3\,V^4 - 32\,734\,022\,U^4\,V^4 + 90\,701\,424\,U^5\,V^4 +$$
$$9\,987\,666\,U^6\,V^4 - 1\,444\,915\,816\,V^5 + 1\,081\,283\,525\,U\,V^5 + 172\,871\,586\,U^2\,V^5 - 108\,103\,153\,U^3\,V^5 -$$
$$17\,498\,380\,U^4\,V^5 - 145\,376\,287\,V^6 + 52\,130\,232\,U\,V^6 + 16\,396\,920\,U^2\,V^6 - 6\,208\,317\,V^7$$

Out[ ]=




```
(*My ToUV is too slow to handle this*)
```

In[ ]:=
```
DK120 = << Theta4DK120.m;
```

Invariance Proof

In[∘]:=
```
δ_{i_,j_} := If[i === j, 1, 0];
```

In[∘]:=
```
gRules_{s_,i_,j_} := {
    g_{ν_,i,β_} :→ δ_{i,β} + T_ν^s g_{ν,i⁺,β} + (1 - T_ν^s) g_{ν,j⁺,β}, g_{ν_,j,β_} :→ δ_{j,β} + g_{ν,j⁺,β},
    g_{ν_,α_,i} :→ T_ν^{-s} (g_{ν,α,i⁺} - δ_{α,i⁺}), g_{ν_,α_,j} :→ g_{ν,α,j⁺} - (1 - T_ν^s) g_{ν,α,i} - δ_{α,j⁺}
  };
gRules[Cs_List] := Union @@ ((gRules_{Sequence@@#}) & /@ Cs)
```

Invariance of $y_{\alpha\beta\gamma}$ under remote R2bs

In[∘]:=
```
Clear[i, j];
Cs = {{1, i, j}, {-1, i⁺, j⁺}}
Z = Module[{s, i, j}, Sum[{s, i, j} = c;
    θ[s, i, j, α, β, γ], {c, Cs}]]
```

Out[∘]=

$\{\{1, i, j\}, \{-1, i^+, j^+\}\}$

Out[∘]=

$$g_{1,\beta,i}\, g_{2,\gamma,i}\, g_{3,j,\alpha} - \frac{g_{1,\beta,j}\, g_{2,\gamma,i}\, g_{3,j,\alpha}}{T_1} - \frac{g_{1,\beta,i^+}\, g_{2,\gamma,i^+}\, g_{3,j^+,\alpha}}{T_1^2\, T_2} + \frac{g_{1,\beta,j^+}\, g_{2,\gamma,i^+}\, g_{3,j^+,\alpha}}{T_1\, T_2}$$

In[∘]:=
```
Expand[Z //. gRules_{1,i,j} ⋃ gRules_{-1,i⁺,j⁺} /. _If → 0]
```

Out[∘]=

0

Invariance of $y_{\alpha\beta\gamma}$ under remote R3s

In[∘]:=
```
Clear[i, j, k];
Cs = {{1, i, j}, {1, i⁺, k}, {1, j⁺, k⁺}}
Z = Module[{s, i, j}, Sum[{s, i, j} = c;
    {s, i, j} = c; θ[s, i, j, α, β, γ],
    {c, Cs}]]
lhs = Simplify[Z //. gRules[Cs] /. _If → 0]
```

Out[∘]=

$\{\{1, i, j\}, \{1, i^+, k\}, \{1, j^+, k^+\}\}$

Out[∘]=

$$g_{1,\beta,i}\, g_{2,\gamma,i}\, g_{3,j,\alpha} - \frac{g_{1,\beta,j}\, g_{2,\gamma,i}\, g_{3,j,\alpha}}{T_1} - \frac{g_{1,\beta,k}\, g_{2,\gamma,i^+}\, g_{3,k,\alpha}}{T_1} + g_{1,\beta,i^+}\, g_{2,\gamma,i^+}\, g_{3,k,\alpha} + g_{1,\beta,j^+}\, g_{2,\gamma,j^+}\, g_{3,k^+,\alpha} - \frac{g_{1,\beta,k^+}\, g_{2,\gamma,j^+}\, g_{3,k^+,\alpha}}{T_1}$$

Out[∘]=

$$\frac{1}{T_1^3 T_2^2}\Big(g_{1,\beta,i^{++}}\, g_{2,\gamma,i^{++}}\, g_{3,k^{++},\alpha} - T_1^2 T_2 \left(g_{1,\beta,j^{++}}\, g_{2,\gamma,i^{++}}\, g_{3,j^{++},\alpha} + g_{1,\beta,k^{++}} \left(g_{2,\gamma,i^{++}} + g_{2,\gamma,j^{++}}\right) g_{3,k^{++},\alpha}\right) + T_1 \left(-g_{1,\beta,j^{++}}\, g_{2,\gamma,i^{++}}\, g_{3,k^{++},\alpha} + T_2 \left(g_{1,\beta,i^{++}}\, g_{2,\gamma,i^{++}}\, g_{3,j^{++},\alpha} + g_{1,\beta,j^{++}} \left(g_{2,\gamma,i^{++}} + g_{2,\gamma,j^{++}}\right) g_{3,k^{++},\alpha}\right)\right)\Big)$$

In[∘]:=
```
Clear[i, j, k];
Cs = {{1, j, k}, {1, i, k⁺}, {1, i⁺, j⁺}}
Z = Module[{s, i, j}, Sum[{s, i, j} = c;
    {s, i, j} = c; θ[s, i, j, α, β, γ],
    {c, Cs}]]
rhs = Simplify[Z //. gRules[Cs] /. _If → 0]
```

Out[∘]=

$\{\{1, j, k\}, \{1, i, k^+\}, \{1, i^+, j^+\}\}$

Out[∘]=

$$g_{1,\beta,j}\, g_{2,\gamma,j}\, g_{3,k,\alpha} - \frac{g_{1,\beta,k}\, g_{2,\gamma,j}\, g_{3,k,\alpha}}{T_1} + g_{1,\beta,i^+}\, g_{2,\gamma,i^+}\, g_{3,j^+,\alpha} - \frac{g_{1,\beta,j^+}\, g_{2,\gamma,i^+}\, g_{3,j^+,\alpha}}{T_1} + g_{1,\beta,i}\, g_{2,\gamma,i}\, g_{3,k^+,\alpha} - \frac{g_{1,\beta,k^+}\, g_{2,\gamma,i}\, g_{3,k^+,\alpha}}{T_1}$$

Out[∘]=

$$\frac{1}{T_1^3 T_2^2}\Big(g_{1,\beta,i^{++}}\, g_{2,\gamma,i^{++}}\, g_{3,k^{++},\alpha} - T_1^2 T_2 \left(g_{1,\beta,j^{++}}\, g_{2,\gamma,i^{++}}\, g_{3,j^{++},\alpha} + g_{1,\beta,k^{++}} \left(g_{2,\gamma,i^{++}} + g_{2,\gamma,j^{++}}\right) g_{3,k^{++},\alpha}\right) + T_1 \left(-g_{1,\beta,j^{++}}\, g_{2,\gamma,i^{++}}\, g_{3,k^{++},\alpha} + T_2 \left(g_{1,\beta,i^{++}}\, g_{2,\gamma,i^{++}}\, g_{3,j^{++},\alpha} + g_{1,\beta,j^{++}} \left(g_{2,\gamma,i^{++}} + g_{2,\gamma,j^{++}}\right) g_{3,k^{++},\alpha}\right)\right)\Big)$$

In[∘]:=
```
lhs == rhs
```

Out[∘]=

True

Invariance of $y_{\alpha\beta\gamma}$ under remote R2cs

In[ ]:=
```
Clear[i, j];
Cs = {{1, i⁺, j}, {-1, i, j⁺}};
Z = Module[{s, i, j}, Sum[{s, i, j} = c; θ[s, i, j, α, β, γ], {c, Cs}]]
Expand[Z //. gRules_{1,i⁺,j} ∪ gRules_{-1,i,j⁺} /. _If → 0]
```

Out[ ]=

$$-\frac{g_{1,\beta,j}\,g_{2,\gamma,i^+}\,g_{3,j,\alpha}}{T_1}+g_{1,\beta,i^+}\,g_{2,\gamma,i^+}\,g_{3,j,\alpha}-\frac{g_{1,\beta,i}\,g_{2,\gamma,i}\,g_{3,j^+,\alpha}}{T_1^2\,T_2}+\frac{g_{1,\beta,j^+}\,g_{2,\gamma,i}\,g_{3,j^+,\alpha}}{T_1\,T_2}$$

Out[ ]=

0

Invariance under R2b

In[ ]:=
```
Y[α_, β_, γ_] := Module[{s, i, j}, Sum[{s, i, j} = c;
    θ[s, i, j, α, β, γ], {c, Cs}]];
yEval[ℰ_] := ℰ /. y_{α_,β_,γ_} :→ Y[α, β, γ];
```

In[ ]:=
```
Clear[i, j];
Cs = {{1, i, j}, {-1, i⁺, j⁺}}
Expand@Together[(Total[R₁ @@@ Cs] // yEval) //. gRules[Cs]]
```

Out[ ]=

$\{\{1, i, j\}, \{-1, i^+, j^+\}\}$

Out[ ]=

0

Invariance under R3b

In[ ]:=
```
Clear[i, j, k];
Cs = {{1, i, j}, {1, i⁺, k}, {1, j⁺, k⁺}}
lhs = Expand@Together[(Total[R₁ @@@ Cs] // yEval) //. gRules[Cs]]
```

Out[ ]=

$\{\{1, i, j\}, \{1, i^+, k\}, \{1, j^+, k^+\}\}$

Out[ ]=

$$\frac{g_{1,j^{++},i^{++}}\,g_{2,i^{++},i^{++}}}{(-1+T_1)\,T_1^2\,(-1+T_1\,T_2)}-\frac{g_{1,j^{++},i^{++}}\,g_{2,i^{++},i^{++}}}{(-1+T_1)\,T_1\,(-1+T_1\,T_2)}-\frac{g_{1,j^{++},j^{++}}\,g_{2,i^{++},i^{++}}}{(-1+T_1)\,T_1\,(-1+T_1\,T_2)}+\cdots 775\cdots+$$
$$\frac{g_{2,k^{++},i^{++}}\,g_{3,k^{++},k^{++}}}{(-1+T_1)\,(-1+T_1\,T_2)}-\frac{g_{2,k^{++},i^{++}}\,g_{3,k^{++},k^{++}}}{(-1+T_1)\,T_1\,(-1+T_1\,T_2)}+\frac{g_{2,k^{++},j^{++}}\,g_{3,k^{++},k^{++}}}{(-1+T_1)\,(-1+T_1\,T_2)}-\frac{g_{2,k^{++},j^{++}}\,g_{3,k^{++},k^{++}}}{(-1+T_1)\,T_1\,(-1+T_1\,T_2)}$$

large output | show less | show more | show all | set size limit...

In[∘]:=
```
Clear[i, j, k];
Cs = {{1, j, k}, {1, i, k⁺}, {1, i⁺, j⁺}}
rhs = Expand@Together[(Total[R₁ @@@ Cs] // yEval) //. gRules[Cs]]
```

Out[∘]=

$\{\{1, j, k\}, \{1, i, k^+\}, \{1, i^+, j^+\}\}$

Out[∘]=

$$\frac{g_{1,j^{++},i^{++}}\, g_{2,i^{++},i^{++}}}{(-1+T_1)\, T_1^2\, (-1+T_1 T_2)} - \frac{g_{1,j^{++},i^{++}}\, g_{2,i^{++},i^{++}}}{(-1+T_1)\, T_1\, (-1+T_1 T_2)} - \frac{g_{1,j^{++},j^{++}}\, g_{2,i^{++},i^{++}}}{(-1+T_1)\, T_1\, (-1+T_1 T_2)} + \cdots 775 \cdots +$$
$$\frac{g_{2,k^{++},i^{++}}\, g_{3,k^{++},k^{++}}}{(-1+T_1)\, (-1+T_1 T_2)} - \frac{g_{2,k^{++},i^{++}}\, g_{3,k^{++},k^{++}}}{(-1+T_1)\, T_1\, (-1+T_1 T_2)} + \frac{g_{2,k^{++},j^{++}}\, g_{3,k^{++},k^{++}}}{(-1+T_1)\, (-1+T_1 T_2)} - \frac{g_{2,k^{++},j^{++}}\, g_{3,k^{++},k^{++}}}{(-1+T_1)\, T_1\, (-1+T_1 T_2)}$$

large output | show less | show more | show all | set size limit...

In[∘]:=
```
lhs == rhs
```

Out[∘]=

True

Invariance under R2c

In[∘]:=
```
Clear[i, j];
Cs = {{1, i⁺, j}, {-1, i, j⁺}};
lhs = Expand@Together[(Total[R₁ @@@ Cs] + Piv_{j⁺} // yEval) //. gRules[Cs]]
rhs = Piv_{j"++"};
lhs == rhs // FullSimplify
```

Out[∘]=

$$-\frac{g_{3,j^{++},j^{++}}}{T_1\, (-1 + T_1 T_2)}$$

Out[∘]=

True

In[∘]:=
```
Solve[1 + h T₁ (-1 + T₁ T₂) == 0, h]
```

Out[∘]=

$$\left\{\left\{h \to -\frac{1}{T_1\, (-1 + T_1 T_2)}\right\}\right\}$$

Invariance under R1l

In[∘]:=
```
Cs = {{1, i⁺, i}};
gRules[Cs]
```

Out[∘]=

$$\{g_{\nu\$\_,i,\beta\$\_} :\to \delta_{i,\beta\$} + g_{\nu\$,i^+,\beta\$},\ g_{\nu\$\_,\alpha\$\_,i} :\to g_{\nu\$,\alpha\$,i^+} - \left(1 - T^1_{\nu\$}\right) g_{\nu\$,\alpha\$,i^+} - \delta_{\alpha\$,i^+},$$
$$g_{\nu\$\_,\alpha\$\_,i^+} :\to T^{-1}_{\nu\$}\left(g_{\nu\$,\alpha\$,(i^+)^+} - \delta_{\alpha\$,(i^+)^+}\right),\ g_{\nu\$\_,i^+,\beta\$\_} :\to \delta_{i^+,\beta\$} + T^1_{\nu\$}\, g_{\nu\$,(i^+)^+,\beta\$} + \left(1 - T^1_{\nu\$}\right) g_{\nu\$,i^+,\beta\$}\}$$

In[∘]:= gr1lRules = $\{g_{\nu\$\_,i,\beta\$\_} \mapsto \delta_{i,\beta\$} + g_{\nu\$,i^+,\beta\$},$
$g_{\nu\$\_,\alpha\$\_,i} \mapsto g_{\nu\$,\alpha\$,i^+} - (1 - T^1_{\nu\$})\, g_{\nu\$,\alpha\$,i^+} - \delta_{\alpha\$,i^+},$
$g_{\nu\$\_,\alpha\$\_,i^+} \mapsto T^{-1}_{\nu\$}\, (g_{\nu\$,\alpha\$,(i^+)^+} - \delta_{\alpha\$,(i^+)^+}),$
$g_{\nu\$\_,i^+,\beta\$\_} \mapsto T^{-1}_{\nu\$}\, (\delta_{i^+,\beta\$} + T^1_{\nu\$}\, g_{\nu\$,(i^+)^+,\beta\$})\}$

Out[∘]=

$$\left\{g_{\nu\$\_,i,\beta\$\_} \mapsto \delta_{i,\beta\$} + g_{\nu\$,i^+,\beta\$},\ g_{\nu\$\_,\alpha\$\_,i} \mapsto g_{\nu\$,\alpha\$,i^+} - (1 - T^1_{\nu\$})\, g_{\nu\$,\alpha\$,i^+} - \delta_{\alpha\$,i^+},\ g_{\nu\$\_,\alpha\$\_,i^+} \mapsto \frac{g_{\nu\$,\alpha\$,(i^+)^+} - \delta_{\alpha\$,(i^+)^+}}{T_{\nu\$}},\ g_{\nu\$\_,i^+,\beta\$\_} \mapsto \frac{\delta_{i^+,\beta\$} + T^1_{\nu\$}\, g_{\nu\$,(i^+)^+,\beta\$}}{T_{\nu\$}}\right\}$$

In[∘]:= Total[R_1 @@@ Cs]

In[∘]:= (Total[R_1 @@@ Cs] + Piv_{i^+} // yEval) //. gr1lRules // Simplify

Out[∘]=

0

Invariance under R1r

In[∘]:= Cs = {{1, i, i^+}};
gRules[Cs]

Out[∘]=

$$\{g_{\nu\$\_,i,\beta\$\_} \mapsto \delta_{i,\beta\$} + T^1_{\nu\$}\, g_{\nu\$,i^+,\beta\$} + (1 - T^1_{\nu\$})\, g_{\nu\$,(i^+)^+,\beta\$},\ g_{\nu\$\_,\alpha\$\_,i} \mapsto T^{-1}_{\nu\$}\, (g_{\nu\$,\alpha\$,i^+} - \delta_{\alpha\$,i^+}),\ g_{\nu\$\_,\alpha\$\_,i^+} \mapsto g_{\nu\$,\alpha\$,(i^+)^+} - (1 - T^1_{\nu\$})\, g_{\nu\$,\alpha\$,i} - \delta_{\alpha\$,(i^+)^+},\ g_{\nu\$\_,i^+,\beta\$\_} \mapsto \delta_{i^+,\beta\$} + g_{\nu\$,(i^+)^+,\beta\$}\}$$

In[∘]:= gr1rRules = {
$g_{\nu\$\_,i,\beta\$\_} \mapsto \delta_{i,\beta\$} + T^1_{\nu\$}\, g_{\nu\$,i^+,\beta\$} + (1 - T^1_{\nu\$})\, g_{\nu\$,(i^+)^+,\beta\$},$
$g_{\nu\$\_,\alpha\$\_,i} \mapsto T^{-1}_{\nu\$}\, (g_{\nu\$,\alpha\$,i^+} - \delta_{\alpha\$,i^+}),$
$g_{\nu\$\_,\alpha\$\_,i} \mapsto (1 - T^1_{\nu\$})^{-1}\, (-g_{\nu\$,\alpha\$,i^+} + g_{\nu\$,\alpha\$,(i^+)^+} - \delta_{\alpha\$,(i^+)^+}),$
$g_{\nu\$\_,i^+,\beta\$\_} \mapsto \delta_{i^+,\beta\$} + g_{\nu\$,(i^+)^+,\beta\$}\};$

In[∘]:= Total[R_1 @@@ Cs]

In[∘]:= (Total[R_1 @@@ Cs] - Piv_{i^+} // yEval) //. gr1rRules // Simplify

Out[∘]=

0

Invariance under Swirl

In[∘]:= Cs = {{1, i, j}};
gRules[Cs]

Out[∘]=

$$\{g_{\nu\$\_,i,\beta\$\_} \mapsto \delta_{i,\beta\$} + T^1_{\nu\$}\, g_{\nu\$,i^+,\beta\$} + (1 - T^1_{\nu\$})\, g_{\nu\$,j^+,\beta\$},\ g_{\nu\$\_,j,\beta\$\_} \mapsto \delta_{j,\beta\$} + g_{\nu\$,j^+,\beta\$},\ g_{\nu\$\_,\alpha\$\_,i} \mapsto T^{-1}_{\nu\$}\, (g_{\nu\$,\alpha\$,i^+} - \delta_{\alpha\$,i^+}),\ g_{\nu\$\_,\alpha\$\_,j} \mapsto g_{\nu\$,\alpha\$,j^+} - (1 - T^1_{\nu\$})\, g_{\nu\$,\alpha\$,i} - \delta_{\alpha\$,j^+}\}$$

In[•]:=

```
rhs = (Total[R_1 @@@ Cs] + Piv_i + Piv_j - Piv_{i+} - Piv_{j+} // yEval) //. gRules[Cs] // Simplify
lhs = (Total[R_1 @@@ Cs] // yEval) //. gRules[Cs];
lhs - rhs // Simplify;
```

Out[•]=

$$
\begin{aligned}
&\frac{1}{(-1+T_1)\, T_1^3\, T_2^2\, (-1+T_1 T_2)}\\
&\Big(T_2\, g_{1,j^+,i^+} (-1+g_{2,i^+,i^+}-g_{2,j^+,i^+})\, g_{3,j^+,i^+} + T_1 \Big(-g_{2,j^+,i^+}\, g_{3,j^+,i^+} + T_2 (-1 + g_{2,j^+,i^+} +\\
&\quad g_{1,j^+,j^+} (1-g_{2,i^+,i^+}+g_{2,j^+,i^+}) + g_{1,j^+,i^+} (2-g_{2,i^+,i^+}+g_{2,j^+,i^+}) + g_{2,j^+,j^+})\, g_{3,j^+,i^+} +\\
&\quad T_2^2\, g_{1,j^+,i^+} (-g_{3,i^+,i^+} - g_{2,i^+,i^+} (-1+g_{3,j^+,i^+}) + g_{2,j^+,i^+} (-1+g_{3,j^+,i^+}) + g_{3,j^+,i^+})\Big) +\\
&T_1^2 \Big(g_{2,j^+,i^+}\, g_{3,j^+,i^+} - T_2 ((-1-g_{1,i^+,i^+}-g_{1,j^+,j^+} (-1+g_{2,i^+,i^+})+g_{2,j^+,j^+})\, g_{3,j^+,i^+} +\\
&\quad g_{1,j^+,i^+} (g_{2,j^+,i^+}+g_{3,j^+,i^+}) + g_{2,j^+,i^+} (g_{3,i^+,i^+}+g_{1,j^+,j^+}\, g_{3,j^+,i^+})) + T_2^2 (g_{2,j^+,i^+}\, g_{3,i^+,i^+} +\\
&\quad g_{2,j^+,j^+}\, g_{3,i^+,i^+} + g_{1,j^+,j^+} (g_{3,i^+,i^+} + g_{2,i^+,i^+} (-1+g_{3,j^+,i^+}) - g_{2,j^+,i^+} (-1+g_{3,j^+,i^+}) - g_{3,j^+,i^+}) -\\
&\quad g_{2,j^+,i^+}\, g_{3,j^+,i^+} - g_{2,j^+,j^+}\, g_{3,j^+,i^+} + g_{1,j^+,i^+} (g_{2,j^+,j^+} + g_{3,i^+,i^+} - g_{2,j^+,i^+} (-2+g_{3,j^+,i^+}) +\\
&\qquad g_{2,i^+,i^+} (-1+g_{3,j^+,i^+}) - 2\, g_{3,j^+,i^+} - g_{3,j^+,j^+}) - g_{2,j^+,i^+}\, g_{3,j^+,j^+})\Big) +\\
&T_1^3\, T_2 (-T_2\, g_{1,j^+,j^+}\, g_{2,j^+,i^+} + g_{2,j^+,i^+}\, g_{3,i^+,i^+} - T_2\, g_{2,j^+,i^+}\, g_{3,i^+,i^+} - T_2\, g_{2,j^+,j^+}\, g_{3,i^+,i^+} +\\
&\quad T_2\, g_{1,j^+,j^+}\, g_{3,j^+,i^+} - T_2\, g_{1,j^+,j^+}\, g_{2,i^+,i^+}\, g_{3,j^+,i^+} - g_{2,j^+,i^+}\, g_{3,j^+,i^+} +\\
&\quad T_2\, g_{2,j^+,i^+}\, g_{3,j^+,i^+} + T_2\, g_{1,j^+,j^+}\, g_{2,j^+,i^+}\, g_{3,j^+,i^+} + T_2\, g_{2,j^+,j^+}\, g_{3,j^+,i^+} +\\
&\quad g_{1,i^+,i^+} ((-1+T_2)\, g_{2,j^+,i^+} + T_2 (g_{2,j^+,j^+} - g_{3,j^+,i^+} - g_{3,j^+,j^+})) + T_2\, g_{2,j^+,i^+}\, g_{3,j^+,j^+} +\\
&\quad g_{1,j^+,i^+} (-((-1+T_2)\, g_{2,j^+,i^+}) + T_2 (-g_{2,j^+,j^+} + g_{3,j^+,i^+} + g_{3,j^+,j^+})))\Big)
\end{aligned}
$$