In[]:=
```
SetDirectory["C:\\Users\\T15Roland\\Wiskunde\\Bn\\HigherRank"];
Once[<< KnotTheory`];
 << Rot.m
<< FormalGaussianIntegration.m
```

ParentDirectory: Argument File should be a positive machine-size integer, a nonempty string, or a File specification.

ParentDirectory: Argument File should be a positive machine-size integer, a nonempty string, or a File specification.

ToFileName: String or list of strings expected at position 1 in ToFileName[{File, WikiLink, mathematica}].

ToFileName: String or list of strings expected at position 1 in ToFileName[{File, QuantumGroups}].

```
Loading KnotTheory` version of September 6, 2014, 13:37:37.2841.
Read more at http://katlas.org/wiki/KnotTheory.
```

```
Loading Rot.m from http://drorbn.net/AP/Projects/HigherRank to compute rotation numbers.
```

In[]:=
```
(*The R3 solutions from UC4A242 (written hard coded below the fold):*)
```

Solve: Equations may not give solutions for all "solve" variables.

```
(*mons0=MonomialList[
    p3x1x2/.
      {(v:p|x)α_:>vα,i+vα,j}
   ]/.c_Integer*mon_:>mon;*)
mons1 = MonomialList[
    1 + p1 x1 + p2 x2 + p3 x3 + p1 p1 x1 x1 + p2 p2 x2 x2 +
      p1 p2 x1 x2 + p1 p3 x1 x3 + p2 p3 x2 x3 + p3 p3 x3 x3 + 0 p1 p2 x3 /.
     {(v : p | x)α_ :> vα,i + vα,j}
   ] /. c_Integer * mon_ :> mon;
(*$A,$B,$C,$D part from DeterminingThePXXandXPPCoefficients.nb file,
sol3b and sol3bα sα3 from UC4A2R42.nb*)
(*Subs={c_->1,BB->BB,α4->BB};*)
r0[1, i_, j_] := ($A p3,j x1,i x2,i - (($A + $B T2) p3,j x1,j x2,i)/T1 + $B p3,j x1,i x2,j)
r0[-1, i_, j_] :=
  (((-$A + $B T1 - $B T2) p3,j x1,i x2,i)/(T1^2 T2) + (($A + $B T2) p3,j x1,j x2,i)/(T1 T2) - ($B p3,j x1,i x2,j)/T1)
r1[1, i_, j_] := (Evaluate[$C p1,j p2,i x3,i + $D p1,i p2,j x3,i +
     (-$C - $D) p1,j p2,j x3,i + Sum[cz mons1[[z]], {z, 1, Length@mons1}]])
r1[-1, i_, j_] :=
  (Evaluate[-($C p1,j p2,i x3,i)/T1 - ($D p1,i p2,j x3,i)/T2 +
      (($D T1 + $C T2) p1,j p2,j x3,i)/(T1 T2) + Sum[dz mons1[[z]], {z, 1, Length@mons1}]]])
r0[1, 4, 7]
r1[1, 4, 7];
```

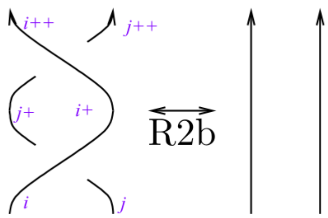
Out[∘]=

$$\$A\, p_{3,7}\, x_{1,4}\, x_{2,4} - \frac{(\$A + \$B\, T_2)\, p_{3,7}\, x_{1,7}\, x_{2,4}}{T_1} + \$B\, p_{3,7}\, x_{1,4}\, x_{2,7}$$

In[∘]:=

```
T3 = T1 T2;
𝒮 = {x__, p__};
q[s_, i_, j_] :=
  Sum[xα,i (pα,i - pα,i+1) + xα,j (pα,j - pα,j+1) + xα,i ((1 - Tα^s) pα,i+1 + (Tα^s - 1) pα,j+1), {α, 3}];
γ1[φ_, k_] := φ (3 / 2 - x1,k p1,k - x2,k p2,k - x3,k p3,k);
ℒ[Xi_,j_[s_]] := T3^s 𝔼[-q[s, i, j] + r0[s, i, j] + ϵ r1[s, i, j] + O[ϵ]^2];
ℒ[Ck_[φ_]] :=
  T3^φ 𝔼[-x1,k (p1,k - p1,k+1) - x2,k (p2,k - p2,k+1) - x3,k (p3,k - p3,k+1) + ϵ γ1[φ, k] + O[ϵ]^2];
ℒ[K_] := (2 π)^(-Features[K][[1]]) CF[ℒ /@ Features[K][[2]]];
vsi_ := Sequence[p1,i, x1,i, p2,i, x2,i, p3,i, x3,i];
vs[K_] := Union @@ Table[{vsi}, {i, Features[K][[1]]}]
```

Reidemeister 2b



Territory: $(x \mid p)_{(i|j)^+}$, $x_{i|j}$.

In[∘]:=
```
{lhs2b} = Cases[
  ∫𝔼[Sum[π_{α,i} p_{α,i} + π_{α,j} p_{α,j}, {α, 3}]] ℒ /@ (X_{i,j}[1] X_{i+1,j+1}[-1]) ⅆ{vs_i, vs_j, vs_{i+1}, vs_{j+1}},
  ϵSeries[ℰ_, ℱ_] :→ ℱ, ∞]
```

Out[∘]=

$$\left\{2\, c_1 + c_5 + c_7 + c_{10} + \cdots 356 \cdots + \frac{\left(d_{77} + c_{77}\, T_1^2\, T_2^2\right) p_{3,2+i}^2\, \pi_{3,j}^2}{T_1^2\, T_2^2} + \frac{\left(-2\, d_{77} + 2\, d_{77}\, T_1\, T_2 + d_{80}\, T_1\, T_2 + c_{80}\, T_1^2\, T_2^2\right) p_{3,2+i}\, p_{3,2+j}\, \pi_{3,j}^2}{T_1^2\, T_2^2} + \frac{\left(d_{77} - 2\, d_{77}\, T_1\, T_2 - d_{80}\, T_1\, T_2 + c_{85}\, T_1^2\, T_2^2 + d_{77}\, T_1^2\, T_2^2 + d_{80}\, T_1^2\, T_2^2 + d_{85}\, T_1^2\, T_2^2\right) p_{3,2+j}^2\, \pi_{3,j}^2}{T_1^2\, T_2^2}\right\}$$

large output | show less | show more | show all | set size limit...

In[∘]:=
```
{rhs2b} = Cases[∫𝔼[Sum[π_{α,i} p_{α,i} + π_{α,j} p_{α,j}, {α, 3}]]
    ℒ /@ (C_i[0] C_{i+1}[0] C_j[0] C_{j+1}[0]) ⅆ{vs_i, vs_j, vs_{i+1}, vs_{j+1}}, ϵSeries[ℰ_, ℱ_] :→ ℱ, ∞]
```

Out[∘]=

{0}

In[∘]:=
```
eqn2b =
 CF[CF[CF[lhs2b - rhs2b] /. {$A $C → α1, $A $D → α2, $B $C → α3, $B $D → α4}] /. {$A | $B → 0}]
```

In[∘]:=
```
cvs2b = Union@Cases[eqn2b, p__ | π__, ∞];
```

In[∘]:=
```
eqns2b = CoefficientRules[eqn2b, cvs2b] /. (_ → c_) :→ (c == 0);
```

In[∘]:=
```
vars2b = Union@Cases[eqn2b, d_, ∞];
```

In[∘]:=
```
eqns2b // Column
```

In[∘]:=
```
{sol2b} = Solve[eqns2b, vars2b]
```

In[∘]:=
```
Cases[sol2b, α_, ∞]
```

Out[∘]=

$\{\alpha_4, \alpha_1, \alpha_3, \alpha_3, \alpha_3, \alpha_4, \alpha_1, \alpha_3, \alpha_1, \alpha_3, \alpha_1, \alpha_2, \alpha_2, \alpha_3, \alpha_4, \alpha_4, \alpha_4, \alpha_4, \alpha_1, \alpha_3, \alpha_4\}$

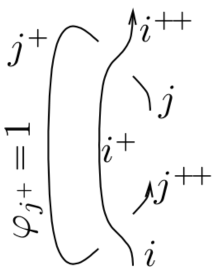
In[∘]:=
```
eqn2b /. sol2b // CF
```

Out[∘]=

0

Verification of Invariance Under Reidemeister 3b

Invariance Under R2c



In[•]:=
```
lhs2c = ∫𝔼[Sum[π_{α,i} p_{α,i} + π_{α,j} p_{α,j}, {α, 3}]]
   ℒ /@ (X_{i+1,j}[1] X_{i,j+2}[-1] C_{j+1}[1]) ⅆ{vs_i, vs_j, vs_{i+1}, vs_{j+1}, vs_{j+2}}
```

In[•]:=
```
rhs2c = ∫𝔼[Sum[π_{α,i} p_{α,i} + π_{α,j} p_{α,j}, {α, 3}]]
   ℒ /@ (C_i[0] C_{i+1}[0] C_j[0] C_{j+1}[1] C_{j+2}[0]) ⅆ{vs_i, vs_j, vs_{i+1}, vs_{j+1}, vs_{j+2}}
```

Out[•]=

$$-32768\, i\, \pi^{15}\, T_1 T_2\, \mathbb{E}\Big[\epsilon\text{Series}\Big[p_{1,2+i}\pi_{1,i} + p_{1,3+j}\pi_{1,j} + p_{2,2+i}\pi_{2,i} + p_{2,3+j}\pi_{2,j} + p_{3,2+i}\pi_{3,i} + p_{3,3+j}\pi_{3,j},\ -\frac{3}{2} - p_{1,3+j}\pi_{1,j} - p_{2,3+j}\pi_{2,j} - p_{3,3+j}\pi_{3,j}\Big]\Big]$$

In[•]:=
```
Cases[Expand[rhs2c], $A $C, ∞]
```

Out[•]=

$$\{\}$$

In[•]:=
```
eqn2c =
  CF[CF[CF[Cases[lhs2c, ϵSeries[ℰ_, ℱ_] :→ ℱ, ∞] - Cases[rhs2c, ϵSeries[ℰ_, ℱ_] :→ ℱ, ∞]] /.
      {$A $C → α_1, $A $D → α_2, $B $C → α_3, $B $D → α_4}] /. {$A | $B → 0}]
```

In[•]:=
```
Cases[eqn2c, α_, ∞]
```

Out[•]=

$$\{\alpha_1, \alpha_3, \alpha_4, \alpha_4, \alpha_1, \alpha_3, \alpha_4, \alpha_1, \alpha_3, \alpha_1, \alpha_3, \alpha_1, \alpha_2, \alpha_2, \alpha_3, \alpha_4, \alpha_4\}$$

In[•]:=
```
eqn2c /. sol2b // CF
```

In[•]:=
```
eqn2cred = eqn2c /. sol2b /. sol3b /. sol3bα /. sα3 // CF
```

Out[•]=

$$\Big\{\frac{(1+c_5)(-1+T_1)\, p_{1,3+j}\pi_{1,i}}{T_1} + \frac{(1+c_{50})(-1+T_2)\, p_{2,3+j}\pi_{2,i}}{T_2} + \Big((-1+T_1T_2)\big(-c_9c_{22} + c_{10}c_{22} + c_{22}c_{28} + c_{22}T_1 + 2c_9c_{22}T_1 - 2c_{10}c_{22}T_1 - c_{22}c_{28}T_1 + c_{22}c_{79}T_1 - c_{22}T_1^2 - c_9c_{22}T_1^2 + c_{10}c_{22}T_1^2 - c_{22}c_{79}T_1^2 - c_{10}c_{22}T_2 - c_{22}c_{28}T_2 - c_{22}T_1T_2 + 2c_{10}c_{22}T_1T_2 + c_{22}c_{28}T_1T_2 - c_{22}c_{79}T_1T_2 + c_{22}T_1^2T_2 - c_{10}c_{22}T_1^2T_2 + c_{22}c_{79}T_1^2T_2 + \alpha_4 + c_9\alpha_4 - c_{10}\alpha_4 - c_{28}\alpha_4 + c_{79}\alpha_4 - T_1\alpha_4 - c_9T_1\alpha_4 + c_{10}T_1\alpha_4 - c_{79}T_1\alpha_4 + c_{10}T_2\alpha_4 + c_{28}T_2\alpha_4 - c_{10}T_1T_2\alpha_4\big)\, p_{3,3+j}\pi_{3,i}\Big) \Big/ \big((-1+T_1)\,T_1T_2\,(-c_{22}T_1 + c_{22}T_1T_2 - \alpha_4)\big)\Big\}$$

In[∘]:= `cvs2c = Union@Cases[eqn2cred, p__ | π__, ∞]`

Out[∘]=

$\{p_{1,3+j}, p_{2,3+j}, p_{3,3+j}, \pi_{1,i}, \pi_{2,i}, \pi_{3,i}\}$

In[∘]:= `{eqns2c} = CoefficientRules[eqn2cred, cvs2c] /. (_ → c_) :→ (c == 0)`

In[∘]:= `vars2b = Union@Cases[eqn2cred, c_, ∞]`

Out[∘]=

$\{c_5, c_9, c_{10}, c_{22}, c_{28}, c_{50}, c_{79}\}$

In[∘]:= `{sol2c} = Solve[eqns2c, {c5, c50, c79}] // Simplify`

Out[∘]=

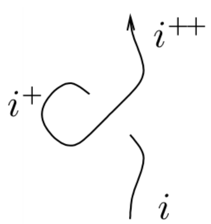
$$\{\{c_5 \to -1, c_{50} \to -1, c_{79} \to (-c_{22} c_{28} - c_{22} T_1 + c_{22} c_{28} T_1 + c_{22} T_1^2 + c_{22} c_{28} T_2 + c_{22} T_1 T_2 - c_{22} c_{28} T_1 T_2 - c_{22} T_1^2 T_2 - \alpha_4 + c_{28} \alpha_4 + T_1 \alpha_4 - c_{28} T_2 \alpha_4 + c_9 (-1 + T_1) (c_{22} (-1 + T_1) + \alpha_4) + c_{10} (-1 + T_1) (-1 + T_2) (c_{22} (-1 + T_1) + \alpha_4)) / ((-1 + T_1) (c_{22} T_1 (-1 + T_2) - \alpha_4))\}\}$$

In[∘]:= `eqn2c /. sol2b /. sol3b /. sol3bα /. sα3 /. sol2c // CF`

Out[∘]=

$\{0\}$

Invariance Under R1l



In[∘]:= `{lhs1l} = Cases[∫𝔼[Sum[π_{α,i} p_{α,i}, {α, 3}]] ℒ /@ (X_{i+2,i}[1] C_{i+1}[1]) ⅆ{vs_i, vs_{i+1}, vs_{i+2}}, ∈Series[ℰ_, ℱ_] :→ ℱ, ∞]`

In[∘]:= `rhs1l = ∫𝔼[Sum[π_{α,i} p_{α,i}, {α, 3}]] ℒ /@ (C_i[0] C_{i+1}[0] C_{i+2}[0]) ⅆ{vs_i, vs_{i+1}, vs_{i+2}}`

Out[∘]=

$-512\, i\, \pi^9\, \mathbb{E}[\in\text{Series}[p_{1,3+i} \pi_{1,i} + p_{2,3+i} \pi_{2,i} + p_{3,3+i} \pi_{3,i}, 0]]$

In[∘]:= `eqn1l = CF[CF[CF[lhs1l] /. {$A $C → α1, $A $D → α2, $B $C → α3, $B $D → α4}] /. {$A | $B → 0}];`

In[∘]:= `eqn1lred = eqn1l /. sol2b /. sol3b /. sol3bα /. sα3 /. sol2c // CF`

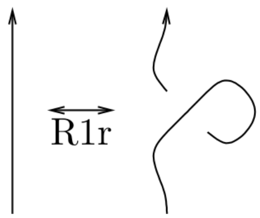
Out[∘]=

$\frac{1}{2} (3 + 2 c_{88})$

In[∘]:=
```
{sol1l} = Solve[eqn1lred == 0, c88]
```
Out[∘]=

$$\left\{\left\{c_{88} \to -\frac{3}{2}\right\}\right\}$$

Invariance Under R1r



In[∘]:=
```
{lhs1r} = Cases[∫𝔼[Sum[π_{α,i} p_{α,i}, {α, 3}]] ℒ /@ (X_{i,i+2}[1] C_{i+1}[-1]) ⅆ{vs_i, vs_{i+1}, vs_{i+2}},
  ϵSeries[ℰ_, ℱ_] :→ ℱ, ∞]
```
In[∘]:=
```
rhs1r = ∫𝔼[Sum[π_{α,i} p_{α,i}, {α, 3}]] ℒ /@ (C_i[0] C_{i+1}[0] C_{i+2}[0]) ⅆ{vs_i, vs_{i+1}, vs_{i+2}}
```
Out[∘]=

$$-512\, i\, \pi^9\, \mathbb{E}[\epsilon\text{Series}[p_{1,3+i}\, \pi_{1,i} + p_{2,3+i}\, \pi_{2,i} + p_{3,3+i}\, \pi_{3,i},\ 0]]$$

In[∘]:=
```
eqn1r = CF[CF[CF[lhs1r] /. {$A $C → α1, $A $D → α2, $B $C → α3, $B $D → α4}] /. {$A | $B → 0}];
```
In[∘]:=
```
eqn1rred = eqn1r /. sol2b /. sol3b /. sol3bα /. sα3 /. sol2c /. sol1l // CF
```
Out[∘]=

$$\Big(c_9 c_{22} - c_{10} c_{22} - c_{22} c_{28} - c_9 c_{22} T_1 + c_{10} c_{22} T_1 + c_{22}^2 T_1 - c_{22} c_{81} T_1 - c_{22} c_{86} T_1 - c_{22} c_{56} T_1^2 - c_{22} c_{69} T_1^2 - c_9 c_{22} T_2 + 2 c_{10} c_{22} T_2 + 2 c_{22} c_{28} T_2 + c_9 c_{22} T_1 T_2 - 2 c_{10} c_{22} T_1 T_2 - c_{15} c_{22} T_1 T_2 - 2 c_{22}^2 T_1 T_2 - c_{22} c_{36} T_1 T_2 + 2 c_{22} c_{81} T_1 T_2 + 2 c_{22} c_{86} T_1 T_2 - 3 c_{22} T_1^2 T_2 + c_{15} c_{22} T_1^2 T_2 + 3 c_{22} c_{56} T_1^2 T_2 + 2 c_{22} c_{69} T_1^2 T_2 + c_{22} c_{81} T_1^2 T_2 - c_{10} c_{22} T_2^2 - c_{22} c_{28} T_2^2 + c_{10} c_{22} T_1 T_2^2 + 2 c_{15} c_{22} T_1 T_2^2 + c_{22}^2 T_1 T_2^2 + 2 c_{22} c_{36} T_1 T_2^2 - c_{22} c_{81} T_1 T_2^2 - c_{22} c_{86} T_1 T_2^2 + 6 c_{22} T_1^2 T_2^2 - 2 c_{15} c_{22} T_1^2 T_2^2 - 3 c_{22} c_{56} T_1^2 T_2^2 - c_{22} c_{69} T_1^2 T_2^2 - 2 c_{22} c_{81} T_1^2 T_2^2 - c_{15} c_{22} T_1 T_2^3 - c_{22} c_{36} T_1 T_2^3 - 3 c_{22} T_1^2 T_2^3 + c_{15} c_{22} T_1^2 T_2^3 + c_{22} c_{56} T_1^2 T_2^3 + c_{22} c_{81} T_1^2 T_2^3 + c_{22} \alpha_4 - c_{81} \alpha_4 - c_{86} \alpha_4 + c_{22} T_1 \alpha_4 - c_{56} T_1 \alpha_4 - c_{69} T_1 \alpha_4 + c_9 T_2 \alpha_4 - c_{10} T_2 \alpha_4 - c_{15} T_2 \alpha_4 - c_{22} T_2 \alpha_4 - c_{28} T_2 \alpha_4 - c_{36} T_2 \alpha_4 + c_{81} T_2 \alpha_4 + c_{86} T_2 \alpha_4 - 3 T_1 T_2 \alpha_4 - c_9 T_1 T_2 \alpha_4 + c_{10} T_1 T_2 \alpha_4 + c_{15} T_1 T_2 \alpha_4 - c_{22} T_1 T_2 \alpha_4 + 2 c_{56} T_1 T_2 \alpha_4 + c_{69} T_1 T_2 \alpha_4 + c_{81} T_1 T_2 \alpha_4 + c_{10} T_2^2 \alpha_4 + c_{15} T_2^2 \alpha_4 + c_{28} T_2^2 \alpha_4 + c_{36} T_2^2 \alpha_4 + 3 T_1 T_2^2 \alpha_4 - c_{10} T_1 T_2^2 \alpha_4 - c_{15} T_1 T_2^2 \alpha_4 - c_{56} T_1 T_2^2 \alpha_4 - c_{81} T_1 T_2^2 \alpha_4 + \alpha_4^2\Big) \Big/ \left(T_1 (-1 + T_2) T_2 (-c_{22} T_1 + c_{22} T_1 T_2 - \alpha_4)\right)$$

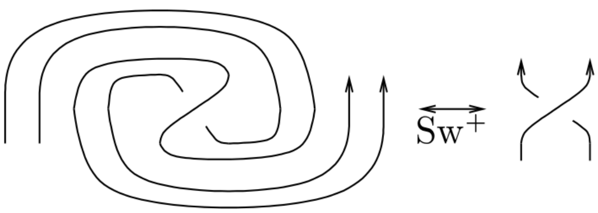
In[∘]:=
```
{sol1r} = Solve[eqn1rred == 0, {c9}] // Simplify
```
Out[∘]=

$$\left\{\left\{c_9 \to \left(-c_{22}^2\, T_1\, (-1+T_2)^2 + \left(c_{56}\, T_1 + c_{69}\, T_1 - c_{86}\, (-1+T_2) + c_{15}\, T_2 + c_{28}\, T_2 + c_{36}\, T_2 + 3\, T_1\, T_2 - c_{15}\, T_1\, T_2 - 2\, c_{56}\, T_1\, T_2 - c_{69}\, T_1\, T_2 - c_{15}\, T_2^2 - c_{28}\, T_2^2 - c_{36}\, T_2^2 - 3\, T_1\, T_2^2 + c_{15}\, T_1\, T_2^2 + c_{56}\, T_1\, T_2^2 + c_{81}\, (-1+T_2)\, (-1+T_1\, T_2) - \alpha_4\right) \alpha_4 + c_{22}\, (-1+T_2) \left(-c_{86}\, T_1 - c_{56}\, T_1^2 - c_{69}\, T_1^2 + c_{28}\, (-1+T_2) - c_{15}\, T_1\, T_2 - c_{36}\, T_1\, T_2 + c_{86}\, T_1\, T_2 - 3\, T_1^2\, T_2 + c_{15}\, T_1^2\, T_2 + 2\, c_{56}\, T_1^2\, T_2 + c_{69}\, T_1^2\, T_2 + c_{15}\, T_1\, T_2^2 + c_{36}\, T_1\, T_2^2 + 3\, T_1^2\, T_2^2 - c_{15}\, T_1^2\, T_2^2 - c_{56}\, T_1^2\, T_2^2 - c_{81}\, T_1\, (-1+T_2)\, (-1+T_1\, T_2) + \alpha_4 + T_1\, \alpha_4\right) - c_{10}\, (-1+T_1)\, (-1+T_2)\, (c_{22}\, (-1+T_2) - T_2\, \alpha_4)\right) \Big/ \left((-1+T_1)\, (c_{22}\, (-1+T_2) - T_2\, \alpha_4)\right)\right\}\right\}$$

In[∘]:=
```
eqn1r /. sol2b /. sol3b /. sol3bα /. sα3 /. sol2c /. sol1l /. sol1r // CF
```
Out[∘]=

0

Invariance Under Sw



In[∘]:=
```
lhssw = ∫𝔼[Sum[π_{α,i} p_{α,i} + π_{α,j} p_{α,j}, {α, 3}]]
  ℒ /@ (X_{i+1,j+1}[1] C_i[-1] C_j[-1] C_{i+2}[1] C_{j+2}[1]) ⅆ{vs_i, vs_j, vs_{i+1}, vs_{j+1}, vs_{i+2}, vs_{j+2}}
```
Out[∘]=

$$262144\, \pi^{18}\, T_1\, T_2\; \mathbb{E}\left[\epsilon\text{Series}\left[T_1\, p_{1,3+i}\, \pi_{1,i} + (1-T_1)\, p_{1,3+j}\, \pi_{1,i} + p_{1,3+j}\, \pi_{1,j} + T_2\, p_{2,3+i}\, \pi_{2,i} + \langle\langle 8 \rangle\rangle + T_1\, T_2\, p_{3,3+i}\, \pi_{3,i} + (1-T_1\, T_2)\, p_{3,3+j}\, \pi_{3,i} + p_{3,3+j}\, \pi_{3,j},\ 2\, c_1 + c_5 + \langle\langle 367 \rangle\rangle + \langle\langle 1 \rangle\rangle\right]\right]$$

large output | show less | show more | show all | set size limit...

In[∘]:=
```
rhssw = ∫𝔼[Sum[π_{α,i} p_{α,i} + π_{α,j} p_{α,j}, {α, 3}]]
  ℒ /@ (X_{i+1,j+1}[1] C_i[0] C_j[0] C_{i+2}[0] C_{j+2}[0]) ⅆ{vs_i, vs_j, vs_{i+1}, vs_{j+1}, vs_{i+2}, vs_{j+2}}
```
Out[∘]=

$$262144\, \pi^{18}\, T_1\, T_2\; \mathbb{E}\left[\epsilon\text{Series}\left[T_1\, p_{1,3+i}\, \pi_{1,i} + (1-T_1)\, p_{1,3+j}\, \pi_{1,i} + p_{1,3+j}\, \pi_{1,j} + T_2\, p_{2,3+i}\, \pi_{2,i} + \langle\langle 8 \rangle\rangle + T_1\, T_2\, p_{3,3+i}\, \pi_{3,i} + (1-T_1\, T_2)\, p_{3,3+j}\, \pi_{3,i} + p_{3,3+j}\, \pi_{3,j},\ 2\, c_1 + c_5 + \langle\langle 367 \rangle\rangle + \langle\langle 1 \rangle\rangle\right]\right]$$

large output | show less | show more | show all | set size limit...

In[∘]:=
```
eqnsw =
  CF[CF[CF[Cases[lhssw, ϵSeries[ℰ_, ℱ_] ⧴ ℱ, ∞] - Cases[rhssw, ϵSeries[ℰ_, ℱ_] ⧴ ℱ, ∞]]]]
```

Out[∘]=

$$\left\{\$A\, p_{3,3+j}\, \pi_{1,i}\, \pi_{2,i} - \frac{\$A\, p_{3,3+j}\, \pi_{1,j}\, \pi_{2,i}}{T_1} - \frac{\$B\, T_2\, p_{3,3+j}\, \pi_{1,j}\, \pi_{2,i}}{T_1} + \$B\, p_{3,3+j}\, \pi_{1,i}\, \pi_{2,j}\right\}$$

```
(*single $A and $B never arise so sw equation holds.*)
```

In[∘]:=
```
eqnsw =
  CF[CF[CF[Cases[lhssw, ϵSeries[ℰ_, ℱ_] ⧴ ℱ, ∞] - Cases[rhssw, ϵSeries[ℰ_, ℱ_] ⧴ ℱ, ∞]] /.
      {$A $C → α1, $A $D → α2, $B $C → α3, $B $D → α4}] /. {$A | $B → 0}]
```

Out[∘]=

{0}

In conclusion: here are the values for the R-matrix we found:

In[∘]:=
```
{r0p, r0m} =
  {r0[1, i, j], r0[-1, i, j]} /. sol2b /. sol3b /. sol3bα /. sα3 /. sol2c /. sol1l /. sol1r // CF
```

Out[∘]=

$$\left\{\frac{\$B\,(-c_{22}\,T_1 + c_{22}\,T_1\,T_2 - T_2\,\alpha_4)\,p_{3,j}\,x_{1,i}\,x_{2,i}}{\alpha_4} - \frac{\$B\,c_{22}\,(-1+T_2)\,p_{3,j}\,x_{1,j}\,x_{2,i}}{\alpha_4} + \$B\,p_{3,j}\,x_{1,i}\,x_{2,j},\right.$$
$$\left. -\frac{\$B\,(-c_{22} + c_{22}\,T_2 - \alpha_4)\,p_{3,j}\,x_{1,i}\,x_{2,i}}{T_1\,T_2\,\alpha_4} + \frac{\$B\,c_{22}\,(-1+T_2)\,p_{3,j}\,x_{1,j}\,x_{2,i}}{T_2\,\alpha_4} - \frac{\$B\,p_{3,j}\,x_{1,i}\,x_{2,j}}{T_1}\right\}$$

In[∘]:=
```
r1p = CF[CF[r1[1, i, j] /. sol3b /. sol3bα] /. sα3 /. sol2c /. sol1l] /. sol1r // CF
```

Out[∘]=

$$-\frac{\cdots 5600 \cdots + \$B\,T_1^3\,T_2^4\,\alpha_4^2\,p_{3,j}^2\,x_{3,i}\,x_{3,j}}{2\,\$B\,(-1+T_1)\,T_1\,(-1+T_2)\,T_2\,(-1+T_1\,T_2)\,(-c_{22}+c_{22}\,T_2-T_2\,\alpha_4)}$$

large output | show less | show more | show all | set size limit...

In[∘]:=
```
r1m =
  CF[CF[CF[r1[-1, i, j] /. sol2b] /. sol3b /. sol3bα] /. sα3 /. sol2c /. sol1l] /. sol1r // CF
```

Out[∘]=

$$\frac{\cdots 8699 \cdots + 3\,\$B\,T_1^3\,T_2^4\,\alpha_4^2\,p_{3,j}^2\,x_{3,i}\,x_{3,j}}{2\,\$B\,(-1+T_1)\,T_1^2\,(-1+T_2)\,T_2^2\,(-1+T_1\,T_2)\,(-c_{22}+c_{22}\,T_2-T_2\,\alpha_4)}$$

large output | show less | show more | show all | set size limit...

In[∘]:=
```
Union@Cases[r1m, c_, ∞]
```

Out[∘]=

$\{c_4, c_{10}, c_{13}, c_{15}, c_{22}, c_{28}, c_{32}, c_{36}, c_{49}, c_{54}, c_{56}, c_{65}, c_{69}, c_{78}, c_{81}, c_{86}\}$

In[∘]:=
```
RandomChoice[{-2, -1, 1, 2}]
```
Out[∘]=

2

In[∘]:=
```
Choices = {-2, -1, 1, 2};
Sub = {c4 → RandomChoice[Choices], c10 → RandomChoice[Choices],
   c13 → RandomChoice[Choices], c15 → RandomChoice[Choices],
   c22 → RandomChoice[Choices], c28 → RandomChoice[{-2, -1, 1, 2}],
   c32 → RandomChoice[Choices], c36 → RandomChoice[Choices], c49 → RandomChoice[Choices],
   c54 → RandomChoice[Choices], c56 → RandomChoice[Choices], c65 → RandomChoice[Choices],
   c69 → RandomChoice[Choices], c78 → RandomChoice[Choices], c81 → RandomChoice[Choices],
   c86 → RandomChoice[Choices], α4 → RandomChoice[Choices], $B → RandomChoice[Choices]}
```
Out[∘]=

$\{c_4 \to -1, c_{10} \to 2, c_{13} \to -2, c_{15} \to -2, c_{22} \to 1, c_{28} \to -2, c_{32} \to 2, c_{36} \to -2, c_{49} \to -2, c_{54} \to -1, c_{56} \to -2, c_{65} \to 1, c_{69} \to 2, c_{78} \to -1, c_{81} \to -2, c_{86} \to 1, \alpha_4 \to -1, \$B \to -2\}$

In[∘]:=
```
r0p /. Sub // CF
r0m /. Sub // CF
```
Out[∘]=

$$2 (-T_1 + T_2 + T_1 T_2)\, p_{3,j} x_{1,i} x_{2,i} - 2 (-1 + T_2)\, p_{3,j} x_{1,j} x_{2,i} - 2 p_{3,j} x_{1,i} x_{2,j}$$

Out[∘]=

$$-\frac{2 p_{3,j} x_{1,i} x_{2,i}}{T_1} + \frac{2 (-1 + T_2)\, p_{3,j} x_{1,j} x_{2,i}}{T_2} + \frac{2 p_{3,j} x_{1,i} x_{2,j}}{T_1}$$

In[∘]:=
```
r1p /. Sub // CF
```
In[∘]:=
```
r1m /. Sub // CF
```
Out[∘]=

$$\frac{3}{2} + 2 p_{1,i} x_{1,i} - \frac{2 \left(-2 + T_1^2\right) p_{1,j} x_{1,i}}{T_1^2} + p_{1,i} p_{1,j} x_{1,i}^2 - \frac{(-1 + T_1)(1 + 2 T_1)\, p_{1,j}^2 x_{1,i}^2}{2 T_1^2} -$$
$$\frac{4 p_{1,j} x_{1,j}}{T_1} + p_{1,i} p_{1,j} x_{1,i} x_{1,j} - \frac{(1 + 3 T_1)\, p_{1,j}^2 x_{1,i} x_{1,j}}{2 T_1} + 2 p_{2,i} x_{2,i} - 2 p_{2,j} x_{2,i} -$$
$$\frac{\left(-2 + 3 T_1 - 4 T_2 + 16 T_1 T_2 - 9 T_1^2 T_2 + 8 T_2^2 - 22 T_1 T_2^2 + 9 T_1^2 T_2^2\right) p_{1,j} p_{2,i} x_{1,i} x_{2,i}}{T_1 (-1 + 2 T_2)} -$$
$$\frac{\left(-1 - T_1 + 8 T_2 - 10 T_1 T_2 + 9 T_1^2 T_2 - 8 T_2^2 + 15 T_1 T_2^2 - 18 T_1^2 T_2^2 - 4 T_1 T_2^3 + 9 T_1^2 T_2^3\right) p_{1,i} p_{2,j} x_{1,i} x_{2,i}}{(-1 + T_1)\, T_2 (-1 + 2 T_2)} +$$
$$\frac{1}{T_1 T_2 (-1 + 2 T_2)} \left(-4 T_1 + 8 T_2 - 14 T_1 T_2 + 18 T_1^2 T_2 - 12 T_2^2 + 35 T_1 T_2^2 - 36 T_1^2 T_2^2 + 8 T_2^3 - 26 T_1 T_2^3 + 18 T_1^2 T_2^3\right)$$
$$p_{1,j} p_{2,j} x_{1,i} x_{2,i} - \frac{\left(-4 + 3 T_1 + 16 T_1 T_2 - 9 T_1^2 T_2 + 8 T_2^2 - 22 T_1 T_2^2 + 9 T_1^2 T_2^2\right) p_{1,j} p_{2,i} x_{1,j} x_{2,i}}{(-1 + T_1)(-1 + 2 T_2)} +$$
$$\frac{(-1 + T_2)\left(2 - 4 T_1 + 4 T_2 - 13 T_1 T_2 + 9 T_1^2 T_2\right) p_{1,j} p_{2,j} x_{1,j} x_{2,i}}{(-1 + T_1)\, T_2} +$$
$$\frac{(1 + T_2)\, p_{2,i} p_{2,j} x_{2,i}^2}{T_2} - \frac{(-1 + T_2)(3 + 2 T_2)\, p_{2,j}^2 x_{2,i}^2}{2 T_2^2} - 2 p_{1,i} p_{2,j} x_{1,i} x_{2,j} +$$

$$\frac{\left(2+T_1-8\,T_2+11\,T_1\,T_2-9\,T_1^2\,T_2-4\,T_1\,T_2^2+9\,T_1^2\,T_2^2\right)p_{1,j}\,p_{2,j}\,x_{1,i}\,x_{2,j}}{T_1\,(-1+2\,T_2)}+p_{2,i}\,p_{2,j}\,x_{2,i}\,x_{2,j}-$$

$$\frac{3\,(1+T_2)\,p_{2,j}^2\,x_{2,i}\,x_{2,j}}{2\,T_2}-\frac{\left(T_1+4\,T_2-11\,T_1\,T_2-4\,T_1\,T_2^2+9\,T_1^2\,T_2^2\right)p_{1,j}\,p_{2,i}\,x_{3,i}}{2\,T_1\,(-1+2\,T_2)}-$$

$$\frac{p_{1,i}\,p_{2,j}\,x_{3,i}}{2\,T_2}+\frac{\left(-T_1+3\,T_1\,T_2+4\,T_2^2-11\,T_1\,T_2^2-4\,T_1\,T_2^3+9\,T_1^2\,T_2^3\right)p_{1,j}\,p_{2,j}\,x_{3,i}}{2\,T_1\,T_2\,(-1+2\,T_2)}+2\,p_{3,i}\,x_{3,i}-$$

$$\frac{\left(-4+9\,T_1-T_1^2+8\,T_2-18\,T_1\,T_2+T_1^2\,T_2-4\,T_1\,T_2^2+15\,T_1^2\,T_2^2-9\,T_1^3\,T_2^2-4\,T_1^2\,T_2^3+9\,T_1^3\,T_2^3\right)p_{3,j}\,x_{3,i}}{T_1^2\,T_2\,(-1+2\,T_2)}-$$

$$\frac{\left(4-4\,T_1+T_1^2+4\,T_2-11\,T_1\,T_2-4\,T_1\,T_2^2+9\,T_1^2\,T_2^2\right)p_{1,j}\,p_{3,i}\,x_{1,i}\,x_{3,i}}{T_1\,(-1+T_1\,T_2)}+$$

$$\frac{\left(-2\,T_1+4\,T_2-7\,T_1\,T_2+10\,T_1^2\,T_2-4\,T_1\,T_2^2+2\,T_1^2\,T_2^2-9\,T_1^3\,T_2^2-4\,T_1^2\,T_2^3+9\,T_1^3\,T_2^3\right)p_{1,i}\,p_{3,j}\,x_{1,i}\,x_{3,i}}{(-1+T_1)\,T_1\,T_2\,(-1+2\,T_2)}-$$

$$\frac{\left(2-6\,T_1+T_1^2+T_1\,T_2+8\,T_1^2\,T_2-4\,T_1\,T_2^2+2\,T_1^2\,T_2^2-9\,T_1^3\,T_2^2-4\,T_1^2\,T_2^3+9\,T_1^3\,T_2^3\right)p_{1,j}\,p_{3,j}\,x_{1,i}\,x_{3,i}}{T_1^2\,T_2\,(-1+2\,T_2)}-$$

$$\frac{\left(6-4\,T_1+T_1^2+4\,T_2-13\,T_1\,T_2-4\,T_1\,T_2^2+9\,T_1^2\,T_2^2\right)p_{1,j}\,p_{3,i}\,x_{1,j}\,x_{3,i}}{(-1+T_1)\,(-1+T_1\,T_2)}-\frac{p_{1,i}\,p_{3,j}\,x_{1,j}\,x_{3,i}}{T_2}+$$

$$\frac{\left(-4+6\,T_1-2\,T_1^2+8\,T_2-11\,T_1\,T_2+4\,T_1^2\,T_2+4\,T_2^2-11\,T_1\,T_2^2-4\,T_1\,T_2^3+9\,T_1^2\,T_2^3\right)p_{1,j}\,p_{3,j}\,x_{1,j}\,x_{3,i}}{(-1+T_1)\,T_1\,T_2\,(-1+2\,T_2)}+$$

$$\Big(\big(-2+T_1-3\,T_2+22\,T_1\,T_2-10\,T_1^2\,T_2+14\,T_2^2-42\,T_1\,T_2^2-T_1^2\,T_2^2+9\,T_1^3\,T_2^2-8\,T_2^3+10\,T_1\,T_2^3+33\,T_1^2\,T_2^3-18\,T_1^3\,T_2^3+8\,T_1\,T_2^4-22\,T_1^2\,T_2^4+9\,T_1^3\,T_2^4\big)\,p_{2,j}\,p_{3,i}\,x_{2,i}\,x_{3,i}\Big)\Big/\big((-1+T_1)\,T_2\,(-1+2\,T_2)\,(-1+T_1\,T_2)\big)+$$

$$\Big((-2+T_1)\,\big(T_1+3\,T_2-12\,T_1\,T_2-6\,T_2^2+17\,T_1\,T_2^2+9\,T_1^2\,T_2^2+4\,T_2^3-3\,T_1\,T_2^3-18\,T_1^2\,T_2^3-4\,T_1\,T_2^4+9\,T_1^2\,T_2^4\big)\,p_{2,i}\,p_{3,j}\,x_{2,i}\,x_{3,i}\Big)\Big/\big((-1+T_1)\,T_1\,(-1+T_2)\,T_2\,(-1+2\,T_2)\big)-\frac{1}{(-1+T_1)\,T_1\,T_2^2}$$

$$\left(1+T_1-T_1^2+5\,T_2-24\,T_1\,T_2+11\,T_1^2\,T_2-8\,T_2^2+18\,T_1\,T_2^2+11\,T_1^2\,T_2^2-9\,T_1^3\,T_2^2+8\,T_1\,T_2^3-22\,T_1^2\,T_2^3+9\,T_1^3\,T_2^3\right)$$

$$p_{2,j}\,p_{3,j}\,x_{2,i}\,x_{3,i}+\Big(\big(-3+2\,T_1-T_2+21\,T_1\,T_2-11\,T_1^2\,T_2+14\,T_2^2-44\,T_1\,T_2^2+T_1^2\,T_2^2+9\,T_1^3\,T_2^2-8\,T_2^3+10\,T_1\,T_2^3+33\,T_1^2\,T_2^3-18\,T_1^3\,T_2^3+8\,T_1\,T_2^4-22\,T_1^2\,T_2^4+9\,T_1^3\,T_2^4\big)\,p_{2,j}\,p_{3,i}\,x_{2,j}\,x_{3,i}\Big)\Big/$$

$$\big((-1+T_1)\,(-1+T_2)\,(-1+2\,T_2)\,(-1+T_1\,T_2)\big)-$$

$$\frac{\left(T_1+4\,T_2-11\,T_1\,T_2-4\,T_1\,T_2^2+9\,T_1^2\,T_2^2\right)p_{2,i}\,p_{3,j}\,x_{2,j}\,x_{3,i}}{(-1+T_1)\,T_1\,(-1+2\,T_2)}-\frac{1}{(-1+T_1)\,T_1\,T_2\,(-1+2\,T_2)}$$

$$\left(1+6\,T_2-25\,T_1\,T_2+10\,T_1^2\,T_2-12\,T_2^2+29\,T_1\,T_2^2+11\,T_1^2\,T_2^2-9\,T_1^3\,T_2^2+12\,T_1\,T_2^3-31\,T_1^2\,T_2^3+9\,T_1^3\,T_2^3\right)$$

$$p_{2,j}\,p_{3,j}\,x_{2,j}\,x_{3,i}-$$

$$\Big(\big(2-T_1+4\,T_2-23\,T_1\,T_2+11\,T_1^2\,T_2-8\,T_2^2+20\,T_1\,T_2^2+9\,T_1^2\,T_2^2-9\,T_1^3\,T_2^2+8\,T_1\,T_2^3-22\,T_1^2\,T_2^3+9\,T_1^3\,T_2^3\big)\,p_{3,i}\,p_{3,j}\,x_{3,i}^2\Big)\Big/\big((-1+T_1)\,T_1\,T_2\,(-1+2\,T_2)\big)+$$

$$\Big((-1+T_1\,T_2)\,\big(1+6\,T_2-28\,T_1\,T_2+13\,T_1^2\,T_2-8\,T_2^2+18\,T_1\,T_2^2+27\,T_1^2\,T_2^2-18\,T_1^3\,T_2^2+16\,T_1\,T_2^3-44\,T_1^2\,T_2^3+18\,T_1^3\,T_2^3\big)\,p_{3,j}^2\,x_{3,i}^2\Big)\Big/\big(2\,(-1+T_1)\,T_1^2\,T_2^2\,(-1+2\,T_2)\big)-\frac{(-4+9\,T_1)\,p_{3,j}\,x_{3,j}}{T_1}+$$

$$\frac{\left(-4+3\,T_1-T_1^2+12\,T_2-17\,T_1\,T_2+12\,T_1^2\,T_2-4\,T_1\,T_2^2+2\,T_1^2\,T_2^2-9\,T_1^3\,T_2^2-4\,T_1^2\,T_2^3+9\,T_1^3\,T_2^3\right)p_{1,i}\,p_{3,j}\,x_{1,i}\,x_{3,j}}{(-1+T_1)\,(-1+2\,T_2)\,(-1+T_1\,T_2)}$$

$$-\frac{\left(2+T_1-8\,T_2+11\,T_1\,T_2-9\,T_1^2\,T_2-4\,T_1\,T_2^2+9\,T_1^2\,T_2^2\right)\,p_{1,j}\,p_{3,j}\,x_{1,i}\,x_{3,j}}{T_1\,(-1+2\,T_2)}-$$

$$\frac{(-1-T_2+T_1\,T_2)\,p_{2,i}\,p_{3,j}\,x_{2,i}\,x_{3,j}}{(-1+T_2)\,(-1+T_1\,T_2)}-\frac{(-2+T_1)\,\left(1+2\,T_2-9\,T_1\,T_2-4\,T_2^2+9\,T_1\,T_2^2\right)\,p_{2,j}\,p_{3,j}\,x_{2,i}\,x_{3,j}}{(-1+T_1)\,(-1+2\,T_2)}-$$

$$\frac{\left(-3+2\,T_1-2\,T_2+18\,T_1\,T_2-9\,T_1^2\,T_2+8\,T_2^2-22\,T_1\,T_2^2+9\,T_1^2\,T_2^2\right)\,p_{3,i}\,p_{3,j}\,x_{3,i}\,x_{3,j}}{(-1+T_1)\,(-1+2\,T_2)}+$$

$$\left(\left(1+6\,T_2-31\,T_1\,T_2+15\,T_1^2\,T_2-8\,T_2^2+16\,T_1\,T_2^2+45\,T_1^2\,T_2^2-27\,T_1^3\,T_2^2+24\,T_1\,T_2^3-66\,T_1^2\,T_2^3+27\,T_1^3\,T_2^3\right)\right.$$
$$\left.p_{3,j}^2\,x_{3,i}\,x_{3,j}\right)\Big/\left(2\,(-1+T_1)\,T_1\,T_2\,(-1+2\,T_2)\right)$$