Pensieve header: Finding the A2 $d=1$ invariant using undetermined coefficients.

Searching for $Q + pxx + \epsilon(ppx + 1 + px + ppxx)$ solutions.

Initialization

In[•]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\HigherRank"];
Once[<< KnotTheory`; << Rot.m];
<< FormalGaussianIntegration.m;
i_⁺ := i + 1;
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

```
Loading Rot.m from http://drorbn.net/AP/Projects/HigherRank to compute rotation numbers.
```

In[•]:=
```
Features[Knot[8, 17]]
```

KnotTheory: Loading precomputed data in PD4Knots`.

Out[•]=
```
Features[18,
 C6[-1] C14[-1] X1,7[1] X3,9[-1] X5,13[-1] X8,16[1] X10,4[-1] X12,18[1] X15,2[-1] X17,11[1]]
```

In[•]:=
```
T3 = T1 T2;
S = {x__, p__};
q[s_, i_, j_] := Sum[
    xv,i (pv,i⁺ - pv,i) + xv,j (pv,j⁺ - pv,j) + (Tv^s - 1) xv,i (pv,i⁺ - pv,j⁺),
    {v, 3}];
ℒ[Xi_,j_[s_]] :=
  T3^s 𝔼[q[s, i, j] + B^-1 r0[s, i, j] + ϵ B r1[s, i, j] + ϵ r42[s, i, j] + O[ϵ]^2];
(*γ1[φ_,k_]:=φ(3/2-x1,kp1,k-x2,kp2,k-x3,kp3,k);*)
ℒ[Ck_[0]] := 𝔼[Sum[xv,k (pv,k⁺ - pv,k), {v, 3}] + O[ϵ]^2];
ℒ[Ck_[φ_]] :=
  T3^φ 𝔼[Sum[xv,k (pv,k⁺ - pv,k), {v, 3}] + B^-1 γ0[φ, k] + ϵ B γ1[φ, k] + ϵ γ42[φ, k] + O[ϵ]^2];
psi_ := Sequence[p1,i, p2,i, p3,i];
xsi_ := Sequence[x1,i, x2,i, x3,i];
vsi_ := Sequence[psi, xsi];
ℱ[is___] := 𝔼[Sum[πv,i pv,i, {i, {is}}, {v, 3}]];
ℒ[K_] := (2 π)^-Features[K][[1]] CF[ℒ /@ Features[K][[2]]];
vs[K_] := Union @@ Table[{vsi}, {i, Features[K][[1]]}]
```

In[•]:=
```
vsi
```

Out[•]=
```
Sequence[p1,i, p2,i, p3,i, x1,i, x2,i, x3,i]
```

The Various Terms (r_0)

The pxx Terms (r_0)

```
In[•]:= κ = 0;
        r0[1, i_, j_] := Evaluate[Sum[
             a++κ p3,k3 x1,k1 x2,k2,
             {k1, {i, j}}, {k2, {i, j}}, {k3, {i, j}}
            ]];
        r0[1, i, j]
```

Out[•]=

$$a_1\, p_{3,i}\, x_{1,i}\, x_{2,i} + a_2\, p_{3,j}\, x_{1,i}\, x_{2,i} + a_5\, p_{3,i}\, x_{1,j}\, x_{2,i} + a_6\, p_{3,j}\, x_{1,j}\, x_{2,i} + a_3\, p_{3,i}\, x_{1,i}\, x_{2,j} + a_4\, p_{3,j}\, x_{1,i}\, x_{2,j} + a_7\, p_{3,i}\, x_{1,j}\, x_{2,j} + a_8\, p_{3,j}\, x_{1,j}\, x_{2,j}$$

```
In[•]:= κ = 0;
        r0[-1, i_, j_] := Evaluate[Sum[
             d++κ p3,k3 x1,k1 x2,k2,
             {k1, {i, j}}, {k2, {i, j}}, {k3, {i, j}}
            ]];
        r0[-1, i, j]
```

Out[•]=

$$d_1\, p_{3,i}\, x_{1,i}\, x_{2,i} + d_2\, p_{3,j}\, x_{1,i}\, x_{2,i} + d_5\, p_{3,i}\, x_{1,j}\, x_{2,i} + d_6\, p_{3,j}\, x_{1,j}\, x_{2,i} + d_3\, p_{3,i}\, x_{1,i}\, x_{2,j} + d_4\, p_{3,j}\, x_{1,i}\, x_{2,j} + d_7\, p_{3,i}\, x_{1,j}\, x_{2,j} + d_8\, p_{3,j}\, x_{1,j}\, x_{2,j}$$

The ppx Terms (r_1)

```
In[•]:= κ = 0;
        r1[1, i_, j_] := Evaluate[Sum[
             b++κ x3,k3 p1,k1 p2,k2,
             {k1, {i, j}}, {k2, {i, j}}, {k3, {i, j}}
            ]];
        r1[1, i, j]
```

Out[•]=

$$b_1\, p_{1,i}\, p_{2,i}\, x_{3,i} + b_5\, p_{1,j}\, p_{2,i}\, x_{3,i} + b_3\, p_{1,i}\, p_{2,j}\, x_{3,i} + b_7\, p_{1,j}\, p_{2,j}\, x_{3,i} + b_2\, p_{1,i}\, p_{2,i}\, x_{3,j} + b_6\, p_{1,j}\, p_{2,i}\, x_{3,j} + b_4\, p_{1,i}\, p_{2,j}\, x_{3,j} + b_8\, p_{1,j}\, p_{2,j}\, x_{3,j}$$

```
In[•]:= κ = 0;
        r1[-1, i_, j_] := Evaluate[Sum[
             e++κ x3,k3 p1,k1 p2,k2,
             {k1, {i, j}}, {k2, {i, j}}, {k3, {i, j}}
            ]];
        r1[-1, i, j]
```

Out[•]=

$$e_1\, p_{1,i}\, p_{2,i}\, x_{3,i} + e_5\, p_{1,j}\, p_{2,i}\, x_{3,i} + e_3\, p_{1,i}\, p_{2,j}\, x_{3,i} + e_7\, p_{1,j}\, p_{2,j}\, x_{3,i} + e_2\, p_{1,i}\, p_{2,i}\, x_{3,j} + e_6\, p_{1,j}\, p_{2,i}\, x_{3,j} + e_4\, p_{1,i}\, p_{2,j}\, x_{3,j} + e_8\, p_{1,j}\, p_{2,j}\, x_{3,j}$$

The ppxx Terms (r_{42})

In[•]:=
```
κ = 0;
Short[r42[1, i_, j_] = Evaluate[Plus[
    Sum[
     c++κ xν1,k1 pν1,k2 xν2,k3 pν2,k4,
     {k1, {i, j}}, {k2, {i, j}}, {k3, {i, j}}, {k4, {i, j}}, {ν1, 2}, {ν2, ν1 + 1, 3}
    ],
    Sum[
     c++κ xν,k1 pν,k2,
     {k1, {i, j}}, {k2, {i, j}}, {ν, 3}
    ]
   ]]]
```

Out[•]//Short=
$$c_{49}\, p_{1,i}\, x_{1,i} + c_{52}\, p_{1,j}\, x_{1,i} + \text{<<57>>} + c_{48}\, p_{2,j}\, p_{3,j}\, x_{2,j}\, x_{3,j}$$

In[•]:=
```
κ = 0;
Short[r42[-1, i_, j_] = Evaluate[Plus[
    Sum[
     f++κ xν1,k1 pν1,k2 xν2,k3 pν2,k4,
     {k1, {i, j}}, {k2, {i, j}}, {k3, {i, j}}, {k4, {i, j}}, {ν1, 2}, {ν2, ν1 + 1, 3}
    ],
    Sum[
     f++κ xν,k1 pν,k2,
     {k1, {i, j}}, {k2, {i, j}}, {ν, 3}
    ]
   ]]]
```

Out[•]//Short=
$$f_{49}\, p_{1,i}\, x_{1,i} + f_{52}\, p_{1,j}\, x_{1,i} + \text{<<57>>} + f_{48}\, p_{2,j}\, p_{3,j}\, x_{2,j}\, x_{3,j}$$

The γ Terms (γ_0, γ_1, γ_{42})

In[•]:=
```
κ = 0;
γ0[1, k_] := Evaluate[g++κ p3,k x1,k x2,k];
γ1[1, k_] := Evaluate[g++κ x3,k p1,k p2,k];
γ42[1, k_] := Evaluate[Plus[
    Sum[g++κ xν,k pν,k, {ν, 3}],
    Sum[g++κ xν1,k pν1,k xν2,k pν2,k, {ν1, 2}, {ν2, ν1 + 1, 3}]
   ]];
{γ0[1, k], γ0[1, k], γ42[1, k]}
```

Out[•]=
$$\{g_1\, p_{3,k}\, x_{1,k}\, x_{2,k},\ g_1\, p_{3,k}\, x_{1,k}\, x_{2,k},$$
$$g_3\, p_{1,k}\, x_{1,k} + g_4\, p_{2,k}\, x_{2,k} + g_6\, p_{1,k}\, p_{2,k}\, x_{1,k}\, x_{2,k} + g_5\, p_{3,k}\, x_{3,k} + g_7\, p_{1,k}\, p_{3,k}\, x_{1,k}\, x_{3,k} + g_8\, p_{2,k}\, p_{3,k}\, x_{2,k}\, x_{3,k}\}$$

```
In[•]:= κ = 0;
        γ0[-1, k_] := Evaluate[h++κ p3,k x1,k x2,k];
        γ1[-1, k_] := Evaluate[h++κ x3,k p1,k p2,k];
        γ42[-1, k_] := Evaluate[Plus[
             Sum[h++κ xν,k pν,k, {ν, 3}],
             Sum[h++κ xν1,k pν1,k xν2,k pν2,k, {ν1, 2}, {ν2, ν1 + 1, 3}]
            ]];
        {γ0[-1, k], γ0[-1, k], γ42[-1, k]}
```

```
Out[•]=
        {h1 p3,k x1,k x2,k, h1 p3,k x1,k x2,k,
         h3 p1,k x1,k + h4 p2,k x2,k + h6 p1,k p2,k x1,k x2,k + h5 p3,k x3,k + h7 p1,k p3,k x1,k x3,k + h8 p2,k p3,k x2,k x3,k}
```

Reidemeister 3b

```
In[•]:= Timing[{LeftR3b} =
          Cases[∫ℱ[i, j, k] × ℒ /@ (Xi,j[1] Xi+,k[1] Xj+,k+[1]) 𝕕{vsi, vsj, vsk, vsi+, vsj+, vsk+},
           𝔼[ℰ_] :↦ ℰ, ∞]]
```

```
Out[•]=
        {34.9844, {∈Series[
           T1^2 p1,2+i π1,i - (-1 + T1) T1 p1,2+j π1,i + (1 - T1) p1,2+k π1,i + T1 p1,2+j π1,j + (1 - T1) p1,2+k π1,j + p1,2+k π1,k + ··· 44 ··· +
            T1^2 T2^2 p3,2+i π3,i - T1 T2 (-1 + T1 T2) p3,2+j π3,i + (1 - T1 T2) p3,2+k π3,i + T1 T2 p3,2+j π3,j + (1 - T1 T2) p3,2+k π3,j + p3,2+k π3,k,
           3 (a1 b1 + a2 b2 + a3 b3 + a4 b4 + a5 b5 + a6 b6 + a7 b7 + a8 b8 + c1 + ··· 8 ··· + c46 + c47 + c48 + c49 + c50 + c51 + c58 + c59 + c60) +
            ··· 406 ··· + ··· 1 ···]}}
```

Full expression not available (original memory size: 3.6 MB)

```
In[•]:= Timing[{RightR3b} =
          Cases[∫ℱ[i, j, k] × ℒ /@ (Xj,k[1] Xi,k+[1] Xi+,j+[1]) 𝕕{vsi, vsj, vsk, vsi+, vsj+, vsk+},
           𝔼[ℰ_] :↦ ℰ, ∞];]
```

```
Out[•]=
        {22.9688, Null}
```

In[•]:=
```
Short[eqn = CF[LeftR3b[[1]] - RightR3b[[1]]]]
cvs = Union@Cases[eqn, p__ | π__, ∞]
vars = Union@Cases[r0[1, i, j], a_, ∞]
Short[eqns = CoefficientRules[eqn, cvs] /. (_ → c_) :↦ (c == 0), 3]
{sol} = Solve[eqns, vars]
```

Out[•]//Short=

$$\frac{T_1\,T_2\,(\text{<<1>>})\ \text{<<1>>}\ \pi_{1,i}\,\pi_{2,i}}{B} - \frac{\text{<<1>>}}{B} + \text{<<32>>} + \frac{a_7\ \text{<<5>>}\ \pi_{\text{<<1>>}}}{B}$$

Out[•]=

$$\{p_{3,2+i},\ p_{3,2+j},\ p_{3,2+k},\ \pi_{1,i},\ \pi_{1,j},\ \pi_{1,k},\ \pi_{2,i},\ \pi_{2,j},\ \pi_{2,k}\}$$

Out[•]=

$$\{a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6,\ a_7,\ a_8\}$$

Out[•]//Short=

$$\left\{-\frac{a_3\,T_1^2\,T_2^2}{B} + \frac{a_3\,T_1^2\,T_2^3}{B} == 0,\ \frac{a_3\,T_1^2\,T_2}{B} - \frac{a_3\,T_1^2\,T_2^2}{B} == 0,\right.$$
$$\left.\text{<<22>>},\ -\frac{a_7}{B} - \frac{a_8}{B} + \frac{a_7\,T_2}{B} + \frac{a_8\,T_2}{B} + \frac{a_7\,T_1\,T_2}{B} - \frac{a_7\,T_1\,T_2^2}{B} == 0\right\}$$

Solve: Equations may not give solutions for all "solve" variables.

Out[•]=

$$\left\{\left\{a_1 \to 0,\ a_3 \to 0,\ a_5 \to 0,\ a_6 \to -\frac{a_2}{T_1} - \frac{a_4\,T_2}{T_1},\ a_7 \to 0,\ a_8 \to 0\right\}\right\}$$

In[•]:=
```
sol /. (v_ → val_) :↦ (v = CF[val]);
r0[1, i, j]
```

Out[•]=

$$a_2\,p_{3,j}\,x_{1,i}\,x_{2,i} - \frac{(a_2 + a_4\,T_2)\,p_{3,j}\,x_{1,j}\,x_{2,i}}{T_1} + a_4\,p_{3,j}\,x_{1,i}\,x_{2,j}$$

```
In[•]:= Short[eqn = CF[Coefficient[
          LeftR3b[[2]] - RightR3b[[2]] /. v : (π | p)__ :> μ v,
          μ^3
         ]], 5]
        cvs = Union@Cases[eqn, p__ | π__, ∞]
        vars = Union@Cases[r1[1, i, j], b_, ∞]
        Short[eqns = CoefficientRules[eqn, cvs] /. (_ → c_) :> (c == 0), 3]
        {sol} = Solve[eqns, vars]
```

```
Out[•]//Short=
B b1 (-1 + T1) T1 T2^2 p1,2+j p2,2+i π3,i - B b1 (-1 + T1) T1 T2^2 p1,2+k p2,2+i π3,i + <<31>> +
 B (-1 + T1) T1 (-b2 - b4 + b2 T2) p1,2+i p2,2+k π3,k - B (-1 + T1) T1 (-b2 - b4 + b2 T2) p1,2+j p2,2+k π3,k
```

```
Out[•]=
{p1,2+i, p1,2+j, p1,2+k, p2,2+i, p2,2+j, p2,2+k, π3,i, π3,j, π3,k}
```

```
Out[•]=
{b1, b2, b3, b4, b5, b6, b7, b8}
```

```
Out[•]//Short=
{-B b2 T1^2 T2^2 + B b2 T1^3 T2^3 == 0, B b2 T1 T2 - B b2 T1^2 T2^2 == 0, <<1>> == 0, <<19>>, <<1>> == 0,
 B b1 + <<47>> == 0, -B b6 T1 - B b8 T1 - B b4 T2 - B b8 T2 + B b2 T1 T2 + <<11>> + B b2 T1^2 T2^2 == 0}
```

Solve: Equations may not give solutions for all "solve" variables.

```
Out[•]=
{{b1 → 0, b2 → 0, b4 → 0, b6 → 0, b7 → -b3 - b5, b8 → 0}}
```

```
In[•]:= sol /. (v_ → val_) :> (v = CF[val]);
        r1[1, i, j]
```

```
Out[•]=
b5 p1,j p2,i x3,i + b3 p1,i p2,j x3,i + (-b3 - b5) p1,j p2,j x3,i
```

In[•]:=
```
Short[eqn = CF[LeftR3b[[2]] - RightR3b[[2]]], 5]
cvs = Union@Cases[eqn, p__ | π__, ∞]
vars = Union@Cases[r42[1, i, j], c_, ∞]
Short[eqns = CoefficientRules[eqn, cvs] /. (_ → c_) :↦ (c == 0), 3]
Short[{sol} = Solve[eqns, vars]]
```

Out[•]//Short=
$$-\left((c_{25}+c_{26}+c_{34}+c_{35}+c_{55})\,(-1+T_1)\,T_1^2\,p_{1,2+j}\,\pi_{1,i}\right)-$$
$$(-1+T_1)\left(c_1+c_2+c_{10}+c_{11}+c_{13}+c_{14}+c_{22}+c_{23}+c_{49}+c_{52}+\text{<<5>>}+c_{38}\,T_1+\right.$$
$$\left.c_{46}\,T_1+c_{47}\,T_1+c_{55}\,T_1+c_{58}\,T_1-c_{25}\,T_1^2-c_{26}\,T_1^2-c_{34}\,T_1^2-c_{35}\,T_1^2-c_{55}\,T_1^2\right)p_{1,\text{<<1>>}}\,\pi_{1,i}+$$
$$\text{<<374>>}+\text{<<1>>}-(-1+T_2)\,T_2\,(-c_{33}-c_{36}+c_{33}\,T_1\,T_2)\,p_{2,2+j}\,p_{3,2+k}\,\pi_{2,k}\,\pi_{3,k}$$

Out[•]=
$$\{p_{1,2+i},\ p_{1,2+j},\ p_{1,2+k},\ p_{2,2+i},\ p_{2,2+j},\ p_{2,2+k},\ p_{3,2+i},$$
$$p_{3,2+j},\ p_{3,2+k},\ \pi_{1,i},\ \pi_{1,j},\ \pi_{1,k},\ \pi_{2,i},\ \pi_{2,j},\ \pi_{2,k},\ \pi_{3,i},\ \pi_{3,j},\ \pi_{3,k}\}$$

Out[•]=
$$\{c_1,\ c_2,\ c_3,\ c_4,\ c_5,\ c_6,\ c_7,\ c_8,\ c_9,\ c_{10},\ c_{11},\ c_{12},\ c_{13},\ c_{14},\ c_{15},\ c_{16},\ c_{17},\ c_{18},\ c_{19},\ c_{20},\ c_{21},\ c_{22},$$
$$c_{23},\ c_{24},\ c_{25},\ c_{26},\ c_{27},\ c_{28},\ c_{29},\ c_{30},\ c_{31},\ c_{32},\ c_{33},\ c_{34},\ c_{35},\ c_{36},\ c_{37},\ c_{38},\ c_{39},\ c_{40},\ c_{41},$$
$$c_{42},\ c_{43},\ c_{44},\ c_{45},\ c_{46},\ c_{47},\ c_{48},\ c_{49},\ c_{50},\ c_{51},\ c_{52},\ c_{53},\ c_{54},\ c_{55},\ c_{56},\ c_{57},\ c_{58},\ c_{59},\ c_{60}\}$$

Out[•]//Short=
$$\{-c_7\,T_1^2\,T_2^2+c_7\,T_1^2\,T_2^3==0,\ \text{<<250>>},\ c_8\,T_1\,T_2+c_9\,T_1\,T_2+c_{44}\,T_1\,T_2+$$
$$c_{45}\,T_1\,T_2+c_{57}\,T_1\,T_2-c_8\,T_1^2\,T_2^2-c_9\,T_1^2\,T_2^2-c_{44}\,T_1^2\,T_2^2-c_{45}\,T_1^2\,T_2^2-c_{57}\,T_1^2\,T_2^2==0\}$$

Solve: Equations may not give solutions for all "solve" variables.

Out[•]//Short=
$$\left\{\left\{c_1\to 0,\ \text{<<46>>},\ c_{60}\to -\frac{c_{51}}{T_1\,T_2}-\frac{c_{\text{<<2>>}}}{\text{<<1>>}\ \text{<<1>>}\ \text{<<1>>}}-\frac{\text{<<8>>}+\text{<<1>>}}{T_1^2\ \text{<<1>>}\ (-1+\text{<<1>>})}\right\}\right\}$$

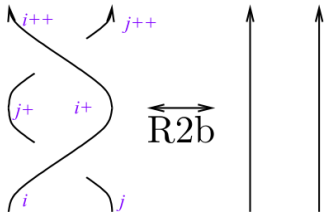
In[•]:=
```
sol /. (v_ → val_) :↦ (v = CF[val]);
```

```
In[•]:= Short[CF[r42[1, i, j]], 20]
```

Out[•]//Short=

$$c_{49}\,p_{1,i}\,x_{1,i}+c_{52}\,p_{1,j}\,x_{1,i}-\frac{(c_{49}+c_{52})\,p_{1,j}\,x_{1,j}}{T_1}+c_{50}\,p_{2,i}\,x_{2,i}+$$
$$c_{53}\,p_{2,j}\,x_{2,i}+c_{13}\,p_{1,j}\,p_{2,i}\,x_{1,i}\,x_{2,i}+c_4\,p_{1,i}\,p_{2,j}\,x_{1,i}\,x_{2,i}+\text{<<25>>}+$$
$$\frac{1}{(-1+T_1)\,(-1+T_2)}\left(-a_4\,b_5-c_{15}+c_{15}\,T_1+a_4\,b_3\,T_2+a_4\,b_5\,T_2-a_4\,b_3\,T_1\,T_2+c_{15}\,T_1\,T_2-c_{15}\,T_1^2\,T_2\right)$$
$$p_{2,j}\,p_{3,j}\,x_{2,j}\,x_{3,i}-\frac{1}{T_1^2\,T_2\,(-1+T_1\,T_2)}$$
$$\left(-a_2\,b_5-a_2\,b_3\,T_1+a_4\,b_3\,T_1-c_{51}\,T_1-c_{54}\,T_1-a_4\,b_5\,T_2-a_4\,b_3\,T_1\,T_2+a_4\,b_5\,T_1\,T_2+c_{51}\,T_1^2\,T_2+c_{54}\,T_1^2\,T_2\right)$$
$$p_{3,j}\,x_{3,j}-\frac{(a_2\,b_5-a_4\,b_5-c_5+c_{14}+c_5\,T_1+a_4\,b_5\,T_2-c_{14}\,T_1\,T_2)\,p_{1,i}\,p_{3,j}\,x_{1,i}\,x_{3,j}}{(-1+T_1)\,(-1+T_1\,T_2)}-$$
$$\left((-a_2\,b_3+c_5+a_2\,b_3\,T_1-a_4\,b_3\,T_1-c_5\,T_1-a_4\,b_3\,T_2-c_5\,T_2+2\,a_4\,b_3\,T_1\,T_2+c_5\,T_1\,T_2)\,p_{1,j}\,p_{3,j}\,x_{1,i}\,x_{3,j}\right)/$$
$$\left((-1+T_2)\,(-1+T_1\,T_2)\right)-$$
$$\left(\left(a_2\,b_3-c_6\,T_1+c_{15}\,T_1+a_4\,b_3\,T_2-a_4\,b_3\,T_1\,T_2+c_6\,T_1\,T_2-c_{15}\,T_1^2\,T_2\right)\,p_{2,i}\,p_{3,j}\,x_{2,i}\,x_{3,j}\right)/$$
$$\left(T_1\,(-1+T_2)\,(-1+T_1\,T_2)\right)-$$
$$\left(\left(c_6\,T_1-c_6\,T_1^2+a_2\,b_5\,T_2-a_2\,b_5\,T_1\,T_2+a_4\,b_5\,T_1\,T_2-c_6\,T_1\,T_2+c_6\,T_1^2\,T_2+a_4\,b_5\,T_2^2-2\,a_4\,b_5\,T_1\,T_2^2\right)\right.$$
$$\left.p_{2,j}\,p_{3,j}\,x_{2,i}\,x_{3,j}\right)/\left((-1+T_1)\,T_1\,(-1+T_1\,T_2)\right)$$

Reidemeister 2b



$\overset{\text{R2b}}{\longleftrightarrow}$

Territory: $(x \mid p)_{(i\,|\,j)^+}$, $x_{i\,|\,j}$.

```
In[•]:= Timing[ Short[LeftR2b = (∫ℱ[i, j] × ℒ /@ (Xi,j[1] Xi+,j+[-1]) dⅆ{vsi, vsj, vsi+, vsj+})[[1]]
        ]]
```

Out[•]=

```
{2.01563, ∈Series[p1,2+i π1,i + p1,2+j π1,j + <<12>> + p3,2+j π3,j, <<1>> ]}
```

```
In[•]:= RightR2b = ∈Series[p1,2+i π1,i + p1,2+j π1,j + p2,2+i π2,i + p2,2+j π2,j + p3,2+i π3,i + p3,2+j π3,j, 0]
```

Out[•]=

```
∈Series[p1,2+i π1,i + p1,2+j π1,j + p2,2+i π2,i + p2,2+j π2,j + p3,2+i π3,i + p3,2+j π3,j, 0]
```

In[•]:=
```
Short[eqn = CF[LeftR2b[[1]] - RightR2b[[1]]]]
cvs = Union@Cases[eqn, p__ | π__, ∞]
vars = Union@Cases[r0[-1, i, j], d_, ∞]
Short[eqns = CoefficientRules[eqn, cvs] /. (_ → c_) :> (c == 0), 3]
{sol} = Solve[eqns, vars]
```

Out[•]//Short=

$$\frac{(d_7 + \text{<<11>>} + d_7\,T_1\,T_2)\ \text{<<1>>}\ \text{<<1>>}\ \text{<<1>>}\ \pi_{\text{<<1>>}}}{B\,T_1\,T_2} + \text{<<7>>} + \frac{\text{<<1>>}}{B\ \text{<<1>>}\ T_2}$$

Out[•]=

$$\{p_{3,2+i},\ p_{3,2+j},\ \pi_{1,i},\ \pi_{1,j},\ \pi_{2,i},\ \pi_{2,j}\}$$

Out[•]=

$$\{d_1,\ d_2,\ d_3,\ d_4,\ d_5,\ d_6,\ d_7,\ d_8\}$$

Out[•]//Short=

$$\left\{\frac{d_1}{B} - \frac{d_3}{B} - \frac{d_5}{B} + \frac{d_7}{B} + \frac{d_5}{B\,T_1} - \frac{d_7}{B\,T_1} + \frac{d_3}{B\,T_2} - \frac{d_7}{B\,T_2} + \frac{d_7}{B\,T_1\,T_2} == 0,\right.$$
$$\left.\frac{d_3}{B\,T_2} - \frac{d_7}{B\,T_2} + \frac{d_7}{B\,T_1\,T_2} == 0,\ \text{<<5>>},\ \frac{d_7}{B} + \frac{d_8}{B} - \frac{d_7}{B\,T_1\,T_2} == 0\right\}$$

Out[•]=

$$\left\{\left\{d_1 \to 0,\ d_2 \to -\frac{a_2 - a_4\,T_1 + a_4\,T_2}{T_1^2\,T_2},\ d_3 \to 0,\ d_4 \to -\frac{a_4}{T_1},\ d_5 \to 0,\ d_6 \to -\frac{-a_2 - a_4\,T_2}{T_1\,T_2},\ d_7 \to 0,\ d_8 \to 0\right\}\right\}$$

In[•]:=
```
sol /. (v_ → val_) :> (v = CF[val]);
r0[-1, i, j]
```

Out[•]=

$$\frac{(-a_2 + a_4\,T_1 - a_4\,T_2)\ p_{3,j}\ x_{1,i}\ x_{2,i}}{T_1^2\,T_2} + \frac{(a_2 + a_4\,T_2)\ p_{3,j}\ x_{1,j}\ x_{2,i}}{T_1\,T_2} - \frac{a_4\ p_{3,j}\ x_{1,i}\ x_{2,j}}{T_1}$$

```
In[•]:= Short[eqn = CF[LeftR2b[[2]] - RightR2b[[2]]]]
        cvs = Union@Cases[eqn, p__ | π__, ∞]
        vars = Union@Cases[r1[-1, i, j] + r42[-1, i, j], e_ | f_, ∞]
        Short[eqns = CoefficientRules[eqn, cvs] /. (_ → c_) :> (c == 0), 3]
        Short[{sol} = Solve[eqns, vars]]
```

Out[•]//Short=

$$\text{<<85>>} + \frac{\text{<<1>>}}{\text{<<1>>}} + \frac{\text{<<1>>}}{T_1\ \text{<<1>>}}$$

Out[•]=

$$\{p_{1,2+i},\ p_{1,2+j},\ p_{2,2+i},\ p_{2,2+j},\ p_{3,2+i},\ p_{3,2+j},\ \pi_{1,i},\ \pi_{1,j},\ \pi_{2,i},\ \pi_{2,j},\ \pi_{3,i},\ \pi_{3,j}\}$$

Out[•]=

$$\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10},$$
$$f_{11}, f_{12}, f_{13}, f_{14}, f_{15}, f_{16}, f_{17}, f_{18}, f_{19}, f_{20}, f_{21}, f_{22}, f_{23}, f_{24}, f_{25}, f_{26},$$
$$f_{27}, f_{28}, f_{29}, f_{30}, f_{31}, f_{32}, f_{33}, f_{34}, f_{35}, f_{36}, f_{37}, f_{38}, f_{39}, f_{40}, f_{41}, f_{42}, f_{43},$$
$$f_{44}, f_{45}, f_{46}, f_{47}, f_{48}, f_{49}, f_{50}, f_{51}, f_{52}, f_{53}, f_{54}, f_{55}, f_{56}, f_{57}, f_{58}, f_{59}, f_{60}\}$$

Out[•]//Short=

$$\left\{f_1 - f_7 - f_{25} + f_{31} + \frac{f_{25}}{T_1} - \frac{f_{31}}{T_1} + \frac{f_7}{T_2} - \frac{f_{31}}{T_2} + \frac{f_{31}}{T_1\,T_2} == 0,\ \frac{f_7}{T_2} - \frac{\text{<<1>>}}{\text{<<1>>}} + \frac{f_{\text{<<2>>}}}{\text{<<1>>}\ \text{<<1>>}} == 0,\right.$$
$$\left.\text{<<66>>},\ \frac{2\,a_4\,b_3}{(1-T_1)\,(1-T_2)} + \frac{2\,c_{49}}{(1-T_1)\,(1-T_2)} + \text{<<204>>} + \frac{f_{60}\,T_1\,T_2}{(1-T_1)\,(1-T_2)} == 0\right\}$$

Out[•]//Short=

$$\left\{\left\{e_1 \to 0,\ e_2 \to 0,\ \text{<<64>>},\ f_{59} \to -\frac{\text{<<1>>}}{\text{<<1>>}},\ f_{60} \to -\frac{a_2\,b_5 + \text{<<13>>}}{T_1^2\,T_2\,(-1 + T_1\ \text{<<1>>})}\right\}\right\}$$

```
In[•]:= sol /. (v_ → val_) :> (v = CF[val]);
```

```
In[•]:= r1[-1, i, j]
        Short[CF[r42[-1, i, j]], 5]
```

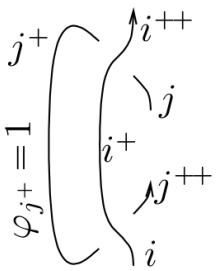
Out[•]=

$$-\frac{b_5\,p_{1,j}\,p_{2,i}\,x_{3,i}}{T_1} - \frac{b_3\,p_{1,i}\,p_{2,j}\,x_{3,i}}{T_2} + \frac{(b_3\,T_1 + b_5\,T_2)\,p_{1,j}\,p_{2,j}\,x_{3,i}}{T_1\,T_2}$$

Out[•]//Short=

$$-c_{49}\,p_{1,i}\,x_{1,i} + \frac{\left(-c_{49} - c_{52} + c_{49}\,T_1^2\right)\,p_{1,j}\,x_{1,i}}{T_1^2} + \text{<<42>>} + \frac{\text{<<1>>}}{\text{<<1>>}} + \frac{(\text{<<1>>})\,p_{2,i}\,p_{3\ \text{<<1>>}\ \text{<<1>>}}\,x_{2,i}\,x_{3,j}}{T_1\,(-1+T_2)\,(-1+T_1\,T_2)} +$$
$$\left(\left(a_2\,b_3 - a_2\,b_3\,T_1 + c_{15}\,T_1 - c_{15}\,T_1^2 + a_4\,b_3\,T_2 + \text{<<6>>} + a_4\,b_3\,T_1^2\,T_2 - c_{15}\,T_1^2\,T_2 + c_{15}\,T_1^3\,T_2 + a_4\,b_5\,T_2^2 - 2\,a_4\,b_5\,T_1\,T_2^2\right)\ \text{<<3>>}\ x_{\text{<<1>>}}\right) \Big/ \left((-1+T_1)\,T_1\,T_2\,(-1+T_1\,T_2)\right)$$

Reidemeister 2c



In[•]:=
```
Timing[ Short[{LeftR2c} = Cases[
    ∫ℱ[i, j] × ℒ /@ (X_{i+1,j}[1] X_{i,j+2}[-1] C_{j+1}[1]) 𝕕{vs_i, vs_j, vs_{i⁺}, vs_{j⁺}, vs_{j+2}}, 𝔼[ℰ_] :↦ ℰ]
 ]]
```

Out[•]=
```
{2.25, {∈Series[p_{1,2+i} π_{1,i} + p_{1,3+j} π_{1,j} + <<9>> + p_{3,3+j} π_{3,j}, g1 g2 + <<41>> + <<1>>]}}
```

In[•]:=
```
Timing[ Short[{RightR2c} =
   Cases[∫ℱ[i, j] × ℒ /@ (C_i[0] C_{i+1}[0] C_j[0] C_{j+1}[1] C_{j+2}[0]) 𝕕{vs_i, vs_j, vs_{i⁺}, vs_{j⁺}, vs_{j+2}},
    𝔼[ℰ_] :↦ ℰ]
 ]]
```

Out[•]=
```
{0., {∈Series[p_{1,2+i} π_{1,i} + p_{1,3+j} π_{1,j} + <<4>> + p_{3,3+j} π_{3,j}, g1 g2 + <<12>> + <<1>>]}}
```

In[•]:=
```
Short[eqn = CF[LeftR2c[[1]] - RightR2c[[1]]]]
cvs = Union@Cases[eqn, p__ | π__, ∞]
vars = Union@Cases[γ0[1, k], g_, ∞]
Short[eqns = CoefficientRules[eqn, cvs] /. (_ → c_) :↦ (c == 0), 3]
{sol} = Solve[eqns, vars]
```

Out[•]//Short=
$$\frac{g_1\ (-1+T_1)\ \ll 1\gg\ \ll 1\gg\ \ll 1\gg\ \pi_{1\ \ll 1\gg\ \ll 1\gg}\ \pi_{2,i}}{B\ T_1\ T_2} - \frac{\ll 1\gg}{B\ \ll 1\gg\ \ll 1\gg} - \frac{g_1\ \ll 3\gg\ \ll 1\gg}{B\ T_1}$$

Out[•]=
```
{p_{3,3+j}, π_{1,i}, π_{1,j}, π_{2,i}, π_{2,j}}
```

Out[•]=
```
{g1}
```

Out[•]//Short=
$$\left\{\frac{g_1}{B} - \frac{g_1}{B\ T_1} - \frac{g_1}{B\ T_2} + \frac{g_1}{B\ T_1\ T_2} == 0,\ -\frac{g_1}{B} + \frac{g_1}{B\ T_1} == 0,\ -\frac{g_1}{B} + \frac{g_1}{B\ T_2} == 0\right\}$$

Out[•]=
```
{{g1 → 0}}
```

```
In[•]:= sol /. (v_ → val_) :↦ (v = CF[val]);
        γ0[1, k]
Out[•]=
        0

In[•]:= Short[eqn = CF[LeftR2c[[2]] - RightR2c[[2]]]]
        cvs = Union@Cases[eqn, p__ | π__, ∞]
        vars = Union@Cases[γ1[1, k] + γ42[1, k], g_, ∞]
        Short[eqns = CoefficientRules[eqn, cvs] /. (_ → c_) :↦ (c == 0), 3]
        Short[{sol} = Solve[eqns, vars]]
Out[•]//Short=
        -((g3 + g6 + g7) (-1 + <<1>>) <<1>> π1,i)/T1 - <<1>>/<<1>> + <<18>>
Out[•]=
        {p1,3+j, p2,3+j, p3,3+j, π1,i, π1,j, π2,i, π2,j, π3,i, π3,j}
Out[•]=
        {g2, g3, g4, g5, g6, g7, g8}
Out[•]//Short=
        {g6 - g6/T1 - g6/T2 + (a2 g2)/(T1 T2) + g6/(T1 T2) == 0, -g6 + (a4 g2)/T1 + g6/T1 == 0, <<1>> == 0, <<7>>, -g8 + <<1>> == 0,
         -g4 - g6 - g8 + g4/T2 + g6/T2 + g8/T2 == 0, (2 a4 b3)/((1 - T1) (1 - T2)) + (a4 b3)/((1 - T1) T1^2 (1 - T2)) + <<50>> == 0}
Out[•]//Short=
        {{g2 → 0, g3 → 0, g4 → 0, g5 → -<<1>>/<<1>>, g6 → 0, g7 → 0, g8 → 0}}

In[•]:= sol /. (v_ → val_) :↦ (v = CF[val]);

In[•]:= γ1[1, k]
        Short[CF[γ42[1, k]], 5]
Out[•]=
        0
Out[•]//Short=
        ((-b3 + b5 + b3 T1 - b5 T2) (-a2 + a2 T1 - a4 T1 - a4 T2 + 2 a4 T1 T2) p3,k x3,k)/((-1 + T1) T1 (-1 + T2) (-1 + T1 T2))
```

$C_k[1]$ and $C_k[-1]$ are inverses

```
In[•]:= Timing[ Short[{LeftCC} = Cases[{∫ℱ[k] × ℒ /@ (Ck[1] Ck+1[-1]) 𝕕{vsk, vsk+}}, 𝔼[ℰ_] :↦ ℰ]
        ]]
Out[•]=
        {0.015625, {∈Series[p1,2+k π1,k + p2,2+k π2,k + <<1>>/B + p3,2+k π3,k, <<1>>/<<1>> + <<6>> + <<1>>]}}
```

```
In[•]:= Timing[ Short[ {RightCC} = Cases[ {∫ℱ[k] × ℒ /@ (C_k[0] C_{k+1}[0]) 𝕕{vs_k, vs_{k⁺}}}, 𝔼[ℰ_] :> ℰ ] ]]
```

```
Out[•]=
{0., {∈Series[p_{1,2+k} π_{1,k} + p_{2,2+k} π_{2,k} + p_{3,2+k} π_{3,k}, 0]}}
```

```
In[•]:= Short[eqn = CF[LeftCC[[1]] - RightCC[[1]]]]
cvs = Union@Cases[eqn, p__ | π__, ∞]
vars = Union@Cases[γ_0[-1, k], h_, ∞]
Short[eqns = CoefficientRules[eqn, cvs] /. (_ → c_) :> (c == 0), 3]
{sol} = Solve[eqns, vars]
```

```
Out[•]//Short=
(h_1 p_3 <<1>> 2 <<1>> <<1>> π_<<1>> π_{2,k}) / B
```

```
Out[•]=
{p_{3,2+k}, π_{1,k}, π_{2,k}}
```

```
Out[•]=
{h_1}
```

```
Out[•]//Short=
{h_1/B == 0}
```

```
Out[•]=
{{h_1 → 0}}
```

```
In[•]:= sol /. (v_ → val_) :> (v = CF[val]);
γ_0[-1, k]
```

```
Out[•]=
0
```

```
In[•]:= Short[eqn = CF[LeftCC⟦2⟧ - RightCC⟦2⟧]]
        cvs = Union@Cases[eqn, p__ | π__, ∞]
        vars = Union@Cases[γ1[-1, k] + γ42[-1, k], h_, ∞]
        Short[eqns = CoefficientRules[eqn, cvs] /. (_ → c_) :↦ (c == 0), 3]
        Short[{sol} = Solve[eqns, vars]]
```

Out[•]//Short=

<<1>>

Out[•]=

$\{p_{1,2+k},\ p_{2,2+k},\ p_{3,2+k},\ \pi_{1,k},\ \pi_{2,k},\ \pi_{3,k}\}$

Out[•]=

$\{h_2,\ h_3,\ h_4,\ h_5,\ h_6,\ h_7,\ h_8\}$

Out[•]//Short=

$$\left\{h_6 == 0,\ B\,h_2 == 0,\ h_7 == 0,\ h_3 + h_6 + h_7 == 0,\ h_8 == 0,\ h_4 + h_6 + h_8 == 0,\ \frac{2\,a_2\,b_3}{(1-T_1)\,(1-T_2)\,(1-T_1\,T_2)} - \frac{a_4\,b_3}{(1-T_1)\,(1-\text{<<1>>})\,(1-T_1\,T_2)} - \frac{\text{<<1>>}\ \text{<<1>>}\ \text{<<1>>}}{\text{<<1>>}} + \text{<<48>>} == 0,\ \text{<<1>>} == 0\right\}$$

Out[•]//Short=

$$\left\{\left\{h_2 \to 0,\ h_3 \to 0,\ h_4 \to 0,\ h_5 \to -\frac{\text{<<1>>}}{\text{<<1>>}},\ h_6 \to 0,\ h_7 \to 0,\ h_8 \to 0\right\}\right\}$$

```
In[•]:= sol /. (v_ → val_) :↦ (v = CF[val]);
```

```
In[•]:= γ1[-1, k]
        Short[CF[γ42[-1, k]], 5]
```

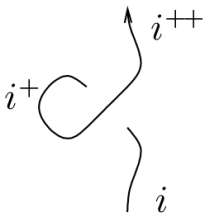
Out[•]=

0

Out[•]//Short=

$$-\frac{(-b_3+b_5+b_3\,T_1-b_5\,T_2)\ (-a_2+a_2\,T_1-a_4\,T_1-a_4\,T_2+2\,a_4\,T_1\,T_2)\ p_{3,k}\,x_{3,k}}{(-1+T_1)\,T_1\,(-1+T_2)\,(-1+T_1\,T_2)}$$

Invariance Under R1l

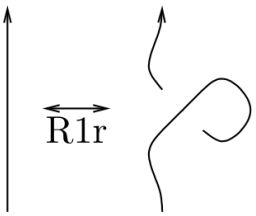



```
In[•]:= {LeftR1l} = Cases[{∫ℱ[i] × ℒ /@ (X_{i+2,i}[1] C_{i+1}[1]) 𝕕{vs_i, vs_{i⁺}, vs_{i+2}}}, 𝔼[ℰ_] :↦ ℰ, ∞]
```

Out[•]=

$\{\in \text{Series}[p_{1,3+i}\,\pi_{1,i} + p_{2,3+i}\,\pi_{2,i} + p_{3,3+i}\,\pi_{3,i},\ 0]\}$

Invariance Under R1r




```
In[•]:= {LeftR1r} = Cases[{∫ℱ[i] × ℒ /@ (X_{i,i+2}[1] C_{i+1}[-1]) 𝕕{vs_i, vs_{i⁺}, vs_{i+2}}}, 𝔼[ℰ_] :↦ ℰ, ∞]
```

```
Out[•]=
{∈Series[p_{1,3+i} π_{1,i} + p_{2,3+i} π_{2,i} + p_{3,3+i} π_{3,i},
  1/(T1^2 T2 (-1 + T1 T2)) (a2 b5 + a2 b3 T1 - a4 b3 T1 + c51 T1 + c54 T1 + c50 T1^2 + c53 T1^2 + a4 b5 T2 +
    a4 b3 T1 T2 - a4 b5 T1 T2 + c49 T1 T2 + c52 T1 T2 - c49 T1^2 T2 - c50 T1^2 T2 - 2 c51 T1^2 T2 -
    c54 T1^2 T2 - c50 T1^3 T2 - c53 T1^3 T2 - c49 T1^2 T2^2 - c52 T1^2 T2^2 + c49 T1^3 T2^2 + c50 T1^3 T2^2 + c51 T1^3 T2^2))]}
```

```
In[•]:= {RightR1r} = Cases[{∫ℱ[i] × ℒ /@ (C_i[0] C_{i+1}[0] C_{i+2}[0]) 𝕕{vs_i, vs_{i⁺}, vs_{i+2}}}, 𝔼[ℰ_] :↦ ℰ, ∞]
```

```
Out[•]=
{∈Series[p_{1,3+i} π_{1,i} + p_{2,3+i} π_{2,i} + p_{3,3+i} π_{3,i}, 0]}
```

```
In[•]:= LeftR1r[[1]] == RightR1r[[1]]
```

```
Out[•]=
True
```

```
In[•]:= Short[eqn = CF[LeftR1r[[2]] - RightR1r[[2]]]]
        cvs = Union@Cases[eqn, p__ | π__, ∞]
        vars = Union@Cases[eqn, (c | d | e | f | g | h)_, ∞]
        Short[eqns = CoefficientRules[eqn, cvs] /. (_ → c_) :↦ (c == 0), 3]
        Short[{sol} = Solve[eqns, vars]]
```

```
Out[•]//Short=
(a2 b5 + <<31>> + c51 T1^3 T2^2) / (T1^2 T2 (-1 + T1 T2))
```

```
Out[•]=
{}
```

```
Out[•]=
{c49, c50, c51, c52, c53, c54}
```

```
Out[•]//Short=
{(a2 b5 + a2 b3 T1 - a4 b3 T1 + <<26>> + c49 T1^3 T2^2 + c50 T1^3 T2^2 + c51 T1^3 T2^2) / (T1^2 T2 (-1 + T1 T2)) == 0}
```

Solve: Equations may not give solutions for all "solve" variables.

```
Out[•]//Short=
{{c54 → <<1>>}}
```

```
In[•]:= sol /. (v_ → val_) :↦ (v = CF[val]);
```