

Pensieve header: Finding the A_2 $\mathcal{S}d=1$ invariant using undetermined coefficients.

Searching for $Q + p_{xx} + \epsilon(ppx + 1 + px + ppx)$ solutions.

Initialization

```
In[*]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\HigherRank"];
Once[<< KnotTheory` ; << Rot.m];
<< FormalGaussianIntegration.m;
i_+ := i + 1;
```

Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.

Read more at <http://katlas.org/wiki/KnotTheory>.

Loading Rot.m from <http://drorbn.net/AP/Projects/HigherRank> to compute rotation numbers.

```
In[*]:= Features[Knot[8, 17]]
```

 KnotTheory: Loading precomputed data in PD4Knots`.

Out[*]=

```
Features[18,
C6[-1] C14[-1] X1,7[1] X3,9[-1] X5,13[-1] X8,16[1] X10,4[-1] X12,18[1] X15,2[-1] X17,11[1]]
```

```
In[*]:= T3 = T1 T2;
S = {x_, p_};
q[s_, i_, j_] := Sum[
  xv,i (pv,i+ - pv,i) + xv,j (pv,j+ - pv,j) + (Tv^S - 1) xv,i (pv,i+ - pv,j+),
  {v, 3}];
L[Xi_,j_[s_]] := T3^S E[q[s, i, j] + r0[s, i, j] + e r1[s, i, j] + O[e]^1];
Y1[phi_, k_] := phi (3/2 - x1,k p1,k - x2,k p2,k - x3,k p3,k);
L[Cr_[phi_]] := T3^phi E[Sum[xv,k (pv,k+ - pv,k), {v, 3}] + e Y1[phi, k] + O[e]^1];
ps_i := Sequence[p1,i, p2,i, p3,i];
xs_i := Sequence[x1,i, x2,i, x3,i];
vs_i := Sequence[ps_i, xs_i];
F[is___] := E[Sum[pi,v,i pv,i, {i, {is}}, {v, 3}]];
L[K_] := (2 pi)^{-Features[K][[1]]} CF[L/@Features[K][[2]]];
vs[K_] := Union@@Table[{vs_i}, {i, Features[K][[1]]}]
```

```
In[*]:= vs_i
```

Out[*]=

```
Sequence[p1,i, p2,i, p3,i, x1,i, x2,i, x3,i]
```

The pxx Terms (r_0)

```

In[*]:=  $x = \mathbf{0}$ ;
r0[1, i_, j_] := Evaluate[Sum[
  a_{++x} p_{3,k3} x_{1,k1} x_{2,k2},
  {k1, {i, j}}, {k2, {i, j}}, {k3, {i, j}}
]];
r0[1, i, j]

Out[*]=
a1 p_{3,i} x_{1,i} x_{2,i} + a2 p_{3,j} x_{1,i} x_{2,i} + a5 p_{3,i} x_{1,j} x_{2,i} + a6 p_{3,j} x_{1,j} x_{2,i} +
a3 p_{3,i} x_{1,i} x_{2,j} + a4 p_{3,j} x_{1,i} x_{2,j} + a7 p_{3,i} x_{1,j} x_{2,j} + a8 p_{3,j} x_{1,j} x_{2,j}

In[*]:=  $\mathcal{L}[X_{i,j}[s]] := T_3^s \mathbb{E}[q[s, i, j] + r_0[s, i, j]]$ ;
 $\mathcal{L}[X_{i,j}[1]]$ 

Out[*]=
T1 T2
 $\mathbb{E}[(-p_{1,i} + p_{1,1+i}) x_{1,i} + (-1 + T_1)(p_{1,1+i} - p_{1,1+j}) x_{1,i} + (-p_{1,j} + p_{1,1+j}) x_{1,j} + (-p_{2,i} + p_{2,1+i}) x_{2,i} +$ 
 $(-1 + T_2)(p_{2,1+i} - p_{2,1+j}) x_{2,i} + a_1 p_{3,i} x_{1,i} x_{2,i} + a_2 p_{3,j} x_{1,i} x_{2,i} + a_5 p_{3,i} x_{1,j} x_{2,i} +$ 
 $a_6 p_{3,j} x_{1,j} x_{2,i} + (-p_{2,j} + p_{2,1+j}) x_{2,j} + a_3 p_{3,i} x_{1,i} x_{2,j} + a_4 p_{3,j} x_{1,i} x_{2,j} + a_7 p_{3,i} x_{1,j} x_{2,j} +$ 
 $a_8 p_{3,j} x_{1,j} x_{2,j} + (-p_{3,i} + p_{3,1+i}) x_{3,i} + (-1 + T_1 T_2)(p_{3,1+i} - p_{3,1+j}) x_{3,i} + (-p_{3,j} + p_{3,1+j}) x_{3,j}]$ 

```

Reidemeister 3 for pxx (r_0)

```

In[*]:= {lhs} = Cases[ $\int \mathcal{F}[i, j, k] \times \mathcal{L} / @ (X_{i,j}[1] X_{i^*,k}[1] X_{j^*,k^*}[1]) d\{vs_i, vs_j, vs_k, vs_{i^*}, vs_{j^*}, vs_{k^*}\}$ ,
   $\mathbb{E}[\mathcal{E}_-] \Rightarrow \mathcal{E}, \infty$ ];

In[*]:= {rhs} = Cases[ $\int \mathcal{F}[i, j, k] \times \mathcal{L} / @ (X_{j,k}[1] X_{i,k^*}[1] X_{i^*,j^*}[1]) d\{vs_i, vs_j, vs_k, vs_{i^*}, vs_{j^*}, vs_{k^*}\}$ ,
   $\mathbb{E}[\mathcal{E}_-] \Rightarrow \mathcal{E}, \infty$ ];

In[*]:= eqn = CF[lhs - rhs];

In[*]:= cvs = Union@Cases[eqn, p__ |  $\pi_{__}, \infty$ ]

Out[*]=
{p_{3,2+i}, p_{3,2+j}, p_{3,2+k},  $\pi_{1,i}, \pi_{1,j}, \pi_{1,k}, \pi_{2,i}, \pi_{2,j}, \pi_{2,k}$ }

In[*]:= eqns = CoefficientRules[eqn, cvs] /. ( $\_ \rightarrow c\_$ ) => (c == 0);

In[*]:= vars = Union@Cases[r0[1, i, j], a_,  $\infty$ ]

Out[*]=
{a1, a2, a3, a4, a5, a6, a7, a8}

```

In[*]:= {sol} = Solve[eqns, vars]

 Solve: Equations may not give solutions for all "solve" variables. 

Out[*]=

$$\left\{ \left\{ a_1 \rightarrow 0, a_3 \rightarrow 0, a_5 \rightarrow 0, a_6 \rightarrow -\frac{a_2}{T_1} - \frac{a_4 T_2}{T_1}, a_7 \rightarrow 0, a_8 \rightarrow 0 \right\} \right\}$$

In[*]:= sol /. (v_ -> val_) -> (v = CF[val]);

In[*]:= r0[1, i, j]

Out[*]=

$$a_2 p_{3,j} x_{1,i} x_{2,i} - \frac{(a_2 + a_4 T_2) p_{3,j} x_{1,j} x_{2,i}}{T_1} + a_4 p_{3,j} x_{1,i} x_{2,j}$$

The ppx Terms (r_1)

In[*]:= x = 0;

r1[1, i_, j_] := Evaluate[Sum[
 b_{++x} x_{3,k3} p_{1,k1} p_{2,k2},
 {k1, {i, j}}, {k2, {i, j}}, {k3, {i, j}}
]];
 r1[1, i, j]

Out[*]=

$$b_1 p_{1,i} p_{2,i} x_{3,i} + b_5 p_{1,j} p_{2,i} x_{3,i} + b_3 p_{1,i} p_{2,j} x_{3,i} + b_7 p_{1,j} p_{2,j} x_{3,i} +$$

$$b_2 p_{1,i} p_{2,i} x_{3,j} + b_6 p_{1,j} p_{2,i} x_{3,j} + b_4 p_{1,i} p_{2,j} x_{3,j} + b_8 p_{1,j} p_{2,j} x_{3,j}$$

In[*]:= L[X_{i_,j}[s_]] := T₃^s E[q[s, i, j] + e r1[s, i, j] + O[e]²];
 L[X_{i,j}[1]]

Out[*]=

$$T_1 T_2 E[\epsilon \text{Series}[(-p_{1,i} + p_{1,1+i}) x_{1,i} + (-1 + T_1) (p_{1,1+i} - p_{1,1+j}) x_{1,i} + (-p_{1,j} + p_{1,1+j}) x_{1,j} +$$

$$(-p_{2,i} + p_{2,1+i}) x_{2,i} + (-1 + T_2) (p_{2,1+i} - p_{2,1+j}) x_{2,i} + (-p_{2,j} + p_{2,1+j}) x_{2,j} +$$

$$(-p_{3,i} + p_{3,1+i}) x_{3,i} + (-1 + T_1 T_2) (p_{3,1+i} - p_{3,1+j}) x_{3,i} + (-p_{3,j} + p_{3,1+j}) x_{3,j},$$

$$b_1 p_{1,i} p_{2,i} x_{3,i} + b_5 p_{1,j} p_{2,i} x_{3,i} + b_3 p_{1,i} p_{2,j} x_{3,i} + b_7 p_{1,j} p_{2,j} x_{3,i} +$$

$$b_2 p_{1,i} p_{2,i} x_{3,j} + b_6 p_{1,j} p_{2,i} x_{3,j} + b_4 p_{1,i} p_{2,j} x_{3,j} + b_8 p_{1,j} p_{2,j} x_{3,j}]]$$

Reidemeister 3 for ppx (r_1)

In[*]:= {lhs} = Cases[
 ∫ F[i, j, k] × L /@ (X_{i,j}[1] X_{i⁺,k}[1] X_{j⁺,k⁺}[1]) d{vs_i, vs_j, vs_k, vs_{i⁺}, vs_{j⁺}, vs_{k⁺}},
 eSeries[_ , ε_] -> ε, ∞];

In[*]:= {rhs} = Cases[
 ∫ F[i, j, k] × L /@ (X_{j,k}[1] X_{i,k⁺}[1] X_{i⁺,j⁺}[1]) d{vs_i, vs_j, vs_k, vs_{i⁺}, vs_{j⁺}, vs_{k⁺}},
 eSeries[_ , ε_] -> ε, ∞];

In[*]:= eqn = CF[lhs - rhs];

```

In[*]:= cvs = Union@Cases[eqn, p__ |  $\pi$ __,  $\infty$ ]
Out[*]=
{p1,2+i, p1,2+j, p1,2+k, p2,2+i, p2,2+j, p2,2+k,  $\pi$ 3,i,  $\pi$ 3,j,  $\pi$ 3,k}

In[*]:= eqns = CoefficientRules[eqn, cvs] /. (_ -> c_) :-> (c == 0);
In[*]:= vars = Union@Cases[r1[1, i, j], b_,  $\infty$ ]
Out[*]=
{b1, b2, b3, b4, b5, b6, b7, b8}

In[*]:= {sol} = Solve[eqns, vars]

(* Solve: Equations may not give solutions for all "solve" variables. ⓘ)

Out[*]=
{{b1 -> 0, b2 -> 0, b4 -> 0, b6 -> 0, b7 -> -b3 - b5, b8 -> 0}}

In[*]:= sol /. (v_ -> val_) :-> (v = CF[val]);
In[*]:= r1[1, i, j]
Out[*]=
b5 p1,j p2,i x3,i + b3 p1,i p2,j x3,i + (-b3 - b5) p1,j p2,j x3,i

```

Reidemeister 3 with pxx and ppx

```

In[*]:=  $\mathcal{L}[X_{i,j}[s_]] := T_3^s \mathbb{E}[q[s, i, j] + r_0[s, i, j] + \epsilon r_1[s, i, j] + 0[\epsilon]^2];$ 
 $\mathcal{L}[X_{i,j}[1]]$ 
Out[*]=
T1 T2  $\mathbb{E}\left[ eSeries\left[ \begin{aligned} &(-p_{1,i} + p_{1,1+i}) x_{1,i} + (-1 + T_1) (p_{1,1+i} - p_{1,1+j}) x_{1,i} + (-p_{1,j} + p_{1,1+j}) x_{1,j} + (-p_{2,i} + p_{2,1+i}) x_{2,i} + \\ &(-1 + T_2) (p_{2,1+i} - p_{2,1+j}) x_{2,i} + a_2 p_{3,j} x_{1,i} x_{2,i} - \frac{(a_2 + a_4 T_2) p_{3,j} x_{1,j} x_{2,i}}{T_1} + (-p_{2,j} + p_{2,1+j}) x_{2,j} + \\ &a_4 p_{3,j} x_{1,i} x_{2,j} + (-p_{3,i} + p_{3,1+i}) x_{3,i} + (-1 + T_1 T_2) (p_{3,1+i} - p_{3,1+j}) x_{3,i} + (-p_{3,j} + p_{3,1+j}) x_{3,j}, \\ &b_5 p_{1,j} p_{2,i} x_{3,i} + b_3 p_{1,i} p_{2,j} x_{3,i} + (-b_3 - b_5) p_{1,j} p_{2,j} x_{3,i} \end{aligned} \right] \right]$ 

In[*]:= {lhs} = Cases[ $\int \mathcal{F}[i, j, k] \times \mathcal{L} / @ (X_{i,j}[1] X_{i^+,k}[1] X_{j^+,k^+}[1]) \, d\{vs_i, vs_j, vs_k, vs_{i^+}, vs_{j^+}, vs_{k^+}\},$ 
 $\mathbb{E}[\mathcal{E}_-] :-> \mathcal{E}, \infty$ ];
In[*]:= {rhs} = Cases[ $\int \mathcal{F}[i, j, k] \times \mathcal{L} / @ (X_{j,k}[1] X_{i,k^+}[1] X_{i^+,j^+}[1]) \, d\{vs_i, vs_j, vs_k, vs_{i^+}, vs_{j^+}, vs_{k^+}\},$ 
 $\mathbb{E}[\mathcal{E}_-] :-> \mathcal{E}, \infty$ ];
In[*]:= Echo /@ Short /@ (CF@CoefficientList[CF[lhs - rhs][[2]] /. {ai ->  $\lambda$  ai, bi ->  $\lambda$  bi},  $\lambda$ );

```

» 0

» 0

$$\begin{aligned}
 & \gg a_2 b_5 T_2 p_{1,2+k} p_{2,2+j} \pi_{1,i} \pi_{2,i} + \ll 45 \gg + \frac{b_5 \ll 5 \gg \pi_{\ll 1 \gg}}{T_1} - \\
 & \frac{(a_2 + a_4 T_2) (-b_3 + b_3 T_2 + b_3 T_1 T_2 + b_5 T_2^2) \ll 1 \gg \ll 1 \gg \pi_{\ll 1 \gg} \pi_{3,j}}{T_1} \\
 & \gg - \frac{a_2 (a_2 b_5 - a_4 b_3 T_1 + a_4 b_5 T_2) p_{3,2+k} \pi_{1,i} \pi_{2,i}}{T_1} - \\
 & a_2 a_4 b_3 (-1 + T_1) T_1 p_{1,2+j} p_{3,2+k} \pi_{1,i}^2 \pi_{2,i} + a_2 \ll 6 \gg \pi_{\ll 1 \gg} + \ll 72 \gg \\
 & \gg \frac{a_2 a_4 (-1 + T_1) (-1 + T_2) (a_2 + \ll 1 \gg) (b_3 + b_5 + b_3 T_1 + b_5 T_2) p_{3,2+k}^2 \pi_{1,i}^2 \pi_{2,i}^2}{T_1} + \\
 & \ll 26 \gg + a_4^3 b_3 p_{3,2+k}^2 \pi_{1,i} \pi_{1,j} \pi_{2,j} \pi_{2,k}
 \end{aligned}$$

In[*]:= **err** = **CF@Coefficient**[**CF**[**lhs** - **rhs**][[2]] /. {**a_i** -> **λ a_i**, **b_i** -> **λ b_i**}, **λ²**]

Out[*]=

$$\begin{aligned}
 & a_2 b_5 T_2 p_{1,2+k} p_{2,2+j} \pi_{1,i} \pi_{2,i} + a_2 b_3 T_1 p_{1,2+j} p_{2,2+k} \pi_{1,i} \pi_{2,i} - a_2 (b_3 T_1 + b_5 T_2) p_{1,2+k} p_{2,2+k} \pi_{1,i} \pi_{2,i} - \\
 & \frac{b_5 T_2 (a_2 + a_4 T_2) p_{1,2+k} p_{2,2+j} \pi_{1,j} \pi_{2,i}}{T_1} - b_3 (a_2 + a_4 T_2) p_{1,2+j} p_{2,2+k} \pi_{1,j} \pi_{2,i} + \\
 & \frac{(a_2 + a_4 T_2) (b_3 T_1 + b_5 T_2) p_{1,2+k} p_{2,2+k} \pi_{1,j} \pi_{2,i}}{T_1} + a_4 b_5 T_2 p_{1,2+k} p_{2,2+j} \pi_{1,i} \pi_{2,j} + \\
 & \frac{a_4 b_3 T_1 p_{1,2+j} p_{2,2+k} \pi_{1,i} \pi_{2,j} - a_4 (b_3 T_1 + b_5 T_2) p_{1,2+k} p_{2,2+k} \pi_{1,i} \pi_{2,j} -}{(a_2 b_5 + a_2 b_3 T_1 - a_4 b_3 T_1 + a_4 b_5 T_2 + a_4 b_3 T_1 T_2 - a_4 b_5 T_1 T_2) p_{3,2+k} \pi_{3,i}} + \\
 & \frac{b_5 T_1 (-a_4 - a_2 T_1 + a_4 T_1) T_2 p_{1,2+k} p_{3,2+j} \pi_{1,i} \pi_{3,i} +}{T_1 (a_2 b_5 - a_4 b_5 - a_4 b_3 T_1 + a_4 b_5 T_2 + a_4 b_3 T_1^2 T_2) p_{1,2+j} p_{3,2+k} \pi_{1,i} \pi_{3,i} -} \\
 & \frac{(-a_4 b_5 - a_2 b_5 T_1 + 2 a_4 b_5 T_1 + a_4 b_3 T_1^2) (-1 + T_1 T_2) p_{1,2+k} p_{3,2+k} \pi_{1,i} \pi_{3,i} +}{b_5 T_1 T_2 (a_2 + a_4 T_2) p_{1,2+k} p_{3,2+j} \pi_{1,j} \pi_{3,i} + b_3 T_1^2 (a_2 + a_4 T_2) p_{1,2+i} p_{3,2+k} \pi_{1,j} \pi_{3,i} +} \\
 & \frac{(-a_2 b_5 + a_4 b_3 T_1 - a_2 b_3 T_1^2 - a_4 b_5 T_2 + a_4 b_5 T_1 T_2 - a_4 b_3 T_1^2 T_2) p_{1,2+j} p_{3,2+k} \pi_{1,j} \pi_{3,i} +}{(a_2 b_5 - a_4 b_5 - a_4 b_3 T_1 + a_4 b_5 T_2 - a_2 b_5 T_1 T_2 - a_4 b_5 T_1 T_2^2) p_{1,2+k} p_{3,2+k} \pi_{1,j} \pi_{3,i} -} \\
 & \frac{b_3 T_1 (a_2 + a_4 T_2) p_{1,2+i} p_{3,2+k} \pi_{1,k} \pi_{3,i} + b_3 T_1 (a_2 + a_4 T_2) p_{1,2+j} p_{3,2+k} \pi_{1,k} \pi_{3,i} -}{b_3 T_2 (-a_2 + a_2 T_2 - a_4 T_2 + a_2 T_1 T_2 + a_4 T_2^2) p_{2,2+k} p_{3,2+j} \pi_{2,i} \pi_{3,i} +} \\
 & \frac{T_2 (a_2 b_3 + a_4 b_3 T_2 + a_2 b_5 T_2 - a_4 b_3 T_1 T_2 + a_4 b_5 T_2^2 - a_2 b_5 T_1 T_2^2 - a_4 b_5 T_1 T_2^3) p_{2,2+j} p_{3,2+k} \pi_{2,i} \pi_{3,i}}{T_1} + \\
 & \frac{1}{T_1} (-1 + T_1 T_2) (-a_2 b_3 + 2 a_2 b_3 T_2 - a_4 b_3 T_2 + a_2 b_3 T_1 T_2 + 2 a_4 b_3 T_2^2 + a_2 b_5 T_2^2 + a_4 b_5 T_2^3) \\
 & \frac{p_{2,2+k} p_{3,2+k} \pi_{2,i} \pi_{3,i} - a_4 b_3 T_1 T_2^2 p_{2,2+k} p_{3,2+j} \pi_{2,j} \pi_{3,i} - a_4 b_5 T_2^3 p_{2,2+i} p_{3,2+k} \pi_{2,j} \pi_{3,i} -}{T_2 (a_2 b_5 + a_2 b_3 T_1 - a_4 b_3 T_1 + a_4 b_5 T_2 + a_4 b_3 T_1 T_2 - a_4 b_5 T_1 T_2^2) p_{2,2+j} p_{3,2+k} \pi_{2,j} \pi_{3,i}} + \\
 & \frac{(a_2 b_3 + a_4 b_3 T_2 + a_2 b_5 T_2 - a_4 b_3 T_1 T_2 + a_4 b_5 T_2^2 + a_4 b_3 T_1^2 T_2^2) p_{2,2+k} p_{3,2+k} \pi_{2,j} \pi_{3,i}}{T_1} + \\
 & \frac{a_4 b_5 T_2^2 p_{2,2+i} p_{3,2+k} \pi_{2,k} \pi_{3,i} - a_4 b_5 T_2^2 p_{2,2+j} p_{3,2+k} \pi_{2,k} \pi_{3,i} - a_4 b_5 T_1 T_2 p_{1,2+k} p_{3,2+j} \pi_{1,i} \pi_{3,j} -}{a_4 b_3 T_1^2 p_{1,2+j} p_{3,2+k} \pi_{1,i} \pi_{3,j} + a_4 (-b_5 + b_5 T_1 + b_3 T_1^2 + b_5 T_1 T_2) p_{1,2+k} p_{3,2+k} \pi_{1,i} \pi_{3,j} +} \\
 & \frac{b_3 T_2 (a_2 + a_4 T_2) p_{2,2+k} p_{3,2+j} \pi_{2,i} \pi_{3,j} + \frac{b_5 T_2^2 (a_2 + a_4 T_2) p_{2,2+j} p_{3,2+k} \pi_{2,i} \pi_{3,j}}{T_1}}{T_1} - \\
 & \frac{(a_2 + a_4 T_2) (-b_3 + b_3 T_2 + b_3 T_1 T_2 + b_5 T_2^2) p_{2,2+k} p_{3,2+k} \pi_{2,i} \pi_{3,j}}{T_1}
 \end{aligned}$$

Reidemeister 3 with ppx and px added

```

In[*]:= x = 0;
r42[1, i_, j_] = Evaluate[Plus[
  Sum[
    C++x Xv1,k1 Pv1,k2 Xv2,k3 Pv2,k4,
    {k1, {i, j}}, {k2, {i, j}}, {k3, {i, j}}, {k4, {i, j}}, {v1, 2}, {v2, v1 + 1, 3}
  ],
  Sum[
    C++x Xv,k1 Pv,k2,
    {k1, {i, j}}, {k2, {i, j}}, {v, 3}
  ]
]]

```

```

Out[*]=
C49 p1,i X1,i + C52 p1,j X1,i + C55 p1,i X1,j + C58 p1,j X1,j + C50 p2,i X2,i + C53 p2,j X2,i +
C1 p1,i p2,i X1,i X2,i + C13 p1,j p2,i X1,i X2,i + C4 p1,i p2,j X1,i X2,i + C16 p1,j p2,j X1,i X2,i +
C25 p1,i p2,i X1,j X2,i + C37 p1,j p2,i X1,j X2,i + C28 p1,i p2,j X1,j X2,i + C40 p1,j p2,j X1,j X2,i +
C56 p2,i X2,j + C59 p2,j X2,j + C7 p1,i p2,i X1,i X2,j + C19 p1,j p2,i X1,i X2,j + C10 p1,i p2,j X1,i X2,j +
C22 p1,j p2,j X1,i X2,j + C31 p1,i p2,i X1,j X2,j + C43 p1,j p2,i X1,j X2,j + C34 p1,i p2,j X1,j X2,j +
C46 p1,j p2,j X1,j X2,j + C51 p3,i X3,i + C54 p3,j X3,i + C2 p1,i p3,i X1,i X3,i + C14 p1,j p3,i X1,i X3,i +
C5 p1,i p3,j X1,i X3,i + C17 p1,j p3,j X1,i X3,i + C26 p1,i p3,i X1,j X3,i + C38 p1,j p3,i X1,j X3,i +
C29 p1,i p3,j X1,j X3,i + C41 p1,j p3,j X1,j X3,i + C3 p2,i p3,i X2,i X3,i + C15 p2,j p3,i X2,i X3,i +
C6 p2,i p3,j X2,i X3,i + C18 p2,j p3,j X2,i X3,i + C27 p2,i p3,i X2,j X3,i + C39 p2,j p3,i X2,j X3,i +
C30 p2,i p3,j X2,j X3,i + C42 p2,j p3,j X2,j X3,i + C57 p3,i X3,j + C60 p3,j X3,j +
C8 p1,i p3,i X1,i X3,j + C20 p1,j p3,i X1,i X3,j + C11 p1,i p3,j X1,i X3,j + C23 p1,j p3,j X1,i X3,j +
C32 p1,i p3,i X1,j X3,j + C44 p1,j p3,i X1,j X3,j + C35 p1,i p3,j X1,j X3,j + C47 p1,j p3,j X1,j X3,j +
C9 p2,i p3,i X2,i X3,j + C21 p2,j p3,i X2,i X3,j + C12 p2,i p3,j X2,i X3,j + C24 p2,j p3,j X2,i X3,j +
C33 p2,i p3,i X2,j X3,j + C45 p2,j p3,i X2,j X3,j + C36 p2,i p3,j X2,j X3,j + C48 p2,j p3,j X2,j X3,j

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In[*]:= $\mathcal{L}[X_{i,j}[s_-]] := T_3^s \mathbb{E}[q[s, i, j] + \epsilon r_{42}[s, i, j] + 0[\epsilon]^2];$
 $\mathcal{L}[X_{i,j}[1]]$


Out[*]=

$T_1 T_2 \mathbb{E}[\epsilon \text{Series}[(-p_{1,i} + p_{1,1+i}) x_{1,i} + (-1 + T_1) (p_{1,1+i} - p_{1,1+j}) x_{1,i} + (-p_{1,j} + p_{1,1+j}) x_{1,j} +$
 $(-p_{2,i} + p_{2,1+i}) x_{2,i} + (-1 + T_2) (p_{2,1+i} - p_{2,1+j}) x_{2,i} + (-p_{2,j} + p_{2,1+j}) x_{2,j} +$
 $(-p_{3,i} + p_{3,1+i}) x_{3,i} + (-1 + T_1 T_2) (p_{3,1+i} - p_{3,1+j}) x_{3,i} + (-p_{3,j} + p_{3,1+j}) x_{3,j},$
 $C_{49} p_{1,i} x_{1,i} + C_{52} p_{1,j} x_{1,i} + C_{55} p_{1,i} x_{1,j} + C_{58} p_{1,j} x_{1,j} + C_{50} p_{2,i} x_{2,i} + C_{53} p_{2,j} x_{2,i} +$
 $C_1 p_{1,i} p_{2,i} x_{1,i} x_{2,i} + C_{13} p_{1,j} p_{2,i} x_{1,i} x_{2,i} + C_4 p_{1,i} p_{2,j} x_{1,i} x_{2,i} + C_{16} p_{1,j} p_{2,j} x_{1,i} x_{2,i} +$
 $C_{25} p_{1,i} p_{2,i} x_{1,j} x_{2,i} + C_{37} p_{1,j} p_{2,i} x_{1,j} x_{2,i} + C_{28} p_{1,i} p_{2,j} x_{1,j} x_{2,i} + C_{40} p_{1,j} p_{2,j} x_{1,j} x_{2,i} +$
 $C_{56} p_{2,i} x_{2,j} + C_{59} p_{2,j} x_{2,j} + C_7 p_{1,i} p_{2,i} x_{1,i} x_{2,j} + C_{19} p_{1,j} p_{2,i} x_{1,i} x_{2,j} + C_{10} p_{1,i} p_{2,j} x_{1,i} x_{2,j} +$
 $C_{22} p_{1,j} p_{2,j} x_{1,i} x_{2,j} + C_{31} p_{1,i} p_{2,i} x_{1,j} x_{2,j} + C_{43} p_{1,j} p_{2,i} x_{1,j} x_{2,j} + C_{34} p_{1,i} p_{2,j} x_{1,j} x_{2,j} +$
 $C_{46} p_{1,j} p_{2,j} x_{1,j} x_{2,j} + C_{51} p_{3,i} x_{3,i} + C_{54} p_{3,j} x_{3,i} + C_2 p_{1,i} p_{3,i} x_{1,i} x_{3,i} + C_{14} p_{1,j} p_{3,i} x_{1,i} x_{3,i} +$
 $C_5 p_{1,i} p_{3,j} x_{1,i} x_{3,i} + C_{17} p_{1,j} p_{3,j} x_{1,i} x_{3,i} + C_{26} p_{1,i} p_{3,i} x_{1,j} x_{3,i} + C_{38} p_{1,j} p_{3,i} x_{1,j} x_{3,i} +$
 $C_{29} p_{1,i} p_{3,j} x_{1,j} x_{3,i} + C_{41} p_{1,j} p_{3,j} x_{1,j} x_{3,i} + C_3 p_{2,i} p_{3,i} x_{2,i} x_{3,i} + C_{15} p_{2,j} p_{3,i} x_{2,i} x_{3,i} +$
 $C_6 p_{2,i} p_{3,j} x_{2,i} x_{3,i} + C_{18} p_{2,j} p_{3,j} x_{2,i} x_{3,i} + C_{27} p_{2,i} p_{3,i} x_{2,j} x_{3,i} + C_{39} p_{2,j} p_{3,i} x_{2,j} x_{3,i} +$
 $C_{30} p_{2,i} p_{3,j} x_{2,j} x_{3,i} + C_{42} p_{2,j} p_{3,j} x_{2,j} x_{3,i} + C_{57} p_{3,i} x_{3,j} + C_{60} p_{3,j} x_{3,j} +$
 $C_8 p_{1,i} p_{3,i} x_{1,i} x_{3,j} + C_{20} p_{1,j} p_{3,i} x_{1,i} x_{3,j} + C_{11} p_{1,i} p_{3,j} x_{1,i} x_{3,j} + C_{23} p_{1,j} p_{3,j} x_{1,i} x_{3,j} +$
 $C_{32} p_{1,i} p_{3,i} x_{1,j} x_{3,j} + C_{44} p_{1,j} p_{3,i} x_{1,j} x_{3,j} + C_{35} p_{1,i} p_{3,j} x_{1,j} x_{3,j} + C_{47} p_{1,j} p_{3,j} x_{1,j} x_{3,j} +$
 $C_9 p_{2,i} p_{3,i} x_{2,i} x_{3,j} + C_{21} p_{2,j} p_{3,i} x_{2,i} x_{3,j} + C_{12} p_{2,i} p_{3,j} x_{2,i} x_{3,j} + C_{24} p_{2,j} p_{3,j} x_{2,i} x_{3,j} +$
 $C_{33} p_{2,i} p_{3,i} x_{2,j} x_{3,j} + C_{45} p_{2,j} p_{3,i} x_{2,j} x_{3,j} + C_{36} p_{2,i} p_{3,j} x_{2,j} x_{3,j} + C_{48} p_{2,j} p_{3,j} x_{2,j} x_{3,j}]]$

In[*]:= $\{\text{lhs}\} = \text{Cases}\left[\int \mathcal{F}[i, j, k] \times \mathcal{L} / @ (X_{i,j}[1] X_{i^+,k}[1] X_{j^+,k^+}[1]) \text{d}\{VS_i, VS_j, VS_k, VS_{i^+}, VS_{j^+}, VS_{k^+}\},\right.$
 $\left. \mathbb{E}[\mathcal{E}_-] \rightarrow \mathcal{E}, \infty\right]$

Out[*]=

$\{\epsilon \text{Series}[T_1^2 p_{1,2+i} \pi_{1,i} - (-1 + T_1) T_1 p_{1,2+j} \pi_{1,i} + (1 - T_1) p_{1,2+k} \pi_{1,i} + T_1 p_{1,2+j} \pi_{1,j} + (1 - T_1) p_{1,2+k} \pi_{1,j} + p_{1,2+k} \pi_{1,k} +$
 $T_2^2 p_{2,2+i} \pi_{2,i} - (-1 + T_2) T_2 p_{2,2+j} \pi_{2,i} + (1 - T_2) p_{2,2+k} \pi_{2,i} + T_2 p_{2,2+j} \pi_{2,j} + (1 - T_2) p_{2,2+k} \pi_{2,j} + p_{2,2+k} \pi_{2,k} +$
 $T_1^2 T_2^2 p_{3,2+i} \pi_{3,i} - T_1 T_2 (-1 + T_1 T_2) p_{3,2+j} \pi_{3,i} + (1 - T_1 T_2) p_{3,2+k} \pi_{3,i} + T_1 T_2 p_{3,2+j} \pi_{3,j} + (1 - T_1 T_2) p_{3,2+k} \pi_{3,j} + p_{3,2+k} \pi_{3,k},$
 $3 (C_1 + C_2 + C_3 + C_{10} + C_{11} + C_{12} + C_{37} + C_{38} + C_{39} + C_{46} + C_{47} + C_{48} + C_{49} + C_{50} + C_{51} + C_{58} + C_{59} + C_{60}) +$
 $2 (C_1 + C_2 + C_{10} + C_{11} + C_{49}) T_1^2 p_{1,2+i} \pi_{1,i} - T_1 (-2 C_1 - 2 C_2 - 2 C_{10} - 2 C_{11} - C_{13} - C_{14} - C_{22} - C_{23} - 2 C_{49} - C_{52} + 2 C_1 T_1 + 2 C_2 T_1 +$
 $2 C_{10} T_1 + 2 C_{11} T_1 - C_{25} T_1 - C_{26} T_1 - C_{34} T_1 - C_{35} T_1 + 2 C_{49} T_1 - C_{55} T_1 + C_{25} T_1^2 + C_{26} T_1^2 + C_{34} T_1^2 + C_{35} T_1^2 + C_{55} T_1^2) p_{1,2+j} \pi_{1,i} +$
 $\dots 338 \dots + 2 (C_{33} + C_{36} + C_{45} + C_{48} - C_{33} T_2 - C_{36} T_2 - C_{33} T_1 T_2 - C_{45} T_1 T_2 + C_{33} T_1 T_2^2) p_{2,2+k} p_{3,2+k} \pi_{2,k} \pi_{3,k}]]$

Full expression not available (original memory size: 0.7 MB) 

In[*]:= {rhs} = Cases [$\int \mathcal{F}[\mathbf{i}, \mathbf{j}, \mathbf{k}] \times \mathcal{L} / @ (X_{j,k}[1] X_{i,k^*}[1] X_{i^*,j^*}[1]) \mathcal{d} \{ \mathbf{vS}_i, \mathbf{vS}_j, \mathbf{vS}_k, \mathbf{vS}_{i^*}, \mathbf{vS}_{j^*}, \mathbf{vS}_{k^*} \},$
 $\mathbb{E}[\mathcal{E}_-] \mapsto \mathcal{E}, \infty]$

Out[*]=

$$\left\{ \text{Series} \left[T_1^2 p_{1,2+i} \pi_{1,i} - (-1 + T_1) T_1 p_{1,2+j} \pi_{1,i} + (1 - T_1) p_{1,2+k} \pi_{1,i} + T_1 p_{1,2+j} \pi_{1,j} + (1 - T_1) p_{1,2+k} \pi_{1,j} + p_{1,2+k} \pi_{1,k} + \right. \right.$$

$$T_2^2 p_{2,2+i} \pi_{2,i} - (-1 + T_2) T_2 p_{2,2+j} \pi_{2,i} + (1 - T_2) p_{2,2+k} \pi_{2,i} + T_2 p_{2,2+j} \pi_{2,j} + (1 - T_2) p_{2,2+k} \pi_{2,j} + p_{2,2+k} \pi_{2,k} +$$

$$T_1^2 T_2^2 p_{3,2+i} \pi_{3,i} - T_1 T_2 (-1 + T_1 T_2) p_{3,2+j} \pi_{3,i} + (1 - T_1 T_2) p_{3,2+k} \pi_{3,i} + T_1 T_2 p_{3,2+j} \pi_{3,j} + (1 - T_1 T_2) p_{3,2+k} \pi_{3,j} + p_{3,2+k} \pi_{3,k},$$

$$3 (C_1 + C_2 + C_3 + C_{10} + C_{11} + C_{12} + C_{37} + C_{38} + C_{39} + C_{46} + C_{47} + C_{48} + C_{49} + C_{50} + C_{51} + C_{58} + C_{59} + C_{60}) +$$

$$2 (C_1 + C_2 + C_{10} + C_{11} + C_{49}) T_1^2 p_{1,2+i} \pi_{1,i} -$$

$$T_1 (-2 C_1 - 2 C_2 - 2 C_{10} - 2 C_{11} - C_{13} - C_{14} - C_{22} - C_{23} - 2 C_{49} - C_{52} + 2 C_1 T_1 + 2 C_2 T_1 + 2 C_{10} T_1 + 2 C_{11} T_1 + 2 C_{49} T_1) p_{1,2+j} \pi_{1,i} +$$

$$\dots 391 \dots + (-2 + T_2) T_2 (-C_{33} - C_{36} + C_{33} T_1 T_2) p_{2,2+j} p_{3,2+k} \pi_{2,k} \pi_{3,k} +$$

$$2 (C_{33} + C_{36} + C_{45} + C_{48} - C_{33} T_2 - C_{36} T_2 - C_{33} T_1 T_2 - C_{45} T_1 T_2 + C_{33} T_1 T_2^2) p_{2,2+k} p_{3,2+k} \pi_{2,k} \pi_{3,k} \left. \right\}$$

Full expression not available (original memory size: 0.6 MB)

In[*]:= eqn = CF [(lhs - rhs) [[2]] - err]

Out[*]=

$$- \left((C_{25} + C_{26} + C_{34} + C_{35} + C_{55}) (-1 + T_1) T_1^2 p_{1,2+j} \pi_{1,i} \right) -$$

$$(-1 + T_1) (C_1 + C_2 + C_{10} + C_{11} + C_{13} + C_{14} + C_{22} + C_{23} + C_{49} + C_{52} + C_{25} T_1 + C_{26} T_1 + C_{34} T_1 + C_{35} T_1 +$$

$$C_{37} T_1 + C_{38} T_1 + C_{46} T_1 + C_{47} T_1 + C_{55} T_1 + C_{58} T_1 - C_{25} T_1^2 - C_{26} T_1^2 - C_{34} T_1^2 - C_{35} T_1^2 - C_{55} T_1^2) p_{1,2+k} \pi_{1,i} +$$

$$(C_{25} + C_{26} + C_{34} + C_{35} + C_{55}) (-1 + T_1) T_1^2 p_{1,2+i} \pi_{1,j} - (C_{25} + C_{26} + C_{34} + C_{35} + C_{55}) (-1 + T_1)^2 T_1 p_{1,2+j} \pi_{1,j} -$$

$$(C_{25} + C_{26} + C_{34} + C_{35} + C_{55}) (-1 + T_1) T_1 p_{1,2+k} \pi_{1,j} - (C_{25} + C_{26} + C_{34} + C_{35} + C_{55}) (-1 + T_1) T_1 p_{1,2+i} \pi_{1,k} +$$

$$\dots 365 \dots + C_{33} T_1^2 (-1 + T_2) T_2^3 p_{2,2+j} p_{3,2+i} \pi_{2,k} \pi_{3,k} + T_1 T_2 (-C_{33} - C_{45} + C_{33} T_2) (-1 + T_1 T_2) p_{2,2+k} p_{3,2+i} \pi_{2,k} \pi_{3,k} +$$

$$C_{33} T_1 T_2^2 (-1 + T_1 T_2) p_{2,2+i} p_{3,2+j} \pi_{2,k} \pi_{3,k} - C_{33} T_1 (-1 + T_2) T_2^2 (-1 + T_1 T_2) p_{2,2+j} p_{3,2+j} \pi_{2,k} \pi_{3,k} -$$

$$T_1 T_2 (-C_{33} - C_{45} + C_{33} T_2) (-1 + T_1 T_2) p_{2,2+k} p_{3,2+j} \pi_{2,k} \pi_{3,k} +$$

$$(-1 + T_2) T_2 (-C_{33} - C_{36} + C_{33} T_1 T_2) p_{2,2+i} p_{3,2+k} \pi_{2,k} \pi_{3,k} - (-1 + T_2) T_2 (-C_{33} - C_{36} + C_{33} T_1 T_2) p_{2,2+j} p_{3,2+k} \pi_{2,k} \pi_{3,k}$$

Full expression not available (original memory size: 0.8 MB)

In[*]:= cvs = Union@Cases [eqn, p_ | \pi_ , \infty]

Out[*]=

{p_{1,2+i}, p_{1,2+j}, p_{1,2+k}, p_{2,2+i}, p_{2,2+j}, p_{2,2+k}, p_{3,2+i},
 p_{3,2+j}, p_{3,2+k}, $\pi_{1,i}$, $\pi_{1,j}$, $\pi_{1,k}$, $\pi_{2,i}$, $\pi_{2,j}$, $\pi_{2,k}$, $\pi_{3,i}$, $\pi_{3,j}$, $\pi_{3,k}$ }

In[*]:= eqns = CoefficientRules [eqn, cvs] /. (_ -> c_) \mapsto (c == 0)

Out[*]=

$$\left\{ -c_7 T_1^2 T_2^2 + c_7 T_1^2 T_2^3 = 0, c_7 T_1^2 T_2 - c_7 T_1^2 T_2^2 = 0, -c_{25} T_1^2 T_2^2 + c_{25} T_1^3 T_2^2 = 0, -c_{31} T_1^2 T_2^2 + c_{31} T_1^3 T_2^2 + c_{31} T_1^2 T_2^3 - c_{31} T_1^3 T_2^3 = 0, \right.$$

$$\dots 245 \dots, -c_8 T_1 T_2 - c_9 T_1 T_2 - c_{44} T_1 T_2 - c_{45} T_1 T_2 - c_{57} T_1 T_2 + c_8 T_1^2 T_2^2 + c_9 T_1^2 T_2^2 + c_{44} T_1^2 T_2^2 + c_{45} T_1^2 T_2^2 + c_{57} T_1^2 T_2^2 = 0,$$

$$a_2 b_3 - a_4 b_3 + c_2 + c_3 + c_5 + c_6 + c_{38} + c_{39} + c_{41} + c_{42} + c_{51} + c_{54} + \frac{a_2 b_5}{T_1} + a_4 b_3 T_2 - a_4 b_5 T_2 + \frac{a_4 b_5 T_2}{T_1} - c_2 T_1 T_2 - c_3 T_1 T_2 - c_5 T_1 T_2 -$$

$$c_6 T_1 T_2 + c_8 T_1 T_2 + c_9 T_1 T_2 + c_{11} T_1 T_2 + c_{12} T_1 T_2 - c_{38} T_1 T_2 - c_{39} T_1 T_2 - c_{41} T_1 T_2 - c_{42} T_1 T_2 + c_{44} T_1 T_2 + c_{45} T_1 T_2 +$$

$$c_{47} T_1 T_2 + c_{48} T_1 T_2 - c_{51} T_1 T_2 - c_{54} T_1 T_2 + c_{57} T_1 T_2 + c_{60} T_1 T_2 - 2 c_8 T_1^2 T_2^2 - 2 c_9 T_1^2 T_2^2 - c_{11} T_1^2 T_2^2 - c_{12} T_1^2 T_2^2 - 2 c_{44} T_1^2 T_2^2 -$$

$$2 c_{45} T_1^2 T_2^2 - c_{47} T_1^2 T_2^2 - c_{48} T_1^2 T_2^2 - 2 c_{57} T_1^2 T_2^2 - c_{60} T_1^2 T_2^2 + c_8 T_1^3 T_2^3 + c_9 T_1^3 T_2^3 + c_{44} T_1^3 T_2^3 + c_{45} T_1^3 T_2^3 + c_{57} T_1^3 T_2^3 = 0,$$

$$c_8 T_1 T_2 + c_9 T_1 T_2 + c_{44} T_1 T_2 + c_{45} T_1 T_2 + c_{57} T_1 T_2 - c_8 T_1^2 T_2^2 - c_9 T_1^2 T_2^2 - c_{44} T_1^2 T_2^2 - c_{45} T_1^2 T_2^2 - c_{57} T_1^2 T_2^2 = 0 \left. \right\}$$

Full expression not available (original memory size: 1 MB)

In[*]:= vars = Union@Cases[r₄₂[1, i, j], c_, ∞]

Out[*]=

{C₁, C₂, C₃, C₄, C₅, C₆, C₇, C₈, C₉, C₁₀, C₁₁, C₁₂, C₁₃, C₁₄, C₁₅, C₁₆, C₁₇, C₁₈, C₁₉, C₂₀, C₂₁, C₂₂,
C₂₃, C₂₄, C₂₅, C₂₆, C₂₇, C₂₈, C₂₉, C₃₀, C₃₁, C₃₂, C₃₃, C₃₄, C₃₅, C₃₆, C₃₇, C₃₈, C₃₉, C₄₀, C₄₁,
C₄₂, C₄₃, C₄₄, C₄₅, C₄₆, C₄₇, C₄₈, C₄₉, C₅₀, C₅₁, C₅₂, C₅₃, C₅₄, C₅₅, C₅₆, C₅₇, C₅₈, C₅₉, C₆₀}

In[*]:= {sol} = Solve[eqns, vars]

 Solve: Equations may not give solutions for all "solve" variables. 

Out[*]=

{ {C₁ → 0, C₂ → 0, C₃ → 0, C₇ → 0, C₈ → 0, C₉ → 0,
C₁₀ → - $\frac{c_{13}}{1 - T_1} - \frac{c_4}{-1 + T_2} - \left(a_2 b_3 - a_4 b_3 - a_2 b_5 + a_4 b_5 - a_2 b_3 T_1 + a_4 b_3 T_1 + a_4 b_3 T_2 + a_2 b_5 T_2 - 2 a_4 b_5 T_2 - a_4 b_3 T_1 T_2 + a_4 b_5 T_2^2 \right) / \left((-1 + T_1) (-1 + T_2) (-1 + T_1 T_2) \right)$,
C₁₁ → - $\frac{c_{14}}{1 - T_1} - \frac{c_5}{-1 + T_1 T_2} - \frac{-a_2 b_5 + a_4 b_5 - a_4 b_5 T_2}{(-1 + T_1) (-1 + T_1 T_2)}$,
C₁₂ → - $\frac{c_{15}}{1 - T_2} - \frac{c_6}{-1 + T_1 T_2} - \frac{-a_2 b_3 - a_4 b_3 T_2 + a_4 b_3 T_1 T_2}{T_1 (-1 + T_2) (-1 + T_1 T_2)}$,
C₁₆ → -c₄ (1 - T₁) - c₁₃ (1 - T₂) - $\frac{a_2 b_3 + a_4 b_3 T_2 + a_2 b_5 T_2 - a_4 b_5 T_2 - a_4 b_3 T_1 T_2 + a_4 b_5 T_2^2}{-1 + T_1 T_2}$,
C₁₇ → -c₅ (1 - T₁) - c₁₄ (1 - T₁ T₂) - $\frac{1}{-1 + T_2}$
 $\left(-a_2 b_3 + a_2 b_5 - a_4 b_5 + a_2 b_3 T_1 - a_4 b_3 T_1 - a_4 b_3 T_2 - a_2 b_5 T_2 + 2 a_4 b_5 T_2 + 2 a_4 b_3 T_1 T_2 - a_4 b_5 T_2^2 \right)$,
C₁₈ → -c₆ (1 - T₂) - c₁₅ (1 - T₁ T₂) - $\frac{1}{(-1 + T_1) T_1}$ $\left(a_2 b_3 - a_2 b_3 T_1 + a_4 b_3 T_2 + a_2 b_5 T_2 - 2 a_4 b_3 T_1 T_2 - a_2 b_5 T_1 T_2 + a_4 b_5 T_1 T_2 + a_4 b_3 T_1^2 T_2 + a_4 b_5 T_2^2 - 2 a_4 b_5 T_1 T_2^2 \right)$,
C₁₉ → 0, C₂₀ → 0, C₂₁ → 0, C₂₂ → - $\frac{a_4 b_3}{-1 + T_2} - \frac{c_4 (-1 + T_1)}{-1 + T_2}$,
C₂₃ → - $\frac{c_5 (-1 + T_1)}{-1 + T_1 T_2} - \frac{a_2 b_3 - a_2 b_3 T_1 + a_4 b_3 T_1 + a_4 b_3 T_2 - 2 a_4 b_3 T_1 T_2}{(-1 + T_2) (-1 + T_1 T_2)}$,
C₂₄ → - $\frac{c_6 (-1 + T_2)}{-1 + T_1 T_2} - \frac{T_2 (a_2 b_5 + a_4 b_5 T_2)}{T_1 (-1 + T_1 T_2)} - \frac{a_4 b_5 (-T_2 + T_2^2)}{(-1 + T_1) (-1 + T_1 T_2)}$,
C₂₅ → 0, C₂₆ → 0, C₂₇ → 0, C₂₈ → 0, C₂₉ → - $\frac{a_2 b_3 + a_4 b_3 T_2}{T_1 (-1 + T_2)}$,
C₃₀ → $\frac{a_4 b_5}{-1 + T_1}$, C₃₁ → 0, C₃₂ → 0, C₃₃ → 0, C₃₄ → 0,
C₃₅ → 0, C₃₆ → 0, C₃₇ → - $\frac{c_{13}}{-1 + T_1} - \frac{c_4}{1 - T_2} -$
 $\left(-a_2 b_3 + a_2 b_5 + a_2 b_3 T_1 - a_4 b_3 T_2 - a_2 b_5 T_2 + a_4 b_5 T_2 + 2 a_4 b_3 T_1 T_2 - a_4 b_5 T_1 T_2 - a_4 b_3 T_1^2 T_2 - a_4 b_5 T_2^2 + a_4 b_5 T_1 T_2^2 \right) / \left((-1 + T_1) T_1 (-1 + T_2) (-1 + T_1 T_2) \right)$, C₃₈ → - $\frac{c_{14}}{-1 + T_1} - \frac{c_5}{1 - T_1 T_2} -$
 $\left(a_2 b_3 - a_2 b_5 - 2 a_2 b_3 T_1 + a_4 b_3 T_1 + a_2 b_3 T_1^2 - a_4 b_3 T_1^2 + a_4 b_3 T_2 + a_2 b_5 T_2 - a_4 b_5 T_2 - 3 a_4 b_3 T_1 T_2 +$

$$\begin{aligned}
 & \frac{a_4 b_5 T_1 T_2 + 2 a_4 b_3 T_1^2 T_2 + a_4 b_5 T_2^2 - a_4 b_5 T_1 T_2^2}{((-1 + T_1) T_1 (-1 + T_2) (-1 + T_1 T_2))}, \\
 c_{39} \rightarrow & -\frac{c_{15}}{-1 + T_2} - \frac{c_6}{1 - T_1 T_2} - \left(-a_2 b_5 - a_2 b_3 T_1 + a_4 b_3 T_1 + a_2 b_5 T_1 - a_4 b_5 T_1 + a_2 b_3 T_1^2 - a_4 b_3 T_1^2 + \right. \\
 & \left. a_2 b_5 T_2 - a_4 b_5 T_2 - a_4 b_3 T_1 T_2 - a_2 b_5 T_1 T_2 + 3 a_4 b_5 T_1 T_2 + a_4 b_3 T_1^2 T_2 + a_4 b_5 T_2^2 - 2 a_4 b_5 T_1 T_2^2 \right) / \\
 & \left((-1 + T_1) T_1 (-1 + T_2) (-1 + T_1 T_2) \right), c_{40} \rightarrow -\frac{c_{13} (-1 + T_2)}{-1 + T_1} - \frac{-a_2 b_5 - a_4 b_5 T_2}{(-1 + T_1) T_1}, \\
 c_{41} \rightarrow & \frac{a_2 b_3 + a_4 b_3 T_2}{T_1 (-1 + T_2)} - \frac{a_2 b_5 + a_4 b_5 T_2}{-1 + T_1} - \frac{c_{14} (-1 + T_1 T_2)}{-1 + T_1}, \\
 c_{42} \rightarrow & -\frac{a_4 b_5}{-1 + T_1} + \frac{a_4 b_3 T_2}{-1 + T_2} - \frac{c_{15} (-1 + T_1 T_2)}{-1 + T_2}, c_{43} \rightarrow \theta, \\
 c_{44} \rightarrow & \theta, c_{45} \rightarrow \theta, c_{46} \rightarrow \theta, c_{47} \rightarrow \theta, c_{48} \rightarrow \theta, c_{55} \rightarrow \theta, \\
 c_{56} \rightarrow & \theta, c_{57} \rightarrow \theta, c_{58} \rightarrow -\frac{c_{49}}{T_1} - \frac{c_{52}}{T_1}, c_{59} \rightarrow -\frac{c_{50}}{T_2} - \frac{c_{53}}{T_2}, \\
 c_{60} \rightarrow & -\left. \frac{c_{51}}{T_1 T_2} - \frac{c_{54}}{T_1 T_2} - \frac{a_2 b_5 + a_2 b_3 T_1 - a_4 b_3 T_1 + a_4 b_5 T_2 + a_4 b_3 T_1 T_2 - a_4 b_5 T_1 T_2}{T_1^2 T_2 (-1 + T_1 T_2)} \right\}
 \end{aligned}$$

In[*]:= sol /. (v_ -> val_) -> (v = CF[val]);

In[*]:= CF[r42[1, i, j]]

Out[*]=

$$\begin{aligned}
 & c_{49} p_{1,i} x_{1,i} + c_{52} p_{1,j} x_{1,i} - \frac{(c_{49} + c_{52}) p_{1,j} x_{1,j}}{T_1} + \\
 & c_{50} p_{2,i} x_{2,i} + c_{53} p_{2,j} x_{2,i} + c_{13} p_{1,j} p_{2,i} x_{1,i} x_{2,i} + c_4 p_{1,i} p_{2,j} x_{1,i} x_{2,i} - \\
 & \frac{1}{-1 + T_1 T_2} \left(a_2 b_3 - c_4 - c_{13} + c_4 T_1 + a_4 b_3 T_2 + a_2 b_5 T_2 - a_4 b_5 T_2 + c_{13} T_2 - a_4 b_3 T_1 T_2 + c_4 T_1 T_2 + c_{13} T_1 T_2 - \right. \\
 & \left. c_4 T_1^2 T_2 + a_4 b_5 T_2^2 - c_{13} T_1 T_2^2 \right) p_{1,j} p_{2,j} x_{1,i} x_{2,i} - \frac{1}{(-1 + T_1) T_1 (-1 + T_2) (-1 + T_1 T_2)} \\
 & \left(-a_2 b_3 + a_2 b_5 + a_2 b_3 T_1 - c_4 T_1 + c_{13} T_1 + c_4 T_1^2 - a_4 b_3 T_2 - a_2 b_5 T_2 + a_4 b_5 T_2 + 2 a_4 b_3 T_1 T_2 - \right. \\
 & \left. a_4 b_5 T_1 T_2 - c_{13} T_1 T_2 - a_4 b_3 T_1^2 T_2 + c_4 T_1^2 T_2 - c_{13} T_1^2 T_2 - c_4 T_1^3 T_2 - a_4 b_5 T_2^2 + a_4 b_5 T_1 T_2^2 + c_{13} T_1^2 T_2^2 \right) \\
 & p_{1,j} p_{2,i} x_{1,j} x_{2,i} - \frac{(-a_2 b_5 - c_{13} T_1 - a_4 b_5 T_2 + c_{13} T_1 T_2) p_{1,j} p_{2,j} x_{1,j} x_{2,i}}{(-1 + T_1) T_1} - \\
 & \frac{(c_{50} + c_{53}) p_{2,j} x_{2,j}}{T_2} + \frac{1}{(-1 + T_1) (-1 + T_2) (-1 + T_1 T_2)} \\
 & \left(-a_2 b_3 + a_4 b_3 + a_2 b_5 - a_4 b_5 - c_4 + c_{13} + a_2 b_3 T_1 - a_4 b_3 T_1 + c_4 T_1 - a_4 b_3 T_2 - a_2 b_5 T_2 + 2 a_4 b_5 T_2 - \right. \\
 & \left. c_{13} T_2 + a_4 b_3 T_1 T_2 + c_4 T_1 T_2 - c_{13} T_1 T_2 - c_4 T_1^2 T_2 - a_4 b_5 T_2^2 + c_{13} T_1 T_2^2 \right) p_{1,i} p_{2,j} x_{1,i} x_{2,j} - \\
 & \frac{(a_4 b_3 - c_4 + c_4 T_1) p_{1,j} p_{2,j} x_{1,i} x_{2,j}}{-1 + T_2} + c_{51} p_{3,i} x_{3,i} + c_{54} p_{3,j} x_{3,i} + \\
 & c_{14} p_{1,j} p_{3,i} x_{1,i} x_{3,i} + c_5 p_{1,i} p_{3,j} x_{1,i} x_{3,i} - \\
 & \frac{1}{-1 + T_2} \left(-a_2 b_3 + a_2 b_5 - a_4 b_5 - c_5 - c_{14} + a_2 b_3 T_1 - a_4 b_3 T_1 + c_5 T_1 - a_4 b_3 T_2 - a_2 b_5 T_2 + 2 a_4 b_5 T_2 + \right. \\
 & \left. c_5 T_2 + c_{14} T_2 + 2 a_4 b_3 T_1 T_2 - c_5 T_1 T_2 + c_{14} T_1 T_2 - a_4 b_5 T_2^2 - c_{14} T_1 T_2^2 \right) p_{1,j} p_{3,j} x_{1,i} x_{3,i} - \\
 & \frac{1}{(-1 + T_1) T_1 (-1 + T_2) (-1 + T_1 T_2)} \left(a_2 b_3 - a_2 b_5 - 2 a_2 b_3 T_1 + a_4 b_3 T_1 - c_5 T_1 + c_{14} T_1 + \right.
 \end{aligned}$$

$$\begin{aligned}
 & a_2 b_3 T_1^2 - a_4 b_3 T_1^2 + c_5 T_1^2 + a_4 b_3 T_2 + a_2 b_5 T_2 - a_4 b_5 T_2 - 3 a_4 b_3 T_1 T_2 + a_4 b_5 T_1 T_2 + c_5 T_1 T_2 - \\
 & c_{14} T_1 T_2 + 2 a_4 b_3 T_1^2 T_2 - c_5 T_1^2 T_2 - c_{14} T_1^2 T_2 + a_4 b_5 T_2^2 - a_4 b_5 T_1 T_2^2 + c_{14} T_1^2 T_2^2) p_{1,j} p_{3,i} x_{1,j} x_{3,i} - \\
 & \frac{b_3 (a_2 + a_4 T_2) p_{1,i} p_{3,j} x_{1,j} x_{3,i}}{T_1 (-1 + T_2)} + \left((-a_2 b_3 + a_2 b_3 T_1 + a_2 b_5 T_1 - c_{14} T_1 - a_4 b_3 T_2 + a_4 b_3 T_1 T_2 - \right. \\
 & \left. a_2 b_5 T_1 T_2 + a_4 b_5 T_1 T_2 + c_{14} T_1 T_2 + c_{14} T_1^2 T_2 - a_4 b_5 T_1 T_2^2 - c_{14} T_1^2 T_2^2) p_{1,j} p_{3,j} x_{1,j} x_{3,i} \right) / \\
 & \frac{1}{(-1 + T_1) T_1 (-1 + T_2)} + c_{15} p_{2,j} p_{3,i} x_{2,i} x_{3,i} + c_6 p_{2,i} p_{3,j} x_{2,i} x_{3,i} + \frac{1}{(-1 + T_1) T_1} \\
 & \left(-a_2 b_3 + a_2 b_3 T_1 + c_6 T_1 + c_{15} T_1 - c_6 T_1^2 - c_{15} T_1^2 - a_4 b_3 T_2 - a_2 b_5 T_2 + 2 a_4 b_3 T_1 T_2 + a_2 b_5 T_1 T_2 - \right. \\
 & \left. a_4 b_5 T_1 T_2 - c_6 T_1 T_2 - a_4 b_3 T_1^2 T_2 + c_6 T_1^2 T_2 - c_{15} T_1^3 T_2 + c_{15} T_1^3 T_2 - a_4 b_5 T_2^2 + 2 a_4 b_5 T_1 T_2^2 \right) \\
 & p_{2,j} p_{3,j} x_{2,i} x_{3,i} - \frac{1}{(-1 + T_1) T_1 (-1 + T_2) (-1 + T_1 T_2)} \\
 & \left(-a_2 b_5 - a_2 b_3 T_1 + a_4 b_3 T_1 + a_2 b_5 T_1 - a_4 b_5 T_1 - c_6 T_1 + c_{15} T_1 + a_2 b_3 T_1^2 - a_4 b_3 T_1^2 + c_6 T_1^2 - c_{15} T_1^2 + \right. \\
 & \left. a_2 b_5 T_2 - a_4 b_5 T_2 - a_4 b_3 T_1 T_2 - a_2 b_5 T_1 T_2 + 3 a_4 b_5 T_1 T_2 + c_6 T_1 T_2 + a_4 b_3 T_1^2 T_2 - c_6 T_1^2 T_2 - \right. \\
 & \left. c_{15} T_1^2 T_2 + c_{15} T_1^3 T_2 + a_4 b_5 T_2^2 - 2 a_4 b_5 T_1 T_2^2 \right) p_{2,j} p_{3,i} x_{2,j} x_{3,i} + \frac{a_4 b_5 p_{2,i} p_{3,j} x_{2,j} x_{3,i}}{-1 + T_1} - \\
 & \frac{1}{(-1 + T_1) (-1 + T_2)} \left(-a_4 b_5 + c_{15} - c_{15} T_1 + a_4 b_3 T_2 + a_4 b_5 T_2 - a_4 b_3 T_1 T_2 - c_{15} T_1 T_2 + c_{15} T_1^2 T_2 \right) \\
 & p_{2,j} p_{3,j} x_{2,j} x_{3,i} - \frac{1}{T_1^2 T_2 (-1 + T_1 T_2)} \\
 & \left(a_2 b_5 + a_2 b_3 T_1 - a_4 b_3 T_1 - c_{51} T_1 - c_{54} T_1 + a_4 b_5 T_2 + a_4 b_3 T_1 T_2 - a_4 b_5 T_1 T_2 + c_{51} T_1^2 T_2 + c_{54} T_1^2 T_2 \right) \\
 & p_{3,j} x_{3,j} + \frac{(a_2 b_5 - a_4 b_5 + c_5 - c_{14} - c_5 T_1 + a_4 b_5 T_2 + c_{14} T_1 T_2) p_{1,i} p_{3,j} x_{1,i} x_{3,j}}{(-1 + T_1) (-1 + T_1 T_2)} + \\
 & \left((-a_2 b_3 - c_5 + a_2 b_3 T_1 - a_4 b_3 T_1 + c_5 T_1 - a_4 b_3 T_2 + c_5 T_2 + 2 a_4 b_3 T_1 T_2 - c_5 T_1 T_2) p_{1,j} p_{3,j} x_{1,i} x_{3,j} \right) / \\
 & \left((-1 + T_2) (-1 + T_1 T_2) \right) - \\
 & \left((-a_2 b_3 - c_6 T_1 + c_{15} T_1 - a_4 b_3 T_2 + a_4 b_3 T_1 T_2 + c_6 T_1 T_2 - c_{15} T_1^2 T_2) p_{2,i} p_{3,j} x_{2,i} x_{3,j} \right) / \\
 & \left(T_1 (-1 + T_2) (-1 + T_1 T_2) \right) - \\
 & \left((c_6 T_1 - c_6 T_1^2 - a_2 b_5 T_2 + a_2 b_5 T_1 T_2 - a_4 b_5 T_1 T_2 - c_6 T_1 T_2 + c_6 T_1^2 T_2 - a_4 b_5 T_2^2 + 2 a_4 b_5 T_1 T_2^2) \right. \\
 & \left. p_{2,j} p_{3,j} x_{2,i} x_{3,j} \right) / \left((-1 + T_1) T_1 (-1 + T_1 T_2) \right)
 \end{aligned}$$

In[*]:= CF[r42[1, i, j] /.

{a4 -> 0, b3 -> 0, b5 -> (T1 - 1) (T2 - 1) (T3 - 1) a2^-1, c4|5|6|13|14|15|49|50|51|52|53|54 -> 0}]

Out[*]=

$$\begin{aligned}
 & - \left((-1 + T_1) (-1 + T_2) T_2 p_{1,j} p_{2,j} x_{1,i} x_{2,i} \right) + \frac{(-1 + T_2) p_{1,j} p_{2,i} x_{1,j} x_{2,i}}{T_1} + \\
 & \frac{(-1 + T_2) (-1 + T_1 T_2) p_{1,j} p_{2,j} x_{1,j} x_{2,i}}{T_1} + (1 - T_2) p_{1,i} p_{2,j} x_{1,i} x_{2,j} + \\
 & (-1 + T_1) (-1 + T_2) (-1 + T_1 T_2) p_{1,j} p_{3,j} x_{1,i} x_{3,i} - \frac{(-1 + T_2) p_{1,j} p_{3,i} x_{1,j} x_{3,i}}{T_1} - \\
 & (-1 + T_2) (-1 + T_1 T_2) p_{1,j} p_{3,j} x_{1,j} x_{3,i} + \frac{(-1 + T_1) (-1 + T_2) T_2 (-1 + T_1 T_2) p_{2,j} p_{3,j} x_{2,i} x_{3,i}}{T_1} + \\
 & \frac{(-1 + T_1) (-1 + T_2) p_{2,j} p_{3,i} x_{2,j} x_{3,i}}{T_1} - \frac{(-1 + T_1) (-1 + T_2) p_{3,j} x_{3,j}}{T_1^2 T_2} + \\
 & (-1 + T_2) p_{1,i} p_{3,j} x_{1,i} x_{3,j} - \frac{(-1 + T_1) (-1 + T_2) T_2 p_{2,j} p_{3,j} x_{2,i} x_{3,j}}{T_1}
 \end{aligned}$$

In[*]:= CF[(r₀[1, i, j] - (r₀[1, i, j] /. {T₁ → T₂, T₂ → T₁, p_{1,i} → p_{2,i}, p_{2,i} → p_{1,i}, x_{1,i} → x_{2,i}, x_{2,i} → x_{1,i}})] /. a₂ → -(T₁ + T₂) a₄]

Out[*]=

0

In[*]:= CF[(r₁[1, i, j] - (r₁[1, i, j] /. {T₁ → T₂, T₂ → T₁, p_{1,i} → p_{2,i}, p_{2,i} → p_{1,i}, x_{1,i} → x_{2,i}, x_{2,i} → x_{1,i}})] /. b₅ → b₃]

Out[*]=

0

In[*]:= CF[(r₄₂[1, i, j] -
 (r₄₂[1, i, j] /. {T₁ → T₂, T₂ → T₁, p_{1,i} → p_{2,i}, p_{2,i} → p_{1,i}, x_{1,i} → x_{2,i}, x_{2,i} → x_{1,i}})] /.
 {a₂ → -(T₁ + T₂) a₄, b₅ → b_{3}}}

Out[*]=

$$\begin{aligned} & (C_{49} - C_{50}) p_{1,i} x_{1,i} + (C_{52} - C_{53}) p_{1,j} x_{1,i} - \frac{(C_{49} - C_{50} + C_{52} - C_{53}) p_{1,j} x_{1,j}}{T_1} + \\ & (-C_{49} + C_{50}) p_{2,i} x_{2,i} + (-C_{52} + C_{53}) p_{2,j} x_{2,i} + (-C_4 + C_{13}) p_{1,j} p_{2,i} x_{1,i} x_{2,i} + \\ & (C_4 - C_{13}) p_{1,i} p_{2,j} x_{1,i} x_{2,i} + (C_4 - C_{13}) (T_1 - T_2) p_{1,j} p_{2,j} x_{1,i} x_{2,i} + \\ & \frac{(C_4 - C_{13}) (-2 + T_1 + T_2) p_{1,j} p_{2,i} x_{1,j} x_{2,i}}{(-1 + T_1) (-1 + T_2)} + \frac{(C_4 - C_{13}) (-1 + T_2) p_{1,j} p_{2,j} x_{1,j} x_{2,i}}{-1 + T_1} + \\ & \frac{(C_{49} - C_{50} + C_{52} - C_{53}) p_{2,j} x_{2,j}}{T_2} - \frac{(C_4 - C_{13}) (-2 + T_1 + T_2) p_{1,i} p_{2,j} x_{1,i} x_{2,j}}{(-1 + T_1) (-1 + T_2)} - \\ & \frac{(C_4 - C_{13}) (-1 + T_1) p_{1,j} p_{2,j} x_{1,i} x_{2,j}}{-1 + T_2} + (C_{14} - C_{15}) p_{1,j} p_{3,i} x_{1,i} x_{3,i} + (C_5 - C_6) p_{1,i} p_{3,j} x_{1,i} x_{3,i} + \\ & (-C_5 + C_6 - C_{14} + C_{15} + C_5 T_1 - C_6 T_1 + C_{14} T_1 T_2 - C_{15} T_1 T_2) p_{1,j} p_{3,j} x_{1,i} x_{3,i} + \\ & \frac{(-C_5 + C_6 + C_{14} - C_{15} + C_5 T_1 - C_6 T_1 - C_{14} T_1 T_2 + C_{15} T_1 T_2) p_{1,j} p_{3,i} x_{1,j} x_{3,i}}{(-1 + T_1) (-1 + T_1 T_2)} - \\ & \frac{(C_{14} - C_{15}) (-1 + T_1 T_2) p_{1,j} p_{3,j} x_{1,j} x_{3,i}}{-1 + T_1} + (-C_{14} + C_{15}) p_{2,j} p_{3,i} x_{2,i} x_{3,i} + \\ & \frac{(-C_5 + C_6) p_{2,i} p_{3,j} x_{2,i} x_{3,i} + (C_5 - C_6 + C_{14} - C_{15} - C_5 T_2 + C_6 T_2 - C_{14} T_1 T_2 + C_{15} T_1 T_2) p_{2,j} p_{3,j} x_{2,i} x_{3,i}}{(-1 + T_2) (-1 + T_1 T_2)} + \\ & \frac{(-C_5 + C_6 + C_{14} - C_{15} + C_5 T_2 - C_6 T_2 - C_{14} T_1 T_2 + C_{15} T_1 T_2) p_{2,j} p_{3,i} x_{2,j} x_{3,i}}{(-1 + T_2) (-1 + T_1 T_2)} + \\ & \frac{(C_{14} - C_{15}) (-1 + T_1 T_2) p_{2,j} p_{3,j} x_{2,j} x_{3,i}}{-1 + T_2} - \\ & \frac{(-C_5 + C_6 + C_{14} - C_{15} + C_5 T_1 - C_6 T_1 - C_{14} T_1 T_2 + C_{15} T_1 T_2) p_{1,i} p_{3,j} x_{1,i} x_{3,j}}{(-1 + T_1) (-1 + T_1 T_2)} - \\ & \frac{(C_5 - C_6) (-1 + T_1) p_{1,j} p_{3,j} x_{1,i} x_{3,j}}{-1 + T_1 T_2} + \\ & \frac{(-C_5 + C_6 + C_{14} - C_{15} + C_5 T_2 - C_6 T_2 - C_{14} T_1 T_2 + C_{15} T_1 T_2) p_{2,i} p_{3,j} x_{2,i} x_{3,j}}{(-1 + T_2) (-1 + T_1 T_2)} + \\ & \frac{(C_5 - C_6) (-1 + T_2) p_{2,j} p_{3,j} x_{2,i} x_{3,j}}{-1 + T_1 T_2} \end{aligned}$$

In[*]:= CF[(r₄₂[1, i, j] -
 (r₄₂[1, i, j] /. {T₁ → T₂, T₂ → T₁, p_{1,i} → p_{2,i}, p_{2,i} → p_{1,i}, x_{1,i} → x_{2,i}, x_{2,i} → x_{1,i}})] /.
 {a₂ → -(T₁ + T₂) a₄, b₅ → b₃, C₅₀ → C₄₉, C₅₃ → C₅₂, C₁₃ → C₄, C₆ → C₅, C₁₅ → C_{14}}}

Out[*]=

0

$$\text{In[*]} := \text{CF}[\mathbf{r}_{42}[1, \mathbf{i}, \mathbf{j}] / \cdot \{ \mathbf{a}_2 \rightarrow -(\mathbf{T}_1 + \mathbf{T}_2) \mathbf{a}_4, \mathbf{b}_5 \rightarrow \mathbf{b}_3, \mathbf{c}_{50} \rightarrow \mathbf{c}_{49}, \mathbf{c}_{53} \rightarrow \mathbf{c}_{52}, \mathbf{c}_{13} \rightarrow \mathbf{c}_4, \mathbf{c}_6 \rightarrow \mathbf{c}_5, \mathbf{c}_{15} \rightarrow \mathbf{c}_{14} \} / \cdot \{ \mathbf{c}_{4|5|14|49|51|52|54} \rightarrow \mathbf{0} \}]$$

Out[*]=

$$\begin{aligned} & \frac{\mathbf{a}_4 \mathbf{b}_3 (\mathbf{T}_1 + \mathbf{T}_2 + 2 \mathbf{T}_1 \mathbf{T}_2) \mathbf{p}_{1,j} \mathbf{p}_{2,j} \mathbf{x}_{1,i} \mathbf{x}_{2,i}}{-1 + \mathbf{T}_1 \mathbf{T}_2} + \frac{\mathbf{a}_4 \mathbf{b}_3 (\mathbf{T}_1 - \mathbf{T}_2) (1 + \mathbf{T}_2) \mathbf{p}_{1,j} \mathbf{p}_{2,i} \mathbf{x}_{1,j} \mathbf{x}_{2,i}}{(-1 + \mathbf{T}_1) (-1 + \mathbf{T}_2) (-1 + \mathbf{T}_1 \mathbf{T}_2)} - \\ & \frac{\mathbf{a}_4 \mathbf{b}_3 \mathbf{p}_{1,j} \mathbf{p}_{2,j} \mathbf{x}_{1,j} \mathbf{x}_{2,i}}{-1 + \mathbf{T}_1} - \frac{\mathbf{a}_4 \mathbf{b}_3 (1 + \mathbf{T}_1) (\mathbf{T}_1 - \mathbf{T}_2) \mathbf{p}_{1,i} \mathbf{p}_{2,j} \mathbf{x}_{1,i} \mathbf{x}_{2,j}}{(-1 + \mathbf{T}_1) (-1 + \mathbf{T}_2) (-1 + \mathbf{T}_1 \mathbf{T}_2)} - \\ & \frac{\mathbf{a}_4 \mathbf{b}_3 \mathbf{p}_{1,j} \mathbf{p}_{2,j} \mathbf{x}_{1,i} \mathbf{x}_{2,j}}{-1 + \mathbf{T}_2} - \frac{\mathbf{a}_4 \mathbf{b}_3 (-1 - \mathbf{T}_1 - \mathbf{T}_1^2 + \mathbf{T}_2 + 2 \mathbf{T}_1 \mathbf{T}_2) \mathbf{p}_{1,j} \mathbf{p}_{3,j} \mathbf{x}_{1,i} \mathbf{x}_{3,i}}{-1 + \mathbf{T}_2} + \\ & \frac{\mathbf{a}_4 \mathbf{b}_3 (-1 - \mathbf{T}_1 + \mathbf{T}_1^2 + \mathbf{T}_2 - \mathbf{T}_1 \mathbf{T}_2 + \mathbf{T}_2^2) \mathbf{p}_{1,j} \mathbf{p}_{3,i} \mathbf{x}_{1,j} \mathbf{x}_{3,i}}{(-1 + \mathbf{T}_1) (-1 + \mathbf{T}_2) (-1 + \mathbf{T}_1 \mathbf{T}_2)} + \frac{\mathbf{a}_4 \mathbf{b}_3 \mathbf{p}_{1,i} \mathbf{p}_{3,j} \mathbf{x}_{1,j} \mathbf{x}_{3,i}}{-1 + \mathbf{T}_2} + \\ & \frac{\mathbf{a}_4 \mathbf{b}_3 (1 - 2 \mathbf{T}_1 + \mathbf{T}_1 \mathbf{T}_2) \mathbf{p}_{1,j} \mathbf{p}_{3,j} \mathbf{x}_{1,j} \mathbf{x}_{3,i}}{(-1 + \mathbf{T}_1) (-1 + \mathbf{T}_2)} - \frac{\mathbf{a}_4 \mathbf{b}_3 (-1 + \mathbf{T}_1 - \mathbf{T}_2 + 2 \mathbf{T}_1 \mathbf{T}_2 - \mathbf{T}_2^2) \mathbf{p}_{2,j} \mathbf{p}_{3,j} \mathbf{x}_{2,i} \mathbf{x}_{3,i}}{-1 + \mathbf{T}_1} + \\ & \frac{\mathbf{a}_4 \mathbf{b}_3 (-1 + \mathbf{T}_1 + \mathbf{T}_1^2 - \mathbf{T}_2 - \mathbf{T}_1 \mathbf{T}_2 + \mathbf{T}_2^2) \mathbf{p}_{2,j} \mathbf{p}_{3,i} \mathbf{x}_{2,j} \mathbf{x}_{3,i}}{(-1 + \mathbf{T}_1) (-1 + \mathbf{T}_2) (-1 + \mathbf{T}_1 \mathbf{T}_2)} + \frac{\mathbf{a}_4 \mathbf{b}_3 \mathbf{p}_{2,i} \mathbf{p}_{3,j} \mathbf{x}_{2,j} \mathbf{x}_{3,i}}{-1 + \mathbf{T}_1} + \\ & \frac{\mathbf{a}_4 \mathbf{b}_3 (1 - 2 \mathbf{T}_2 + \mathbf{T}_1 \mathbf{T}_2) \mathbf{p}_{2,j} \mathbf{p}_{3,j} \mathbf{x}_{2,j} \mathbf{x}_{3,i}}{(-1 + \mathbf{T}_1) (-1 + \mathbf{T}_2)} + \frac{\mathbf{a}_4 \mathbf{b}_3 (2 + \mathbf{T}_1 + \mathbf{T}_2) \mathbf{p}_{3,j} \mathbf{x}_{3,j}}{\mathbf{T}_1 \mathbf{T}_2 (-1 + \mathbf{T}_1 \mathbf{T}_2)} - \\ & \frac{\mathbf{a}_4 \mathbf{b}_3 (1 + \mathbf{T}_1) \mathbf{p}_{1,i} \mathbf{p}_{3,j} \mathbf{x}_{1,i} \mathbf{x}_{3,j}}{(-1 + \mathbf{T}_1) (-1 + \mathbf{T}_1 \mathbf{T}_2)} - \frac{\mathbf{a}_4 \mathbf{b}_3 \mathbf{T}_1 (\mathbf{T}_1 - \mathbf{T}_2) \mathbf{p}_{1,j} \mathbf{p}_{3,j} \mathbf{x}_{1,i} \mathbf{x}_{3,j}}{(-1 + \mathbf{T}_2) (-1 + \mathbf{T}_1 \mathbf{T}_2)} - \\ & \frac{\mathbf{a}_4 \mathbf{b}_3 (1 + \mathbf{T}_2) \mathbf{p}_{2,i} \mathbf{p}_{3,j} \mathbf{x}_{2,i} \mathbf{x}_{3,j}}{(-1 + \mathbf{T}_2) (-1 + \mathbf{T}_1 \mathbf{T}_2)} + \frac{\mathbf{a}_4 \mathbf{b}_3 (\mathbf{T}_1 - \mathbf{T}_2) \mathbf{T}_2 \mathbf{p}_{2,j} \mathbf{p}_{3,j} \mathbf{x}_{2,i} \mathbf{x}_{3,j}}{(-1 + \mathbf{T}_1) (-1 + \mathbf{T}_1 \mathbf{T}_2)} \end{aligned}$$

$$\text{In[*]} := \text{CF}[\mathbf{r}_{42}[1, \mathbf{i}, \mathbf{j}] / \cdot \{ \mathbf{a}_2 \rightarrow -(\mathbf{T}_1 + \mathbf{T}_2) \mathbf{a}_4, \mathbf{b}_5 \rightarrow \mathbf{b}_3, \mathbf{c}_{50} \rightarrow \mathbf{c}_{49}, \mathbf{c}_{53} \rightarrow \mathbf{c}_{52}, \mathbf{c}_{13} \rightarrow \mathbf{c}_4, \mathbf{c}_6 \rightarrow \mathbf{c}_5, \mathbf{c}_{15} \rightarrow \mathbf{c}_{14} \} / \cdot \{ \mathbf{c}_{4|5|14|49|51|52|54} \rightarrow \mathbf{0} \} / \cdot \{ \mathbf{a}_4 \rightarrow \mathbf{b}_3^{-1} (-1 + \mathbf{T}_1) (-1 + \mathbf{T}_2) (-1 + \mathbf{T}_1 \mathbf{T}_2) \}]$$

Out[*]=

$$\begin{aligned} & (-1 + \mathbf{T}_1) (-1 + \mathbf{T}_2) (\mathbf{T}_1 + \mathbf{T}_2 + 2 \mathbf{T}_1 \mathbf{T}_2) \mathbf{p}_{1,j} \mathbf{p}_{2,j} \mathbf{x}_{1,i} \mathbf{x}_{2,i} + \\ & (\mathbf{T}_1 - \mathbf{T}_2) (1 + \mathbf{T}_2) \mathbf{p}_{1,j} \mathbf{p}_{2,i} \mathbf{x}_{1,j} \mathbf{x}_{2,i} - (-1 + \mathbf{T}_2) (-1 + \mathbf{T}_1 \mathbf{T}_2) \mathbf{p}_{1,j} \mathbf{p}_{2,j} \mathbf{x}_{1,j} \mathbf{x}_{2,i} - \\ & (1 + \mathbf{T}_1) (\mathbf{T}_1 - \mathbf{T}_2) \mathbf{p}_{1,i} \mathbf{p}_{2,j} \mathbf{x}_{1,i} \mathbf{x}_{2,j} - (-1 + \mathbf{T}_1) (-1 + \mathbf{T}_1 \mathbf{T}_2) \mathbf{p}_{1,j} \mathbf{p}_{2,j} \mathbf{x}_{1,i} \mathbf{x}_{2,j} - \\ & (-1 + \mathbf{T}_1) (-1 + \mathbf{T}_1 \mathbf{T}_2) (-1 - \mathbf{T}_1 - \mathbf{T}_1^2 + \mathbf{T}_2 + 2 \mathbf{T}_1 \mathbf{T}_2) \mathbf{p}_{1,j} \mathbf{p}_{3,j} \mathbf{x}_{1,i} \mathbf{x}_{3,i} + \\ & (-1 - \mathbf{T}_1 + \mathbf{T}_1^2 + \mathbf{T}_2 - \mathbf{T}_1 \mathbf{T}_2 + \mathbf{T}_2^2) \mathbf{p}_{1,j} \mathbf{p}_{3,i} \mathbf{x}_{1,j} \mathbf{x}_{3,i} + \\ & (-1 + \mathbf{T}_1) (-1 + \mathbf{T}_1 \mathbf{T}_2) \mathbf{p}_{1,i} \mathbf{p}_{3,j} \mathbf{x}_{1,j} \mathbf{x}_{3,i} + (-1 + \mathbf{T}_1 \mathbf{T}_2) (1 - 2 \mathbf{T}_1 + \mathbf{T}_1 \mathbf{T}_2) \mathbf{p}_{1,j} \mathbf{p}_{3,j} \mathbf{x}_{1,j} \mathbf{x}_{3,i} - \\ & (-1 + \mathbf{T}_2) (-1 + \mathbf{T}_1 \mathbf{T}_2) (-1 + \mathbf{T}_1 - \mathbf{T}_2 + 2 \mathbf{T}_1 \mathbf{T}_2 - \mathbf{T}_2^2) \mathbf{p}_{2,j} \mathbf{p}_{3,j} \mathbf{x}_{2,i} \mathbf{x}_{3,i} + \\ & (-1 + \mathbf{T}_1 + \mathbf{T}_1^2 - \mathbf{T}_2 - \mathbf{T}_1 \mathbf{T}_2 + \mathbf{T}_2^2) \mathbf{p}_{2,j} \mathbf{p}_{3,i} \mathbf{x}_{2,j} \mathbf{x}_{3,i} + (-1 + \mathbf{T}_2) (-1 + \mathbf{T}_1 \mathbf{T}_2) \mathbf{p}_{2,i} \mathbf{p}_{3,j} \mathbf{x}_{2,j} \mathbf{x}_{3,i} + \\ & (-1 + \mathbf{T}_1 \mathbf{T}_2) (1 - 2 \mathbf{T}_2 + \mathbf{T}_1 \mathbf{T}_2) \mathbf{p}_{2,j} \mathbf{p}_{3,j} \mathbf{x}_{2,j} \mathbf{x}_{3,i} + \frac{(-1 + \mathbf{T}_1) (-1 + \mathbf{T}_2) (2 + \mathbf{T}_1 + \mathbf{T}_2) \mathbf{p}_{3,j} \mathbf{x}_{3,j}}{\mathbf{T}_1 \mathbf{T}_2} - \\ & (1 + \mathbf{T}_1) (-1 + \mathbf{T}_2) \mathbf{p}_{1,i} \mathbf{p}_{3,j} \mathbf{x}_{1,i} \mathbf{x}_{3,j} - (-1 + \mathbf{T}_1) \mathbf{T}_1 (\mathbf{T}_1 - \mathbf{T}_2) \mathbf{p}_{1,j} \mathbf{p}_{3,j} \mathbf{x}_{1,i} \mathbf{x}_{3,j} - \\ & (-1 + \mathbf{T}_1) (1 + \mathbf{T}_2) \mathbf{p}_{2,i} \mathbf{p}_{3,j} \mathbf{x}_{2,i} \mathbf{x}_{3,j} + (\mathbf{T}_1 - \mathbf{T}_2) (-1 + \mathbf{T}_2) \mathbf{T}_2 \mathbf{p}_{2,j} \mathbf{p}_{3,j} \mathbf{x}_{2,i} \mathbf{x}_{3,j} \end{aligned}$$

Reidemeister 2b for r_0

```
In[*]:= x = 0;
r0[-1, i_, j_] := Evaluate[Sum[
  d_{++x} p_{3,k3} x_{1,k1} x_{2,k2},
  {k1, {i, j}}, {k2, {i, j}}, {k3, {i, j}}
]];
r0[-1, i, j]
```

```
Out[*]=
d1 p_{3,i} x_{1,i} x_{2,i} + d2 p_{3,j} x_{1,i} x_{2,i} + d5 p_{3,i} x_{1,j} x_{2,i} + d6 p_{3,j} x_{1,j} x_{2,i} +
d3 p_{3,i} x_{1,i} x_{2,j} + d4 p_{3,j} x_{1,i} x_{2,j} + d7 p_{3,i} x_{1,j} x_{2,j} + d8 p_{3,j} x_{1,j} x_{2,j}
```

```
In[*]:= L[X_{i_,j_}[s_]] := T3^s E[q[s, i, j] + r0[s, i, j] + 0[e]];
L[X_{i,j}[-1]]
```

```
Out[*]=
\frac{1}{T_1 T_2} E[Series[
  (-p_{1,i} + p_{1,1+i}) x_{1,i} + \left(-1 + \frac{1}{T_1}\right) (p_{1,1+i} - p_{1,1+j}) x_{1,i} + (-p_{1,j} + p_{1,1+j}) x_{1,j} + (-p_{2,i} + p_{2,1+i}) x_{2,i} +
  \left(-1 + \frac{1}{T_2}\right) (p_{2,1+i} - p_{2,1+j}) x_{2,i} + d_1 p_{3,i} x_{1,i} x_{2,i} + d_2 p_{3,j} x_{1,i} x_{2,i} + d_5 p_{3,i} x_{1,j} x_{2,i} +
  d_6 p_{3,j} x_{1,j} x_{2,i} + (-p_{2,j} + p_{2,1+j}) x_{2,j} + d_3 p_{3,i} x_{1,i} x_{2,j} + d_4 p_{3,j} x_{1,i} x_{2,j} + d_7 p_{3,i} x_{1,j} x_{2,j} +
  d_8 p_{3,j} x_{1,j} x_{2,j} + (-p_{3,i} + p_{3,1+i}) x_{3,i} + \left(-1 + \frac{1}{T_1 T_2}\right) (p_{3,1+i} - p_{3,1+j}) x_{3,i} + (-p_{3,j} + p_{3,1+j}) x_{3,j}
]]
```


$$In[] := \{lhs\} = \text{Cases} \left[\int \mathcal{F}[i, j] \times \mathcal{L} / @ (X_{i,j}[1] X_{i^+,j^+}[-1]) \, d\{vs_i, vs_j, vs_{i^+}, vs_{j^+}\}, \text{eSeries}[\mathcal{E}_-] \Rightarrow \mathcal{E}, \infty \right]$$

Out[] =

$$\left\{ \begin{aligned} & p_{1,2+i} \pi_{1,i} + p_{1,2+j} \pi_{1,j} + p_{2,2+i} \pi_{2,i} + \\ & \frac{1}{T_1 T_2} (d_7 + d_3 T_1 - d_7 T_1 + d_5 T_2 - d_7 T_2 + d_1 T_1 T_2 - d_3 T_1 T_2 - d_5 T_1 T_2 + d_7 T_1 T_2) p_{3,2+i} \pi_{1,i} \pi_{2,i} + \\ & \frac{1}{T_1 T_2} \left(-d_7 - d_3 T_1 + d_7 T_1 - d_5 T_2 + d_7 T_2 + a_2 T_1 T_2 - d_1 T_1 T_2 + d_3 T_1 T_2 + d_5 T_1 T_2 + d_8 T_1 T_2 + \right. \\ & \quad d_3 T_1^2 T_2 + d_4 T_1^2 T_2 - d_7 T_1^2 T_2 - d_8 T_1^2 T_2 + d_5 T_1 T_2^2 + d_6 T_1 T_2^2 - d_7 T_1 T_2^2 - d_8 T_1 T_2^2 + \\ & \quad \left. d_1 T_1^2 T_2^2 + d_2 T_1^2 T_2^2 - d_3 T_1^2 T_2^2 - d_4 T_1^2 T_2^2 - d_5 T_1^2 T_2^2 - d_6 T_1^2 T_2^2 + d_7 T_1^2 T_2^2 + d_8 T_1^2 T_2^2 \right) \\ & p_{3,2+j} \pi_{1,i} \pi_{2,i} + \frac{(d_7 + d_5 T_2 - d_7 T_2) p_{3,2+i} \pi_{1,j} \pi_{2,i}}{T_1 T_2} - \\ & \frac{1}{T_1 T_2} (d_7 + a_2 T_2 + d_5 T_2 - d_7 T_2 - d_7 T_1 T_2 - d_8 T_1 T_2 + a_4 T_2^2 - d_5 T_1 T_2^2 - d_6 T_1 T_2^2 + d_7 T_1 T_2^2 + d_8 T_1 T_2^2) \\ & p_{3,2+j} \pi_{1,j} \pi_{2,i} + p_{2,2+j} \pi_{2,j} + \frac{(d_7 + d_3 T_1 - d_7 T_1) p_{3,2+i} \pi_{1,i} \pi_{2,j}}{T_1 T_2} + \\ & \frac{1}{T_1 T_2} \left(-d_7 - d_3 T_1 + d_7 T_1 + a_4 T_1 T_2 + d_7 T_1 T_2 + d_8 T_1 T_2 + d_3 T_1^2 T_2 + d_4 T_1^2 T_2 - d_7 T_1^2 T_2 - d_8 T_1^2 T_2 \right) \\ & p_{3,2+j} \pi_{1,i} \pi_{2,j} + \frac{d_7 p_{3,2+i} \pi_{1,j} \pi_{2,j}}{T_1 T_2} + \\ & \left. \frac{(-d_7 + d_7 T_1 T_2 + d_8 T_1 T_2) p_{3,2+j} \pi_{1,j} \pi_{2,j}}{T_1 T_2} + p_{3,2+i} \pi_{3,i} + p_{3,2+j} \pi_{3,j} \right\} \end{aligned} \right.$$

$$In[] := \text{eqn} = \text{CF}[\text{lhs} - (p_{1,2+i} \pi_{1,i} + p_{1,2+j} \pi_{1,j} + p_{2,2+i} \pi_{2,i} + p_{2,2+j} \pi_{2,j} + p_{3,2+i} \pi_{3,i} + p_{3,2+j} \pi_{3,j})]$$

Out[] =

$$\left\{ \begin{aligned} & \frac{1}{T_1 T_2} (d_7 + d_3 T_1 - d_7 T_1 + d_5 T_2 - d_7 T_2 + d_1 T_1 T_2 - d_3 T_1 T_2 - d_5 T_1 T_2 + d_7 T_1 T_2) p_{3,2+i} \pi_{1,i} \pi_{2,i} + \\ & \frac{1}{T_1 T_2} \left(-d_7 - d_3 T_1 + d_7 T_1 - d_5 T_2 + d_7 T_2 + a_2 T_1 T_2 - d_1 T_1 T_2 + d_3 T_1 T_2 + d_5 T_1 T_2 + \right. \\ & \quad d_8 T_1 T_2 + d_3 T_1^2 T_2 + d_4 T_1^2 T_2 - d_7 T_1^2 T_2 - d_8 T_1^2 T_2 + d_5 T_1 T_2^2 + d_6 T_1 T_2^2 - d_7 T_1 T_2^2 - d_8 T_1 T_2^2 + \\ & \quad \left. d_1 T_1^2 T_2^2 + d_2 T_1^2 T_2^2 - d_3 T_1^2 T_2^2 - d_4 T_1^2 T_2^2 - d_5 T_1^2 T_2^2 - d_6 T_1^2 T_2^2 + d_7 T_1^2 T_2^2 + d_8 T_1^2 T_2^2 \right) \\ & p_{3,2+j} \pi_{1,i} \pi_{2,i} + \frac{(d_7 + d_5 T_2 - d_7 T_2) p_{3,2+i} \pi_{1,j} \pi_{2,i}}{T_1 T_2} - \\ & \frac{1}{T_1 T_2} (d_7 + a_2 T_2 + d_5 T_2 - d_7 T_2 - d_7 T_1 T_2 - d_8 T_1 T_2 + a_4 T_2^2 - d_5 T_1 T_2^2 - d_6 T_1 T_2^2 + d_7 T_1 T_2^2 + d_8 T_1 T_2^2) \\ & p_{3,2+j} \pi_{1,j} \pi_{2,i} + \frac{(d_7 + d_3 T_1 - d_7 T_1) p_{3,2+i} \pi_{1,i} \pi_{2,j}}{T_1 T_2} + \\ & \frac{1}{T_1 T_2} \left(-d_7 - d_3 T_1 + d_7 T_1 + a_4 T_1 T_2 + d_7 T_1 T_2 + d_8 T_1 T_2 + d_3 T_1^2 T_2 + d_4 T_1^2 T_2 - d_7 T_1^2 T_2 - d_8 T_1^2 T_2 \right) \\ & p_{3,2+j} \pi_{1,i} \pi_{2,j} + \frac{d_7 p_{3,2+i} \pi_{1,j} \pi_{2,j}}{T_1 T_2} + \frac{(-d_7 + d_7 T_1 T_2 + d_8 T_1 T_2) p_{3,2+j} \pi_{1,j} \pi_{2,j}}{T_1 T_2} \end{aligned} \right.$$

In[*]:= **cvs** = **Union@Cases** [eqn, p__ | $\pi_{,,}$, ∞]

Out[*]=
{p_{3,2+i}, p_{3,2+j}, $\pi_{1,i}$, $\pi_{1,j}$, $\pi_{2,i}$, $\pi_{2,j}$ }

In[*]:= **eqns** = **CoefficientRules** [eqn, cvs] /. ($_ \rightarrow c_$) \Rightarrow (**c** == 0)

Out[*]=

$$\left\{ d_1 - d_3 - d_5 + d_7 + \frac{d_5}{T_1} - \frac{d_7}{T_1} + \frac{d_3}{T_2} - \frac{d_7}{T_2} + \frac{d_7}{T_1 T_2} = 0, \frac{d_3}{T_2} - \frac{d_7}{T_2} + \frac{d_7}{T_1 T_2} = 0, \frac{d_5}{T_1} - \frac{d_7}{T_1} + \frac{d_7}{T_1 T_2} = 0, \frac{d_7}{T_1 T_2} = 0, \right.$$

$$a_2 - d_1 + d_3 + d_5 + d_8 - \frac{d_5}{T_1} + \frac{d_7}{T_1} + d_3 T_1 + d_4 T_1 - d_7 T_1 - d_8 T_1 - \frac{d_3}{T_2} + \frac{d_7}{T_2} - \frac{d_7}{T_1 T_2} + d_5 T_2 + d_6 T_2 - d_7 T_2 -$$

$$d_8 T_2 + d_1 T_1 T_2 + d_2 T_1 T_2 - d_3 T_1 T_2 - d_4 T_1 T_2 - d_5 T_1 T_2 - d_6 T_1 T_2 + d_7 T_1 T_2 + d_8 T_1 T_2 = 0,$$

$$a_4 + d_7 + d_8 + d_3 T_1 + d_4 T_1 - d_7 T_1 - d_8 T_1 - \frac{d_3}{T_2} + \frac{d_7}{T_2} - \frac{d_7}{T_1 T_2} = 0,$$

$$\left. d_7 + d_8 - \frac{a_2}{T_1} - \frac{d_5}{T_1} + \frac{d_7}{T_1} - \frac{d_7}{T_1 T_2} + d_5 T_2 + d_6 T_2 - d_7 T_2 - d_8 T_2 - \frac{a_4 T_2}{T_1} = 0, d_7 + d_8 - \frac{d_7}{T_1 T_2} = 0 \right\}$$

In[*]:= **vars** = **Union@Cases** [r₀[-1, i, j], d_, ∞]

Out[*]=
{d₁, d₂, d₃, d₄, d₅, d₆, d₇, d₈}

In[*]:= **{sol}** = **Solve** [eqns, vars]

Out[*]=

$$\left\{ \left\{ d_1 \rightarrow 0, d_2 \rightarrow -\frac{a_2 - a_4 T_1 + a_4 T_2}{T_1^2 T_2}, d_3 \rightarrow 0, d_4 \rightarrow -\frac{a_4}{T_1}, d_5 \rightarrow 0, d_6 \rightarrow -\frac{-a_2 - a_4 T_2}{T_1 T_2}, d_7 \rightarrow 0, d_8 \rightarrow 0 \right\} \right\}$$

In[*]:= **sol** /. (**v**_ \rightarrow **val**_) \Rightarrow (**v** = **CF** [**val**]);

In[*]:= **r₀[-1, i, j]**

Out[*]=

$$\frac{(-a_2 + a_4 T_1 - a_4 T_2) p_{3,j} x_{1,i} x_{2,i}}{T_1^2 T_2} + \frac{(a_2 + a_4 T_2) p_{3,j} x_{1,j} x_{2,i}}{T_1 T_2} - \frac{a_4 p_{3,j} x_{1,i} x_{2,j}}{T_1}$$

Reidemeister 2b for r₁

In[*]:= **x** = 0;

r₁[-1, i_, j_] := Evaluate [Sum [
 $e_{++x} x_{3,k3} p_{1,k1} p_{2,k2},$
 {k1, {i, j}}, {k2, {i, j}}, {k3, {i, j}}
]];
r₁[-1, i, j]

Out[*]=

$$e_1 p_{1,i} p_{2,i} x_{3,i} + e_5 p_{1,j} p_{2,i} x_{3,i} + e_3 p_{1,i} p_{2,j} x_{3,i} + e_7 p_{1,j} p_{2,j} x_{3,i} +$$

$$e_2 p_{1,i} p_{2,i} x_{3,j} + e_6 p_{1,j} p_{2,i} x_{3,j} + e_4 p_{1,i} p_{2,j} x_{3,j} + e_8 p_{1,j} p_{2,j} x_{3,j}$$

In[*]:= $\mathcal{L}[X_{i,j}[s_-]] := T_3^s \mathbb{E}[q[s, i, j] + e r_1[s, i, j] + 0[e]^2];$
 $\mathcal{L}[X_{i,j}[1]]$
 $\mathcal{L}[X_{i,j}[-1]]$

Out[*]=

$T_1 T_2 \mathbb{E}[\text{Series}[(-p_{1,i} + p_{1,1+i}) x_{1,i} + (-1 + T_1) (p_{1,1+i} - p_{1,1+j}) x_{1,i} + (-p_{1,j} + p_{1,1+j}) x_{1,j} +$
 $(-p_{2,i} + p_{2,1+i}) x_{2,i} + (-1 + T_2) (p_{2,1+i} - p_{2,1+j}) x_{2,i} + (-p_{2,j} + p_{2,1+j}) x_{2,j} +$
 $(-p_{3,i} + p_{3,1+i}) x_{3,i} + (-1 + T_1 T_2) (p_{3,1+i} - p_{3,1+j}) x_{3,i} + (-p_{3,j} + p_{3,1+j}) x_{3,j},$
 $b_5 p_{1,j} p_{2,i} x_{3,i} + b_3 p_{1,i} p_{2,j} x_{3,i} + (-b_3 - b_5) p_{1,j} p_{2,j} x_{3,i}]]$

Out[*]=

$\frac{1}{T_1 T_2} \mathbb{E}[\text{Series}[(-p_{1,i} + p_{1,1+i}) x_{1,i} + (-1 + \frac{1}{T_1}) (p_{1,1+i} - p_{1,1+j}) x_{1,i} + (-p_{1,j} + p_{1,1+j}) x_{1,j} +$
 $(-p_{2,i} + p_{2,1+i}) x_{2,i} + (-1 + \frac{1}{T_2}) (p_{2,1+i} - p_{2,1+j}) x_{2,i} + (-p_{2,j} + p_{2,1+j}) x_{2,j} +$
 $(-p_{3,i} + p_{3,1+i}) x_{3,i} + (-1 + \frac{1}{T_1 T_2}) (p_{3,1+i} - p_{3,1+j}) x_{3,i} + (-p_{3,j} + p_{3,1+j}) x_{3,j},$
 $e_1 p_{1,i} p_{2,i} x_{3,i} + e_5 p_{1,j} p_{2,i} x_{3,i} + e_3 p_{1,i} p_{2,j} x_{3,i} + e_7 p_{1,j} p_{2,j} x_{3,i} +$
 $e_2 p_{1,i} p_{2,i} x_{3,j} + e_6 p_{1,j} p_{2,i} x_{3,j} + e_4 p_{1,i} p_{2,j} x_{3,j} + e_8 p_{1,j} p_{2,j} x_{3,j}]]]$

In[*]:= {eqn} = {lhs} =

Cases[$\int \mathcal{F}[i, j] \times \mathcal{L} / @ (X_{i,j}[1] X_{i^+,j^+}[-1]) \text{d}\{vs_i, vs_j, vs_{i^+}, vs_{j^+}\}, \text{eSeries}[_ , \varepsilon_-] \Rightarrow \varepsilon, \infty]$

Out[*]=

$\left\{ \frac{(e_2 + e_1 T_1 T_2 - e_2 T_1 T_2) p_{1,2+i} p_{2,2+i} \pi_{3,i}}{T_1 T_2} + \right.$
 $\frac{1}{T_1 T_2} (-e_2 + e_2 T_1 + e_6 T_1 + b_5 T_1 T_2 - e_1 T_1 T_2 + e_2 T_1 T_2 + e_1 T_1^2 T_2 - e_2 T_1^2 T_2 + e_5 T_1^2 T_2 - e_6 T_1^2 T_2)$
 $p_{1,2+j} p_{2,2+i} \pi_{3,i} +$
 $\frac{1}{T_1 T_2} (-e_2 + e_2 T_2 + e_4 T_2 + b_3 T_1 T_2 - e_1 T_1 T_2 + e_2 T_1 T_2 + e_1 T_1 T_2^2 - e_2 T_1 T_2^2 + e_3 T_1 T_2^2 - e_4 T_1 T_2^2) p_{1,2+i}$
 $p_{2,2+j} \pi_{3,i} - \frac{1}{T_1 T_2} (-e_2 + e_2 T_1 + e_6 T_1 + e_2 T_2 + e_4 T_2 + b_3 T_1 T_2 + b_5 T_1 T_2 - e_1 T_1 T_2 - e_4 T_1 T_2 -$
 $e_6 T_1 T_2 - e_8 T_1 T_2 + e_1 T_1^2 T_2 - e_2 T_1^2 T_2 + e_5 T_1^2 T_2 - e_6 T_1^2 T_2 + e_1 T_1 T_2^2 - e_2 T_1 T_2^2 + e_3 T_1 T_2^2 - e_4 T_1 T_2^2 -$
 $e_1 T_1^2 T_2^2 + e_2 T_1^2 T_2^2 - e_3 T_1^2 T_2^2 + e_4 T_1^2 T_2^2 - e_5 T_1^2 T_2^2 + e_6 T_1^2 T_2^2 - e_7 T_1^2 T_2^2 + e_8 T_1^2 T_2^2) p_{1,2+j} p_{2,2+j} \pi_{3,i} +$
 $\frac{e_2 p_{1,2+i} p_{2,2+i} \pi_{3,j}}{T_1 T_2} + \frac{(-e_2 + e_2 T_1 + e_6 T_1) p_{1,2+j} p_{2,2+i} \pi_{3,j}}{T_1 T_2} +$
 $\frac{(-e_2 + e_2 T_2 + e_4 T_2) p_{1,2+i} p_{2,2+j} \pi_{3,j}}{T_1 T_2} +$
 $\left. \frac{1}{T_1 T_2} (e_2 - e_2 T_1 - e_6 T_1 - e_2 T_2 - e_4 T_2 + e_2 T_1 T_2 + e_4 T_1 T_2 + e_6 T_1 T_2 + e_8 T_1 T_2) p_{1,2+j} p_{2,2+j} \pi_{3,j} \right\}$

In[*]:= cvs = Union@Cases[eqn, p__ | \pi__, \infty]

Out[*]=

{p_{1,2+i}, p_{1,2+j}, p_{2,2+i}, p_{2,2+j}, \pi_{3,i}, \pi_{3,j}}

In[*]:= eqns = CoefficientRules[eqn, cvs] /. (_ -> c_) :-> (c == 0)

Out[*]=

$$\left\{ \begin{aligned} e_1 - e_2 + \frac{e_2}{T_1 T_2} &= 0, \quad \frac{e_2}{T_1 T_2} = 0, \\ b_3 - e_1 + e_2 + \frac{e_2}{T_1} + \frac{e_4}{T_1} - \frac{e_2}{T_1 T_2} + e_1 T_2 - e_2 T_2 + e_3 T_2 - e_4 T_2 &= 0, \quad \frac{e_2}{T_1} + \frac{e_4}{T_1} - \frac{e_2}{T_1 T_2} = 0, \\ b_5 - e_1 + e_2 + e_1 T_1 - e_2 T_1 + e_5 T_1 - e_6 T_1 + \frac{e_2}{T_2} + \frac{e_6}{T_2} - \frac{e_2}{T_1 T_2} &= 0, \quad \frac{e_2}{T_2} + \frac{e_6}{T_2} - \frac{e_2}{T_1 T_2} = 0, \\ -b_3 - b_5 + e_1 + e_4 + e_6 + e_8 - \frac{e_2}{T_1} - \frac{e_4}{T_1} - e_1 T_1 + e_2 T_1 - e_5 T_1 + e_6 T_1 - \frac{e_2}{T_2} - \frac{e_6}{T_2} + \frac{e_2}{T_1 T_2} - e_1 T_2 + e_2 T_2 - \\ e_3 T_2 + e_4 T_2 + e_1 T_1 T_2 - e_2 T_1 T_2 + e_3 T_1 T_2 - e_4 T_1 T_2 + e_5 T_1 T_2 - e_6 T_1 T_2 + e_7 T_1 T_2 - e_8 T_1 T_2 &= 0, \\ e_2 + e_4 + e_6 + e_8 - \frac{e_2}{T_1} - \frac{e_4}{T_1} - \frac{e_2}{T_2} - \frac{e_6}{T_2} + \frac{e_2}{T_1 T_2} &= 0 \end{aligned} \right\}$$

In[*]:= vars = Union@Cases[r1[-1, i, j], e_, ∞]

Out[*]=

{e1, e2, e3, e4, e5, e6, e7, e8}

In[*]:= {sol} = Solve[eqns, vars]

Out[*]=

$$\left\{ \left\{ e_1 \rightarrow 0, e_2 \rightarrow 0, e_3 \rightarrow -\frac{b_3}{T_2}, e_4 \rightarrow 0, e_5 \rightarrow -\frac{b_5}{T_1}, e_6 \rightarrow 0, e_7 \rightarrow -\frac{-b_3 T_1 - b_5 T_2}{T_1 T_2}, e_8 \rightarrow 0 \right\} \right\}$$

In[*]:= sol /. (v_ -> val_) :-> (v = CF[val]);

In[*]:= r1[-1, i, j]

Out[*]=

$$-\frac{b_5 p_{1,j} p_{2,i} x_{3,i}}{T_1} - \frac{b_3 p_{1,i} p_{2,j} x_{3,i}}{T_2} + \frac{(b_3 T_1 + b_5 T_2) p_{1,j} p_{2,j} x_{3,i}}{T_1 T_2}$$

Reidemeister 2b in full

$$\text{In[*]} := \mathcal{L}[\mathbf{X}_{i,j}[s_-]] := T_3^5 \mathbb{E}[\mathbf{q}[s, i, j] + r_0[s, i, j] + \epsilon r_1[s, i, j] + \mathbf{0}[\epsilon]^2];$$

$$\mathcal{L}[\mathbf{X}_{i,j}[\mathbf{1}]]$$

$$\mathcal{L}[\mathbf{X}_{i,j}[-\mathbf{1}]]$$

Out[*]=

$$T_1 T_2 \mathbb{E} \left[\epsilon \text{Series} \left[\begin{aligned} &(-p_{1,i} + p_{1,1+i}) x_{1,i} + (-1 + T_1) (p_{1,1+i} - p_{1,1+j}) x_{1,i} + (-p_{1,j} + p_{1,1+j}) x_{1,j} + (-p_{2,i} + p_{2,1+i}) x_{2,i} + \\ &(-1 + T_2) (p_{2,1+i} - p_{2,1+j}) x_{2,i} + a_2 p_{3,j} x_{1,i} x_{2,i} - \frac{(a_2 + a_4 T_2) p_{3,j} x_{1,j} x_{2,i}}{T_1} + (-p_{2,j} + p_{2,1+j}) x_{2,j} + \\ &a_4 p_{3,j} x_{1,i} x_{2,j} + (-p_{3,i} + p_{3,1+i}) x_{3,i} + (-1 + T_1 T_2) (p_{3,1+i} - p_{3,1+j}) x_{3,i} + (-p_{3,j} + p_{3,1+j}) x_{3,j}, \\ &b_5 p_{1,j} p_{2,i} x_{3,i} + b_3 p_{1,i} p_{2,j} x_{3,i} + (-b_3 - b_5) p_{1,j} p_{2,j} x_{3,i} \end{aligned} \right] \right]$$

Out[*]=

$$\frac{1}{T_1 T_2} \mathbb{E} \left[\epsilon \text{Series} \left[\begin{aligned} &(-p_{1,i} + p_{1,1+i}) x_{1,i} + \left(-1 + \frac{1}{T_1}\right) (p_{1,1+i} - p_{1,1+j}) x_{1,i} + \\ &(-p_{1,j} + p_{1,1+j}) x_{1,j} + (-p_{2,i} + p_{2,1+i}) x_{2,i} + \left(-1 + \frac{1}{T_2}\right) (p_{2,1+i} - p_{2,1+j}) x_{2,i} + \\ &\frac{(-a_2 + a_4 T_1 - a_4 T_2) p_{3,j} x_{1,i} x_{2,i}}{T_1^2 T_2} + \frac{(a_2 + a_4 T_2) p_{3,j} x_{1,j} x_{2,i}}{T_1 T_2} + (-p_{2,j} + p_{2,1+j}) x_{2,j} - \\ &\frac{a_4 p_{3,j} x_{1,i} x_{2,j}}{T_1} + (-p_{3,i} + p_{3,1+i}) x_{3,i} + \left(-1 + \frac{1}{T_1 T_2}\right) (p_{3,1+i} - p_{3,1+j}) x_{3,i} + (-p_{3,j} + p_{3,1+j}) x_{3,j}, \\ &\frac{(-b_5 T_2 p_{1,j} p_{2,i} - b_3 T_1 p_{1,i} p_{2,j} + b_3 T_1 p_{1,j} p_{2,j} + b_5 T_2 p_{1,j} p_{2,j}) x_{3,i}}{T_1 T_2} \end{aligned} \right] \right]$$

$$\text{In[*]} := \int \mathcal{F}[\mathbf{i}, \mathbf{j}] \times \mathcal{L} / @ (\mathbf{X}_{i,j}[\mathbf{1}] \mathbf{X}_{i^+,j^+}[-\mathbf{1}]) \, d\{\mathbf{vs}_i, \mathbf{vs}_j, \mathbf{vs}_{i^+}, \mathbf{vs}_{j^+}\}$$

Out[*]=

$$\mathbb{E} \left[\epsilon \text{Series} \left[\begin{aligned} &p_{1,2+i} \pi_{1,i} + p_{1,2+j} \pi_{1,j} + p_{2,2+i} \pi_{2,i} + p_{2,2+j} \pi_{2,j} + p_{3,2+i} \pi_{3,i} + p_{3,2+j} \pi_{3,j}, \\ &\frac{(-a_2 b_5 + a_4 b_3 T_1 - a_4 b_5 T_2) p_{3,2+j} \pi_{3,i}}{T_1} + a_4 b_3 p_{1,2+i} p_{3,2+j} \pi_{1,i} \pi_{3,i} + (-a_4 b_3 + a_2 b_5 - a_4 b_5) p_{1,2+j} \\ &p_{3,2+j} \pi_{1,i} \pi_{3,i} - \frac{b_5 (a_2 + a_4 T_2) p_{1,2+j} p_{3,2+j} \pi_{1,j} \pi_{3,i}}{T_1} - \frac{b_5 (a_2 + a_4 T_2) p_{2,2+i} p_{3,2+j} \pi_{2,i} \pi_{3,i}}{T_1} + \\ &\frac{(a_2 b_3 + a_2 b_5 + a_2 b_3 T_1 + a_4 b_3 T_2 + a_4 b_5 T_2) p_{2,2+j} p_{3,2+j} \pi_{2,i} \pi_{3,i}}{T_1} + \\ &\frac{1}{T_1} (-a_2 a_4 b_3 + a_2^2 b_5 - a_2 a_4 b_5 - a_2 a_4 b_3 T_1 - a_4^2 b_3 T_2 + a_2 a_4 b_5 T_2 - a_4^2 b_5 T_2) p_{3,2+j}^2 \pi_{1,i} \pi_{2,i} \pi_{3,i} - \\ &\frac{b_5 (a_2 + a_4 T_2)^2 p_{3,2+j}^2 \pi_{1,j} \pi_{2,i} \pi_{3,i}}{T_1^2} + a_4 b_3 p_{2,2+j} p_{3,2+j} \pi_{2,j} \pi_{3,i} - a_4^2 b_3 p_{3,2+j}^2 \pi_{1,i} \pi_{2,j} \pi_{3,i} \end{aligned} \right] \right]$$

`In[*]:= Echo /@ (CF@CoefficientList [Cases [∫ ℱ[i, j] × ℒ /@ (Xi,j[1] Xi+,j+[-1]) d{vsi, vsj, vsi+, vsj+},
eSeries[_ , ε-] ⇒ ε , ∞] [[1]] /. {ai- ⇒ λ ai, bi- ⇒ λ bi}, λ]);`

» 0

» 0

$$\begin{aligned} & \gg \frac{(-a_2 b_5 + a_4 b_3 T_1 - a_4 b_5 T_2) p_{3,2+j} \pi_{3,i}}{T_1} + a_4 b_3 p_{1,2+i} p_{3,2+j} \pi_{1,i} \pi_{3,i} + (-a_4 b_3 + a_2 b_5 - a_4 b_5) p_{1,2+j} p_{3,2+j} \pi_{1,i} \pi_{3,i} - \\ & \frac{b_5 (a_2 + a_4 T_2) p_{1,2+j} p_{3,2+j} \pi_{1,j} \pi_{3,i}}{T_1} - \frac{b_5 (a_2 + a_4 T_2) p_{2,2+i} p_{3,2+j} \pi_{2,i} \pi_{3,i}}{T_1} + \\ & \frac{(a_2 b_3 + a_2 b_5 + a_2 b_3 T_1 + a_4 b_3 T_2 + a_4 b_5 T_2) p_{2,2+j} p_{3,2+j} \pi_{2,i} \pi_{3,i}}{T_1} + a_4 b_3 p_{2,2+j} p_{3,2+j} \pi_{2,j} \pi_{3,i} \\ & \gg \frac{1}{T_1} (-a_2 a_4 b_3 + a_2^2 b_5 - a_2 a_4 b_5 - a_2 a_4 b_3 T_1 - a_4^2 b_3 T_2 + a_2 a_4 b_5 T_2 - a_4^2 b_5 T_2) p_{3,2+j}^2 \pi_{1,i} \pi_{2,i} \pi_{3,i} - \\ & \frac{b_5 (a_2 + a_4 T_2)^2 p_{3,2+j}^2 \pi_{1,j} \pi_{2,i} \pi_{3,i}}{T_1^2} - a_4^2 b_3 p_{3,2+j}^2 \pi_{1,i} \pi_{2,j} \pi_{3,i} \end{aligned}$$

`In[*]:= err = CF@Coefficient [Cases [∫ ℱ[i, j] × ℒ /@ (Xi,j[1] Xi+,j+[-1]) d{vsi, vsj, vsi+, vsj+}, eSeries[_ , ε-] ⇒ ε , ∞] [[1]] /. {ai- ⇒ λ ai, bi- ⇒ λ bi}, λ2]`

`Out[*]=`

$$\begin{aligned} & \frac{(-a_2 b_5 + a_4 b_3 T_1 - a_4 b_5 T_2) p_{3,2+j} \pi_{3,i}}{T_1} + \\ & a_4 b_3 p_{1,2+i} p_{3,2+j} \pi_{1,i} \pi_{3,i} + (-a_4 b_3 + a_2 b_5 - a_4 b_5) p_{1,2+j} p_{3,2+j} \pi_{1,i} \pi_{3,i} - \\ & \frac{b_5 (a_2 + a_4 T_2) p_{1,2+j} p_{3,2+j} \pi_{1,j} \pi_{3,i}}{T_1} - \frac{b_5 (a_2 + a_4 T_2) p_{2,2+i} p_{3,2+j} \pi_{2,i} \pi_{3,i}}{T_1} + \\ & \frac{(a_2 b_3 + a_2 b_5 + a_2 b_3 T_1 + a_4 b_3 T_2 + a_4 b_5 T_2) p_{2,2+j} p_{3,2+j} \pi_{2,i} \pi_{3,i}}{T_1} + a_4 b_3 p_{2,2+j} p_{3,2+j} \pi_{2,j} \pi_{3,i} \end{aligned}$$

```
In[*]:= x = 0;
r42[-1, i_, j_] = Evaluate[Plus[
  Sum[
    f_{++x} x_{v1,k1} p_{v1,k2} x_{v2,k3} p_{v2,k4},
    {k1, {i, j}}, {k2, {i, j}}, {k3, {i, j}}, {k4, {i, j}}, {v1, 2}, {v2, v1 + 1, 3}
  ],
  Sum[
    f_{++x} x_{v,k1} p_{v,k2},
    {k1, {i, j}}, {k2, {i, j}}, {v, 3}
  ]
]]
```

```
Out[*]=
f49 p1,i x1,i + f52 p1,j x1,i + f55 p1,i x1,j + f58 p1,j x1,j + f50 p2,i x2,i + f53 p2,j x2,i +
f1 p1,i p2,i x1,i x2,i + f13 p1,j p2,i x1,i x2,i + f4 p1,i p2,j x1,i x2,i + f16 p1,j p2,j x1,i x2,i +
f25 p1,i p2,i x1,j x2,i + f37 p1,j p2,i x1,j x2,i + f28 p1,i p2,j x1,j x2,i + f40 p1,j p2,j x1,j x2,i +
f56 p2,i x2,j + f59 p2,j x2,j + f7 p1,i p2,i x1,i x2,j + f19 p1,j p2,i x1,i x2,j + f10 p1,i p2,j x1,i x2,j +
f22 p1,j p2,j x1,i x2,j + f31 p1,i p2,i x1,j x2,j + f43 p1,j p2,i x1,j x2,j + f34 p1,i p2,j x1,j x2,j +
f46 p1,j p2,j x1,j x2,j + f51 p3,i x3,i + f54 p3,j x3,i + f2 p1,i p3,i x1,i x3,i + f14 p1,j p3,i x1,i x3,i +
f5 p1,i p3,j x1,i x3,i + f17 p1,j p3,j x1,i x3,i + f26 p1,i p3,i x1,j x3,i + f38 p1,j p3,i x1,j x3,i +
f29 p1,i p3,j x1,j x3,i + f41 p1,j p3,j x1,j x3,i + f3 p2,i p3,i x2,i x3,i + f15 p2,j p3,i x2,i x3,i +
f6 p2,i p3,j x2,i x3,i + f18 p2,j p3,j x2,i x3,i + f27 p2,i p3,i x2,j x3,i + f39 p2,j p3,i x2,j x3,i +
f30 p2,i p3,j x2,j x3,i + f42 p2,j p3,j x2,j x3,i + f57 p3,i x3,j + f60 p3,j x3,j +
f8 p1,i p3,i x1,i x3,j + f20 p1,j p3,i x1,i x3,j + f11 p1,i p3,j x1,i x3,j + f23 p1,j p3,j x1,i x3,j +
f32 p1,i p3,i x1,j x3,j + f44 p1,j p3,i x1,j x3,j + f35 p1,i p3,j x1,j x3,j + f47 p1,j p3,j x1,j x3,j +
f9 p2,i p3,i x2,i x3,j + f21 p2,j p3,i x2,i x3,j + f12 p2,i p3,j x2,i x3,j + f24 p2,j p3,j x2,i x3,j +
f33 p2,i p3,i x2,j x3,j + f45 p2,j p3,i x2,j x3,j + f36 p2,i p3,j x2,j x3,j + f48 p2,j p3,j x2,j x3,j
```

```
In[*]:= L[X_{i_,j}[s_]] := T3^s E[q[s, i, j] + e r42[s, i, j] + O[e]^2];
L[X_{i,j}[1]]
L[X_{i,j}[-1]]
```

```
Out[*]=
T1 T2 E[Series[(-p1,i + p1,1+i) x1,i + (-1 + T1) (p1,1+i - p1,1+j) x1,i + (-p1,j + p1,1+j) x1,j +
(-p2,i + p2,1+i) x2,i + (-1 + T2) (p2,1+i - p2,1+j) x2,i + (-p2,j + p2,1+j) x2,j +
(-p3,i + p3,1+i) x3,i + (-1 + T1 T2) (p3,1+i - p3,1+j) x3,i + (-p3,j + p3,1+j) x3,j,
C49 p1,i x1,i + C52 p1,j x1,i - (C49 + C52) p1,j x1,j / T1 + C50 p2,i x2,i + C53 p2,j x2,i + C13 p1,j p2,i x1,i x2,i +
C4 p1,i p2,j x1,i x2,i + 1 / (-1 + T1 T2) (-a2 b3 + c4 + c13 - c4 T1 - a4 b3 T2 - a2 b5 T2 + a4 b5 T2 - c13 T2 +
a4 b3 T1 T2 - c4 T1 T2 - c13 T1 T2 + c4 T1^2 T2 - a4 b5 T2^2 + c13 T1 T2^2) p1,j p2,j x1,i x2,i -
((-a2 b3 + a2 b5 + a2 b3 T1 - c4 T1 + c13 T1 + c4 T1^2 - a4 b3 T2 - a2 b5 T2 + a4 b5 T2 + 2 a4 b3 T1 T2 - a4 b5
T1 T2 - c13 T1 T2 - a4 b3 T1^2 T2 + c4 T1^2 T2 - c13 T1^2 T2 - c4 T1^3 T2 - a4 b5 T2^2 + a4 b5 T1 T2^2 + c13 T1^2 T2^2)
p1,j p2,i x1,j x2,i) / ((-1 + T1) T1 (-1 + T2) (-1 + T1 T2)) -
(-a2 b5 - c13 T1 - a4 b5 T2 + c13 T1 T2) p1,j p2,j x1,j x2,i / (-1 + T1) T1 - (C50 + C53) p2,j x2,j / T2
```

$$\begin{aligned}
 & \left((-a_2 b_3 + a_4 b_3 + a_2 b_5 - a_4 b_5 - c_4 + c_{13} + a_2 b_3 T_1 - a_4 b_3 T_1 + c_4 T_1 - a_4 b_3 T_2 - a_2 b_5 T_2 + 2 a_4 b_5 T_2 - \right. \\
 & \quad \left. c_{13} T_2 + a_4 b_3 T_1 T_2 + c_4 T_1 T_2 - c_{13} T_1 T_2 - c_4 T_1^2 T_2 - a_4 b_5 T_2^2 + c_{13} T_1 T_2^2) p_{1,i} p_{2,j} x_{1,i} x_{2,j} \right) / \\
 & \quad \left((-1 + T_1) (-1 + T_2) (-1 + T_1 T_2) \right) - \frac{(a_4 b_3 - c_4 + c_4 T_1) p_{1,j} p_{2,j} x_{1,i} x_{2,j}}{-1 + T_2} + \\
 & c_{51} p_{3,i} x_{3,i} + c_{54} p_{3,j} x_{3,i} + c_{14} p_{1,j} p_{3,i} x_{1,i} x_{3,i} + c_5 p_{1,i} p_{3,j} x_{1,i} x_{3,i} + \\
 & \frac{1}{-1 + T_2} \left(a_2 b_3 - a_2 b_5 + a_4 b_5 + c_5 + c_{14} - a_2 b_3 T_1 + a_4 b_3 T_1 - c_5 T_1 + a_4 b_3 T_2 + a_2 b_5 T_2 - 2 a_4 b_5 T_2 - \right. \\
 & \quad \left. c_5 T_2 - c_{14} T_2 - 2 a_4 b_3 T_1 T_2 + c_5 T_1 T_2 - c_{14} T_1 T_2 + a_4 b_5 T_2^2 + c_{14} T_1 T_2^2 \right) p_{1,j} p_{3,j} x_{1,i} x_{3,i} - \\
 & \left((a_2 b_3 - a_2 b_5 - 2 a_2 b_3 T_1 + a_4 b_3 T_1 - c_5 T_1 + c_{14} T_1 + a_2 b_3 T_1^2 - a_4 b_3 T_1^2 + c_5 T_1^2 + a_4 b_3 T_2 + \right. \\
 & \quad \left. a_2 b_5 T_2 - a_4 b_5 T_2 - 3 a_4 b_3 T_1 T_2 + a_4 b_5 T_1 T_2 + c_5 T_1 T_2 - c_{14} T_1 T_2 + 2 a_4 b_3 T_1^2 T_2 - \right. \\
 & \quad \left. c_5 T_1^2 T_2 - c_{14} T_1^2 T_2 + a_4 b_5 T_2^2 - a_4 b_5 T_1 T_2^2 + c_{14} T_1^2 T_2^2) p_{1,j} p_{3,i} x_{1,j} x_{3,i} \right) / \\
 & \quad \left((-1 + T_1) T_1 (-1 + T_2) (-1 + T_1 T_2) \right) - \frac{b_3 (a_2 + a_4 T_2) p_{1,i} p_{3,j} x_{1,j} x_{3,i}}{T_1 (-1 + T_2)} - \\
 & \left((a_2 b_3 - a_2 b_3 T_1 - a_2 b_5 T_1 + c_{14} T_1 + a_4 b_3 T_2 - a_4 b_3 T_1 T_2 + a_2 b_5 T_1 T_2 - a_4 b_5 T_1 T_2 - \right. \\
 & \quad \left. c_{14} T_1 T_2 - c_{14} T_1^2 T_2 + a_4 b_5 T_1 T_2^2 + c_{14} T_1^2 T_2^2) p_{1,j} p_{3,j} x_{1,j} x_{3,i} \right) / \\
 & \quad \left((-1 + T_1) T_1 (-1 + T_2) \right) + c_{15} p_{2,j} p_{3,i} x_{2,i} x_{3,i} + c_6 p_{2,i} p_{3,j} x_{2,i} x_{3,i} + \\
 & \frac{1}{(-1 + T_1) T_1} \left(-a_2 b_3 + a_2 b_3 T_1 + c_6 T_1 + c_{15} T_1 - c_6 T_1^2 - c_{15} T_1^2 - a_4 b_3 T_2 - \right. \\
 & \quad \left. a_2 b_5 T_2 + 2 a_4 b_3 T_1 T_2 + a_2 b_5 T_1 T_2 - a_4 b_5 T_1 T_2 - c_6 T_1 T_2 - a_4 b_3 T_1^2 T_2 + \right. \\
 & \quad \left. c_6 T_1^2 T_2 - c_{15} T_1^2 T_2 + c_{15} T_1^3 T_2 - a_4 b_5 T_2^2 + 2 a_4 b_5 T_1 T_2^2 \right) p_{2,j} p_{3,j} x_{2,i} x_{3,i} - \\
 & \left((-a_2 b_5 - a_2 b_3 T_1 + a_4 b_3 T_1 + a_2 b_5 T_1 - a_4 b_5 T_1 - c_6 T_1 + c_{15} T_1 + a_2 b_3 T_1^2 - a_4 b_3 T_1^2 + \right. \\
 & \quad \left. c_6 T_1^2 - c_{15} T_1^2 + a_2 b_5 T_2 - a_4 b_5 T_2 - a_4 b_3 T_1 T_2 - a_2 b_5 T_1 T_2 + 3 a_4 b_5 T_1 T_2 + c_6 T_1 T_2 + \right. \\
 & \quad \left. a_4 b_3 T_1^2 T_2 - c_6 T_1^2 T_2 - c_{15} T_1^2 T_2 + c_{15} T_1^3 T_2 + a_4 b_5 T_2^2 - 2 a_4 b_5 T_1 T_2^2) p_{2,j} p_{3,i} x_{2,j} x_{3,i} \right) / \\
 & \quad \left((-1 + T_1) T_1 (-1 + T_2) (-1 + T_1 T_2) \right) + \frac{a_4 b_5 p_{2,i} p_{3,j} x_{2,j} x_{3,i}}{-1 + T_1} - \frac{1}{(-1 + T_1) (-1 + T_2)} \\
 & \left(-a_4 b_5 + c_{15} - c_{15} T_1 + a_4 b_3 T_2 + a_4 b_5 T_2 - a_4 b_3 T_1 T_2 - c_{15} T_1 T_2 + c_{15} T_1^2 T_2 \right) p_{2,j} p_{3,j} x_{2,j} x_{3,i} - \\
 & \frac{1}{T_1^2 T_2 (-1 + T_1 T_2)} \left(a_2 b_5 + a_2 b_3 T_1 - a_4 b_3 T_1 - c_{51} T_1 - c_{54} T_1 + a_4 b_5 T_2 + \right. \\
 & \quad \left. a_4 b_3 T_1 T_2 - a_4 b_5 T_1 T_2 + c_{51} T_1^2 T_2 + c_{54} T_1^2 T_2 \right) p_{3,j} x_{3,j} + \\
 & \frac{(a_2 b_5 - a_4 b_5 + c_5 - c_{14} - c_5 T_1 + a_4 b_5 T_2 + c_{14} T_1 T_2) p_{1,i} p_{3,j} x_{1,i} x_{3,j}}{(-1 + T_1) (-1 + T_1 T_2)} - \\
 & \left((a_2 b_3 + c_5 - a_2 b_3 T_1 + a_4 b_3 T_1 - c_5 T_1 + a_4 b_3 T_2 - c_5 T_2 - 2 a_4 b_3 T_1 T_2 + c_5 T_1 T_2) p_{1,j} p_{3,j} x_{1,i} x_{3,j} \right) / \\
 & \quad \left((-1 + T_2) (-1 + T_1 T_2) \right) + \\
 & \left((a_2 b_3 + c_6 T_1 - c_{15} T_1 + a_4 b_3 T_2 - a_4 b_3 T_1 T_2 - c_6 T_1 T_2 + c_{15} T_1^2 T_2) p_{2,i} p_{3,j} x_{2,i} x_{3,j} \right) / \\
 & \quad \left(T_1 (-1 + T_2) (-1 + T_1 T_2) \right) - \\
 & \left((c_6 T_1 - c_6 T_1^2 - a_2 b_5 T_2 + a_2 b_5 T_1 T_2 - a_4 b_5 T_1 T_2 - c_6 T_1 T_2 + c_6 T_1^2 T_2 - a_4 b_5 T_2^2 + 2 a_4 b_5 T_1 T_2^2) \right. \\
 & \quad \left. p_{2,j} p_{3,j} x_{2,i} x_{3,j} \right) / \left((-1 + T_1) T_1 (-1 + T_1 T_2) \right) \Big]
 \end{aligned}$$

Out[*]=

$$\frac{1}{T_1 T_2} \mathbb{E} \left[\text{Series} \left[\left(-p_{1,i} + p_{1,1+i} \right) x_{1,i} + \left(-1 + \frac{1}{T_1} \right) \left(p_{1,1+i} - p_{1,1+j} \right) x_{1,i} + \left(-p_{1,j} + p_{1,1+j} \right) x_{1,j} + \right. \right. \\ \left. \left(-p_{2,i} + p_{2,1+i} \right) x_{2,i} + \left(-1 + \frac{1}{T_2} \right) \left(p_{2,1+i} - p_{2,1+j} \right) x_{2,i} + \left(-p_{2,j} + p_{2,1+j} \right) x_{2,j} + \right. \\ \left. \left(-p_{3,i} + p_{3,1+i} \right) x_{3,i} + \left(-1 + \frac{1}{T_1 T_2} \right) \left(p_{3,1+i} - p_{3,1+j} \right) x_{3,i} + \left(-p_{3,j} + p_{3,1+j} \right) x_{3,j}, \right. \\ f_{49} p_{1,i} x_{1,i} + f_{52} p_{1,j} x_{1,i} + f_{55} p_{1,i} x_{1,j} + f_{58} p_{1,j} x_{1,j} + f_{50} p_{2,i} x_{2,i} + f_{53} p_{2,j} x_{2,i} + \\ f_1 p_{1,i} p_{2,i} x_{1,i} x_{2,i} + f_{13} p_{1,j} p_{2,i} x_{1,i} x_{2,i} + f_4 p_{1,i} p_{2,j} x_{1,i} x_{2,i} + f_{16} p_{1,j} p_{2,j} x_{1,i} x_{2,i} + \\ f_{25} p_{1,i} p_{2,i} x_{1,j} x_{2,i} + f_{37} p_{1,j} p_{2,i} x_{1,j} x_{2,i} + f_{28} p_{1,i} p_{2,j} x_{1,j} x_{2,i} + f_{40} p_{1,j} p_{2,j} x_{1,j} x_{2,i} + \\ f_{56} p_{2,i} x_{2,j} + f_{59} p_{2,j} x_{2,j} + f_7 p_{1,i} p_{2,i} x_{1,i} x_{2,j} + f_{19} p_{1,j} p_{2,i} x_{1,i} x_{2,j} + f_{10} p_{1,i} p_{2,j} x_{1,i} x_{2,j} + \\ f_{22} p_{1,j} p_{2,j} x_{1,i} x_{2,j} + f_{31} p_{1,i} p_{2,i} x_{1,j} x_{2,j} + f_{43} p_{1,j} p_{2,i} x_{1,j} x_{2,j} + f_{34} p_{1,i} p_{2,j} x_{1,j} x_{2,j} + \\ f_{46} p_{1,j} p_{2,j} x_{1,j} x_{2,j} + f_{51} p_{3,i} x_{3,i} + f_{54} p_{3,j} x_{3,i} + f_2 p_{1,i} p_{3,i} x_{1,i} x_{3,i} + f_{14} p_{1,j} p_{3,i} x_{1,i} x_{3,i} + \\ f_5 p_{1,i} p_{3,j} x_{1,i} x_{3,i} + f_{17} p_{1,j} p_{3,j} x_{1,i} x_{3,i} + f_{26} p_{1,i} p_{3,i} x_{1,j} x_{3,i} + f_{38} p_{1,j} p_{3,i} x_{1,j} x_{3,i} + \\ f_{29} p_{1,i} p_{3,j} x_{1,j} x_{3,i} + f_{41} p_{1,j} p_{3,j} x_{1,j} x_{3,i} + f_3 p_{2,i} p_{3,i} x_{2,i} x_{3,i} + f_{15} p_{2,j} p_{3,i} x_{2,i} x_{3,i} + \\ f_6 p_{2,i} p_{3,j} x_{2,i} x_{3,i} + f_{18} p_{2,j} p_{3,j} x_{2,i} x_{3,i} + f_{27} p_{2,i} p_{3,i} x_{2,j} x_{3,i} + f_{39} p_{2,j} p_{3,i} x_{2,j} x_{3,i} + \\ f_{30} p_{2,i} p_{3,j} x_{2,j} x_{3,i} + f_{42} p_{2,j} p_{3,j} x_{2,j} x_{3,i} + f_{57} p_{3,i} x_{3,j} + f_{60} p_{3,j} x_{3,j} + \\ f_8 p_{1,i} p_{3,i} x_{1,i} x_{3,j} + f_{20} p_{1,j} p_{3,i} x_{1,i} x_{3,j} + f_{11} p_{1,i} p_{3,j} x_{1,i} x_{3,j} + f_{23} p_{1,j} p_{3,j} x_{1,i} x_{3,j} + \\ f_{32} p_{1,i} p_{3,i} x_{1,j} x_{3,j} + f_{44} p_{1,j} p_{3,i} x_{1,j} x_{3,j} + f_{35} p_{1,i} p_{3,j} x_{1,j} x_{3,j} + f_{47} p_{1,j} p_{3,j} x_{1,j} x_{3,j} + \\ f_9 p_{2,i} p_{3,i} x_{2,i} x_{3,j} + f_{21} p_{2,j} p_{3,i} x_{2,i} x_{3,j} + f_{12} p_{2,i} p_{3,j} x_{2,i} x_{3,j} + f_{24} p_{2,j} p_{3,j} x_{2,i} x_{3,j} + \\ f_{33} p_{2,i} p_{3,i} x_{2,j} x_{3,j} + f_{45} p_{2,j} p_{3,i} x_{2,j} x_{3,j} + f_{36} p_{2,i} p_{3,j} x_{2,j} x_{3,j} + f_{48} p_{2,j} p_{3,j} x_{2,j} x_{3,j} \left. \right]$$

In[*]:= **lhs** =

Cases [$\int \mathcal{F}[i, j] \times \mathcal{L} / @ (X_{i,j}[1] X_{i^+,j^+}[-1]) \text{d} \{ \mathbf{vs}_i, \mathbf{vs}_j, \mathbf{vs}_{i^+}, \mathbf{vs}_{j^+} \}, \text{eSeries}[_ , \mathcal{E}_-] \Rightarrow \mathcal{E}, \infty$]

Out[*]=

$$\left\{ \frac{a_2 b_5 + a_2 b_3 T_1 - a_4 b_3 T_1 - a_2 b_5 T_1 - c_{51} T_1 - c_{54} T_1 - a_2 b_3 T_1^2 + a_4 b_3 T_1^2 - c_{50} T_1^2 + \dots 151 \dots + f_{47} T_1^2 T_2^2 + f_{48} T_1^2 T_2^2 + f_{49} T_1^2 T_2^2 + f_{50} T_1^2 T_2^2 + f_{51} T_1^2 T_2^2 + f_{58} T_1^2 T_2^2 + f_{59} T_1^2 T_2^2 + f_{60} T_1^2 T_2^2}{(-1+T_1) T_1^2 (-1+T_2) T_2} \right. \\ \left. + \frac{\dots 163 \dots + f_{49} T_1^2 T_2^2 - f_{55} T_1^2 T_2^2}{(-1+T_1) T_1 (-1+T_2) (-1+T_1 T_2)} p_{1,i} \pi_{1,i} + \frac{\dots 1 \dots}{\dots 1 \dots} + \frac{\dots 1 \dots}{\dots 1 \dots} + \frac{\dots 70 \dots}{\dots 1 \dots} + \frac{\dots 1 \dots}{T_1 T_2^2} + \frac{\dots 1 \dots}{T_1 T_2^2} p_{2,2+j} p_{3,2+j} \pi_{2,j} \pi_{3,j} \right\}$$

Full expression not available (original memory size: 1 MB) ⚙️

In[*]:= **eqn** = **CF** [**lhs** - **err**]

Out[*]=

$$\frac{-a_2 b_5 - a_2 b_3 T_1 + a_4 b_3 T_1 + a_2 b_5 T_1 + c_{51} T_1 + c_{54} T_1 + a_2 b_3 T_1^2 - a_4 b_3 T_1^2 + \dots 163 \dots}{(-1+T_1) T_1^2 (-1+T_2) T_2} + \frac{(\dots 163 \dots + f_{49} T_1^2 T_2^2 - f_{55} T_1^2 T_2^2) p_{1,i} \pi_{1,i}}{(-1+T_1) T_1 (-1+T_2) (-1+T_1 T_2)} + \frac{\dots 1 \dots}{\dots 1 \dots} + \dots 75 \dots + \\ \frac{\dots 1 \dots}{T_1 T_2^2} + \frac{(\dots 1 \dots) \pi_{3,i}}{T_1 T_2^2} + \frac{(\dots 3 \dots) \pi_{1,i}}{T_1 T_2^2} + \frac{(f_{33} - f_{33} T_2 - f_{45} T_2 - f_{33} T_1 T_2 - f_{36} T_1 T_2 + f_{33} T_1 T_2^2 + f_{36} T_1 T_2^2 + f_{45} T_1 T_2^2 + f_{48} T_1 T_2^2) p_{2,i} p_{3,i} \pi_{2,j} \pi_{3,j}}{T_1 T_2^2}$$

Full expression not available (original memory size: 1 MB) ⚙️

In[*]:= **cvs** = **Union**@**Cases** [**eqn**, **p**__ | π __ , ∞]

Out[*]=

{ $p_{1,2+i}, p_{1,2+j}, p_{2,2+i}, p_{2,2+j}, p_{3,2+i}, p_{3,2+j}, \pi_{1,i}, \pi_{1,j}, \pi_{2,i}, \pi_{2,j}, \pi_{3,i}, \pi_{3,j}$ }

In[*]:= eqns = CoefficientRules[eqn, cvs] /. (_ -> c_) :-> (c == 0)

Out[*]=

$$\left\{ \begin{aligned} & f_1 - f_7 - f_{25} + f_{31} + \frac{f_{25}}{T_1} - \frac{f_{31}}{T_1} + \frac{f_7}{T_2} - \frac{f_{31}}{T_2} + \frac{f_{31}}{T_1 T_2} = 0, \frac{f_7}{T_2} - \frac{f_{31}}{T_2} + \frac{f_{31}}{T_1 T_2} = 0, \dots 1 \dots = 0, \dots 55 \dots, \\ & \dots 1 \dots = 0, \dots 1 \dots = 0, -\frac{2 a_4 b_3}{(1-T_1)(1-T_2)} + \frac{2 c_{49}}{(1-T_1)(1-T_2)} + \frac{2 c_{50}}{(1-T_1)(1-T_2)} + \frac{c_{52}}{(1-T_1)(1-T_2)} + \frac{c_{53}}{(1-T_1)(1-T_2)} - \frac{c_{54}}{(1-T_1)(1-T_2)} + \\ & \frac{f_1}{(1-T_1)(1-T_2)} + \dots 159 \dots + \frac{f_{49} T_1 T_2}{(1-T_1)(1-T_2)} + \frac{f_{50} T_1 T_2}{(1-T_1)(1-T_2)} + \frac{f_{51} T_1 T_2}{(1-T_1)(1-T_2)} + \frac{f_{58} T_1 T_2}{(1-T_1)(1-T_2)} + \frac{f_{59} T_1 T_2}{(1-T_1)(1-T_2)} + \frac{f_{60} T_1 T_2}{(1-T_1)(1-T_2)} = 0 \end{aligned} \right\}$$

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In[*]:= vars = Union@Cases[r42[-1, i, j], f_, ∞]

Out[*]=

- {f₁, f₂, f₃, f₄, f₅, f₆, f₇, f₈, f₉, f₁₀, f₁₁, f₁₂, f₁₃, f₁₄, f₁₅, f₁₆, f₁₇, f₁₈, f₁₉, f₂₀, f₂₁, f₂₂, f₂₃, f₂₄, f₂₅, f₂₆, f₂₇, f₂₈, f₂₉, f₃₀, f₃₁, f₃₂, f₃₃, f₃₄, f₃₅, f₃₆, f₃₇, f₃₈, f₃₉, f₄₀, f₄₁, f₄₂, f₄₃, f₄₄, f₄₅, f₄₆, f₄₇, f₄₈, f₄₉, f₅₀, f₅₁, f₅₂, f₅₃, f₅₄, f₅₅, f₅₆, f₅₇, f₅₈, f₅₉, f₆₀}

In[*]:= {sol} = Solve[eqns, vars]

Out[*]=

$$\left\{ \begin{aligned} & f_1 \rightarrow 0, f_2 \rightarrow 0, f_3 \rightarrow 0, \\ & f_4 \rightarrow -\left(\frac{(-a_2 b_3 + a_4 b_3 + a_2 b_5 - a_4 b_5 + c_{13} + a_2 b_3 T_1 - a_4 b_3 T_1 - a_4 b_3 T_2 - a_2 b_5 T_2 + 2 a_4 b_5 T_2 - c_{13} T_2 + a_4 b_3 T_1 T_2 - c_{13} T_1 T_2 - a_4 b_5 T_2^2 + c_{13} T_1 T_2^2)}{((-1 + T_1) T_2 (-1 + T_1 T_2))} \right), \\ & f_5 \rightarrow -\left(\frac{(-a_2 b_3 + 2 a_2 b_3 T_1 - a_4 b_3 T_1 - a_2 b_5 T_1 + a_4 b_5 T_1 + c_{14} T_1 - a_2 b_3 T_1^2 + a_4 b_3 T_1^2 - a_4 b_3 T_2 + 3 a_4 b_3 T_1 T_2 + a_2 b_5 T_1 T_2 - 2 a_4 b_5 T_1 T_2 - c_{14} T_1 T_2 - 2 a_4 b_3 T_1^2 T_2 - c_{14} T_1^2 T_2 + a_4 b_5 T_1 T_2^2 + c_{14} T_1^2 T_2^2)}{((-1 + T_1) T_1^2 (-1 + T_2) T_2)} \right), \\ & f_6 \rightarrow -\left(\frac{(-a_2 b_3 + a_2 b_5 + a_2 b_3 T_1 - a_2 b_5 T_1 + a_4 b_5 T_1 + c_{15} T_1 - c_{15} T_1^2 - a_4 b_3 T_2 - a_2 b_5 T_2 + a_4 b_5 T_2 + 2 a_4 b_3 T_1 T_2 + a_2 b_5 T_1 T_2 - 3 a_4 b_5 T_1 T_2 - a_4 b_3 T_1^2 T_2 - c_{15} T_1^2 T_2 + c_{15} T_1^3 T_2 - a_4 b_5 T_2^2 + 2 a_4 b_5 T_1 T_2^2)}{((-1 + T_1) T_1^2 (-1 + T_2) T_2)} \right), f_7 \rightarrow 0, f_8 \rightarrow 0, f_9 \rightarrow 0, \\ & f_{10} \rightarrow -\left(\frac{(-a_2 b_3 + a_4 b_3 + a_2 b_5 - a_4 b_5 - c_4 + c_{13} + a_2 b_3 T_1 - a_4 b_3 T_1 + c_4 T_1 - a_4 b_3 T_2 - a_2 b_5 T_2 + 2 a_4 b_5 T_2 - c_{13} T_2 + a_4 b_3 T_1 T_2 + c_4 T_1 T_2 - c_{13} T_1 T_2 - c_4 T_1^2 T_2 - a_4 b_5 T_2^2 + c_{13} T_1 T_2^2)}{((-1 + T_1) (-1 + T_2) (-1 + T_1 T_2))} \right), f_{11} \rightarrow -\frac{a_2 b_5 - a_4 b_5 + c_5 - c_{14} - c_5 T_1 + a_4 b_5 T_2 + c_{14} T_1 T_2}{(-1 + T_1) (-1 + T_1 T_2)}, \\ & f_{12} \rightarrow -\frac{a_2 b_3 + c_6 T_1 - c_{15} T_1 + a_4 b_3 T_2 - a_4 b_3 T_1 T_2 - c_6 T_1 T_2 + c_{15} T_1^2 T_2}{T_1 (-1 + T_2) (-1 + T_1 T_2)}, \\ & f_{13} \rightarrow -\left(\frac{(a_2 b_3 - a_2 b_5 - a_2 b_3 T_1 + c_4 T_1 - c_4 T_1^2 + a_4 b_3 T_2 + a_2 b_5 T_2 - a_4 b_5 T_2 - 2 a_4 b_3 T_1 T_2 + a_4 b_5 T_1 T_2 + a_4 b_3 T_1^2 T_2 - c_4 T_1^2 T_2 + c_4 T_1^3 T_2 + a_4 b_5 T_2^2 - a_4 b_5 T_1 T_2^2)}{(T_1^2 (-1 + T_2) (-1 + T_1 T_2))} \right), \\ & f_{14} \rightarrow -\left(\frac{(-a_2 b_3 + a_2 b_5 + 2 a_2 b_3 T_1 - a_4 b_3 T_1 + c_5 T_1 - a_2 b_3 T_1^2 + a_4 b_3 T_1^2 - c_5 T_1^2 - a_4 b_3 T_2 - a_2 b_5 T_2 + a_4 b_5 T_2 + 3 a_4 b_3 T_1 T_2 - a_4 b_5 T_1 T_2 - c_5 T_1 T_2 - 2 a_4 b_3 T_1^2 T_2 + c_5 T_1^2 T_2 - a_4 b_5 T_2^2 + a_4 b_5 T_1 T_2^2)}{(T_1^2 (-1 + T_2) (-1 + T_1 T_2))} \right), \\ & f_{15} \rightarrow -\left(\frac{(a_2 b_5 + a_2 b_3 T_1 - a_4 b_3 T_1 - a_2 b_5 T_1 + a_4 b_5 T_1 + c_6 T_1 - a_2 b_3 T_1^2 + a_4 b_3 T_1^2 - c_6 T_1^2 - a_2 b_5 T_2 + a_4 b_5 T_2 + a_4 b_3 T_1 T_2 + a_2 b_5 T_1 T_2 - 3 a_4 b_5 T_1 T_2 - c_6 T_1 T_2 - a_4 b_3 T_1^2 T_2 + c_6 T_1^2 T_2 - a_4 b_5 T_2^2 + 2 a_4 b_5 T_1 T_2^2)}{((-1 + T_1) T_1 T_2 (-1 + T_1 T_2))} \right), \\ & f_{16} \rightarrow -\frac{1}{T_1^2 T_2 (-1 + T_1 T_2)} \left(-a_2 b_3 + a_2 b_3 T_1 - a_2 b_5 T_1 + a_4 b_5 T_1 - c_4 T_1 - c_{13} T_1 - a_2 b_3 T_1^2 + a_4 b_3 T_1^2 + c_4 T_1^2 - a_4 b_3 T_2 - a_2 b_5 T_2 + 2 a_4 b_3 T_1 T_2 + a_2 b_5 T_1 T_2 - 2 a_4 b_5 T_1 T_2 + c_{13} T_1 T_2 - \right) \end{aligned} \right.$$

$$\begin{aligned}
 & 2 a_4 b_3 T_1^2 T_2 + c_4 T_1^2 T_2 + c_{13} T_1^2 T_2 - c_4 T_1^3 T_2 - a_4 b_5 T_2^2 + 2 a_4 b_5 T_1 T_2^2 - c_{13} T_1^2 T_2^2), \\
 f_{17} \rightarrow & -\frac{1}{T_1^3 (-1 + T_2) T_2} (a_2 b_3 - 3 a_2 b_3 T_1 + a_4 b_3 T_1 + a_2 b_5 T_1 - a_4 b_5 T_1 - c_5 T_1 - c_{14} T_1 + \\
 & 2 a_2 b_3 T_1^2 - 2 a_4 b_3 T_1^2 + c_5 T_1^2 + a_4 b_3 T_2 - 4 a_4 b_3 T_1 T_2 - a_2 b_5 T_1 T_2 + 2 a_4 b_5 T_1 T_2 + \\
 & c_5 T_1 T_2 + c_{14} T_1 T_2 + 4 a_4 b_3 T_1^2 T_2 - c_5 T_1^2 T_2 + c_{14} T_1^2 T_2 - a_4 b_5 T_1 T_2^2 - c_{14} T_1^2 T_2^2), \\
 f_{18} \rightarrow & -\frac{1}{(-1 + T_1) T_1^2 T_2^2} (a_2 b_3 - a_2 b_5 - a_2 b_3 T_1 + a_2 b_5 T_1 - a_4 b_5 T_1 - c_6 T_1 - c_{15} T_1 + c_6 T_1^2 + \\
 & c_{15} T_1^2 + a_4 b_3 T_2 + 2 a_2 b_5 T_2 - a_4 b_5 T_2 - 2 a_4 b_3 T_1 T_2 - 2 a_2 b_5 T_1 T_2 + 4 a_4 b_5 T_1 T_2 + c_6 T_1 T_2 + \\
 & a_4 b_3 T_1^2 T_2 - c_6 T_1^2 T_2 + c_{15} T_1^2 T_2 - c_{15} T_1^3 T_2 + 2 a_4 b_5 T_2^2 - 4 a_4 b_5 T_1 T_2^2), f_{19} \rightarrow 0, f_{20} \rightarrow 0, \\
 f_{21} \rightarrow & 0, f_{22} \rightarrow -\left((a_2 b_3 - a_2 b_5 + a_4 b_5 - c_{13} - a_2 b_3 T_1 + a_4 b_3 T_1 + a_4 b_3 T_2 + a_2 b_5 T_2 - 2 a_4 b_5 T_2 + \right. \\
 & \left. c_{13} T_2 - 2 a_4 b_3 T_1 T_2 + c_{13} T_1 T_2 + a_4 b_5 T_2^2 - c_{13} T_1 T_2^2) / (T_1 (-1 + T_2) (-1 + T_1 T_2)) \right), \\
 f_{23} \rightarrow & -\left((-a_2 b_3 + a_2 b_5 - a_4 b_5 - c_{14} + a_2 b_3 T_1 - a_4 b_3 T_1 - a_4 b_3 T_2 - a_2 b_5 T_2 + 2 a_4 b_5 T_2 + \right. \\
 & \left. c_{14} T_2 + 2 a_4 b_3 T_1 T_2 + c_{14} T_1 T_2 - a_4 b_5 T_2^2 - c_{14} T_1 T_2^2) / (T_1 (-1 + T_2) (-1 + T_1 T_2)) \right), \\
 f_{24} \rightarrow & -\left((a_2 b_3 - a_2 b_3 T_1 - c_{15} T_1 + c_{15} T_1^2 + a_4 b_3 T_2 + a_2 b_5 T_2 - 2 a_4 b_3 T_1 T_2 - a_2 b_5 T_1 T_2 + a_4 b_5 T_1 T_2 + \right. \\
 & \left. a_4 b_3 T_1^2 T_2 + c_{15} T_1^2 T_2 - c_{15} T_1^3 T_2 + a_4 b_5 T_2^2 - 2 a_4 b_5 T_1 T_2^2) / ((-1 + T_1) T_1 T_2 (-1 + T_1 T_2)) \right), \\
 f_{25} \rightarrow & 0, f_{26} \rightarrow 0, f_{27} \rightarrow 0, f_{28} \rightarrow 0, f_{29} \rightarrow -\frac{-a_2 b_3 - a_4 b_3 T_2}{T_1 (-1 + T_2) T_2}, \\
 f_{30} \rightarrow & -\frac{a_4 b_5}{(-1 + T_1) T_1}, \\
 f_{31} \rightarrow & 0, \\
 f_{32} \rightarrow & 0, \\
 f_{33} \rightarrow & 0, \\
 f_{34} \rightarrow & 0, \\
 f_{35} \rightarrow & 0, \\
 f_{36} \rightarrow & 0, \\
 f_{37} \rightarrow & -\left((a_2 b_3 - a_2 b_5 - a_2 b_3 T_1 + c_4 T_1 - c_{13} T_1 - c_4 T_1^2 + a_4 b_3 T_2 + a_2 b_5 T_2 - a_4 b_5 T_2 - \right. \\
 & \left. 2 a_4 b_3 T_1 T_2 + a_4 b_5 T_1 T_2 + c_{13} T_1 T_2 + a_4 b_3 T_1^2 T_2 - c_4 T_1^2 T_2 + c_{13} T_1^2 T_2 + c_4 T_1^3 T_2 + \right. \\
 & \left. a_4 b_5 T_2^2 - a_4 b_5 T_1 T_2^2 - c_{13} T_1^2 T_2^2) / ((-1 + T_1) T_1 (-1 + T_2) (-1 + T_1 T_2)) \right), \\
 f_{38} \rightarrow & -\left((-a_2 b_3 + a_2 b_5 + 2 a_2 b_3 T_1 - a_4 b_3 T_1 + c_5 T_1 - c_{14} T_1 - a_2 b_3 T_1^2 + a_4 b_3 T_1^2 - c_5 T_1^2 - \right. \\
 & \left. a_4 b_3 T_2 - a_2 b_5 T_2 + a_4 b_5 T_2 + 3 a_4 b_3 T_1 T_2 - a_4 b_5 T_1 T_2 - c_5 T_1 T_2 + c_{14} T_1 T_2 - 2 a_4 b_3 T_1^2 T_2 + \right. \\
 & \left. c_5 T_1^2 T_2 + c_{14} T_1^2 T_2 - a_4 b_5 T_2^2 + a_4 b_5 T_1 T_2^2 - c_{14} T_1^2 T_2^2) / ((-1 + T_1) T_1 (-1 + T_2) (-1 + T_1 T_2)) \right), \\
 f_{39} \rightarrow & -\left((a_2 b_5 + a_2 b_3 T_1 - a_4 b_3 T_1 - a_2 b_5 T_1 + a_4 b_5 T_1 + c_6 T_1 - c_{15} T_1 - a_2 b_3 T_1^2 + a_4 b_3 T_1^2 - c_6 T_1^2 + \right. \\
 & \left. c_{15} T_1^2 - a_2 b_5 T_2 + a_4 b_5 T_2 + a_4 b_3 T_1 T_2 + a_2 b_5 T_1 T_2 - 3 a_4 b_5 T_1 T_2 - c_6 T_1 T_2 - a_4 b_3 T_1^2 T_2 + c_6 T_1^2 \right. \\
 & \left. T_2 + c_{15} T_1^2 T_2 - c_{15} T_1^3 T_2 - a_4 b_5 T_2^2 + 2 a_4 b_5 T_1 T_2^2) / ((-1 + T_1) T_1 (-1 + T_2) (-1 + T_1 T_2)) \right), \\
 f_{40} \rightarrow & -\left((-a_2 b_3 + a_2 b_3 T_1 - c_4 T_1 + c_4 T_1^2 - a_4 b_3 T_2 - a_2 b_5 T_2 + 2 a_4 b_3 T_1 T_2 + a_2 b_5 T_1 T_2 - a_4 b_5 T_1 T_2 - \right. \\
 & \left. a_4 b_3 T_1^2 T_2 + c_4 T_1^2 T_2 - c_4 T_1^3 T_2 - a_4 b_5 T_2^2 + 2 a_4 b_5 T_1 T_2^2) / ((-1 + T_1) T_1 T_2 (-1 + T_1 T_2)) \right), \\
 f_{41} \rightarrow & -\left((a_2 b_3 - 3 a_2 b_3 T_1 + a_4 b_3 T_1 - c_5 T_1 + 2 a_2 b_3 T_1^2 - a_4 b_3 T_1^2 + c_5 T_1^2 + a_4 b_3 T_2 - 4 a_4 b_3 T_1 T_2 + \right. \\
 & \left. a_4 b_5 T_1 T_2 + c_5 T_1 T_2 + 3 a_4 b_3 T_1^2 T_2 - c_5 T_1^2 T_2 - a_4 b_5 T_1 T_2^2) / ((-1 + T_1) T_1^2 (-1 + T_2) T_2) \right), \\
 f_{42} \rightarrow & -\left((-a_2 b_5 - a_2 b_3 T_1 + a_2 b_5 T_1 - a_4 b_5 T_1 - c_6 T_1 + a_2 b_3 T_1^2 + c_6 T_1^2 + a_2 b_5 T_2 - \right. \\
 & \left. a_4 b_5 T_2 - a_4 b_3 T_1 T_2 - a_2 b_5 T_1 T_2 + 4 a_4 b_5 T_1 T_2 + c_6 T_1 T_2 + a_4 b_3 T_1^2 T_2 - \right. \\
 & \left. c_6 T_1^2 T_2 + a_4 b_5 T_2^2 - 3 a_4 b_5 T_1 T_2^2) / ((-1 + T_1) T_1^2 (-1 + T_2) T_2) \right), \\
 f_{43} \rightarrow & 0, f_{44} \rightarrow 0, f_{45} \rightarrow 0, f_{46} \rightarrow 0, f_{47} \rightarrow 0, f_{48} \rightarrow 0, f_{49} \rightarrow -c_{49}, \\
 f_{50} \rightarrow & -c_{50},
 \end{aligned}$$

$$\begin{aligned}
 f_{51} &\rightarrow -c_{51}, \\
 f_{52} &\rightarrow -\frac{c_{49} + c_{52} - c_{49} T_1^2}{T_1^2}, \\
 f_{53} &\rightarrow -\frac{c_{50} + c_{53} - c_{50} T_2^2}{T_2^2}, \\
 f_{54} &\rightarrow -\frac{1}{T_1^3 T_2^2} \left(-a_2 b_5 - a_2 b_3 T_1 + a_4 b_3 T_1 + c_{51} T_1 + c_{54} T_1 - a_4 b_5 T_2 - \right. \\
 &\quad \left. a_4 b_3 T_1 T_2 - a_2 b_5 T_1 T_2 + a_4 b_5 T_1 T_2 + a_4 b_3 T_1^2 T_2 - a_4 b_5 T_1 T_2^2 - c_{51} T_1^3 T_2^2 \right), \\
 f_{55} &\rightarrow 0, f_{56} \rightarrow 0, f_{57} \rightarrow 0, f_{58} \rightarrow -\frac{-c_{49} - c_{52}}{T_1}, f_{59} \rightarrow -\frac{-c_{50} - c_{53}}{T_2}, \\
 f_{60} &\rightarrow -\frac{1}{T_1^2 T_2 (-1 + T_1 T_2)} \left(-a_2 b_5 - a_2 b_3 T_1 + a_4 b_3 T_1 + c_{51} T_1 + \right. \\
 &\quad \left. c_{54} T_1 - a_4 b_5 T_2 - a_4 b_3 T_1 T_2 + a_4 b_5 T_1 T_2 - c_{51} T_1^2 T_2 - c_{54} T_1^2 T_2 \right) \} \}
 \end{aligned}$$

In[*]:= Length[sol]

Out[*]=
60

In[*]:= sol /. (v_ -> val_) :-> (v = CF[val]);

In[*]:= CF[r42[-1, i, j]]

Out[*]=

$$\begin{aligned}
 &-c_{49} p_{1,i} x_{1,i} + \frac{(-c_{49} - c_{52} + c_{49} T_1^2) p_{1,j} x_{1,i}}{T_1^2} + \\
 &\frac{(c_{49} + c_{52}) p_{1,j} x_{1,j}}{T_1} - c_{50} p_{2,i} x_{2,i} + \frac{(-c_{50} - c_{53} + c_{50} T_2^2) p_{2,j} x_{2,i}}{T_2^2} - \\
 &\left((a_2 b_3 - a_2 b_5 - a_2 b_3 T_1 + c_4 T_1 - c_4 T_1^2 + a_4 b_3 T_2 + a_2 b_5 T_2 - a_4 b_5 T_2 - 2 a_4 b_3 T_1 T_2 + a_4 b_5 T_1 T_2 + a_4 b_3 T_1^2 T_2 - \right. \\
 &\quad \left. c_4 T_1^2 T_2 + c_4 T_1^3 T_2 + a_4 b_5 T_2^2 - a_4 b_5 T_1 T_2^2) p_{1,j} p_{2,i} x_{1,i} x_{2,i} \right) / (T_1^2 (-1 + T_2) (-1 + T_1 T_2)) - \\
 &\left((-a_2 b_3 + a_4 b_3 + a_2 b_5 - a_4 b_5 + c_{13} + a_2 b_3 T_1 - a_4 b_3 T_1 - a_4 b_3 T_2 - a_2 b_5 T_2 + 2 a_4 b_5 T_2 - c_{13} T_2 + \right. \\
 &\quad \left. a_4 b_3 T_1 T_2 - c_{13} T_1 T_2 - a_4 b_5 T_2^2 + c_{13} T_1 T_2^2) p_{1,i} p_{2,j} x_{1,i} x_{2,i} \right) / ((-1 + T_1) T_2 (-1 + T_1 T_2)) + \\
 &\frac{1}{T_1^2 T_2 (-1 + T_1 T_2)} \left(a_2 b_3 - a_2 b_3 T_1 + a_2 b_5 T_1 - a_4 b_5 T_1 + c_4 T_1 + c_{13} T_1 + a_2 b_3 T_1^2 - a_4 b_3 T_1^2 - \right. \\
 &\quad c_4 T_1^2 + a_4 b_3 T_2 + a_2 b_5 T_2 - 2 a_4 b_3 T_1 T_2 - a_2 b_5 T_1 T_2 + 2 a_4 b_5 T_1 T_2 - c_{13} T_1 T_2 + 2 a_4 b_3 T_1^2 T_2 - \\
 &\quad \left. c_4 T_1^2 T_2 - c_{13} T_1^2 T_2 + c_4 T_1^3 T_2 + a_4 b_5 T_2^2 - 2 a_4 b_5 T_1 T_2^2 + c_{13} T_1^2 T_2^2 \right) p_{1,j} p_{2,j} x_{1,i} x_{2,i} + \\
 &\left((-a_2 b_3 + a_2 b_5 + a_2 b_3 T_1 - c_4 T_1 + c_{13} T_1 + c_4 T_1^2 - a_4 b_3 T_2 - a_2 b_5 T_2 + a_4 b_5 T_2 + 2 a_4 b_3 T_1 T_2 - \right. \\
 &\quad \left. a_4 b_5 T_1 T_2 - c_{13} T_1 T_2 - a_4 b_3 T_1^2 T_2 + c_4 T_1^2 T_2 - c_{13} T_1^2 T_2 - c_4 T_1^3 T_2 - a_4 b_5 T_2^2 + a_4 b_5 T_1 T_2^2 + c_{13} T_1^2 T_2^2) \right. \\
 &\quad \left. p_{1,j} p_{2,i} x_{1,j} x_{2,i} \right) / ((-1 + T_1) T_1 (-1 + T_2) (-1 + T_1 T_2)) + \\
 &\left((a_2 b_3 - a_2 b_3 T_1 + c_4 T_1 - c_4 T_1^2 + a_4 b_3 T_2 + a_2 b_5 T_2 - 2 a_4 b_3 T_1 T_2 - a_2 b_5 T_1 T_2 + a_4 b_5 T_1 T_2 + \right. \\
 &\quad \left. a_4 b_3 T_1^2 T_2 - c_4 T_1^2 T_2 + c_4 T_1^3 T_2 + a_4 b_5 T_2^2 - 2 a_4 b_5 T_1 T_2^2) p_{1,j} p_{2,j} x_{1,j} x_{2,i} \right) / \\
 &\left((-1 + T_1) T_1 T_2 (-1 + T_1 T_2) \right) + \frac{(c_{50} + c_{53}) p_{2,j} x_{2,j}}{T_2} - \\
 &\left((-a_2 b_3 + a_4 b_3 + a_2 b_5 - a_4 b_5 - c_4 + c_{13} + a_2 b_3 T_1 - a_4 b_3 T_1 + c_4 T_1 - a_4 b_3 T_2 - a_2 b_5 T_2 + \right.
 \end{aligned}$$

$$\begin{aligned}
 & (2 a_4 b_5 T_2 - c_{13} T_2 + a_4 b_3 T_1 T_2 + c_4 T_1 T_2 - c_{13} T_1 T_2 - c_4 T_1^2 T_2 - a_4 b_5 T_2^2 + c_{13} T_1 T_2^2) \\
 & p_{1,i} p_{2,j} x_{1,i} x_{2,j} \Big/ \left((-1 + T_1) (-1 + T_2) (-1 + T_1 T_2) \right) + \\
 & \left((-a_2 b_3 + a_2 b_5 - a_4 b_5 + c_{13} + a_2 b_3 T_1 - a_4 b_3 T_1 - a_4 b_3 T_2 - a_2 b_5 T_2 + 2 a_4 b_5 T_2 - c_{13} T_2 + \right. \\
 & \quad \left. 2 a_4 b_3 T_1 T_2 - c_{13} T_1 T_2 - a_4 b_5 T_2^2 + c_{13} T_1 T_2^2) p_{1,j} p_{2,j} x_{1,i} x_{2,j} \right) \Big/ (T_1 (-1 + T_2) (-1 + T_1 T_2)) - \\
 & c_{51} p_{3,i} x_{3,i} + \frac{1}{T_1^3 T_2^2} (a_2 b_5 + a_2 b_3 T_1 - a_4 b_3 T_1 - c_{51} T_1 - c_{54} T_1 + a_4 b_5 T_2 + a_4 b_3 T_1 T_2 + \\
 & \quad a_2 b_5 T_1 T_2 - a_4 b_5 T_1 T_2 - a_4 b_3 T_1^2 T_2 + a_4 b_5 T_1 T_2^2 + c_{51} T_1^3 T_2^2) p_{3,j} x_{3,i} + \\
 & \left((a_2 b_3 - a_2 b_5 - 2 a_2 b_3 T_1 + a_4 b_3 T_1 - c_5 T_1 + a_2 b_3 T_1^2 - a_4 b_3 T_1^2 + c_5 T_1^2 + a_4 b_3 T_2 + a_2 b_5 T_2 - \right. \\
 & \quad \left. a_4 b_5 T_2 - 3 a_4 b_3 T_1 T_2 + a_4 b_5 T_1 T_2 + c_5 T_1 T_2 + 2 a_4 b_3 T_1^2 T_2 - c_5 T_1^2 T_2 + a_4 b_5 T_2^2 - a_4 b_5 T_1 T_2^2) \right. \\
 & \quad \left. p_{1,j} p_{3,i} x_{1,i} x_{3,i} \right) \Big/ (T_1^2 (-1 + T_2) (-1 + T_1 T_2)) + \\
 & \left((a_2 b_3 - 2 a_2 b_3 T_1 + a_4 b_3 T_1 + a_2 b_5 T_1 - a_4 b_5 T_1 - c_{14} T_1 + a_2 b_3 T_1^2 - a_4 b_3 T_1^2 + a_4 b_3 T_2 - 3 a_4 b_3 T_1 T_2 - \right. \\
 & \quad \left. a_2 b_5 T_1 T_2 + 2 a_4 b_5 T_1 T_2 + c_{14} T_1 T_2 + 2 a_4 b_3 T_1^2 T_2 + c_{14} T_1^2 T_2 - a_4 b_5 T_1 T_2^2 - c_{14} T_1^2 T_2^2) \right. \\
 & \quad \left. p_{1,i} p_{3,j} x_{1,i} x_{3,i} \right) \Big/ \left((-1 + T_1) T_1^2 (-1 + T_2) T_2 \right) - \frac{1}{T_1^3 (-1 + T_2) T_2} \\
 & (a_2 b_3 - 3 a_2 b_3 T_1 + a_4 b_3 T_1 + a_2 b_5 T_1 - a_4 b_5 T_1 - c_5 T_1 - c_{14} T_1 + 2 a_2 b_3 T_1^2 - 2 a_4 b_3 T_1^2 + \\
 & \quad c_5 T_1^2 + a_4 b_3 T_2 - 4 a_4 b_3 T_1 T_2 - a_2 b_5 T_1 T_2 + 2 a_4 b_5 T_1 T_2 + c_5 T_1 T_2 + c_{14} T_1 T_2 + \\
 & \quad 4 a_4 b_3 T_1^2 T_2 - c_5 T_1^2 T_2 + c_{14} T_1^2 T_2 - a_4 b_5 T_1 T_2^2 - c_{14} T_1^2 T_2^2) p_{1,j} p_{3,j} x_{1,i} x_{3,i} + \\
 & \left((a_2 b_3 - a_2 b_5 - 2 a_2 b_3 T_1 + a_4 b_3 T_1 - c_5 T_1 + c_{14} T_1 + a_2 b_3 T_1^2 - a_4 b_3 T_1^2 + c_5 T_1^2 + a_4 b_3 T_2 + \right. \\
 & \quad \left. a_2 b_5 T_2 - a_4 b_5 T_2 - 3 a_4 b_3 T_1 T_2 + a_4 b_5 T_1 T_2 + c_5 T_1 T_2 - c_{14} T_1 T_2 + 2 a_4 b_3 T_1^2 T_2 - \right. \\
 & \quad \left. c_5 T_1^2 T_2 - c_{14} T_1^2 T_2 + a_4 b_5 T_2^2 - a_4 b_5 T_1 T_2^2 + c_{14} T_1^2 T_2^2) p_{1,j} p_{3,i} x_{1,j} x_{3,i} \right) \Big/ \\
 & \left((-1 + T_1) T_1 (-1 + T_2) (-1 + T_1 T_2) \right) + \frac{b_3 (a_2 + a_4 T_2) p_{1,i} p_{3,j} x_{1,j} x_{3,i}}{T_1 (-1 + T_2) T_2} - \\
 & \left((a_2 b_3 - 3 a_2 b_3 T_1 + a_4 b_3 T_1 - c_5 T_1 + 2 a_2 b_3 T_1^2 - a_4 b_3 T_1^2 + c_5 T_1^2 + a_4 b_3 T_2 - 4 a_4 b_3 T_1 T_2 + a_4 b_5 T_1 T_2 + \right. \\
 & \quad \left. c_5 T_1 T_2 + 3 a_4 b_3 T_1^2 T_2 - c_5 T_1^2 T_2 - a_4 b_5 T_1 T_2^2) p_{1,j} p_{3,j} x_{1,j} x_{3,i} \right) \Big/ \left((-1 + T_1) T_1^2 (-1 + T_2) T_2 \right) + \\
 & \left((-a_2 b_5 - a_2 b_3 T_1 + a_4 b_3 T_1 + a_2 b_5 T_1 - a_4 b_5 T_1 - c_6 T_1 + a_2 b_3 T_1^2 - a_4 b_3 T_1^2 + c_6 T_1^2 + a_2 b_5 T_2 - a_4 b_5 T_2 - \right. \\
 & \quad \left. a_4 b_3 T_1 T_2 - a_2 b_5 T_1 T_2 + 3 a_4 b_5 T_1 T_2 + c_6 T_1 T_2 + a_4 b_3 T_1^2 T_2 - c_6 T_1^2 T_2 + a_4 b_5 T_2^2 - 2 a_4 b_5 T_1 T_2^2) \right. \\
 & \quad \left. p_{2,j} p_{3,i} x_{2,i} x_{3,i} \right) \Big/ \left((-1 + T_1) T_1 T_2 (-1 + T_1 T_2) \right) + \\
 & \left((a_2 b_3 - a_2 b_5 - a_2 b_3 T_1 + a_2 b_5 T_1 - a_4 b_5 T_1 - c_{15} T_1 + c_{15} T_1^2 + a_4 b_3 T_2 + a_2 b_5 T_2 - a_4 b_5 T_2 - \right. \\
 & \quad \left. 2 a_4 b_3 T_1 T_2 - a_2 b_5 T_1 T_2 + 3 a_4 b_5 T_1 T_2 + a_4 b_3 T_1^2 T_2 + c_{15} T_1^2 T_2 - c_{15} T_1^3 T_2 + a_4 b_5 T_2^2 - 2 a_4 b_5 T_1 T_2^2) \right. \\
 & \quad \left. p_{2,i} p_{3,j} x_{2,i} x_{3,i} \right) \Big/ \left((-1 + T_1) T_1^2 (-1 + T_2) T_2 \right) + \frac{1}{(-1 + T_1) T_1^2 T_2^2} \\
 & (-a_2 b_3 + a_2 b_5 + a_2 b_3 T_1 - a_2 b_5 T_1 + a_4 b_5 T_1 + c_6 T_1 + c_{15} T_1 - c_6 T_1^2 - c_{15} T_1^2 - a_4 b_3 T_2 - \\
 & \quad 2 a_2 b_5 T_2 + a_4 b_5 T_2 + 2 a_4 b_3 T_1 T_2 + 2 a_2 b_5 T_1 T_2 - 4 a_4 b_5 T_1 T_2 - c_6 T_1 T_2 - a_4 b_3 T_1^2 T_2 + \\
 & \quad c_6 T_1^2 T_2 - c_{15} T_1^2 T_2 + c_{15} T_1^3 T_2 - 2 a_4 b_5 T_2^2 + 4 a_4 b_5 T_1 T_2^2) p_{2,j} p_{3,j} x_{2,i} x_{3,i} + \\
 & \left((-a_2 b_5 - a_2 b_3 T_1 + a_4 b_3 T_1 + a_2 b_5 T_1 - a_4 b_5 T_1 - c_6 T_1 + c_{15} T_1 + a_2 b_3 T_1^2 - a_4 b_3 T_1^2 + c_6 T_1^2 - \right. \\
 & \quad \left. c_{15} T_1^2 + a_2 b_5 T_2 - a_4 b_5 T_2 - a_4 b_3 T_1 T_2 - a_2 b_5 T_1 T_2 + 3 a_4 b_5 T_1 T_2 + c_6 T_1 T_2 + \right. \\
 & \quad \left. a_4 b_3 T_1^2 T_2 - c_6 T_1^2 T_2 - c_{15} T_1^2 T_2 + c_{15} T_1^3 T_2 + a_4 b_5 T_2^2 - 2 a_4 b_5 T_1 T_2^2) p_{2,j} p_{3,i} x_{2,j} x_{3,i} \right) \Big/ \\
 & \left((-1 + T_1) T_1 (-1 + T_2) (-1 + T_1 T_2) \right) - \frac{a_4 b_5 p_{2,i} p_{3,j} x_{2,j} x_{3,i}}{(-1 + T_1) T_1} - \\
 & \left((-a_2 b_5 - a_2 b_3 T_1 + a_2 b_5 T_1 - a_4 b_5 T_1 - c_6 T_1 + a_2 b_3 T_1^2 + c_6 T_1^2 + a_2 b_5 T_2 - a_4 b_5 T_2 - a_4 b_3 T_1 T_2 - \right. \\
 & \quad \left. a_2 b_5 T_1 T_2 + 4 a_4 b_5 T_1 T_2 + c_6 T_1 T_2 + a_4 b_3 T_1^2 T_2 - c_6 T_1^2 T_2 + a_4 b_5 T_2^2 - 3 a_4 b_5 T_1 T_2^2) \right.
 \end{aligned}$$

$$\begin{aligned} & p_{2,j} p_{3,j} x_{2,j} x_{3,i} \Big/ \left((-1 + T_1) T_1^2 (-1 + T_2) T_2 \right) + \frac{1}{T_1^2 T_2 (-1 + T_1 T_2)} \\ & \left(a_2 b_5 + a_2 b_3 T_1 - a_4 b_3 T_1 - c_{51} T_1 - c_{54} T_1 + a_4 b_5 T_2 + a_4 b_3 T_1 T_2 - a_4 b_5 T_1 T_2 + c_{51} T_1^2 T_2 + c_{54} T_1^2 T_2 \right) \\ & p_{3,j} x_{3,j} - \\ & \frac{\left(a_2 b_5 - a_4 b_5 + c_5 - c_{14} - c_5 T_1 + a_4 b_5 T_2 + c_{14} T_1 T_2 \right) p_{1,i} p_{3,j} x_{1,i} x_{3,j}}{(-1 + T_1) (-1 + T_1 T_2)} - \\ & \left(-a_2 b_3 + a_2 b_5 - a_4 b_5 - c_{14} + a_2 b_3 T_1 - a_4 b_3 T_1 - a_4 b_3 T_2 - a_2 b_5 T_2 + 2 a_4 b_5 T_2 + c_{14} T_2 + 2 a_4 b_3 T_1 T_2 + \right. \\ & \quad \left. c_{14} T_1 T_2 - a_4 b_5 T_2^2 - c_{14} T_1 T_2^2 \right) p_{1,j} p_{3,j} x_{1,i} x_{3,j} \Big/ \left(T_1 (-1 + T_2) (-1 + T_1 T_2) \right) + \\ & \left(-a_2 b_3 - c_6 T_1 + c_{15} T_1 - a_4 b_3 T_2 + a_4 b_3 T_1 T_2 + c_6 T_1 T_2 - c_{15} T_1^2 T_2 \right) p_{2,i} p_{3,j} x_{2,i} x_{3,j} \Big/ \\ & \left(T_1 (-1 + T_2) (-1 + T_1 T_2) \right) - \\ & \left(a_2 b_3 - a_2 b_3 T_1 - c_{15} T_1 + c_{15} T_1^2 + a_4 b_3 T_2 + a_2 b_5 T_2 - 2 a_4 b_3 T_1 T_2 - a_2 b_5 T_1 T_2 + a_4 b_5 T_1 T_2 + a_4 b_3 T_1^2 T_2 + \right. \\ & \quad \left. c_{15} T_1^2 T_2 - c_{15} T_1^3 T_2 + a_4 b_5 T_2^2 - 2 a_4 b_5 T_1 T_2^2 \right) p_{2,j} p_{3,j} x_{2,i} x_{3,j} \Big/ \left((-1 + T_1) T_1 T_2 (-1 + T_1 T_2) \right) \end{aligned}$$

In[*]:= CF[r₄₂[-1, i, j] / .

$$\{a_4 \rightarrow \theta, b_3 \rightarrow \theta, b_5 \rightarrow (T_1 - 1) (T_2 - 1) (T_3 - 1) a_2^{-1}, c_{4|5|6|13|14|15|49|50|51|52|53|54} \rightarrow \theta\}$$

Out[*]=

$$\begin{aligned} & - \frac{(-1 + T_1) (-1 + T_2) p_{1,j} p_{2,i} x_{1,i} x_{2,i}}{T_1^2} + \frac{(-1 + T_2)^2 p_{1,i} p_{2,j} x_{1,i} x_{2,i}}{T_2} - \\ & \frac{(-1 + T_1) (-1 + T_2) (-T_1 - T_2 + T_1 T_2) p_{1,j} p_{2,j} x_{1,i} x_{2,i}}{T_1^2 T_2} - \frac{(-1 + T_2) p_{1,j} p_{2,i} x_{1,j} x_{2,i}}{T_1} - \\ & \frac{(-1 + T_1) (-1 + T_2) p_{1,j} p_{2,j} x_{1,j} x_{2,i}}{T_1} + (-1 + T_2) p_{1,i} p_{2,j} x_{1,i} x_{2,j} - \\ & \frac{(-1 + T_1) (-1 + T_2) p_{1,j} p_{2,j} x_{1,i} x_{2,j}}{T_1} + \frac{(-1 + T_1) (-1 + T_2) (-1 + T_1 T_2) (1 + T_1 T_2) p_{3,j} x_{3,i}}{T_1^3 T_2^2} + \\ & \frac{(-1 + T_1) (-1 + T_2) p_{1,j} p_{3,i} x_{1,i} x_{3,i}}{T_1^2} - \frac{(-1 + T_2) (-1 + T_1 T_2) p_{1,i} p_{3,j} x_{1,i} x_{3,i}}{T_1 T_2} + \\ & \frac{(-1 + T_1) (-1 + T_2) (-1 + T_1 T_2) p_{1,j} p_{3,j} x_{1,i} x_{3,i}}{T_1^2 T_2} + \frac{(-1 + T_2) p_{1,j} p_{3,i} x_{1,j} x_{3,i}}{T_1} - \\ & \frac{(-1 + T_1) (-1 + T_2)^2 p_{2,j} p_{3,i} x_{2,i} x_{3,i}}{T_1 T_2} - \frac{(-1 + T_1) (-1 + T_2) (-1 + T_1 T_2) p_{2,i} p_{3,j} x_{2,i} x_{3,i}}{T_1^2 T_2} + \\ & \frac{(-1 + T_1) (-1 + T_2) (-1 + 2 T_2) (-1 + T_1 T_2) p_{2,j} p_{3,j} x_{2,i} x_{3,i}}{T_1^2 T_2^2} - \\ & \frac{(-1 + T_1) (-1 + T_2) p_{2,j} p_{3,i} x_{2,j} x_{3,i}}{T_1} + \frac{(-1 + T_1) (-1 + T_2) (-1 + T_1 T_2) p_{2,j} p_{3,j} x_{2,j} x_{3,i}}{T_1^2 T_2} + \\ & \frac{(-1 + T_1) (-1 + T_2) p_{3,j} x_{3,j}}{T_1^2 T_2} + (1 - T_2) p_{1,i} p_{3,j} x_{1,i} x_{3,j} + \\ & \frac{(-1 + T_1) (-1 + T_2) p_{1,j} p_{3,j} x_{1,i} x_{3,j}}{T_1} + \frac{(-1 + T_1) (-1 + T_2) p_{2,j} p_{3,j} x_{2,i} x_{3,j}}{T_1} \end{aligned}$$