Pensieve header: Proof of invariance of ρ_2 using integration techniques.

Initialization

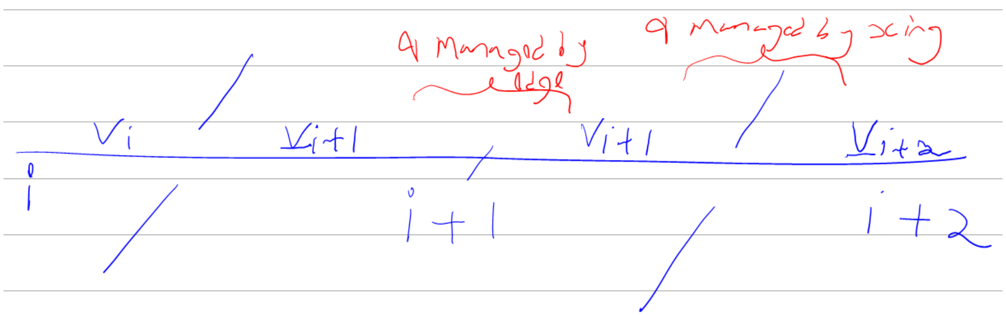
In[•]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\HigherRank"];
Once[<< KnotTheory`; << Rot.m];
<< NilpotentIntegration.m;
$π = Normal[# + O[ϵ]^3] &;
```

In[•]:=
```
T2z[p_] := Module[{q = Expand[p], n, c},
   If[q === 0, 0, c = Coefficient[q, T, n = Exponent[q, T]];
    c z^(2 n) + T2z[q - c (T^(1/2) - T^(-1/2))^(2 n)]]];
```

The ρ_2 Integrand

Adopted from pensieve://Talks//Oaxaca-2210/Rho.nb.

Variable convention:



In[∘]:=
```
q[s_, i_, j_] := x_i (p_i - p̲_{i+1}) + x_j (p_j - p̲_{j+1}) + x_i ((1 - T^s) p̲_{i+1} + (T^s - 1) p̲_{j+1});
r_1[s_, i_, j_] :=
  s (-1 + 2 p_i x_i - 2 p_j x_i + (T^s - 1) p_i p_j x_i^2 + (1 - T^s) p_j^2 x_i^2 - 2 p_i p_j x_i x_j + 2 p_j^2 x_i x_j) / 2;
r_2[1, i_, j_] := (-6 p_i x_i + 6 p_j x_i - 3 (-1 + 3 T) p_i p_j x_i^2 + 3 (-1 + 3 T) p_j^2 x_i^2 + 4 (-1 + T) p_i^2 p_j x_i^3 -
      2 (-1 + T) (5 + T) p_i p_j^2 x_i^3 + 2 (-1 + T) (3 + T) p_j^3 x_i^3 + 18 p_i p_j x_i x_j - 18 p_j^2 x_i x_j -
      6 p_i^2 p_j x_i^2 x_j + 6 (2 + T) p_i p_j^2 x_i^2 x_j - 6 (1 + T) p_j^3 x_i^2 x_j - 6 p_i p_j^2 x_i x_j^2 + 6 p_j^3 x_i x_j^2) / 12;
r_2[-1, i_, j_] :=
  (-6 T^2 p_i x_i + 6 T^2 p_j x_i + 3 (-3 + T) T p_i p_j x_i^2 - 3 (-3 + T) T p_j^2 x_i^2 - 4 (-1 + T) T p_i^2 p_j x_i^3 +
      2 (-1 + T) (1 + 5 T) p_i p_j^2 x_i^3 - 2 (-1 + T) (1 + 3 T) p_j^3 x_i^3 + 18 T^2 p_i p_j x_i x_j -
      18 T^2 p_j^2 x_i x_j - 6 T^2 p_i^2 p_j x_i^2 x_j + 6 T (1 + 2 T) p_i p_j^2 x_i^2 x_j -
      6 T (1 + T) p_j^3 x_i^2 x_j - 6 T^2 p_i p_j^2 x_i x_j^2 + 6 T^2 p_j^3 x_i x_j^2) / (12 T^2);
γ_1[φ_, k_] := φ (1 / 2 - x̲_k p̲_k);
γ_2[φ_, k_] := -φ^2 p̲_k x̲_k / 2;
ℒ[s_, i_, j_] := T^{s/2} 𝔼[-q[s, i, j] + ϵ r_1[s, i, j] + ϵ^2 r_2[s, i, j]];
ℒ[φ_, k_] := T^{φ/2} 𝔼[-x̲_k (p̲_k - p_k) + ϵ γ_1[φ, k] + ϵ^2 γ_2[φ, k]];
ℒ[Dot, i_] := 𝔼[-x_i (p_i - p̲_{i+1})];
ℒ[End, k_] := 𝔼[-x̲_k (p̲_k - p_k) - x_k p_k];
ℒ[K_] := Module[{Cs, φ, n, c, k, ℰ},
   {Cs, φ} = Rot[K]; n = Length[Cs];
   ℰ = (2 π)^{4 n+2} ℒ[End, 2 n + 1];
   Do[ℰ *= ℒ @@ c, {c, Cs}];
   Do[ℰ *= ℒ[φ⟦k⟧, k], {k, 2 n}];
   CF@ℰ
  ];
vs_i_ := Sequence[x̲_i, p̲_i, x_i, p_i]
ρ2vs[K_] := Union @@ Table[{vs_i}, {i, 2 Crossings[K] + 1}]
```

```
ℒ[Knot[3, 1]]
```

Out[∘]=

$$
\begin{aligned}
&\frac{1}{T^2}\, 16384\, \pi^{14}\, \mathbb{E}\Big[-p_1 x_1 - p_2 x_2 - \in p_2 x_2 - \frac{1}{2} \in^2 p_2 x_2 + \in p_5 x_2 + \frac{1}{2} \in^2 p_5 x_2 + \frac{(-1+T) \in p_2 p_5 x_2^2}{2\,T} + \\
&\frac{(-3+T) \in^2 p_2 p_5 x_2^2}{4\,T} - \frac{(-1+T) \in p_5^2 x_2^2}{2\,T} - \frac{(-3+T) \in^2 p_5^2 x_2^2}{4\,T} - \frac{(-1+T) \in^2 p_2^2 p_5 x_2^3}{3\,T} + \\
&\frac{(-1+T)(1+5\,T) \in^2 p_2 p_5^2 x_2^3}{6\,T^2} - \frac{(-1+T)(1+3\,T) \in^2 p_5^3 x_2^3}{6\,T^2} - p_3 x_3 + \in p_1 x_4 + \frac{1}{2} \in^2 p_1 x_4 - \\
&p_4 x_4 - \in p_4 x_4 - \frac{1}{2} \in^2 p_4 x_4 - \in p_1^2 x_1 x_4 - \frac{3}{2} \in^2 p_1^2 x_1 x_4 + \in p_1 p_4 x_1 x_4 + \frac{3}{2} \in^2 p_1 p_4 x_1 x_4 + \\
&\frac{1}{2} \in^2 p_1^3 x_1^2 x_4 - \frac{1}{2} \in^2 p_1^2 p_4 x_1^2 x_4 - \frac{(-1+T) \in p_1^2 x_4^2}{2\,T} - \frac{(-3+T) \in^2 p_1^2 x_4^2}{4\,T} + \frac{(-1+T) \in p_1 p_4 x_4^2}{2\,T} + \\
&\frac{(-3+T) \in^2 p_1 p_4 x_4^2}{4\,T} - \frac{(1+T) \in^2 p_1^3 x_1 x_4^2}{2\,T} + \frac{(1+2\,T) \in^2 p_1^2 p_4 x_1 x_4^2}{2\,T} - \frac{1}{2} \in^2 p_1 p_4^2 x_1 x_4^2 - \\
&\frac{(-1+T)(1+3\,T) \in^2 p_1^3 x_4^3}{6\,T^2} + \frac{(-1+T)(1+5\,T) \in^2 p_1^2 p_4 x_4^3}{6\,T^2} - \frac{(-1+T) \in^2 p_1 p_4^2 x_4^3}{3\,T} - p_5 x_5 + \\
&\in p_2 p_5 x_2 x_5 + \frac{3}{2} \in^2 p_2 p_5 x_2 x_5 - \in p_5^2 x_2 x_5 - \frac{3}{2} \in^2 p_5^2 x_2 x_5 - \frac{1}{2} \in^2 p_2^2 p_5 x_2^2 x_5 + \frac{(1+2\,T) \in^2 p_2 p_5^2 x_2^2 x_5}{2\,T} - \\
&\frac{(1+T) \in^2 p_5^3 x_2^2 x_5}{2\,T} - \frac{1}{2} \in^2 p_2 p_5^2 x_2 x_5^2 + \frac{1}{2} \in^2 p_5^3 x_2 x_5^2 + \in p_3 x_6 + \frac{1}{2} \in^2 p_3 x_6 - p_6 x_6 - \in p_6 x_6 - \\
&\frac{1}{2} \in^2 p_6 x_6 - \in p_3^2 x_3 x_6 - \frac{3}{2} \in^2 p_3^2 x_3 x_6 + \in p_3 p_6 x_3 x_6 + \frac{3}{2} \in^2 p_3 p_6 x_3 x_6 + \frac{1}{2} \in^2 p_3^3 x_3^2 x_6 - \\
&\frac{1}{2} \in^2 p_3^2 p_6 x_3^2 x_6 - \frac{(-1+T) \in p_3^2 x_6^2}{2\,T} - \frac{(-3+T) \in^2 p_3^2 x_6^2}{4\,T} + \frac{(-1+T) \in p_3 p_6 x_6^2}{2\,T} + \frac{(-3+T) \in^2 p_3 p_6 x_6^2}{4\,T} - \\
&\frac{(1+T) \in^2 p_3^3 x_3 x_6^2}{2\,T} + \frac{(1+2\,T) \in^2 p_3^2 p_6 x_3 x_6^2}{2\,T} - \frac{1}{2} \in^2 p_3 p_6^2 x_3 x_6^2 - \frac{(-1+T)(1+3\,T) \in^2 p_3^3 x_6^3}{6\,T^2} + \\
&\frac{(-1+T)(1+5\,T) \in^2 p_3^2 p_6 x_6^3}{6\,T^2} - \frac{(-1+T) \in^2 p_3 p_6^2 x_6^3}{3\,T} - p_7 x_7 + x_1 \underline{p}_2 + x_3 \underline{p}_4 + \frac{x_4 \left(-\underline{p}_2 + T\,\underline{p}_2 + \underline{p}_5\right)}{T} + \\
&x_5 \underline{p}_6 + \frac{x_2 \left(\underline{p}_3 - \underline{p}_6 + T\,\underline{p}_6\right)}{T} + \frac{x_6 \left(-\underline{p}_4 + T\,\underline{p}_4 + \underline{p}_7\right)}{T} + p_1 \underline{x}_1 - \underline{p}_1 \underline{x}_1 + p_2 \underline{x}_2 - \underline{p}_2 \underline{x}_2 + p_3 \underline{x}_3 - \\
&\underline{p}_3 \underline{x}_3 + p_4 \underline{x}_4 - \underline{p}_4 \underline{x}_4 - \frac{1}{2} \in^2 \underline{p}_4 \underline{x}_4 + \in \left(1 + \underline{p}_4 \underline{x}_4\right) + p_5 \underline{x}_5 - \underline{p}_5 \underline{x}_5 + p_6 \underline{x}_6 - \underline{p}_6 \underline{x}_6 + p_7 \underline{x}_7 - \underline{p}_7 \underline{x}_7 \Big]
\end{aligned}
$$

In[∘]:=

```
ρ2vs[Knot[3, 1]]
```

Out[∘]=

$$
\{p_1, p_2, p_3, p_4, p_5, p_6, p_7, x_1, x_2, x_3, x_4, x_5, x_6, x_7, \underline{p}_1, \underline{p}_2, \underline{p}_3, \underline{p}_4, \underline{p}_5, \underline{p}_6, \underline{p}_7, \underline{x}_1, \underline{x}_2, \underline{x}_3, \underline{x}_4, \underline{x}_5, \underline{x}_6, \underline{x}_7\}
$$

```
K = Knot[3, 1]; ∫ℒ[K] ⅆ (ρ2vs@K)
```

Out[∘]=

$$\frac{T\,\mathbb{E}\left[\frac{(-1+T)^2\,(1+T^2)\,\epsilon}{(1-T+T^2)^2}-\frac{T^2\,(1-4\,T^2+T^4)\,\epsilon^2}{2\,(1-T+T^2)^4}\right]}{1-T+T^2}$$

In[∘]:= `T2z[T^-2 (1 - 4 T^2 + T^4)]`

Out[∘]=

$$-2+4\,z^2+z^4$$

In[∘]:= `Factor@(2 - 4 z^2 + 3 z^4 + 4 z^6 + z^8)`

Out[∘]=

$$2-4\,z^2+3\,z^4+4\,z^6+z^8$$

```
K = Knot[5, 2]; ∫ℒ[K] ⅆ (ρ2vs@K)
```

Out[∘]=

$$\frac{T\,\mathbb{E}\left[\frac{(-1+T)^2\,(5-4\,T+5\,T^2)\,\epsilon}{(2-3\,T+2\,T^2)^2}+\frac{(1-4\,T+11\,T^2-44\,T^3+76\,T^4-44\,T^5+11\,T^6-4\,T^7+T^8)\,\epsilon^2}{2\,(2-3\,T+2\,T^2)^4}\right]}{2-3\,T+2\,T^2}$$

In[∘]:= `T2z[(1 - 4 T + 11 T^2 - 44 T^3 + 76 T^4 - 44 T^5 + 11 T^6 - 4 T^7 + T^8) / T^4]`

Out[∘]=

$$4-20\,z^2+7\,z^4+4\,z^6+z^8$$

```
K = Knot[8, 19]; ∫ℒ[K] ⅆ (ρ2vs@K)
```

Out[∘]=

$$\frac{T^3\,\mathbb{E}\left[-\frac{(-1+T)^2\,(1+T^4)\,(3+4\,T^3+3\,T^6)\,\epsilon}{(1-T+T^2)^2\,(1-T^2+T^4)^2}\right]}{17\,179\,869\,184\,\pi^{34}\,(1-T+T^3-T^5+T^6)}$$

Concatenating edges

In[∘]:= `ℒ[Dot, i]`

Out[∘]=

$$\mathbb{E}\left[-x_i\,(p_i-\underline{p}_{1+i})\right]$$

In[∘]:= `ℒ[φ2, i + 1] /. ϵ → 0`

Out[∘]=

$$T^{\varphi 2/2}\,\mathbb{E}\left[-\left((-p_{1+i}+\underline{p}_{1+i})\,\underline{x}_{1+i}\right)\right]$$

In[∘]:= `(𝔼[π_i p_i] × ℒ[φ1, i] × ℒ[Dot, i] × ℒ[φ2, i + 1]) /. ϵ → 0`

Out[∘]=

$$T^{\frac{\varphi 1}{2}+\frac{\varphi 2}{2}}\,\mathbb{E}\left[p_i\,\pi_i-x_i\,(p_i-\underline{p}_{1+i})-(-p_i+\underline{p}_i)\,\underline{x}_i-(-p_{1+i}+\underline{p}_{1+i})\,\underline{x}_{1+i}\right]$$

In[∘]:= lhs = CF[∫∫(𝔼[π_i p_i] × ℒ[φ1, i] × ℒ[Dot, i] × ℒ[φ2, i + 1]) ⅆ{x_i, p_i, x̲_{i+1}, p̲_{i+1}}]

rhs = ∫(𝔼[π_i p_i] × ℒ[φ1 + φ2, i]) ⅆ{x̲_i, p̲_i}

Out[∘]=

$$\frac{1}{4\pi^2} T^{\frac{\varphi 1}{2}+\frac{\varphi 2}{2}} \mathbb{E}\left[-\underline{p}_i \underline{x}_i - \frac{1}{2}\epsilon^2 \varphi 1^2 \underline{p}_i \underline{x}_i + p_{1+i}(\pi_i + \underline{x}_i) - \epsilon\varphi 2\, p_{1+i}(\pi_i + \underline{x}_i) + \frac{1}{2}\epsilon^2\varphi 2^2 p_{1+i}(\pi_i + \underline{x}_i) + \frac{1}{2}\epsilon\left(\varphi 1 - \varphi 2 - 2\varphi 1\, \underline{p}_i \underline{x}_i\right)\right]$$

Out[∘]=

$$-\frac{i\, T^{\frac{\varphi 1}{2}+\frac{\varphi 2}{2}} \mathbb{E}\left[\frac{1}{2}\epsilon(-\varphi 1 - \varphi 2) + p_i \pi_i\right]}{2\pi}$$

In[∘]:= CF[𝔼[π_i p_i] × ℒ[φ1 + φ2, i]]

Out[∘]=

$$T^{\frac{\varphi 1}{2}+\frac{\varphi 2}{2}} \mathbb{E}\left[-\underline{p}_i \underline{x}_i - \frac{1}{2}\epsilon^2(\varphi 1 + \varphi 2)^2 \underline{p}_i \underline{x}_i + p_i(\pi_i + \underline{x}_i) - \frac{1}{2}\epsilon(\varphi 1 + \varphi 2)\left(-1 + 2\underline{p}_i \underline{x}_i\right)\right]$$

Invariance Under Reidemeister 3b

(𝔼[π_i p_i + π_j p_j + π_k p_k] × ℒ[1, i, j] × ℒ[1, i + 1, k] × ℒ[1, j + 1, k + 1] × ℒ[0, i + 1] × ℒ[0, j + 1] × ℒ[0, k + 1]) /. {ϵ → 0, T → 1}

Out[∘]=

$$\mathbb{E}\left[p_i\pi_i + p_j\pi_j + p_k\pi_k - x_i\left(p_i - \underline{p}_{1+i}\right) - x_{1+i}\left(p_{1+i} - \underline{p}_{2+i}\right) - x_j\left(p_j - \underline{p}_{1+j}\right) - x_{1+j}\left(p_{1+j} - \underline{p}_{2+j}\right) - x_k\left(p_k - \underline{p}_{1+k}\right) - x_{1+k}\left(p_{1+k} - \underline{p}_{2+k}\right) - \left(-p_{1+i} + \underline{p}_{1+i}\right)\underline{x}_{1+i} - \left(-p_{1+j} + \underline{p}_{1+j}\right)\underline{x}_{1+j} - \left(-p_{1+k} + \underline{p}_{1+k}\right)\underline{x}_{1+k}\right]$$

In[∘]:= {x_i, x_j, x_k, p_i, p_j, p_k, vs_{i+1}, vs_{j+1}, vs_{k+1}, x̲_{i+2}, x̲_{j+2}, x̲_{k+2}, p̲_{i+2}, p̲_{j+2}, p̲_{k+2}}

Out[∘]=

$$\{x_i, x_j, x_k, p_i, p_j, p_k, \underline{x}_{1+i}, \underline{p}_{1+i}, x_{1+i}, p_{1+i}, \underline{x}_{1+j}, \underline{p}_{1+j}, x_{1+j}, p_{1+j}, \underline{x}_{1+k}, \underline{p}_{1+k}, x_{1+k}, p_{1+k}, \underline{x}_{2+i}, \underline{x}_{2+j}, \underline{x}_{2+k}, \underline{p}_{2+i}, \underline{p}_{2+j}, \underline{p}_{2+k}\}$$

∫(𝔼[π_i p_i + π_j p_j + π_k p_k] × ℒ[1, i, j] × ℒ[1, i + 1, k] × ℒ[1, j + 1, k + 1] × ℒ[0, i + 1] × ℒ[0, j + 1] × ℒ[0, k + 1]) ⅆ{x_i, x_j, x_k, p_i, p_j, p_k, vs_{i+1}, vs_{j+1}, vs_{k+1}}

Out[∘]=

$Aborted

lhs = ∫(𝔼[π_i p_i + π_j p_j + π_k p_k] × ℒ[1, i, j] × ℒ[1, i + 1, k] × ℒ[1, j + 1, k + 1] × ℒ[0, i + 1] × ℒ[0, j + 1] × ℒ[0, k + 1]) ⅆ{x_i, x_j, x_k, p_i, p_j, p_k, vs_{i+1}, vs_{j+1}, vs_{k+1}}

rhs = ∫(𝔼[π_i p_i + π_j p_j + π_k p_k] × ℒ[1, j, k] × ℒ[1, i, k + 1] × ℒ[1, i + 1, j + 1] × ℒ[0, i + 1] × ℒ[0, j + 1] × ℒ[0, k + 1]) ⅆ{x_i, x_j, x_k, p_i, p_j, p_k, vs_{i+1}, vs_{j+1}, vs_{k+1}};

lhs == rhs

Out[∘]=

$$-\frac{1}{512\pi^9}$$

$$\mathbb{i}\, T^{3/2}\, \mathbb{E}\Big[T^2 \pi_i \underline{p}_{2+i} + T \pi_i \underline{p}_{2+j} - T^2 \pi_i \underline{p}_{2+j} + T \pi_j \underline{p}_{2+j} + \pi_i \underline{p}_{2+k} - T \pi_i \underline{p}_{2+k} + \pi_j \underline{p}_{2+k} - T \pi_j \underline{p}_{2+k} + \pi_k \underline{p}_{2+k} +$$

$$\frac{1}{2} \epsilon \Big(-3 + 2 T^2 \pi_i \underline{p}_{2+j} - 2 T \pi_j \underline{p}_{2+j} - T^3 \pi_i^2 \underline{p}_{2+i} \underline{p}_{2+j} + T^4 \pi_i^2 \underline{p}_{2+i} \underline{p}_{2+j} - 2 T^3 \pi_i \pi_j \underline{p}_{2+i} \underline{p}_{2+j} +$$

$$T^3 \pi_i^2 \underline{p}_{2+j}^2 - T^4 \pi_i^2 \underline{p}_{2+j}^2 + 2 T^3 \pi_i \pi_j \underline{p}_{2+j}^2 + 2 T \pi_i \underline{p}_{2+k} - 2 \pi_j \underline{p}_{2+k} + 4 T \pi_j \underline{p}_{2+k} -$$

$$4 \pi_k \underline{p}_{2+k} - T^2 \pi_i^2 \underline{p}_{2+i} \underline{p}_{2+k} + T^3 \pi_i^2 \underline{p}_{2+i} \underline{p}_{2+k} - 2 T^2 \pi_i \pi_j \underline{p}_{2+i} \underline{p}_{2+k} + 2 T^3 \pi_i \pi_j \underline{p}_{2+i} \underline{p}_{2+k} -$$

$$2 T^2 \pi_i \pi_k \underline{p}_{2+i} \underline{p}_{2+k} - T \pi_i^2 \underline{p}_{2+j} \underline{p}_{2+k} + 2 T^2 \pi_i^2 \underline{p}_{2+j} \underline{p}_{2+k} - T^3 \pi_i^2 \underline{p}_{2+j} \underline{p}_{2+k} - 2 T \pi_i \pi_j \underline{p}_{2+j} \underline{p}_{2+k} +$$

$$4 T^2 \pi_i \pi_j \underline{p}_{2+j} \underline{p}_{2+k} - 2 T^3 \pi_i \pi_j \underline{p}_{2+j} \underline{p}_{2+k} - T \pi_j^2 \underline{p}_{2+j} \underline{p}_{2+k} + T^2 \pi_j^2 \underline{p}_{2+j} \underline{p}_{2+k} -$$

$$2 T \pi_i \pi_k \underline{p}_{2+j} \underline{p}_{2+k} + 2 T^2 \pi_i \pi_k \underline{p}_{2+j} \underline{p}_{2+k} - 2 T \pi_j \pi_k \underline{p}_{2+j} \underline{p}_{2+k} + T \pi_i^2 \underline{p}_{2+k}^2 - T^2 \pi_i^2 \underline{p}_{2+k}^2 +$$

$$2 T \pi_i \pi_j \underline{p}_{2+k}^2 - 2 T^2 \pi_i \pi_j \underline{p}_{2+k}^2 + T \pi_j^2 \underline{p}_{2+k}^2 - T^2 \pi_j^2 \underline{p}_{2+k}^2 + 2 T \pi_i \pi_k \underline{p}_{2+k}^2 + 2 T \pi_j \pi_k \underline{p}_{2+k}^2\Big) +$$

$$\frac{1}{12} \epsilon^2 \Big(-6 T^2 \pi_i \underline{p}_{2+j} + 6 T \pi_j \underline{p}_{2+j} + 3 T^3 \pi_i^2 \underline{p}_{2+i} \underline{p}_{2+j} - 9 T^4 \pi_i^2 \underline{p}_{2+i} \underline{p}_{2+j} + 18 T^3 \pi_i \pi_j \underline{p}_{2+i} \underline{p}_{2+j} +$$

$$2 T^5 \pi_i^3 \underline{p}_{2+i}^2 \underline{p}_{2+j} - 2 T^6 \pi_i^3 \underline{p}_{2+i}^2 \underline{p}_{2+j} + 6 T^5 \pi_i^2 \pi_j \underline{p}_{2+i}^2 \underline{p}_{2+j} - 9 T^3 \pi_i^2 \underline{p}_{2+j}^2 + 15 T^4 \pi_i^2 \underline{p}_{2+j}^2 -$$

$$30 T^3 \pi_i \pi_j \underline{p}_{2+j}^2 + 2 T^4 \pi_i^3 \underline{p}_{2+i} \underline{p}_{2+j}^2 - 10 T^5 \pi_i^3 \underline{p}_{2+i} \underline{p}_{2+j}^2 + 8 T^6 \pi_i^3 \underline{p}_{2+i} \underline{p}_{2+j}^2 + 6 T^4 \pi_i^2 \pi_j \underline{p}_{2+i} \underline{p}_{2+j}^2 -$$

$$24 T^5 \pi_i^2 \pi_j \underline{p}_{2+i} \underline{p}_{2+j}^2 + 6 T^4 \pi_i \pi_j^2 \underline{p}_{2+i} \underline{p}_{2+j}^2 - 2 T^4 \pi_i^3 \underline{p}_{2+j}^3 + 8 T^5 \pi_i^3 \underline{p}_{2+j}^3 - 6 T^6 \pi_i^3 \underline{p}_{2+j}^3 -$$

$$6 T^4 \pi_i^2 \pi_j \underline{p}_{2+j}^3 + 18 T^5 \pi_i^2 \pi_j \underline{p}_{2+j}^3 - 6 T^4 \pi_i \pi_j^2 \underline{p}_{2+j}^3 - 6 T \pi_i \underline{p}_{2+k} + 6 \pi_j \underline{p}_{2+k} - 24 T \pi_j \underline{p}_{2+k} +$$

$$24 \pi_k \underline{p}_{2+k} + 3 T^2 \pi_i^2 \underline{p}_{2+i} \underline{p}_{2+k} - 9 T^3 \pi_i^2 \underline{p}_{2+i} \underline{p}_{2+k} + 18 T^2 \pi_i \pi_j \underline{p}_{2+i} \underline{p}_{2+k} - 30 T^3 \pi_i \pi_j \underline{p}_{2+i} \underline{p}_{2+k} +$$

$$30 T^2 \pi_i \pi_k \underline{p}_{2+i} \underline{p}_{2+k} + 2 T^4 \pi_i^3 \underline{p}_{2+i}^2 \underline{p}_{2+k} - 2 T^5 \pi_i^3 \underline{p}_{2+i}^2 \underline{p}_{2+k} + 6 T^4 \pi_i^2 \pi_j \underline{p}_{2+i}^2 \underline{p}_{2+k} -$$

$$6 T^5 \pi_i^2 \pi_j \underline{p}_{2+i}^2 \underline{p}_{2+k} + 6 T^4 \pi_i^2 \pi_k \underline{p}_{2+i}^2 \underline{p}_{2+k} + 3 T \pi_i^2 \underline{p}_{2+j} \underline{p}_{2+k} - 18 T^2 \pi_i^2 \underline{p}_{2+j} \underline{p}_{2+k} +$$

$$15 T^3 \pi_i^2 \underline{p}_{2+j} \underline{p}_{2+k} + 18 T \pi_i \pi_j \underline{p}_{2+j} \underline{p}_{2+k} - 60 T^2 \pi_i \pi_j \underline{p}_{2+j} \underline{p}_{2+k} + 42 T^3 \pi_i \pi_j \underline{p}_{2+j} \underline{p}_{2+k} +$$

$$15 T \pi_j^2 \underline{p}_{2+j} \underline{p}_{2+k} - 21 T^2 \pi_j^2 \underline{p}_{2+j} \underline{p}_{2+k} + 30 T \pi_i \pi_k \underline{p}_{2+j} \underline{p}_{2+k} - 42 T^2 \pi_i \pi_k \underline{p}_{2+j} \underline{p}_{2+k} +$$

$$42 T \pi_j \pi_k \underline{p}_{2+j} \underline{p}_{2+k} + 10 T^3 \pi_i^3 \underline{p}_{2+i} \underline{p}_{2+j} \underline{p}_{2+k} - 20 T^4 \pi_i^3 \underline{p}_{2+i} \underline{p}_{2+j} \underline{p}_{2+k} + 10 T^5 \pi_i^3 \underline{p}_{2+i} \underline{p}_{2+j} \underline{p}_{2+k} +$$

$$30 T^3 \pi_i^2 \pi_j \underline{p}_{2+i} \underline{p}_{2+j} \underline{p}_{2+k} - 54 T^4 \pi_i^2 \pi_j \underline{p}_{2+i} \underline{p}_{2+j} \underline{p}_{2+k} + 24 T^5 \pi_i^2 \pi_j \underline{p}_{2+i} \underline{p}_{2+j} \underline{p}_{2+k} +$$

$$24 T^3 \pi_i \pi_j^2 \underline{p}_{2+i} \underline{p}_{2+j} \underline{p}_{2+k} - 24 T^4 \pi_i \pi_j^2 \underline{p}_{2+i} \underline{p}_{2+j} \underline{p}_{2+k} + 24 T^3 \pi_i^2 \pi_k \underline{p}_{2+i} \underline{p}_{2+j} \underline{p}_{2+k} -$$

$$24 T^4 \pi_i^2 \pi_k \underline{p}_{2+i} \underline{p}_{2+j} \underline{p}_{2+k} + 36 T^3 \pi_i \pi_j \pi_k \underline{p}_{2+i} \underline{p}_{2+j} \underline{p}_{2+k} + 2 T^2 \pi_i^3 \underline{p}_{2+j}^2 \underline{p}_{2+k} - 12 T^3 \pi_i^3 \underline{p}_{2+j}^2 \underline{p}_{2+k} +$$

$$18 T^4 \pi_i^3 \underline{p}_{2+j}^2 \underline{p}_{2+k} - 8 T^5 \pi_i^3 \underline{p}_{2+j}^2 \underline{p}_{2+k} + 6 T^2 \pi_i^2 \pi_j \underline{p}_{2+j}^2 \underline{p}_{2+k} - 36 T^3 \pi_i^2 \pi_j \underline{p}_{2+j}^2 \underline{p}_{2+k} +$$

$$48 T^4 \pi_i^2 \pi_j \underline{p}_{2+j}^2 \underline{p}_{2+k} - 18 T^5 \pi_i^2 \pi_j \underline{p}_{2+j}^2 \underline{p}_{2+k} + 6 T^2 \pi_i \pi_j^2 \underline{p}_{2+j}^2 \underline{p}_{2+k} - 30 T^3 \pi_i \pi_j^2 \underline{p}_{2+j}^2 \underline{p}_{2+k} +$$

$$24 T^4 \pi_i \pi_j^2 \underline{p}_{2+j}^2 \underline{p}_{2+k} + 2 T^2 \pi_j^3 \underline{p}_{2+j}^2 \underline{p}_{2+k} - 2 T^3 \pi_j^3 \underline{p}_{2+j}^2 \underline{p}_{2+k} + 6 T^2 \pi_i^2 \pi_k \underline{p}_{2+j}^2 \underline{p}_{2+k} -$$

$$24 T^3 \pi_i^2 \pi_k \underline{p}_{2+j}^2 \underline{p}_{2+k} + 18 T^4 \pi_i^2 \pi_k \underline{p}_{2+j}^2 \underline{p}_{2+k} + 12 T^2 \pi_i \pi_j \pi_k \underline{p}_{2+j}^2 \underline{p}_{2+k} - 36 T^3 \pi_i \pi_j \pi_k \underline{p}_{2+j}^2 \underline{p}_{2+k} +$$

$$6 T^2 \pi_j^2 \pi_k \underline{p}_{2+j}^2 \underline{p}_{2+k} - 9 T \pi_i^2 \underline{p}_{2+k}^2 + 15 T^2 \pi_i^2 \underline{p}_{2+k}^2 - 30 T \pi_i \pi_j \underline{p}_{2+k}^2 + 42 T^2 \pi_i \pi_j \underline{p}_{2+k}^2 - 21 T \pi_j^2 \underline{p}_{2+k}^2 +$$

$$27 T^2 \pi_j^2 \underline{p}_{2+k}^2 - 42 T \pi_i \pi_k \underline{p}_{2+k}^2 - 54 T \pi_j \pi_k \underline{p}_{2+k}^2 + 2 T^2 \pi_i^3 \underline{p}_{2+i} \underline{p}_{2+k}^2 - 10 T^3 \pi_i^3 \underline{p}_{2+i} \underline{p}_{2+k}^2 +$$

$$8 T^4 \pi_i^3 \underline{p}_{2+i} \underline{p}_{2+k}^2 + 6 T^2 \pi_i^2 \pi_j \underline{p}_{2+i} \underline{p}_{2+k}^2 - 30 T^3 \pi_i^2 \pi_j \underline{p}_{2+i} \underline{p}_{2+k}^2 + 24 T^4 \pi_i^2 \pi_j \underline{p}_{2+i} \underline{p}_{2+k}^2 +$$

$$6 T^2 \pi_i \pi_j^2 \underline{p}_{2+i} \underline{p}_{2+k}^2 - 24 T^3 \pi_i \pi_j^2 \underline{p}_{2+i} \underline{p}_{2+k}^2 + 18 T^4 \pi_i \pi_j^2 \underline{p}_{2+i} \underline{p}_{2+k}^2 + 6 T^2 \pi_i^2 \pi_k \underline{p}_{2+i} \underline{p}_{2+k}^2 -$$

$$24 T^3 \pi_i^2 \pi_k \underline{p}_{2+i} \underline{p}_{2+k}^2 + 12 T^2 \pi_i \pi_j \pi_k \underline{p}_{2+i} \underline{p}_{2+k}^2 - 36 T^3 \pi_i \pi_j \pi_k \underline{p}_{2+i} \underline{p}_{2+k}^2 + 6 T^2 \pi_i \pi_k^2 \underline{p}_{2+i} \underline{p}_{2+k}^2 +$$

$$2 T \pi_i^3 \underline{p}_{2+j} \underline{p}_{2+k}^2 - 12 T^2 \pi_i^3 \underline{p}_{2+j} \underline{p}_{2+k}^2 + 18 T^3 \pi_i^3 \underline{p}_{2+j} \underline{p}_{2+k}^2 - 8 T^4 \pi_i^3 \underline{p}_{2+j} \underline{p}_{2+k}^2 + 6 T \pi_i^2 \pi_j \underline{p}_{2+j} \underline{p}_{2+k}^2 -$$

$$36 T^2 \pi_i^2 \pi_j \underline{p}_{2+j} \underline{p}_{2+k}^2 + 54 T^3 \pi_i^2 \pi_j \underline{p}_{2+j} \underline{p}_{2+k}^2 - 24 T^4 \pi_i^2 \pi_j \underline{p}_{2+j} \underline{p}_{2+k}^2 + 6 T \pi_i \pi_j^2 \underline{p}_{2+j} \underline{p}_{2+k}^2 -$$

$$36 T^2 \pi_i \pi_j^2 \underline{p}_{2+j} \underline{p}_{2+k}^2 + 48 T^3 \pi_i \pi_j^2 \underline{p}_{2+j} \underline{p}_{2+k}^2 - 18 T^4 \pi_i \pi_j^2 \underline{p}_{2+j} \underline{p}_{2+k}^2 + 2 T \pi_j^3 \underline{p}_{2+j} \underline{p}_{2+k}^2 -$$

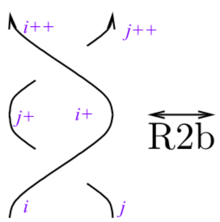
$$10 T^2 \pi_j^3 \underline{p}_{2+j} \underline{p}_{2+k}^2 + 8 T^3 \pi_j^3 \underline{p}_{2+j} \underline{p}_{2+k}^2 + 6 T \pi_i^2 \pi_k \underline{p}_{2+j} \underline{p}_{2+k}^2 - 30 T^2 \pi_i^2 \pi_k \underline{p}_{2+j} \underline{p}_{2+k}^2 +$$

$$24\,T^3\pi_i^2\pi_k\,\underline{p}_{2+j}\,\underline{p}^2_{2+k}+12\,T\,\pi_i\pi_j\pi_k\,\underline{p}_{2+j}\,\underline{p}^2_{2+k}-60\,T^2\pi_i\pi_j\pi_k\,\underline{p}_{2+j}\,\underline{p}^2_{2+k}+36\,T^3\pi_i\pi_j\pi_k\,\underline{p}_{2+j}\,\underline{p}^2_{2+k}+$$
$$6\,T\,\pi_j^2\pi_k\,\underline{p}_{2+j}\,\underline{p}^2_{2+k}-24\,T^2\pi_j^2\pi_k\,\underline{p}_{2+j}\,\underline{p}^2_{2+k}+6\,T\,\pi_i\pi_k^2\,\underline{p}_{2+j}\,\underline{p}^2_{2+k}-6\,T^2\pi_i\pi_k^2\,\underline{p}_{2+j}\,\underline{p}^2_{2+k}+$$
$$6\,T\,\pi_j\pi_k^2\,\underline{p}_{2+j}\,\underline{p}^2_{2+k}-2\,T\,\pi_i^3\,\underline{p}^3_{2+k}+8\,T^2\pi_i^3\,\underline{p}^3_{2+k}-6\,T^3\pi_i^3\,\underline{p}^3_{2+k}-6\,T\,\pi_i^2\pi_j\,\underline{p}^3_{2+k}+24\,T^2\pi_i^2\pi_j\,\underline{p}^3_{2+k}-$$
$$18\,T^3\pi_i^2\pi_j\,\underline{p}^3_{2+k}-6\,T\,\pi_i\pi_j^2\,\underline{p}^3_{2+k}+24\,T^2\pi_i\pi_j^2\,\underline{p}^3_{2+k}-18\,T^3\pi_i\pi_j^2\,\underline{p}^3_{2+k}-2\,T\,\pi_j^3\,\underline{p}^3_{2+k}+$$
$$8\,T^2\pi_j^3\,\underline{p}^3_{2+k}-6\,T^3\pi_j^3\,\underline{p}^3_{2+k}-6\,T\,\pi_i^2\pi_k\,\underline{p}^3_{2+k}+18\,T^2\pi_i^2\pi_k\,\underline{p}^3_{2+k}-12\,T\,\pi_i\pi_j\pi_k\,\underline{p}^3_{2+k}+$$
$$36\,T^2\pi_i\pi_j\pi_k\,\underline{p}^3_{2+k}-6\,T\,\pi_j^2\pi_k\,\underline{p}^3_{2+k}+18\,T^2\pi_j^2\pi_k\,\underline{p}^3_{2+k}-6\,T\,\pi_i\pi_k^2\,\underline{p}^3_{2+k}-6\,T\,\pi_j\pi_k^2\,\underline{p}^3_{2+k}\Big)\Big]$$

Out[∘]=

```
True
```

Invariance Under Reidemeister 2b



Territory: $(x \mid p)_{(i\mid j)^+},\ x_{i\mid j}$.

In[∘]:=
```
lhs = ∫(𝔼[π_i p_i + π_j p_j] × ℒ[1, i, j] × ℒ[-1, i + 1, j + 1] × ℒ[0, i + 1] × ℒ[0, j + 1])
    ⅆ{x_i, x_j, p_i, p_j, vs_{i+1}, vs_{j+1}}
```

Out[∘]=

$$\frac{\mathbb{E}\left[\pi_i\,\underline{p}_{2+i}+\pi_j\,\underline{p}_{2+j}\right]}{64\,\pi^6}$$

In[∘]:=
```
rhs = ∫(𝔼[π_i p_i + π_j p_j] × ℒ[0, i + 1] × ℒ[0, j + 1]) ℒ[Dot, i] ×
    ℒ[Dot, j] × ℒ[Dot, i + 1] × ℒ[Dot, j + 1] ⅆ{x_i, x_j, p_i, p_j, vs_{i+1}, vs_{j+1}}
lhs == rhs
```

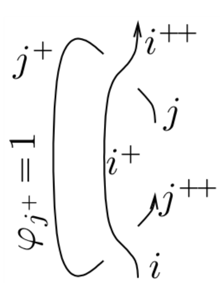
Out[∘]=

$$\frac{\mathbb{E}\left[\pi_i\,\underline{p}_{2+i}+\pi_j\,\underline{p}_{2+j}\right]}{64\,\pi^6}$$

Out[∘]=

```
True
```

Invariance Under R2c



In[∘]:=
```
lhs = ∫(𝔼[π_i p_i + π_j p_j] × ℒ[-1, i, j + 1] × ℒ[1, i + 1, j] × ℒ[0, i + 1] × ℒ[1, j + 1])
   ⅆ{x_i, x_j, p_i, p_j, vs_{i+1}, vs_{j+1}}
```

Out[∘]=

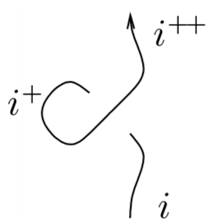
$$\frac{\sqrt{T}\ \mathbb{E}\left[\pi_i\, \underline{p}_{2+i} + \pi_j\, \underline{p}_{2+j} + \frac{1}{2}\, \epsilon^2\, \pi_j\, \underline{p}_{2+j} + \frac{1}{2}\, \epsilon\, \left(-1 - 2\, \pi_j\, \underline{p}_{2+j}\right)\right]}{64\, \pi^6}$$

In[∘]:=
```
rhs = ∫(𝔼[π_i p_i + π_j p_j] × ℒ[Dot, i] × ℒ[Dot, j] × ℒ[Dot, i + 1] ×
     ℒ[Dot, j + 1] × ℒ[0, i + 1] × ℒ[1, j + 1]) ⅆ{x_i, x_j, p_i, p_j, vs_{i+1}, vs_{j+1}};
lhs == rhs
```

Out[∘]=

True

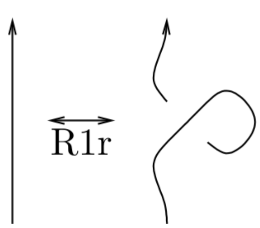
Invariance Under R1l




```
lhs = ∫(𝔼[π_i p_i] × ℒ[1, i + 1, i] × ℒ[0, i] × ℒ[1, i + 1]) ⅆ{x_i, p_i, x_{i+1}, p_{i+1}}

rhs = ∫(𝔼[π_i p_i] × ℒ[0, i] × ℒ[0, i + 1]) ⅆ{x_i, p_i, x_{i+1}, p_{i+1}};
lhs == rhs
```

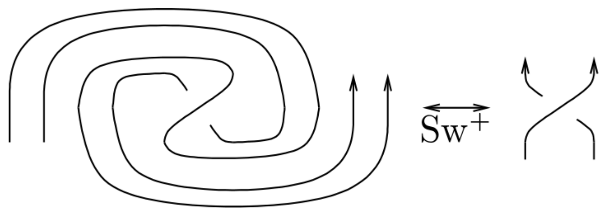
Invariance Under R1r




```
lhs = ∫(𝔼[π_i p_i] × ℒ[1, i, i + 1] × ℒ[0, i] × ℒ[-1, i + 1]) ⅆ{x_i, p_i, x_{i+1}, p_{i+1}}

rhs = ∫(𝔼[π_i p_i] × ℒ[0, i] × ℒ[0, i + 1]) ⅆ{x_i, p_i, x_{i+1}, p_{i+1}};
lhs == rhs
```

Invariance Under Sw




```
CF /@ {ℒ[1, j], ℒ[1, i, j]}
```

$$\text{lhs} = \int \left(\mathbb{E}[\pi_i p_i + \pi_j p_j + \epsilon\, \pi_{i+1} p_{i+1} + \epsilon\, \pi_{j+1} p_{j+1} + \xi_{i+1} x_{i+1} + \xi_{j+1} x_{j+1}] \times \mathcal{L}[1, i, j] \times \mathcal{L}[-1, i] \times \mathcal{L}[1, i+1] \times \mathcal{L}[-1, j] \times \mathcal{L}[1, j+1]\right) d\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}\}$$

$$\text{rhs} = \int \left(\mathbb{E}[\pi_i p_i + \pi_j p_j + \epsilon\, \pi_{i+1} p_{i+1} + \epsilon\, \pi_{j+1} p_{j+1} + \xi_{i+1} x_{i+1} + \xi_{j+1} x_{j+1}] \times \mathcal{L}[1, i, j] \times \mathcal{L}[0, i] \times \mathcal{L}[0, i+1] \times \mathcal{L}[0, j] \times \mathcal{L}[0, j+1]\right) d\{x_i, x_j, p_i, p_j, x_{i+1}, x_{j+1}, p_{i+1}, p_{j+1}\};$$

```
lhs == rhs
```