

Pensieve header: Mathematica notebook for the 2-variable perturbed Alexander invariant.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\HigherRank"];
Once[<< KnotTheory` ; << ../APAI/Rot.m];
TS = T S;
( $\alpha_{-}^{+}$ )+ :=  $\alpha^{++}$ ; (* this is for cosmetic reasons only *)
 $\delta_{i,j}$  := If[i == j, 1, 0];
```

Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.

Read more at <http://katlas.org/wiki/KnotTheory>.

Loading Rot.m from <http://drorbn.net/APAI> to compute rotation numbers.

Lower Rank 2 Gassner: (Modified from Rank2Gassner.nb)

```
In[ ]:= LR2G $_{i,j}$ [ $\mathcal{E}$ ] := Expand[ $\mathcal{E}$  /. { $e_i$  => T  $e_i$  + (1 - T)  $e_j$ ,  $f_i$  => S  $f_i$  + (1 - S)  $f_j$ ,
   $g_i$  => T S  $g_i$  + (1 - T S)  $g_j$ 
  + S (T - 1)  $e_j f_i$  + (S - 1) T  $e_i f_j$  + (S + T - 2 S T)  $e_j f_j$  }];
LR2G $_{i,j}$ [ $\mathcal{E}$ ] := Expand[ $\mathcal{E}$  /. { $e_i$  => T-1  $e_i$  + (1 - T-1)  $e_j$ ,  $f_i$  => S-1  $f_i$  + (1 - S-1)  $f_j$ ,
   $g_i$  => T-1 S-1  $g_i$  + (1 - T-1 S-1)  $g_j$ 
  - S-1 (1 - T-1)  $e_j f_i$  - T-1 (1 - S-1)  $e_i f_j$  + (S-1 + T-1 - 2 S-1 T-1)  $e_j f_j$  }];
bas =
  { $e_1$ ,  $e_2$ ,  $e_3$ ,  $f_1$ ,  $f_2$ ,  $f_3$ ,  $e_1 f_1$ ,  $e_1 f_2$ ,  $e_1 f_3$ ,  $e_2 f_1$ ,  $e_2 f_2$ ,  $e_2 f_3$ ,  $e_3 f_1$ ,  $e_3 f_2$ ,  $e_3 f_3$ ,  $g_1$ ,  $g_2$ ,  $g_3$ };
(bas // LR2G $_{1,2}$  // LR2G $_{1,2}$ ) == bas
(bas // LR2G $_{3,2}$  // LR2G $_{3,2}$ ) == bas
(1hs = bas // LR2G $_{1,2}$  // LR2G $_{1,3}$  // LR2G $_{2,3}$ ) == (bas // LR2G $_{2,3}$  // LR2G $_{1,3}$  // LR2G $_{1,2}$ )

Out[ ]:=
True

Out[ ]:=
True

Out[ ]:=
True
```

```

In[*]:= R1[s_, i_, j_] := TBD; (* a sign(s)-dependent quadratic polynomial in  $g_{1\alpha\beta}$ ,  $g_{2\alpha\beta}$ ,  $g_{3\alpha\beta}$ ,
and  $y_{\alpha\beta\gamma}$ , where  $\deg y_{\alpha\beta\gamma}=2$  and  $\alpha, \beta, \gamma \in \{i, j\}$ , with coefficients in  $\mathbb{Z}[T^{\pm 1}, S^{\pm 1}]$ . *)
CF[ε_] := Factor@Together[ε];
λ[K_] := Module[{Cs, φ, n, A, s, i, j, k, Δ, G, gEval, Y, yEval, c, λ1},
  {Cs, φ} = Rot[K]; n = Length[Cs];
  A = IdentityMatrix[2 n + 1];
  Cases[Cs, {s_, i_, j_} => (A[[{i, j}, {i + 1, j + 1}]] += (

$$\begin{pmatrix} -T^s & T^s - 1 \\ 0 & -1 \end{pmatrix}$$

))];
  Δ = T(-Total[φ]-Total[Cs[[All,1]])/2 Det[A];
  G = Inverse[A];
  gEval[ε_] := CF[ε /. α_+ => α + 1 /.
    {g1,α,β_ => G[[α, β]], g2,α,β_ => (G[[α, β]] /. T -> S), g3,α,β_ => (G[[α, β]] /. T -> TS)}];
  Y[α_, β_, γ_] := Sum[{s, i, j} = c;
    g3,α,i (Ss (Ts - 1) g1,j+,β g2,i+,γ + (Ss - 1) Ts g1,i+,β g2,j+,γ + (Ss + Ts - 2 TSs) g1,j+,β g2,j+,γ),
    {c, Cs}];
  yEval[ε_] := CF[ε /. y_α,β,γ_ => Y[α, β, γ]];
  λ1 = Sumk=1n R1@@Cs[[k]] - Sumk=12n φ[[k]] (g1,k,k + g2,k,k + g3,k,k);
  {Δ, Δ (Δ /. T -> S) (Δ /. T -> TS) λ1} // yEval // gEval
];

```

Step-by-step Run-Through

```

In[*]:= CF[ε_] := Factor@Together[ε];

```

```

In[*]:= K = {
  PD[X[4, 2, 5, 1], X[2, 6, 3, 5], X[6, 4, 7, 3]],
  Knot[6, 2]
}[[1]]

```

```

Out[*]=
PD[X[4, 2, 5, 1], X[2, 6, 3, 5], X[6, 4, 7, 3]]

```

```

In[*]:= {Cs, φ} = Rot[K]

```

```

Out[*]=
{{{1, 1, 4}, {1, 5, 2}, {1, 3, 6}}, {0, 0, 0, -1, 0, 0}}

```

```

In[*]:= n = Length[Cs];

```

```

A = IdentityMatrix[2 n + 1];

```

```

Cases[Cs, {s_, i_, j_} => (A[[{i, j}, {i + 1, j + 1}]] += (

$$\begin{pmatrix} -T^s & T^s - 1 \\ 0 & -1 \end{pmatrix}$$

))];

```

In[*]:= **A // MatrixForm**

Out[*]//MatrixForm=

$$\begin{pmatrix} 1 & -T & 0 & 0 & -1+T & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -T & 0 & 0 & -1+T \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1+T & 0 & 1 & -T & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In[*]:= $\Delta = T^{(-Total[\varnothing]-Total[Cs[All,1]])/2} \text{Det}[A]$

Out[*]=

$$\frac{1 - T + T^2}{T}$$

In[*]:= **G = Inverse[A];**
G // MatrixForm

Out[*]//MatrixForm=

$$\begin{pmatrix} 1 & \frac{T-T^2+T^3}{1-T+T^2} & 1 & \frac{T-T^2+T^3}{1-T+T^2} & 1 & \frac{T-T^2+T^3}{1-T+T^2} & 1 \\ 0 & 1 & \frac{1}{1-T+T^2} & \frac{T}{1-T+T^2} & \frac{T}{1-T+T^2} & \frac{T^2}{1-T+T^2} & 1 \\ 0 & 0 & \frac{1}{1-T+T^2} & \frac{T}{1-T+T^2} & \frac{T}{1-T+T^2} & \frac{T^2}{1-T+T^2} & 1 \\ 0 & 0 & \frac{1-T}{1-T+T^2} & \frac{1}{1-T+T^2} & \frac{1}{1-T+T^2} & \frac{T}{1-T+T^2} & 1 \\ 0 & 0 & \frac{1-T}{1-T+T^2} & \frac{T-T^2}{1-T+T^2} & \frac{1}{1-T+T^2} & \frac{T}{1-T+T^2} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In[*]:= **gEval[ε_] := CF[ε /. α_+ := α + 1 /.**

{g1,α,β := G[α, β], g2,α,β := (G[α, β] /. T → S), g3,α,β := (G[α, β] /. T → TS)}];

In[*]:= **Clear[Y]**

Y[α_, β_, γ_] := Y[α, β, γ] = Sum[{s, i, j} = c;

g3,α,i (S^s (T^s - 1) g1,j^{+,β} g2,i^{+,γ} + (S^s - 1) T^s g1,i^{+,β} g2,j^{+,γ} + (S^s + T^s - 2 TS^s) g1,j^{+,β} g2,j^{+,γ}),
{c, Cs}];

Column[MatrixForm/@Table[Y[α, β, γ], {α, 3}, {β, 3}, {γ, 3}]]

Out[*]=

$$\begin{pmatrix} (S(-1+T) g_{1,4^+,1} g_{2,1^+,1} + (-1+S) T g_{1,1^+,1} g_{2,4^+,1} + (S+T-2ST) g_{1,4^+,1} g_{2,4^+,1}) g_{3,1,1} + (S(-1+T) g_{1,4^+,2} g_{2,1^+,1} + (-1+S) T g_{1,1^+,2} g_{2,4^+,1} + (S+T-2ST) g_{1,4^+,2} g_{2,4^+,1}) g_{3,1,1} + (S(-1+T) g_{1,4^+,3} g_{2,1^+,1} + (-1+S) T g_{1,1^+,3} g_{2,4^+,1} + (S+T-2ST) g_{1,4^+,3} g_{2,4^+,1}) g_{3,1,1} + (S(-1+T) g_{1,4^+,1} g_{2,1^+,1} + (-1+S) T g_{1,1^+,1} g_{2,4^+,1} + (S+T-2ST) g_{1,4^+,1} g_{2,4^+,1}) g_{3,2,1} + (S(-1+T) g_{1,4^+,2} g_{2,1^+,1} + (-1+S) T g_{1,1^+,2} g_{2,4^+,1} + (S+T-2ST) g_{1,4^+,2} g_{2,4^+,1}) g_{3,2,1} + (S(-1+T) g_{1,4^+,3} g_{2,1^+,1} + (-1+S) T g_{1,1^+,3} g_{2,4^+,1} + (S+T-2ST) g_{1,4^+,3} g_{2,4^+,1}) g_{3,2,1} + (S(-1+T) g_{1,4^+,1} g_{2,1^+,1} + (-1+S) T g_{1,1^+,1} g_{2,4^+,1} + (S+T-2ST) g_{1,4^+,1} g_{2,4^+,1}) g_{3,3,1} + (S(-1+T) g_{1,4^+,2} g_{2,1^+,1} + (-1+S) T g_{1,1^+,2} g_{2,4^+,1} + (S+T-2ST) g_{1,4^+,2} g_{2,4^+,1}) g_{3,3,1} + (S(-1+T) g_{1,4^+,3} g_{2,1^+,1} + (-1+S) T g_{1,1^+,3} g_{2,4^+,1} + (S+T-2ST) g_{1,4^+,3} g_{2,4^+,1}) g_{3,3,1} + (S(-1+T) g_{1,4^+,1} g_{2,1^+,1} + (-1+S) T g_{1,1^+,1} g_{2,4^+,1} + (S+T-2ST) g_{1,4^+,1} g_{2,4^+,1}) g_{3,3,1} \end{pmatrix}$$

```
In[*]:= Column[MatrixForm/@Table[Y[α, β, γ] // gEval, {α, 3}, {β, 3}, {γ, 3}]]
```

```
Out[*]=
```

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{(-1+S)^2 T}{1-S+S^2} \\ 0 & -\frac{S(-1+T)^2}{1-T+T^2} & -\frac{-S+S^2-T+2ST-2S^2T+T^2-2ST^2+2S^2T^2}{(1-S+S^2)(1-T+T^2)} \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{ST(-S-T+2ST)}{(1-S+S^2)(1-T+T^2)(1-S+T+S^2T^2)} \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{ST(-S-T+2ST)}{(1-S+S^2)(1-T+T^2)(1-S+T+S^2T^2)} \end{pmatrix}$$

```
In[*]:= yEval[ε_] := CF[ε /. y_{α,β,γ} -> Y[α, β, γ]]
```

The g-Rules

```
In[*]:= gRules_{s_,i_,j_} := {
  g_{1,i,β} -> δ_{i,β} + T^S g_{1,i^+,β} + (1 - T^S) g_{1,j^+,β}, g_{1,j,β} -> δ_{j,β} + g_{1,j^+,β},
  g_{1,α,i} -> T^{-S} (g_{1,α,i^+} - δ_{α,i^+}), g_{1,α,j} -> g_{1,α,j^+} - (1 - T^S) g_{1,α,i} - δ_{α,j^+},
  g_{2,i,β} -> δ_{i,β} + S^S g_{2,i^+,β} + (1 - S^S) g_{2,j^+,β}, g_{2,j,β} -> δ_{j,β} + g_{2,j^+,β},
  g_{2,α,i} -> S^{-S} (g_{2,α,i^+} - δ_{α,i^+}), g_{2,α,j} -> g_{2,α,j^+} - (1 - S^S) g_{2,α,i} - δ_{α,j^+},
  g_{3,i,β} -> δ_{i,β} + TS^S g_{3,i^+,β} + (1 - TS^S) g_{3,j^+,β}, g_{3,j,β} -> δ_{j,β} + g_{3,j^+,β},
  g_{3,α,i} -> TS^{-S} (g_{3,α,i^+} - δ_{α,i^+}), g_{3,α,j} -> g_{3,α,j^+} - (1 - TS^S) g_{3,α,i} - δ_{α,j^+}
}
```

```
In[*]:= (gs = Table[{g_{1,α,β}, g_{2,α,β}, g_{3,α,β}}, {α, 2n+1}, {β, 2n+1}]) [[1 ;; 2, 1 ;; 2, 1 ;; 2]]
```

```
Out[*]=
```

```
{{{g_{1,1,1}, g_{2,1,1}}, {g_{1,1,2}, g_{2,1,2}}, {{g_{1,2,1}, g_{2,2,1}}, {g_{1,2,2}, g_{2,2,2}}}}
```

```
In[*]:= Cs
```

```
Out[*]=
```

```
{{1, 1, 4}, {1, 5, 2}, {1, 3, 6}}
```

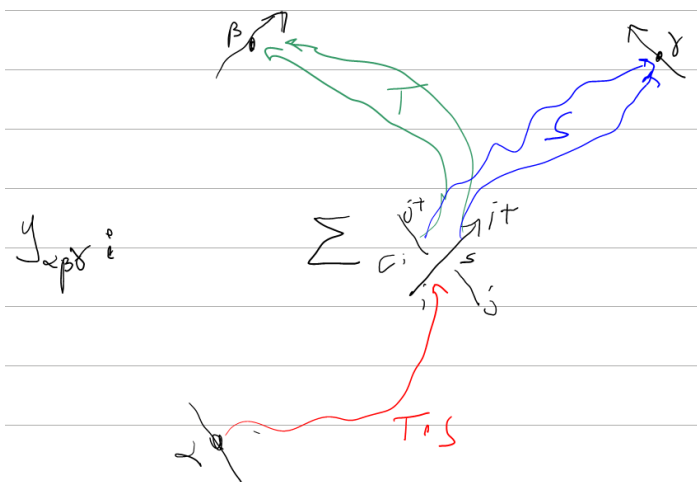
```
In[*]:= (gs /. gRules_{Sequence@@Cs[[1]]) [[1 ;; 2, 1 ;; 2, 1 ;; 2]]
```

```
Out[*]=
```

```
{
  {{(1 + T) g_{1,1^+,1} + (1 - T) g_{1,4^+,1}, (1 + S) g_{2,1^+,1} + (1 - S) g_{2,4^+,1}},
  {T g_{1,1^+,2} + (1 - T) g_{1,4^+,2}, S g_{2,1^+,2} + (1 - S) g_{2,4^+,2}},
  {{-If[2 == 1^+, 1, 0] + g_{1,2,1^+}, -If[2 == 1^+, 1, 0] + g_{2,2,1^+}}, {g_{1,2,2}, g_{2,2,2}}}
}
```

```
In[ ]:= gEval[gs - (gs /. Flatten@Table[gRulesSequence@@c, {c, Cs}]]]
Out[ ]=
{{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}}}
```

The y-Rules



```
In[ ]:= yRuless, i, j := {
  yi, beta, gamma -> TSs yi+, beta, gamma + (1 - TSs) yj+, beta, gamma +
    (Ss (Ts - 1) g1, j+, beta g2, i+, gamma + (Ss - 1) Ts g1, i+, beta g2, j+, gamma + (Ss + Ts - 2 TSs) g1, j+, beta g2, j+, gamma),
  yj, beta, gamma -> yj+, beta, gamma,
  yalpha, i, gamma -> T-s yalpha, i+, gamma + (1 - Ss) g3, alpha, i g2, j+, gamma,
  yalpha, j, gamma -> yalpha, j+, gamma - (1 - Ts) yalpha, i, gamma - g3, alpha, i (Ss (Ts - 1) g2, i+, gamma + (Ss + Ts - 2 TSs) g2, j+, gamma),
  yalpha, beta, i -> S-s yalpha, beta, i+ + (1 - Ts) g3, alpha, i g1, j+, beta,
  yalpha, beta, j -> yalpha, beta, j+ - (1 - Ss) yalpha, beta, i - g3, alpha, i (Ts (Ss - 1) g1, i+, beta + (Ss + Ts - 2 TSs) g1, beta, j+)
}
```

```
In[ ]:= (ys = Table[yalpha, beta, gamma, {alpha, 2 n + 1}, {beta, 2 n + 1}, {gamma, 2 n + 1}]) [[1 ;; 2, 1 ;; 2, 1 ;; 2]]
```

```
Out[ ]=
{{{y1,1,1, y1,1,2}, {y1,2,1, y1,2,2}, {{y2,1,1, y2,1,2}, {y2,2,1, y2,2,2}}}
```

```
In[ ]:= Cs
```

```
Out[ ]=
{{1, 1, 4}, {1, 5, 2}, {1, 3, 6}}
```

In[*]:= (ys /. yRules_{Sequence@@Cs[[1]]}) [[1 ;; 2, 1 ;; 2, 1 ;; 2]]

Out[*]=

$$\left\{ \left\{ \left\{ S(-1+T)g_{1,4^+}g_{2,1^+} + (-1+S)Tg_{1,1^+}g_{2,4^+} + (S+T-2ST)g_{1,4^+}g_{2,4^+} + STy_{1^+,1,1} + (1-ST)y_{4^+,1,1}, S(-1+T)g_{1,4^+}g_{2,1^+,2} + (-1+S)Tg_{1,1^+,2}g_{2,4^+,2} + (S+T-2ST)g_{1,4^+,2}g_{2,4^+,2} + STy_{1^+,1,2} + (1-ST)y_{4^+,1,2} \right\}, \left\{ S(-1+T)g_{1,4^+,2}g_{2,1^+,1} + (-1+S)Tg_{1,1^+,2}g_{2,4^+,1} + (S+T-2ST)g_{1,4^+,2}g_{2,4^+,1} + STy_{1^+,2,1} + (1-ST)y_{4^+,2,1}, S(-1+T)g_{1,4^+,2}g_{2,1^+,2} + (-1+S)Tg_{1,1^+,2}g_{2,4^+,2} + (S+T-2ST)g_{1,4^+,2}g_{2,4^+,2} + STy_{1^+,2,2} + (1-ST)y_{4^+,2,2} \right\} \right\}, \left\{ \left\{ (1-S)g_{2,4^+,1}g_{3,2,1} + \frac{y_{2,1^+,1}}{T}, (1-S)g_{2,4^+,2}g_{3,2,1} + \frac{y_{2,1^+,2}}{T} \right\}, \left\{ (1-T)g_{1,4^+,2}g_{3,2,1} + \frac{y_{2,2,1}}{S}, y_{2,2,2} \right\} \right\} \right\}$$

In[*]:= yEval[ys - (ys /. yRules_{Sequence@@Cs[[1]]})] [[1, 1, 1]]

Out[*]=

$$\begin{aligned} & Sg_{1,4^+}g_{2,1^+} - STg_{1,4^+}g_{2,1^+} + Tg_{1,1^+}g_{2,4^+} - STg_{1,1^+}g_{2,4^+} - Sg_{1,4^+}g_{2,4^+} - \\ & Tg_{1,4^+}g_{2,4^+} + 2STg_{1,4^+}g_{2,4^+} - Sg_{1,4^+}g_{2,1^+}g_{3,1,1} + STg_{1,4^+}g_{2,1^+}g_{3,1,1} - \\ & Tg_{1,1^+}g_{2,4^+}g_{3,1,1} + STg_{1,1^+}g_{2,4^+}g_{3,1,1} + Sg_{1,4^+}g_{2,4^+}g_{3,1,1} + Tg_{1,4^+}g_{2,4^+}g_{3,1,1} - \\ & 2STg_{1,4^+}g_{2,4^+}g_{3,1,1} - Sg_{1,6^+}g_{2,3^+}g_{3,1,3} + STg_{1,6^+}g_{2,3^+}g_{3,1,3} - Tg_{1,3^+}g_{2,6^+}g_{3,1,3} + \\ & STg_{1,3^+}g_{2,6^+}g_{3,1,3} + Sg_{1,6^+}g_{2,6^+}g_{3,1,3} + Tg_{1,6^+}g_{2,6^+}g_{3,1,3} - 2STg_{1,6^+}g_{2,6^+}g_{3,1,3} + \\ & Sg_{1,2^+}g_{2,2^+}g_{3,1,5} + Tg_{1,2^+}g_{2,2^+}g_{3,1,5} - 2STg_{1,2^+}g_{2,2^+}g_{3,1,5} - Tg_{1,5^+}g_{2,2^+}g_{3,1,5} + \\ & STg_{1,5^+}g_{2,2^+}g_{3,1,5} - Sg_{1,2^+}g_{2,5^+}g_{3,1,5} + STg_{1,2^+}g_{2,5^+}g_{3,1,5} + S^2Tg_{1,4^+}g_{2,1^+}g_{3,1^+} - \\ & S^2T^2g_{1,4^+}g_{2,1^+}g_{3,1^+} + S^2T^2g_{1,1^+}g_{2,4^+}g_{3,1^+} - S^2T^2g_{1,1^+}g_{2,4^+}g_{3,1^+} - \\ & S^2Tg_{1,4^+}g_{2,4^+}g_{3,1^+} - S^2T^2g_{1,4^+}g_{2,4^+}g_{3,1^+} + 2S^2T^2g_{1,4^+}g_{2,4^+}g_{3,1^+} + \\ & S^2Tg_{1,6^+}g_{2,3^+}g_{3,1^+,3} - S^2T^2g_{1,6^+}g_{2,3^+}g_{3,1^+,3} + S^2T^2g_{1,3^+}g_{2,6^+}g_{3,1^+,3} - \\ & S^2T^2g_{1,3^+}g_{2,6^+}g_{3,1^+,3} - S^2Tg_{1,6^+}g_{2,6^+}g_{3,1^+,3} - S^2T^2g_{1,6^+}g_{2,6^+}g_{3,1^+,3} + \\ & 2S^2T^2g_{1,6^+}g_{2,6^+}g_{3,1^+,3} - S^2Tg_{1,2^+}g_{2,2^+}g_{3,1^+,5} - S^2T^2g_{1,2^+}g_{2,2^+}g_{3,1^+,5} + \\ & 2S^2T^2g_{1,2^+}g_{2,2^+}g_{3,1^+,5} + S^2T^2g_{1,5^+}g_{2,2^+}g_{3,1^+,5} - S^2T^2g_{1,5^+}g_{2,2^+}g_{3,1^+,5} + \\ & S^2Tg_{1,2^+}g_{2,5^+}g_{3,1^+,5} - S^2T^2g_{1,2^+}g_{2,5^+}g_{3,1^+,5} + Sg_{1,4^+}g_{2,1^+}g_{3,4^+} - STg_{1,4^+}g_{2,1^+}g_{3,4^+} - \\ & S^2Tg_{1,4^+}g_{2,1^+}g_{3,4^+} + S^2T^2g_{1,4^+}g_{2,1^+}g_{3,4^+} + Tg_{1,1^+}g_{2,4^+}g_{3,4^+} - STg_{1,1^+}g_{2,4^+}g_{3,4^+} - \\ & S^2Tg_{1,1^+}g_{2,4^+}g_{3,4^+} + S^2T^2g_{1,1^+}g_{2,4^+}g_{3,4^+} - Sg_{1,4^+}g_{2,4^+}g_{3,4^+} - Tg_{1,4^+}g_{2,4^+}g_{3,4^+} + \\ & 2STg_{1,4^+}g_{2,4^+}g_{3,4^+} + S^2Tg_{1,4^+}g_{2,4^+}g_{3,4^+} + S^2T^2g_{1,4^+}g_{2,4^+}g_{3,4^+} - \\ & 2S^2T^2g_{1,4^+}g_{2,4^+}g_{3,4^+} + Sg_{1,6^+}g_{2,3^+}g_{3,4^+,3} - STg_{1,6^+}g_{2,3^+}g_{3,4^+,3} - S^2Tg_{1,6^+}g_{2,3^+}g_{3,4^+,3} + \\ & S^2T^2g_{1,6^+}g_{2,3^+}g_{3,4^+,3} + Tg_{1,3^+}g_{2,6^+}g_{3,4^+,3} - STg_{1,3^+}g_{2,6^+}g_{3,4^+,3} - S^2Tg_{1,3^+}g_{2,6^+}g_{3,4^+,3} + \\ & S^2T^2g_{1,3^+}g_{2,6^+}g_{3,4^+,3} - Sg_{1,6^+}g_{2,6^+}g_{3,4^+,3} - Tg_{1,6^+}g_{2,6^+}g_{3,4^+,3} + 2STg_{1,6^+}g_{2,6^+}g_{3,4^+,3} + \\ & S^2Tg_{1,6^+}g_{2,6^+}g_{3,4^+,3} + S^2T^2g_{1,6^+}g_{2,6^+}g_{3,4^+,3} - 2S^2T^2g_{1,6^+}g_{2,6^+}g_{3,4^+,3} - \\ & Sg_{1,2^+}g_{2,2^+}g_{3,4^+,5} - Tg_{1,2^+}g_{2,2^+}g_{3,4^+,5} + 2STg_{1,2^+}g_{2,2^+}g_{3,4^+,5} + S^2Tg_{1,2^+}g_{2,2^+}g_{3,4^+,5} + \\ & S^2T^2g_{1,2^+}g_{2,2^+}g_{3,4^+,5} - 2S^2T^2g_{1,2^+}g_{2,2^+}g_{3,4^+,5} + Tg_{1,5^+}g_{2,2^+}g_{3,4^+,5} - \\ & STg_{1,5^+}g_{2,2^+}g_{3,4^+,5} - S^2Tg_{1,5^+}g_{2,2^+}g_{3,4^+,5} + S^2T^2g_{1,5^+}g_{2,2^+}g_{3,4^+,5} + \\ & Sg_{1,2^+}g_{2,5^+}g_{3,4^+,5} - STg_{1,2^+}g_{2,5^+}g_{3,4^+,5} - S^2Tg_{1,2^+}g_{2,5^+}g_{3,4^+,5} + S^2T^2g_{1,2^+}g_{2,5^+}g_{3,4^+,5} \end{aligned}$$

Invariance of $y_{\alpha\beta\gamma}$ under remote R3s

```

In[ ]:= Clear[i, j, k];
Cs = {{1, i, j}, {1, i+, k}, {1, j+, k+}}
Z = Module[{s, i, j}, Sum[{s, i, j} = c;
  g3,α,i (S^s (T^s - 1) g1,j+,β g2,i+,γ + (S^s - 1) T^s g1,i+,β g2,j+,γ + (S^s + T^s - 2 T S^s) g1,j+,β g2,j+,γ),
  {c, Cs}]]]
lhs = Simplify[Z //. gRules1,i,j ∪ gRules1,i+,k ∪ gRules1,j+,k+ /. _If → 0]

Out[ ]:=
{{1, i, j}, {1, i+, k}, {1, j+, k+}}

Out[ ]:=
(S (-1 + T) g1,j+,β g2,i+,γ + (-1 + S) T g1,i+,β g2,j+,γ + (S + T - 2 S T) g1,j+,β g2,j+,γ) g3,α,i +
(S (-1 + T) g1,k+,β g2,i+,γ + (-1 + S) T g1,i+,β g2,k+,γ + (S + T - 2 S T) g1,k+,β g2,k+,γ) g3,α,i+ +
(S (-1 + T) g1,k+,β g2,j+,γ + (-1 + S) T g1,j+,β g2,k+,γ + (S + T - 2 S T) g1,k+,β g2,k+,γ) g3,α,j+

Out[ ]:=
1
S^2 T^2 ((S (-1 + T) (T g1,j+,β - (-1 + T) g1,k+,β) (S g2,i+,γ - (-1 + S) g2,k+,γ) +
(-1 + S) T (T g1,i+,β - (-1 + T) g1,k+,β) (S g2,j+,γ - (-1 + S) g2,k+,γ) +
(S + T - 2 S T) (T g1,j+,β - (-1 + T) g1,k+,β) (S g2,j+,γ - (-1 + S) g2,k+,γ)) g3,α,i++ +
S T ((-1 + S) T g1,i+,β g2,k+,γ + g1,k+,β (S (-1 + T) g2,i+,γ + (S + T - 2 S T) g2,k+,γ)) g3,α,i++ +
S T ((-1 + S) T g1,j+,β g2,k+,γ + g1,k+,β (S (-1 + T) g2,j+,γ + (S + T - 2 S T) g2,k+,γ)) g3,α,j++)

In[ ]:= Clear[i, j, k];
Cs = {{1, j, k}, {1, i, k+}, {1, i+, j+}}
Z = Module[{s, i, j}, Sum[{s, i, j} = c;
  g3,α,i (S^s (T^s - 1) g1,j+,β g2,i+,γ + (S^s - 1) T^s g1,i+,β g2,j+,γ + (S^s + T^s - 2 T S^s) g1,j+,β g2,j+,γ),
  {c, Cs}]]]
rhs = Simplify[Z //. gRules1,j,k ∪ gRules1,i,k+ ∪ gRules1,i+,j+ /. _If → 0]

Out[ ]:=
{{1, j, k}, {1, i, k+}, {1, i+, j+}}

Out[ ]:=
((S + T - 2 S T) g1,k+,β g2,k+,γ + (-1 + S) T g1,i+,β g2,k+,γ + S (-1 + T) g1,k+,β g2,i+,γ) g3,α,i+ +
(S (-1 + T) g1,k+,β g2,j+,γ + (-1 + S) T g1,j+,β g2,k+,γ + (S + T - 2 S T) g1,k+,β g2,k+,γ) g3,α,j+ +
(S (-1 + T) g1,j+,β g2,i+,γ + (-1 + S) T g1,i+,β g2,j+,γ + (S + T - 2 S T) g1,j+,β g2,j+,γ) g3,α,i+

Out[ ]:=
1
S^2 T^2 (S T ((-1 + S) T g1,i+,β g2,j+,γ + g1,j+,β (S (-1 + T) g2,i+,γ + (S + T - 2 S T) g2,j+,γ)) g3,α,i++ +
(S (-1 + T) g1,k+,β (S g2,i+,γ - (-1 + S) g2,j+,γ) +
(-1 + S) T (T g1,i+,β - (-1 + T) g1,j+,β) g2,k+,γ + (S + T - 2 S T) g1,k+,β g2,k+,γ) g3,α,i++ +
((-1 + S) T g1,j+,β g2,k+,γ + g1,k+,β (S (-1 + T) g2,j+,γ + (S + T - 2 S T) g2,k+,γ))
((-1 + S T) g3,α,i++ + S T g3,α,j++))

```



```
In[*]:= Simplify[lhs == rhs]
Out[*]=
True
```

Setting up R_1

```
In[*]:= gs = Flatten@Table[{gp,i,i, gp,i,j, gp,j,i, gp,j,j}, {p, 3}];
bas = {1} ∪ gs ∪ Flatten@Table[gs[[p] × gs[[q]], {p, Length[gs]}, {q, p, Length[gs]}] ∪
  {yi,i,i, yi,i,j, yi,j,i, yi,j,j, yj,i,i, yj,i,j, yj,j,i, yj,j,j}
len = Length[bas];
vars = Table[cp = 0; cp = .; cp, {p, 2 len}]
R1[1, i_, j_] := Evaluate[Sum[cp bas[[p]], {p, len}]];
R1[-1, i_, j_] := Evaluate[Sum[c1len+p bas[[p]], {p, Length[bas]}]];
```

```
Out[*]=
{1, g1,i,i, g1,i,i2, g1,i,j, g1,i,i g1,i,j, g1,i,j2, g1,j,i, g1,i,i g1,j,i, g1,i,j g1,j,i, g1,j,i2, g1,j,j,
g1,i,i g1,j,j, g1,i,j g1,j,j, g1,j,i g1,j,j, g1,j,j2, g2,i,i, g1,i,i g2,i,i, g1,i,j g2,i,i, g1,j,i g2,i,i,
g1,j,j g2,i,i, g2,i,i2, g2,i,j, g1,i,i g2,i,j, g1,i,j g2,i,j, g1,j,i g2,i,j, g1,j,j g2,i,j, g2,i,i g2,i,j,
g2,j,i, g1,i,i g2,j,i, g1,i,j g2,j,i, g1,j,i g2,j,i, g1,j,j g2,j,i, g2,i,i g2,j,i, g2,i,j g2,j,i,
g2,j,i2, g2,j,j, g1,i,i g2,j,j, g1,i,j g2,j,j, g1,j,i g2,j,j, g1,j,j g2,j,j, g2,i,i g2,j,j, g2,i,j g2,j,j,
g2,j,i g2,j,j, g2,j,j2, g3,i,i, g1,i,i g3,i,i, g1,i,j g3,i,i, g1,j,i g3,i,i, g1,j,j g3,i,i, g2,i,i g3,i,i,
g2,i,j g3,i,i, g2,j,i g3,i,i, g2,j,j g3,i,i, g3,i,i2, g3,i,j, g1,i,i g3,i,j, g1,i,j g3,i,j, g1,j,i g3,i,j,
g1,j,j g3,i,j, g2,i,i g3,i,j, g2,i,j g3,i,j, g2,j,i g3,i,j, g2,j,j g3,i,j, g3,i,i g3,i,j, g3,i,j2,
g3,j,i, g1,i,i g3,j,i, g1,i,j g3,j,i, g1,j,i g3,j,i, g1,j,j g3,j,i, g2,i,i g3,j,i, g2,i,j g3,j,i,
g2,j,i g3,j,i, g2,j,j g3,j,i, g3,i,i g3,j,i, g3,i,j g3,j,i, g3,j,i2, g3,j,j, g1,i,i g3,j,j, g1,i,j g3,j,j,
g1,j,i g3,j,j, g1,j,j g3,j,j, g2,i,i g3,j,j, g2,i,j g3,j,j, g2,j,i g3,j,j, g2,j,j g3,j,j, g3,i,i g3,j,j,
g3,i,j g3,j,j, g3,j,i g3,j,j, g3,j,j2, yi,i,i, yi,i,j, yi,j,i, yi,j,j, yj,i,i, yj,i,j, yj,j,i, yj,j,j}
```

```
Out[*]=
{c1, c2, c3, c4, c5, c6, c7, c8, c9, c10, c11, c12, c13, c14, c15, c16, c17, c18, c19, c20, c21, c22,
c23, c24, c25, c26, c27, c28, c29, c30, c31, c32, c33, c34, c35, c36, c37, c38, c39, c40, c41,
c42, c43, c44, c45, c46, c47, c48, c49, c50, c51, c52, c53, c54, c55, c56, c57, c58, c59, c60,
c61, c62, c63, c64, c65, c66, c67, c68, c69, c70, c71, c72, c73, c74, c75, c76, c77, c78, c79,
c80, c81, c82, c83, c84, c85, c86, c87, c88, c89, c90, c91, c92, c93, c94, c95, c96, c97, c98,
c99, c100, c101, c102, c103, c104, c105, c106, c107, c108, c109, c110, c111, c112, c113, c114, c115,
c116, c117, c118, c119, c120, c121, c122, c123, c124, c125, c126, c127, c128, c129, c130, c131, c132,
c133, c134, c135, c136, c137, c138, c139, c140, c141, c142, c143, c144, c145, c146, c147, c148, c149,
c150, c151, c152, c153, c154, c155, c156, c157, c158, c159, c160, c161, c162, c163, c164, c165, c166,
c167, c168, c169, c170, c171, c172, c173, c174, c175, c176, c177, c178, c179, c180, c181, c182,
c183, c184, c185, c186, c187, c188, c189, c190, c191, c192, c193, c194, c195, c196, c197, c198}
```

In[*]:= **R₁**[-1, i⁺, 7]

Out[*]=

$$\begin{aligned}
 & C_{100} + C_{110} g_{1,7,7} + C_{114} g_{1,7,7}^2 + C_{106} g_{1,7,i^+} + C_{113} g_{1,7,7} g_{1,7,i^+} + C_{109} g_{1,7,i^+}^2 + C_{103} g_{1,i^+,7} + C_{112} g_{1,7,7} g_{1,i^+,7} + \\
 & C_{108} g_{1,7,i^+} g_{1,i^+,7} + C_{105} g_{1,i^+,7}^2 + C_{101} g_{1,i^+,i^+} + C_{111} g_{1,7,7} g_{1,i^+,i^+} + C_{107} g_{1,7,i^+} g_{1,i^+,i^+} + \\
 & C_{104} g_{1,i^+,7} g_{1,i^+,i^+} + C_{102} g_{1,i^+,i^+}^2 + C_{136} g_{2,7,7} + C_{140} g_{1,7,7} g_{2,7,7} + C_{139} g_{1,7,i^+} g_{2,7,7} + C_{138} g_{1,i^+,7} g_{2,7,7} + \\
 & C_{137} g_{1,i^+,i^+} g_{2,7,7} + C_{144} g_{2,7,7}^2 + C_{128} g_{2,7,i^+} + C_{132} g_{1,7,7} g_{2,7,i^+} + C_{131} g_{1,7,i^+} g_{2,7,i^+} + C_{130} g_{1,i^+,7} g_{2,7,i^+} + \\
 & C_{129} g_{1,i^+,i^+} g_{2,7,i^+} + C_{143} g_{2,7,7} g_{2,7,i^+} + C_{135} g_{2,7,i^+}^2 + C_{121} g_{2,i^+,7} + C_{125} g_{1,7,7} g_{2,i^+,7} + \\
 & C_{124} g_{1,7,i^+} g_{2,i^+,7} + C_{123} g_{1,i^+,7} g_{2,i^+,7} + C_{122} g_{1,i^+,i^+} g_{2,i^+,7} + C_{142} g_{2,7,7} g_{2,i^+,7} + C_{134} g_{2,7,i^+} g_{2,i^+,7} + \\
 & C_{127} g_{2,i^+,7}^2 + C_{115} g_{2,i^+,i^+} + C_{119} g_{1,7,7} g_{2,i^+,i^+} + C_{118} g_{1,7,i^+} g_{2,i^+,i^+} + C_{117} g_{1,i^+,7} g_{2,i^+,i^+} + \\
 & C_{116} g_{1,i^+,i^+} g_{2,i^+,i^+} + C_{141} g_{2,7,7} g_{2,i^+,i^+} + C_{133} g_{2,7,i^+} g_{2,i^+,i^+} + C_{126} g_{2,i^+,7} g_{2,i^+,i^+} + C_{120} g_{2,i^+,i^+}^2 + \\
 & C_{178} g_{3,7,7} + C_{182} g_{1,7,7} g_{3,7,7} + C_{181} g_{1,7,i^+} g_{3,7,7} + C_{180} g_{1,i^+,7} g_{3,7,7} + C_{179} g_{1,i^+,i^+} g_{3,7,7} + \\
 & C_{186} g_{2,7,7} g_{3,7,7} + C_{185} g_{2,7,i^+} g_{3,7,7} + C_{184} g_{2,i^+,7} g_{3,7,7} + C_{183} g_{2,i^+,i^+} g_{3,7,7} + C_{190} g_{3,7,7}^2 + C_{166} g_{3,7,i^+} + \\
 & C_{170} g_{1,7,7} g_{3,7,i^+} + C_{169} g_{1,7,i^+} g_{3,7,i^+} + C_{168} g_{1,i^+,7} g_{3,7,i^+} + C_{167} g_{1,i^+,i^+} g_{3,7,i^+} + C_{174} g_{2,7,7} g_{3,7,i^+} + \\
 & C_{173} g_{2,7,i^+} g_{3,7,i^+} + C_{172} g_{2,i^+,7} g_{3,7,i^+} + C_{171} g_{2,i^+,i^+} g_{3,7,i^+} + C_{189} g_{3,7,7} g_{3,7,i^+} + C_{177} g_{3,7,i^+}^2 + \\
 & C_{155} g_{3,i^+,7} + C_{159} g_{1,7,7} g_{3,i^+,7} + C_{158} g_{1,7,i^+} g_{3,i^+,7} + C_{157} g_{1,i^+,7} g_{3,i^+,7} + C_{156} g_{1,i^+,i^+} g_{3,i^+,7} + \\
 & C_{163} g_{2,7,7} g_{3,i^+,7} + C_{162} g_{2,7,i^+} g_{3,i^+,7} + C_{161} g_{2,i^+,7} g_{3,i^+,7} + C_{160} g_{2,i^+,i^+} g_{3,i^+,7} + C_{188} g_{3,7,7} g_{3,i^+,7} + \\
 & C_{176} g_{3,7,i^+} g_{3,i^+,7} + C_{165} g_{3,i^+,7}^2 + C_{145} g_{3,i^+,i^+} + C_{149} g_{1,7,7} g_{3,i^+,i^+} + C_{148} g_{1,7,i^+} g_{3,i^+,i^+} + \\
 & C_{147} g_{1,i^+,7} g_{3,i^+,i^+} + C_{146} g_{1,i^+,i^+} g_{3,i^+,i^+} + C_{153} g_{2,7,7} g_{3,i^+,i^+} + C_{152} g_{2,7,i^+} g_{3,i^+,i^+} + C_{151} g_{2,i^+,7} g_{3,i^+,i^+} + \\
 & C_{150} g_{2,i^+,i^+} g_{3,i^+,i^+} + C_{187} g_{3,7,7} g_{3,i^+,i^+} + C_{175} g_{3,7,i^+} g_{3,i^+,i^+} + C_{164} g_{3,i^+,7} g_{3,i^+,i^+} + C_{154} g_{3,i^+,i^+}^2 + \\
 & C_{198} y_{7,7,7} + C_{197} y_{7,7,i^+} + C_{196} y_{7,i^+,7} + C_{195} y_{7,i^+,i^+} + C_{194} y_{i^+,7,7} + C_{193} y_{i^+,7,i^+} + C_{192} y_{i^+,i^+,7} + C_{191} y_{i^+,i^+,i^+}
 \end{aligned}$$

Solving R2b

In[*]:= **Clear**[i, j];

eqn = **Expand**[**R₁**[1, i, j] + **R₁**[-1, i⁺, j⁺]] //.

gRules_{1,i,j} **U** **gRules**_{-1,i⁺,j⁺} **U** **yRules**_{1,i,j} **U** **yRules**_{-1,i⁺,j⁺} /. **_If** → 0]

Out[*]=

$$\begin{aligned}
 & C_1 + C_{100} + C_2 g_{1,i^{**},i^{**}} - C_{93} g_{1,i^{**},i^{**}} + 2 S C_{93} g_{1,i^{**},i^{**}} + \frac{S C_{93} g_{1,i^{**},i^{**}}}{T^2} + \frac{C_{93} g_{1,i^{**},i^{**}}}{T} - \frac{3 S C_{93} g_{1,i^{**},i^{**}}}{T} + \\
 & \dots 2412 \dots + C_{194} y_{j^{**},j^{**},i^{**}} - S C_{194} y_{j^{**},j^{**},i^{**}} + \frac{C_{194} y_{j^{**},j^{**},i^{**}}}{T} - \frac{C_{194} y_{j^{**},j^{**},i^{**}}}{S T} + S C_{197} y_{j^{**},j^{**},i^{**}} + \\
 & C_{198} y_{j^{**},j^{**},i^{**}} - S C_{198} y_{j^{**},j^{**},i^{**}} + C_{99} y_{j^{**},j^{**},j^{**}} + C_{194} y_{j^{**},j^{**},j^{**}} - \frac{C_{194} y_{j^{**},j^{**},j^{**}}}{S T} + C_{198} y_{j^{**},j^{**},j^{**}}
 \end{aligned}$$

Full expression not available (original memory size: 1.2 MB) ⚙️

```
In[*]:= covars = Cases[eqn, g_., ∞] ∪ Cases[eqn, c_. * (g1 : g_.,_) (g2 : g_.,_) ⇒ g1 g2, ∞] ∪
Cases[eqn, g_., ∞] ∪ Cases[eqn, y_., ∞]
```

Out[*]=

$$\left\{ \begin{aligned} &g_{1,i^{++},i^{++}}, g_{1,i^{++},i^{++}}^2, g_{1,i^{++},j^{++}}, g_{1,i^{++},i^{++}} g_{1,i^{++},j^{++}}, g_{1,i^{++},j^{++}}^2, g_{1,j^{++},i^{++}}, g_{1,i^{++},i^{++}} g_{1,j^{++},i^{++}}, \\ &g_{1,i^{++},j^{++}} g_{1,j^{++},i^{++}}, g_{1,j^{++},i^{++}}^2, g_{1,j^{++},j^{++}}, g_{1,i^{++},i^{++}} g_{1,j^{++},j^{++}}, g_{1,i^{++},j^{++}} g_{1,j^{++},j^{++}}, \\ &g_{1,j^{++},i^{++}} g_{1,j^{++},j^{++}}, g_{1,j^{++},j^{++}}^2, g_{2,i^{++},i^{++}}, g_{1,i^{++},i^{++}} g_{2,i^{++},i^{++}}, g_{1,i^{++},j^{++}} g_{2,i^{++},i^{++}}, g_{1,j^{++},i^{++}} g_{2,i^{++},i^{++}}, \\ &g_{1,j^{++},j^{++}} g_{2,i^{++},i^{++}}, g_{2,i^{++},i^{++}}^2, g_{2,i^{++},j^{++}}, g_{1,i^{++},i^{++}} g_{2,i^{++},j^{++}}, g_{1,i^{++},j^{++}} g_{2,i^{++},j^{++}}, g_{1,j^{++},i^{++}} g_{2,i^{++},j^{++}}, \\ &g_{1,j^{++},j^{++}} g_{2,i^{++},j^{++}}, g_{2,i^{++},i^{++}} g_{2,i^{++},j^{++}}, g_{2,i^{++},j^{++}}^2, g_{2,j^{++},i^{++}}, g_{1,i^{++},i^{++}} g_{2,j^{++},i^{++}}, g_{1,i^{++},j^{++}} g_{2,j^{++},i^{++}}, \\ &g_{1,j^{++},i^{++}} g_{2,j^{++},i^{++}}, g_{1,j^{++},j^{++}} g_{2,j^{++},i^{++}}, g_{2,i^{++},i^{++}} g_{2,j^{++},i^{++}}, g_{2,i^{++},j^{++}} g_{2,j^{++},i^{++}}, g_{2,j^{++},i^{++}}^2, g_{2,j^{++},j^{++}}, \\ &g_{1,i^{++},i^{++}} g_{2,j^{++},j^{++}}, g_{1,i^{++},j^{++}} g_{2,j^{++},j^{++}}, g_{1,j^{++},i^{++}} g_{2,j^{++},j^{++}}, g_{1,j^{++},j^{++}} g_{2,j^{++},j^{++}}, g_{2,i^{++},i^{++}} g_{2,j^{++},j^{++}}, \\ &g_{2,i^{++},j^{++}} g_{2,j^{++},j^{++}}, g_{2,j^{++},i^{++}} g_{2,j^{++},j^{++}}, g_{2,j^{++},j^{++}}^2, g_{3,i^{++},i^{++}}, g_{1,i^{++},i^{++}} g_{3,i^{++},i^{++}}, g_{1,i^{++},j^{++}} g_{3,i^{++},i^{++}}, \\ &g_{1,j^{++},i^{++}} g_{3,i^{++},i^{++}}, g_{1,j^{++},j^{++}} g_{3,i^{++},i^{++}}, g_{2,i^{++},i^{++}} g_{3,i^{++},i^{++}}, g_{2,i^{++},j^{++}} g_{3,i^{++},i^{++}}, g_{2,j^{++},i^{++}} g_{3,i^{++},i^{++}}, \\ &g_{2,j^{++},j^{++}} g_{3,i^{++},i^{++}}, g_{3,i^{++},i^{++}}^2, g_{3,i^{++},j^{++}}, g_{1,i^{++},i^{++}} g_{3,i^{++},j^{++}}, g_{1,i^{++},j^{++}} g_{3,i^{++},j^{++}}, g_{1,j^{++},i^{++}} g_{3,i^{++},j^{++}}, \\ &g_{1,j^{++},j^{++}} g_{3,i^{++},j^{++}}, g_{2,i^{++},i^{++}} g_{3,i^{++},j^{++}}, g_{2,i^{++},j^{++}} g_{3,i^{++},j^{++}}, g_{2,j^{++},i^{++}} g_{3,i^{++},j^{++}}, g_{2,j^{++},j^{++}} g_{3,i^{++},j^{++}}, \\ &g_{3,i^{++},i^{++}} g_{3,i^{++},j^{++}}, g_{3,i^{++},j^{++}}^2, g_{3,j^{++},i^{++}}, g_{1,i^{++},i^{++}} g_{3,j^{++},i^{++}}, g_{1,i^{++},j^{++}} g_{3,j^{++},i^{++}}, g_{1,j^{++},i^{++}} g_{3,j^{++},i^{++}}, \\ &g_{1,j^{++},j^{++}} g_{3,j^{++},i^{++}}, g_{2,i^{++},i^{++}} g_{3,j^{++},i^{++}}, g_{2,i^{++},j^{++}} g_{3,j^{++},i^{++}}, g_{2,j^{++},i^{++}} g_{3,j^{++},i^{++}}, g_{2,j^{++},j^{++}} g_{3,j^{++},i^{++}}, \\ &g_{3,i^{++},i^{++}} g_{3,j^{++},i^{++}}, g_{3,i^{++},j^{++}} g_{3,j^{++},i^{++}}, g_{3,j^{++},i^{++}}^2, g_{3,j^{++},j^{++}}, g_{1,i^{++},i^{++}} g_{3,j^{++},j^{++}}, g_{1,i^{++},j^{++}} g_{3,j^{++},j^{++}}, \\ &g_{1,j^{++},i^{++}} g_{3,j^{++},j^{++}}, g_{1,j^{++},j^{++}} g_{3,j^{++},j^{++}}, g_{2,i^{++},i^{++}} g_{3,j^{++},j^{++}}, g_{2,i^{++},j^{++}} g_{3,j^{++},j^{++}}, g_{2,j^{++},i^{++}} g_{3,j^{++},j^{++}}, \\ &g_{2,j^{++},j^{++}} g_{3,j^{++},j^{++}}, g_{3,i^{++},i^{++}} g_{3,j^{++},j^{++}}, g_{3,i^{++},j^{++}} g_{3,j^{++},j^{++}}, g_{3,j^{++},i^{++}} g_{3,j^{++},j^{++}}, g_{3,j^{++},j^{++}}^2, \\ &y_{i^{++},i^{++},i^{++}}, y_{i^{++},i^{++},j^{++}}, y_{i^{++},j^{++},i^{++}}, y_{i^{++},j^{++},j^{++}}, y_{j^{++},i^{++},i^{++}}, y_{j^{++},i^{++},j^{++}}, y_{j^{++},j^{++},i^{++}}, y_{j^{++},j^{++},j^{++}} \end{aligned} \right\}$$

```
In[*]:= Short[eqns = {eqn == 0} ∪ Union@Table[Coefficient[eqn, cv] == 0, {cv, covars}] /. {(g | y) _.,_ → 0}, 5]
```

Out[*]//Short=

$$\left\{ \begin{aligned} &c_6 + \frac{c_{105}}{T^2} = 0, c_3 + c_{102} - c_{104} + \frac{c_{104}}{T} + c_{105} + \frac{c_{105}}{T^2} - \frac{2 c_{105}}{T} = 0, \langle\langle 95 \rangle\rangle, \\ &c_{67} + 2 c_{99} - S c_{99} - T c_{99} - c_{145} + S T c_{145} + 2 c_{155} - \frac{c_{155}}{S T} - S T c_{155} + S T c_{166} + c_{178} - S T c_{178} + c_{192} - \frac{c_{192}}{S} + \\ &T c_{192} - S T c_{192} + c_{193} + S c_{193} - \frac{c_{193}}{T} - S T c_{193} + \frac{2 c_{194}}{S} - 2 S c_{194} + \frac{2 c_{194}}{T} - \frac{2 c_{194}}{S T} - 2 T c_{194} + \\ &2 S T c_{194} + T c_{196} - S T c_{196} + S c_{197} - S T c_{197} + 2 c_{198} - 2 S c_{198} - 2 T c_{198} + 2 S T c_{198} = 0, c_1 + c_{100} = 0 \end{aligned} \right\}$$

In[*]:= Short[{sol} = Solve[eqns, vars], 5]

 Solve: Equations may not give solutions for all "solve" variables. 

Out[*]//Short=

$$\left\{ \left\{ c_2 \rightarrow -c_{101} - \left(-1 + \frac{1}{T}\right) c_{103} - \frac{(-S - T + 3ST + T^2 - 2ST^2) c_{192}}{ST^2}, \right. \right.$$

$$c_3 \rightarrow -c_{102} - \left(-1 + \frac{1}{T}\right) c_{104} - \frac{(1 - 2T + T^2) c_{105}}{T^2}, c_4 \rightarrow -\frac{c_{103}}{T} - \frac{(-S - T + 2ST) c_{192}}{ST^2},$$

$$c_5 \rightarrow -\frac{c_{104}}{T} + \frac{2(-1 + T) c_{105}}{T^2}, c_6 \rightarrow -\frac{c_{105}}{T^2}, \ll 90 \gg,$$

$$c_{97} \rightarrow -\frac{(-1 + ST) c_{192}}{S} - \frac{(-1 + T + ST - ST^2) c_{194}}{ST} - T c_{196} - (1 - T) c_{198},$$

$$c_{98} \rightarrow -\frac{(-1 + ST) c_{193}}{T} - \frac{(-1 + S + ST - S^2 T) c_{194}}{ST} - S c_{197} - (1 - S) c_{198},$$

$$c_{99} \rightarrow -\frac{(-1 + ST) c_{194}}{ST} - c_{198}, c_{100} \rightarrow -c_1 \left. \right\}$$

In[*]:= Short[sol /. Rule -> Set]

Out[*]//Short=

$$\left\{ -c_{101} - \left(-1 + \frac{1}{T}\right) c_{103} - \frac{(-S - T + 3ST + T^2 - 2ST^2) c_{192}}{ST^2}, \ll 97 \gg, -c_1 \right\}$$

In[*]:= Short[R1[1, i, j], 5]

Out[*]//Short=

$$c_1 + \left(-c_{101} - \left(-1 + \frac{1}{T}\right) c_{103} - \frac{(-S - T + 3ST + T^2 - 2ST^2) c_{192}}{ST^2} \right) g_{1,i,i} +$$

$$\left(-c_{102} - \left(-1 + \frac{1}{T}\right) c_{104} - \frac{(1 - 2T + T^2) c_{105}}{T^2} \right) g_{1,i,i}^2 + \left(-\frac{c_{103}}{T} - \frac{(-S - T + 2ST) c_{192}}{ST^2} \right) g_{1,i,j} +$$

$$\ll 101 \gg + \left(-\frac{(-1 + ST) c_{192}}{S} - \frac{(-1 + T + ST - ST^2) c_{194}}{ST} - T c_{196} - (1 - T) c_{198} \right) y_{j,i,j} +$$

$$\left(-\frac{(-1 + ST) c_{193}}{T} - \frac{(-1 + S + ST - S^2 T) c_{194}}{ST} - S c_{197} - (1 - S) c_{198} \right) y_{j,j,i} +$$

$$\left(-\frac{(-1 + ST) c_{194}}{ST} - c_{198} \right) y_{j,j,j}$$

Solving R3

```
In[*]:= Clear [i, j, k];
lhs = Expand [R1 [1, i, j] + R1 [1, i+, k] + R1 [1, j+, k+] // . gRules1,i,j ∪
  gRules1,i+,k ∪ gRules1,j+,k+ ∪ yRules1,i,j ∪ yRules1,i+,k ∪ yRules1,j+,k+ /. _If → 0];
rhs = Expand [R1 [1, j, k] + R1 [1, i, k+] + R1 [1, i+, j+] // . gRules1,j,k ∪
  gRules1,i,k+ ∪ gRules1,i+,j+ ∪ yRules1,j,k ∪ yRules1,i,k+ ∪ yRules1,i+,j+ /. _If → 0];
eqn = lhs - rhs
```

Out[*]=

$$-2 C_{194} + \frac{C_{194}}{S} + \frac{C_{194}}{T} - C_{103} g_{1,i^{++},j^{++}} + \frac{C_{103} g_{1,i^{++},j^{++}}}{T} - \frac{C_{192} g_{1,i^{++},j^{++}}}{T^2} + \frac{2 C_{192} g_{1,i^{++},j^{++}}}{T} -$$

$$\frac{C_{192} g_{1,i^{++},j^{++}}}{ST} - C_{104} g_{1,i^{++},i^{++}} g_{1,i^{++},j^{++}} + \dots 3208 \dots + \frac{C_{198} y_{k^{++},j^{++},j^{++}}}{S} + \frac{C_{198} y_{k^{++},j^{++},j^{++}}}{T} - \frac{2 C_{198} y_{k^{++},j^{++},j^{++}}}{ST} -$$

$$C_{198} y_{k^{++},j^{++},k^{++}} + \frac{C_{198} y_{k^{++},j^{++},k^{++}}}{T} - \frac{C_{198} y_{k^{++},k^{++},i^{++}}}{S^2} + \frac{C_{198} y_{k^{++},k^{++},i^{++}}}{S} - C_{198} y_{k^{++},k^{++},j^{++}} + \frac{C_{198} y_{k^{++},k^{++},j^{++}}}{S}$$

Full expression not available (original memory size: 1.6 MB)

```
In[*]:= vars = Union@Cases [eqn, c_., ∞]
```

Out[*]=

- {C₁₀₁, C₁₀₂, C₁₀₃, C₁₀₄, C₁₀₅, C₁₀₆, C₁₀₇, C₁₀₈, C₁₀₉, C₁₁₀, C₁₁₁, C₁₁₂, C₁₁₃, C₁₁₄, C₁₁₅, C₁₁₆, C₁₁₇,
- C₁₁₈, C₁₁₉, C₁₂₀, C₁₂₁, C₁₂₂, C₁₂₃, C₁₂₄, C₁₂₅, C₁₂₆, C₁₂₇, C₁₂₈, C₁₂₉, C₁₃₀, C₁₃₁, C₁₃₂, C₁₃₃, C₁₃₄,
- C₁₃₅, C₁₃₆, C₁₃₇, C₁₃₈, C₁₃₉, C₁₄₀, C₁₄₁, C₁₄₂, C₁₄₃, C₁₄₄, C₁₄₅, C₁₄₆, C₁₄₇, C₁₄₈, C₁₄₉, C₁₅₀,
- C₁₅₁, C₁₅₂, C₁₅₃, C₁₅₄, C₁₅₅, C₁₅₆, C₁₅₇, C₁₅₈, C₁₅₉, C₁₆₀, C₁₆₁, C₁₆₂, C₁₆₃, C₁₆₄, C₁₆₅, C₁₆₆,
- C₁₆₇, C₁₆₈, C₁₆₉, C₁₇₀, C₁₇₁, C₁₇₂, C₁₇₃, C₁₇₄, C₁₇₅, C₁₇₆, C₁₇₇, C₁₇₈, C₁₇₉, C₁₈₀, C₁₈₁, C₁₈₂,
- C₁₈₃, C₁₈₄, C₁₈₅, C₁₈₆, C₁₈₇, C₁₈₈, C₁₈₉, C₁₉₀, C₁₉₁, C₁₉₂, C₁₉₃, C₁₉₄, C₁₉₅, C₁₉₆, C₁₉₇, C₁₉₈}

```
In[*]:= Short [
  covars = Cases [eqn, c_. * (g1 : g_.,_) (g2 : g_.,_) ⇒ g1 g2, ∞] ∪
  Cases [eqn, g^2, ∞] ∪ Cases [eqn, y_., ∞],
  5]
```

Out[*]//Short=

- {g_{1,i^{++},i^{++}}} g_{1,i^{++},j^{++}}}, g_{1,i^{++},j^{++}}^2, g_{1,i^{++},i^{++}}} g_{1,i^{++},k^{++}}}, g_{1,i^{++},j^{++}}} g_{1,i^{++},k^{++}}},}
- g_{1,i^{++},k^{++}}}^2, g_{1,i^{++},i^{++}}} g_{1,j^{++},i^{++}}}, g_{1,i^{++},k^{++}}} g_{1,j^{++},i^{++}}}, <<298>>, y_{k^{++},i^{++},j^{++}}},
- y_{k^{++},i^{++},k^{++}}}, y_{k^{++},j^{++},i^{++}}}, y_{k^{++},j^{++},j^{++}}}, y_{k^{++},j^{++},k^{++}}}, y_{k^{++},k^{++},i^{++}}}, y_{k^{++},k^{++},j^{++}}}}

In[*]:= Short[eqns = {eqn == 0} ∪ Union@Table[Coefficient[eqn, cv] == 0, {cv, covars}] /. {(g | y)_,_ ,_ → 0}, 5]

Out[*]//Short=

$$\left\{ \begin{aligned} &2 c_{102} - \frac{2 c_{102}}{T} == 0, -2 c_{102} + \frac{2 c_{102}}{T} == 0, c_{104} - \frac{c_{104}}{T} == 0, \\ &-c_{104} + \frac{c_{104}}{T} == 0, \ll 293 \gg, -c_{182} + \frac{c_{182}}{S T} + 2 c_{196} - 6 T c_{196} + \frac{2 T c_{196}}{S} + 4 T^2 c_{196} - \\ &\frac{2 T^2 c_{196}}{S} - 5 c_{198} + \frac{c_{198}}{S} + \frac{c_{198}}{T} + \frac{c_{198}}{S T} + 5 T c_{198} - \frac{2 T c_{198}}{S} - 2 T^2 c_{198} + \frac{T^2 c_{198}}{S} == 0, \\ &-c_{191} + \frac{c_{191}}{S} + \frac{c_{192}}{S^2} - \frac{c_{192}}{S} - 2 c_{196} + \frac{c_{196}}{S} + \frac{c_{196}}{T} - T c_{196} + 2 T^2 c_{196} - \frac{T^2 c_{196}}{S} - c_{198} + \\ &3 T c_{198} - \frac{T c_{198}}{S} - 2 T^2 c_{198} + \frac{T^2 c_{198}}{S} == 0, -2 c_{194} + \frac{c_{194}}{S} + \frac{c_{194}}{T} == 0 \end{aligned} \right\}$$

In[*]:= Short[{sol} = Solve[eqns, vars], 5]

 Solve: Equations may not give solutions for all "solve" variables. 

Out[*]//Short=

$$\left\{ \left\{ \begin{aligned} &c_{102} \rightarrow 0, c_{104} \rightarrow 0, c_{105} \rightarrow 0, c_{108} \rightarrow -\frac{T c_{107}}{1+T} - \frac{T c_{113}}{1+T}, \\ &c_{109} \rightarrow -\frac{(-1+T) c_{107}}{1+T} - \frac{(1-T) c_{113}}{T(1+T)}, c_{111} \rightarrow -\frac{T c_{107}}{1+T} - \frac{T c_{113}}{1+T}, c_{112} \rightarrow 0, c_{114} \rightarrow 0, c_{116} \rightarrow 0, \\ &c_{117} \rightarrow 0, c_{119} \rightarrow \frac{T c_{118}}{-1+T} - \frac{S c_{129}}{-1+S}, c_{120} \rightarrow 0, c_{122} \rightarrow 0, c_{123} \rightarrow 0, c_{124} \rightarrow 0, \ll 45 \gg, \\ &c_{184} \rightarrow 0, c_{185} \rightarrow -\frac{(-1+S) T c_{171}}{-1+ST}, c_{186} \rightarrow 0, c_{187} \rightarrow -\frac{ST c_{175}}{1+ST} - \frac{ST c_{189}}{1+ST}, c_{188} \rightarrow 0, \\ &c_{190} \rightarrow 0, c_{191} \rightarrow 0, c_{192} \rightarrow 0, c_{193} \rightarrow 0, c_{194} \rightarrow 0, c_{195} \rightarrow 0, c_{196} \rightarrow 0, c_{197} \rightarrow 0, c_{198} \rightarrow 0 \end{aligned} \right\} \right\}$$

In[*]:= Short[sol /. Rule → Set, 5]

Out[*]//Short=

$$\left\{ \begin{aligned} &0, 0, 0, -\frac{T c_{107}}{1+T} - \frac{T c_{113}}{1+T}, -\frac{(-1+T) c_{107}}{1+T} - \frac{(1-T) c_{113}}{T(1+T)}, -\frac{T c_{107}}{1+T} - \frac{T c_{113}}{1+T}, 0, \\ &0, 0, 0, \frac{T c_{118}}{-1+T} - \frac{S c_{129}}{-1+S}, 0, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{(-1+S) c_{118}}{S} - \frac{(-1+T) c_{129}}{T}, \\ &\ll 35 \gg, -\frac{T c_{148}}{-1+T} + \frac{ST c_{167}}{-1+ST}, 0, -\frac{S(-1+T) c_{167}}{-1+ST}, 0, -\frac{S c_{152}}{-1+S} + \frac{ST c_{171}}{-1+ST}, \\ &0, -\frac{(-1+S) T c_{171}}{-1+ST}, 0, -\frac{ST c_{175}}{1+ST} - \frac{ST c_{189}}{1+ST}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \end{aligned} \right\}$$

In[*]:= Short[R₁[1, i, j], 5]

Out[*]//Short=

$$\begin{aligned} & c_1 + \left(-c_{101} - \left(-1 + \frac{1}{T} \right) c_{103} \right) g_{1,i,i} - \frac{c_{103} g_{1,i,j}}{T} + \\ & \left(- \left((-1+T) c_{101} \right) - \left(2 - \frac{1}{T} - T \right) c_{103} - T c_{106} - (1-T) c_{110} \right) g_{1,j,i} + \ll 43 \gg + \\ & \left(\frac{S c_{152}}{-1+S} - \frac{S T c_{171}}{-1+S T} \right) g_{2,i,i} g_{3,j,j} + \left(\frac{(-1+S) S T c_{171}}{-1+S T} - (-1+S) \left(-\frac{S c_{152}}{-1+S} + \frac{S T c_{171}}{-1+S T} \right) \right) g_{2,j,i} g_{3,j,j} + \\ & \left(\frac{S T c_{175}}{1+S T} + \frac{S T c_{189}}{1+S T} \right) g_{3,i,i} g_{3,j,j} + \left(-S T c_{189} - 2(-1+S T) \left(-\frac{S T c_{175}}{1+S T} - \frac{S T c_{189}}{1+S T} \right) \right) g_{3,j,i} g_{3,j,j} \end{aligned}$$

In[*]:= R₁[1, i, j]

Out[*]=

$$\begin{aligned} & c_1 + \left(-c_{101} - \left(-1 + \frac{1}{T} \right) c_{103} \right) g_{1,i,i} - \frac{c_{103} g_{1,i,j}}{T} + \\ & \left(- \left((-1+T) c_{101} \right) - \left(2 - \frac{1}{T} - T \right) c_{103} - T c_{106} - (1-T) c_{110} \right) g_{1,j,i} + \\ & \left(-T c_{107} - 2(1-T) \left(-\frac{T c_{107}}{1+T} - \frac{T c_{113}}{1+T} \right) \right) g_{1,i,i} g_{1,j,i} + \left(\frac{T c_{107}}{1+T} + \frac{T c_{113}}{1+T} \right) g_{1,i,j} g_{1,j,i} + \\ & \left(- \left((-T+T^2) c_{107} \right) - (T-T^2) c_{113} - T^2 \left(-\frac{(-1+T) c_{107}}{1+T} - \frac{(1-T) c_{113}}{T(1+T)} \right) - \right. \\ & \quad \left. 2(-1+2T-T^2) \left(-\frac{T c_{107}}{1+T} - \frac{T c_{113}}{1+T} \right) \right) g_{1,j,i}^2 + \left(-\frac{(-1+T) c_{103}}{T} - c_{110} \right) g_{1,j,j} + \\ & \left(\frac{T c_{107}}{1+T} + \frac{T c_{113}}{1+T} \right) g_{1,i,i} g_{1,j,j} + \left(-T c_{113} - 2(-1+T) \left(-\frac{T c_{107}}{1+T} - \frac{T c_{113}}{1+T} \right) \right) g_{1,j,i} g_{1,j,j} + \\ & \left(-c_{115} - \left(-1 + \frac{1}{S} \right) c_{121} \right) g_{2,i,i} + \left(-T c_{118} - (1-T) \left(\frac{T c_{118}}{-1+T} - \frac{S c_{129}}{-1+S} \right) \right) g_{1,j,i} g_{2,i,i} + \\ & \left(-\frac{T c_{118}}{-1+T} + \frac{S c_{129}}{-1+S} \right) g_{1,j,j} g_{2,i,i} - \frac{c_{121} g_{2,i,j}}{S} + \\ & \left(- \left((-1+S) c_{115} \right) - \left(2 - \frac{1}{S} - S \right) c_{121} - S c_{128} - (1-S) c_{136} \right) g_{2,j,i} + \\ & \left(-S c_{129} - (1-S) \left(-\frac{T c_{118}}{-1+T} + \frac{S c_{129}}{-1+S} \right) \right) g_{1,i,i} g_{2,j,i} + \\ & \left(\frac{(-1+S) T (S-S T) c_{118}}{S(-1+T)} - (-T+S T) c_{118} + \frac{S(-1+T)(T-S T) c_{129}}{(-1+S) T} - (-S+S T) c_{129} - \right. \\ & \quad \left. (-1+S+T-S T) \left(\frac{T c_{118}}{-1+T} - \frac{S c_{129}}{-1+S} \right) - (-1+S+T-S T) \left(-\frac{T c_{118}}{-1+T} + \frac{S c_{129}}{-1+S} \right) - \right. \\ & \quad \left. S T \left(-\frac{(-1+S) c_{118}}{S} - \frac{(-1+T) c_{129}}{T} \right) \right) g_{1,j,i} g_{2,j,i} + \\ & \left(\frac{(-1+S) T c_{118}}{-1+T} - (-1+S) \left(\frac{T c_{118}}{-1+T} - \frac{S c_{129}}{-1+S} \right) \right) g_{1,j,j} g_{2,j,i} + \\ & \left(-S c_{133} - 2(1-S) \left(-\frac{S c_{133}}{1+S} - \frac{S c_{143}}{1+S} \right) \right) g_{2,i,i} g_{2,j,i} + \left(\frac{S c_{133}}{1+S} + \frac{S c_{143}}{1+S} \right) g_{2,i,j} g_{2,j,i} + \end{aligned}$$

$$\begin{aligned}
 & \left(- \left((-S + S^2) c_{133} \right) - \left(S - S^2 \right) c_{143} - S^2 \left(- \frac{(-1 + S) c_{133}}{1 + S} - \frac{(1 - S) c_{143}}{S(1 + S)} \right) - \right. \\
 & \quad \left. 2(-1 + 2S - S^2) \left(- \frac{S c_{133}}{1 + S} - \frac{S c_{143}}{1 + S} \right) \right) g_{2,j,i}^2 + \left(- \frac{(-1 + S) c_{121}}{S} - c_{136} \right) g_{2,j,j} + \\
 & \left(\frac{T c_{118}}{-1 + T} - \frac{S c_{129}}{-1 + S} \right) g_{1,i,i} g_{2,j,j} + \left(\frac{S(-1 + T) c_{129}}{-1 + S} - (-1 + T) \left(- \frac{T c_{118}}{-1 + T} + \frac{S c_{129}}{-1 + S} \right) \right) g_{1,j,i} g_{2,j,j} + \\
 & \left(\frac{S c_{133}}{1 + S} + \frac{S c_{143}}{1 + S} \right) g_{2,i,i} g_{2,j,j} + \left(-S c_{143} - 2(-1 + S) \left(- \frac{S c_{133}}{1 + S} - \frac{S c_{143}}{1 + S} \right) \right) g_{2,j,i} g_{2,j,j} + \\
 & \left(-c_{145} - \left(-1 + \frac{1}{S T} \right) c_{155} \right) g_{3,i,i} + \left(-T c_{148} - (1 - T) \left(\frac{T c_{148}}{-1 + T} - \frac{S T c_{167}}{-1 + S T} \right) \right) g_{1,j,i} g_{3,i,i} + \\
 & \left(- \frac{T c_{148}}{-1 + T} + \frac{S T c_{167}}{-1 + S T} \right) g_{1,j,j} g_{3,i,i} + \left(-S c_{152} - (1 - S) \left(\frac{S c_{152}}{-1 + S} - \frac{S T c_{171}}{-1 + S T} \right) \right) g_{2,j,i} g_{3,i,i} + \\
 & \left(- \frac{S c_{152}}{-1 + S} + \frac{S T c_{171}}{-1 + S T} \right) g_{2,j,j} g_{3,i,i} - \frac{c_{155} g_{3,i,j}}{S T} + \\
 & \left(- \left((-1 + S T) c_{145} \right) - \left(2 - \frac{1}{S T} - S T \right) c_{155} - S T c_{166} - (1 - S T) c_{178} \right) g_{3,j,i} + \\
 & \left(-S T c_{167} - (1 - S T) \left(- \frac{T c_{148}}{-1 + T} + \frac{S T c_{167}}{-1 + S T} \right) \right) g_{1,i,i} g_{3,j,i} + \\
 & \left(\frac{(-1 + S T) (S T - S T^2) c_{148}}{S(-1 + T)} - (-T + S T^2) c_{148} + \frac{S(-1 + T) (T - S T^2) c_{167}}{-1 + S T} - (-S T + S T^2) c_{167} - \right. \\
 & \quad \left. S T^2 \left(- \frac{(-1 + S T) c_{148}}{S T} - \frac{(-1 + T) c_{167}}{T} \right) - (-1 + T + S T - S T^2) \left(\frac{T c_{148}}{-1 + T} - \frac{S T c_{167}}{-1 + S T} \right) - \right. \\
 & \quad \left. (-1 + T + S T - S T^2) \left(- \frac{T c_{148}}{-1 + T} + \frac{S T c_{167}}{-1 + S T} \right) \right) g_{1,j,i} g_{3,j,i} + \\
 & \left(\frac{T(-1 + S T) c_{148}}{-1 + T} - (-1 + S T) \left(\frac{T c_{148}}{-1 + T} - \frac{S T c_{167}}{-1 + S T} \right) \right) g_{1,j,j} g_{3,j,i} + \\
 & \left(-S T c_{171} - (1 - S T) \left(- \frac{S c_{152}}{-1 + S} + \frac{S T c_{171}}{-1 + S T} \right) \right) g_{2,i,i} g_{3,j,i} + \\
 & \left(\frac{(-1 + S T) (S T - S^2 T) c_{152}}{(-1 + S) T} - (-S + S^2 T) c_{152} + \frac{(-1 + S) T (S - S^2 T) c_{171}}{-1 + S T} - (-S T + S^2 T) c_{171} - \right. \\
 & \quad \left. S^2 T \left(- \frac{(-1 + S T) c_{152}}{S T} - \frac{(-1 + S) c_{171}}{S} \right) - (-1 + S + S T - S^2 T) \left(\frac{S c_{152}}{-1 + S} - \frac{S T c_{171}}{-1 + S T} \right) - \right. \\
 & \quad \left. (-1 + S + S T - S^2 T) \left(- \frac{S c_{152}}{-1 + S} + \frac{S T c_{171}}{-1 + S T} \right) \right) g_{2,j,i} g_{3,j,i} + \\
 & \left(\frac{S(-1 + S T) c_{152}}{-1 + S} - (-1 + S T) \left(\frac{S c_{152}}{-1 + S} - \frac{S T c_{171}}{-1 + S T} \right) \right) g_{2,j,j} g_{3,j,i} + \\
 & \left(-S T c_{175} - 2(1 - S T) \left(- \frac{S T c_{175}}{1 + S T} - \frac{S T c_{189}}{1 + S T} \right) \right) g_{3,i,i} g_{3,j,i} + \\
 & \left(\frac{S T c_{175}}{1 + S T} + \frac{S T c_{189}}{1 + S T} \right) g_{3,i,j} g_{3,j,i} + \left(- \left((-S T + S^2 T^2) c_{175} \right) - (S T - S^2 T^2) c_{189} - \right. \\
 & \quad \left. 2(-1 + 2S T - S^2 T^2) \left(- \frac{S T c_{175}}{1 + S T} - \frac{S T c_{189}}{1 + S T} \right) - S^2 T^2 \left(- \frac{(-1 + S T) c_{175}}{1 + S T} - \frac{(1 - S T) c_{189}}{S T(1 + S T)} \right) \right) g_{3,j,i}^2 +
 \end{aligned}$$

$$\begin{aligned}
& \left(-\frac{(-1+ST) c_{155}}{ST} - c_{178} \right) g_{3,j,j} + \left(\frac{T c_{148}}{-1+T} - \frac{ST c_{167}}{-1+ST} \right) g_{1,i,i} g_{3,j,j} + \\
& \left(\frac{S(-1+T) T c_{167}}{-1+ST} - (-1+T) \left(-\frac{T c_{148}}{-1+T} + \frac{ST c_{167}}{-1+ST} \right) \right) g_{1,j,i} g_{3,j,j} + \\
& \left(\frac{S c_{152}}{-1+S} - \frac{ST c_{171}}{-1+ST} \right) g_{2,i,i} g_{3,j,j} + \left(\frac{(-1+S) ST c_{171}}{-1+ST} - (-1+S) \left(-\frac{S c_{152}}{-1+S} + \frac{ST c_{171}}{-1+ST} \right) \right) g_{2,j,i} g_{3,j,j} + \\
& \left(\frac{ST c_{175}}{1+ST} + \frac{ST c_{189}}{1+ST} \right) g_{3,i,i} g_{3,j,j} + \left(-ST c_{189} - 2(-1+ST) \left(-\frac{ST c_{175}}{1+ST} - \frac{ST c_{189}}{1+ST} \right) \right) g_{3,j,i} g_{3,j,j}
\end{aligned}$$